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Model reduction for stochastic systems

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Abstract To solve a stochastic linear evolution equation numerically, finite dimensional approximations are commonly used. If one uses the well-known Galerkin scheme, one might end up with a sequence of ordinary stochastic linear equations of high order. To reduce the high dimension for practical computations we consider balanced truncation as a model order reduction technique. This approach is well-known from deterministic control theory and successfully employed in practice for decades. So, we generalize balanced truncation for controlled linear systems with Levy noise, discuss properties of the reduced order model, provide an error bound, and give some examples.

Keywords Model order reduction · Balanced truncation · Numerical solutions for SPDEs · Jump processes

Mathematics Subject Classification 78M34 · 60J75 · 60H15 · 60H10 · 34K28

1 Introduction

Model order reduction is of major importance, for example, in the field of system and control theory. A commonly used method is balanced truncation, which was first introduced by Moore [19] for linear deterministic systems. A good overview containing all results of this scheme is stated in Antoulas [1]. Balanced truncation also works for

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deterministic bilinear equations (see Benner and Damm [5] and Zhang et al. [26]). Benner and Damm additionally pointed out the relation between balanced truncation for deterministic bilinear control systems and linear stochastic systems with Wiener noise. So, in both cases the reachability and observability Gramians are solutions of generalized Lyapunov equations under certain conditions. We resume working on balanced truncation for stochastic systems and want to generalize the results known for the Wiener case. The main idea is to allow the states to have jumps. Furthermore, we want to ensure that the Gramians we define are still solutions of generalized Lyapunov equations. So, a convenient noise process is given by a square integrable Levy process.

In Sect. 2, we provide the necessary background on semimartingales, square integrable Levy processes, and stochastic calculus in order to render this paper as self-contained as possible. Detailed information regarding general Levy processes one can find in Bertoin [8] and Sato [24], and we refer to Applebaum [2] and Kuo [17] for an extended version of stochastic integration theory. In Sect. 3, we focus on a linear controlled state equation driven by uncorrelated Levy processes, which is asymptotically mean square stable and equipped with an output equation. We introduce the fundamental solution Φ of the state equation and point out the differences compared to fundamental solutions of deterministic systems. Using Φ we introduce reachability and observability Gramians the same way like Benner and Damm [5]. We prove that the observable states and the corresponding energies are characterized by the observabilty Gramian and that the reachability Gramian provides partial information about the degree of reachability of a state. In Sect. 4, we describe the procedure of balanced truncation for the linear system with Levy noise, which is similar to the procedure known from the deterministic case (see Antoulas [1] and Obinata and Anderson [20]). We discuss properties of the resulting reduced order model (ROM). We will show that it is mean square stable, not balanced, that the Hankel singular values (HV) of the ROM are not a subset of the HVs of the original system, and that one can lose complete observability and reachability. Finally, we provide an error bound for balanced truncation of the Levy driven system. This error bound has the same structure as the $\mathscr{H}_2$ error bound of linear deterministic systems. In Sect. 5, we deal with a linear controlled stochastic evolution equation with Levy noise (compare Da Prato and Zabczyk [10], Peszat and Zabczyk [21], Prévôt and Röckner [22]). To solve such a problem numerically, finite dimensional approximations are commonly used. The scheme we state here is the well-known Galerkin method (see Grecksch and Kloeden [12]), leading to a sequence of ordinary stochastic differential equations of the kind we considered in Sect. 4. For a particular case, we apply balanced truncation to that Galerkin solution and compute the error bounds and the exact errors of the approximation.

2 Basics from stochastics

Let all stochastic processes appearing in this section be defined on a filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\right)$[1]. We denote the set of all cadlag[2] square integrable $\mathbb{R}$-valued martingales with respect to $(\mathscr{F}_t)_{t\geq 0}$ by $\mathscr{M}^2(\mathbb{R})$.

[1] $(\mathscr{F}_t)_{t\geq 0}$ shall be right continuous and complete.

[2] Cadlag means that $\mathbb{P}$-almost all paths are right continuous and the left limits exist.

2.1 Semimartingales and Ito's formula

Below, we introduce the class of semimartingales.

Definition 2.1 (i) An $(\mathscr{F}_t)_{t\geq 0}$-adapted cadlag process X with values in $\mathbb{R}$ is called *semimartingale* if it has the representation $X = X_0 + M + A$. Here, X_0 is an $\mathscr{F}_0$-measurable random variable, $M \in \mathscr{M}^2(\mathbb{R})$ and A is a cadlag process of bounded variation.[3]

(ii) An $\mathbb{R}^d$-valued process $\mathbf{X}$ is called *semimartingale* if all components are real-valued semimartingales.

The following is based on Proposition 17.2 in [18].

Proposition 2.2 *Let $M, N \in \mathscr{M}^2(\mathbb{R})$, then there exists a unique predictable[4] process $\langle M, N\rangle$ of bounded variation such that $MN - \langle M, N\rangle$ is a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.*

Next, we consider a decomposition of square integrable martingales (see Theorem 4.18 in [15]).

Theorem 2.3 *A process $M \in \mathscr{M}^2(\mathbb{R})$ has the following representation:*

$$M(t) = M_0 + M^c(t) + M^d(t), \quad t \geq 0,$$

where $M^c(0) = M^d(0) = 0$, M_0 is an $\mathscr{F}_0$-measurable random variable, M^c is a continuous process in $\mathscr{M}^2(\mathbb{R})$ and $M^d \in \mathscr{M}^2(\mathbb{R})$.

We need the quadratic covariation $[Z_1, Z_2]$ of two real-valued semimartingales Z_1 and Z_2, which can be introduced by

$$[Z_1, Z_2]_t := Z_1(t)Z_2(t) - Z_1(0)Z_2(0) - \int_0^t Z_1(s-)dZ_2(s) - \int_0^t Z_2(s-)dZ_1(s) \tag{1}$$

for $t \geq 0$. By the linearity of the integrals in (1) we obtain the property

$$[Z_1, Z_2]_t = \frac{1}{2}\left([Z_1 + Z_2, Z_1 + Z_2]_t - [Z_1, Z_1]_t - [Z_2, Z_2]_t\right), \quad t \geq 0.$$

From Theorem 4.52 in [15], we know that $[Z_1, Z_2]$ is also given by

$$[Z_1, Z_2]_t = \left\langle M_1^c, M_2^c\right\rangle_t + \sum_{0\leq s\leq t} \Delta Z_1(s)\Delta Z_2(s) \tag{2}$$

for $t \geq 0$, where M_1^c and M_2^c are the continuous martingale parts of Z_1 and Z_2. Furthermore, we set $\Delta Z(s) := Z(s) - Z(s-)$ with $Z(s-) := \lim_{t\uparrow s} Z(t)$ for a

[3] This means that $\mathbb{P}$-almost all paths are of bounded variation.

[4] The process $\langle M, N\rangle$ is measurable with respect to $\mathscr{P}$, which we characterize below Definition 2.10.

real-valued semimartingale Z. If we rearrange Eq. (1), we obtain the Ito product formula

$$Z_1(t)Z_2(t) = Z_1(0)Z_2(0) + \int_0^t Z_1(s-)dZ_2(s) + \int_0^t Z_2(s-)dZ_1(s) + [Z_1, Z_2]_t \tag{3}$$

for $t \geq 0$, which we use for the following corollaries:

Corollary 2.4 *Let Y and Z be two $\mathbb{R}^d$-valued semimartingales, then*

$$Y^T(t)Z(t) = Y^T(0)Z(0) + \int_0^t Z^T(s-)dY(s) + \int_0^t Y^T(s-)dZ(s) + \sum_{i=1}^d [Y_i, Z_i]_t$$

for all $t \geq 0$.

Proof We have

$$\begin{aligned} Y^T(t)Z(t) &= \sum_{i=1}^d Y_i(t)Z_i(t) \\ &= \sum_{i=1}^d \left(Y_i(0)Z_i(0) + \int_0^t Z_i(s-)dY_i(s) + \int_0^t Y_i(s-)dZ_i(s) + [Y_i, Z_i]_t \right) \\ &= Y^T(0)Z(0) + \int_0^t Z^T(s-)dY(s) + \int_0^t Y^T(s-)dZ(s) + \sum_{i=1}^d [Y_i, Z_i]_t \end{aligned}$$

by applying the product formula in (3). □

Corollary 2.5 *Let Y be an $\mathbb{R}^d$-valued and Z be an $\mathbb{R}^n$-valued semimartingale, then*

$$Y(t)Z^T(t) = Y(0)Z^T(0) + \int_0^t dY(s)Z^T(s-) + \int_0^t Y(s-)dZ^T(s) + \left([Y_i, Z_j]_t\right)_{\substack{i=1,\ldots,d \\ j=1,\ldots,n}}$$

for all $t \geq 0$.

Proof We consider the stochastic differential of the ij-th component of the matrix-valued process $Y(t)Z^T(t)$, $t \geq 0$, and obtain the following via the product formula in (3):

$$
\begin{aligned}
e_i^T Y(t) Z^T(t) e_j &= e_i^T Y(0) Z^T(0) e_j + \int_0^t Z^T(s-) e_j d(e_i^T Y(s)) \\
&\quad + \int_0^t e_i^T Y(s-) d(Z^T(s) e_j) + [e_i^T Y, Z^T e_j]_t \\
&= e_i^T Y(0) Z^T(0) e_j + e_i^T \int_0^t d(Y(s)) Z^T(s-) e_j \\
&\quad + e_i^T \int_0^t Y(s-) d(Z^T(s)) e_j + [Y_i, Z_j]_t
\end{aligned}
$$

for all $t \geq 0$, $i \in \{1, \ldots, d\}$, and $j \in \{1, \ldots, n\}$, where e_i is the i-th unit vector in $\mathbb{R}^d$ or in $\mathbb{R}^n$, respectively. Hence, in compact form we have

$$
Y(t) Z^T(t) = Y(0) Z^T(0) + \int_0^t dY(s) Z^T(s-) + \int_0^t Y(s-) dZ^T(s) + \big([Y_i, Z_j]_t\big)_{\substack{i=1,\ldots,d \\ j=1,\ldots,n}}
$$

for all $t \geq 0$. □

2.2 Levy processes

Definition 2.6 Let $L = (L(t))_{t \geq 0}$ be a cadlag stochastic process with values in $\mathbb{R}$ having independent and homogeneous increments. If, furthermore, $L(0) = 0$ $\mathbb{P}$-almost surely, and L is continuous in probability, then L is called (real-valued) Levy process.

Below, we focus on Levy processes L being square integrable. The following theorem is proven analogously to Theorem 4.44 in [21].

Theorem 2.7 *We set $\tilde{m} = \mathbb{E}[L(1)]$. For square integrable Levy processes L and $t, s \geq 0$ it holds*

$$
\begin{aligned}
&\mathbb{E}[L(t)] = t\mathbb{E}[L(1)] \text{ and} \\
&\mathrm{Cov}(L(s), L(t)) = \mathbb{E}\left[(L(t) - \tilde{m}t)(L(s) - \tilde{m}s)\right] = \min\{t, s\} \ \mathrm{Var}(L(1)).
\end{aligned}
$$

Proposition 2.8 *Let L be a square integrable Levy process adapted to a filtration $(\mathscr{F}_t)_{t \geq 0}$, such that the increments $L(t+h) - L(t)$ are independent of $\mathscr{F}_t$ $(t, h \geq 0)$, then L is a martingale with respect to $(\mathscr{F}_t)_{t \geq 0}$ if and only if L has mean zero.*

Proof First, we assume that L has mean zero, then the conditional expectation $\mathbb{E}\{L(t) | \mathscr{F}_s\}$ fulfills

$$
\begin{aligned}
\mathbb{E}\{L(t) | \mathscr{F}_s\} &= \mathbb{E}\{L(t) - L(s) + L(s) | \mathscr{F}_s\} = \mathbb{E}\{L(t) - L(s) | \mathscr{F}_s\} + L(s) \\
&= \mathbb{E}[L(t) - L(s)] + L(s) = L(s)
\end{aligned}
$$

for $0 \leq s < t$. If we know that L is a martingale, then it easily follows that it has a constant mean function, since

$$
\mathbb{E}[L(t)] = \mathbb{E}[\mathbb{E}\{L(t) | \mathscr{F}_s\}] = \mathbb{E}[L(s)]
$$

for $0 \leq s < t$. But by Theorem 2.7, we know that the mean function is linear. Thus, $\mathbb{E}[L(t)] = 0$ for all $t \geq 0$. □

We set $M(t) := L(t) - t\mathbb{E}[L(1)]$, $t \geq 0$, where L is square integrable. By Proposition 2.8, M is a square integrable martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$ and a Levy process as well. So, we have the following representation for square integrable Levy processes L:

$$L(t) = M(t) + \mathbb{E}[L(1)]t, \quad t \geq 0.$$

The compensator $\langle M, M\rangle$ of M is deterministic and continuous and given by

$$\langle M, M\rangle_t = \mathbb{E}\left[M^2(t)\right] = \mathbb{E}\left[M^2(1)\right]t,$$

because $M^2(t) - \mathbb{E}\left[M^2(1)\right]t$, $t \geq 0$, is a martingale with respect to $(\mathscr{F}_t)_{t\geq 0}$.

2.3 Stochastic integration

We assume that $M \in \mathscr{M}^2(\mathbb{R})$. The definition of an integral with respect to M is similar to that with respect to a Wiener process W. This makes things comfortable. A definition for an integral based on W can for example be found in Applebaum [2], Arnold [3] and Kloeden and Platen [16]. Furthermore, Applebaum [2] gives a definition of an integral with respect to the so-called "martingale-valued measures", which is a generalization of the integral introduced here. We take the definition of the integral with respect to M from Chap. 6.5 in the book of Kuo [17].

Fist of all, we characterize the class of simple processes.

Definition 2.9 A process $\Psi = (\Psi(t))_{t\in[0,T]}$ is called simple if it has the following representation:

$$\Psi(s) = \sum_{i=0}^{m} \chi_{(t_i,t_{i+1}]}(s)\Psi_i, \quad s \in [0, T], \tag{4}$$

for $0 = t_0 < t_1 < \cdots < t_{m+1} = T$. Here, the random variables Ψ_i are $\mathscr{F}_{t_i}$-measurable and bounded, $i \in \{0, 1, \ldots, m\}$.

For simple processes Ψ, we define

$$I_T^M(\Psi) := \int_0^T \Psi(s)dM(s) := \sum_{i=0}^{m} \Psi_i\left(M(t_{i+1}) - M(t_i)\right)$$

and for $0 \leq t_0 \leq t \leq T$, we set

$$I_{t_0,t}^M(\Psi) := \int_{t_0}^t \Psi(s)dM(s) := \int_0^T \chi_{[t_0,t]}(s)\Psi(s)dM(s).$$

Definition 2.10 Let $(F(t))_{t\in[0,T]}$ be adapted to the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ with left continuous trajectories. We define $\mathscr{P}_T$ as the smallest sub σ-algebra of $\mathscr{B}([0,T])\otimes\mathscr{F}$ with respect to which all mappings $F:[0,T]\times\Omega\to\mathbb{R}$ are measurable. We call $\mathscr{P}_T$ *predictable* σ-algebra.

Remark $\mathscr{P}_T$ is generated as follows:

$$\mathscr{P}_T=\sigma\left(\{(s,t]\times A\colon 0\le s\le t\le T, A\in\mathscr{F}_s\}\cup\{\{0\}\times A\colon A\in\mathscr{F}_0\}\right). \quad (5)$$

In Definition 2.10, we can replace the time interval $[0,T]$ by $\mathbb{R}_+$. Then the predictable σ-algebra is denoted by $\mathscr{P}$. $\mathscr{P}_T$- or $\mathscr{P}$-measurable processes we call predictable.

We want to extend the set of all integrable processes with respect to M. Therefore, we introduce $\mathscr{L}_T^2$ as the space of all predictable mappings Ψ on $[0,T]\times\Omega$ with $\|\Psi\|_T<\infty$, where

$$\|\Psi\|_T^2:=\mathbb{E}\int_0^T|\Psi(s)|^2\,d\langle M,M\rangle_s \quad (6)$$

and $\langle M,M\rangle$ is the compensator of M introduced in Proposition 2.2.

By Chap. 6.5 in Kuo [17], we can choose a sequence $(\Psi_n)_{n\in\mathbb{N}}\subset\mathscr{L}_T^2$ of simple processes, such that

$$\|\Psi_n-\Psi\|_T\to 0$$

for $\Psi\in\mathscr{L}_T^2$ and $n\to\infty$. Hence, we obtain that $\left(I_T^M(\Psi_n)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega,\mathscr{F},\mathbb{P})$. Therefore, we can define

$$\int_0^T\Psi(s)dM(s):=L^2-\lim_{n\to\infty}\int_0^T\Psi_n(s)dM(s)$$

and for $0\le t_0\le t\le T$ we set

$$\int_{t_0}^t\Psi(s)dM(s):=L^2-\lim_{n\to\infty}\int_{t_0}^t\Psi_n(s)dM(s).$$

Here, "$L^2-\lim_{n\to\infty}$" denotes the limit in $L^2(\Omega,\mathscr{F},\mathbb{P})$.

By Theorem 6.5.8 in Kuo [17], the integral with respect to M has the following properties:

Theorem 2.11 *If* $\Psi\in\mathscr{L}_T^2$ *for* $T>0$, *then*

(i) *the integral with respect to* M *has mean zero:*

$$\mathbb{E}\left[\int_0^T\Psi(s)dM(s)\right]=0,$$

(ii) *the second moment of* $I_T^M(\Psi)$ *is given by*

$$\mathbb{E}\left|\int_0^T \Psi(s)dM(s)\right|^2 = \mathbb{E}\int_0^T |\Psi(s)|^2 \, d\,\langle M, M\rangle_s\,,$$

(iii) *and the process*

$$\left(\int_0^t \Psi(s)dM(s)\right)_{t\in[0,T]}$$

is a martingale with respect to $(\mathscr{F}_t)_{t\in[0,T]}$.

2.4 Levy type integrals

Below, we want to determine the mean of the quadratic covariation of the following Levy type integrals:

$$\tilde{Z}_1(t) = \tilde{Z}_1(0) + \int_0^t A_1(s)ds + \sum_{i=1}^q \int_0^t B_1^i(s)dM^i(s), \quad t \geq 0,$$
$$\tilde{Z}_2(t) = \tilde{Z}_2(0) + \int_0^t A_2(s)ds + \sum_{i=1}^q \int_0^t B_2^i(s)dM^i(s), \quad t \geq 0,$$

where the processes M^i $(i = 1, \ldots, q)$ are uncorrelated scalar square integrable Levy processes with mean zero. In addition, the processes B_1^i, B_2^i are integrable with respect to M^i $(i = 1, \ldots, q)$, which by Sect. 2.3 means that they are predictable with

$$\mathbb{E}\int_0^t \left|B^i(s)\right|^2 ds < \infty, \quad t \geq 0,$$

considering (6) with $\langle M, M\rangle_t = \mathbb{E}\left[M^2(1)\right] t$. Furthermore, A_1, A_2 are $\mathbb{P}$-almost surely Lebesgue integrable and $(\mathscr{F}_t)_{t\geq 0}$-adapted.

We set $b_1(t) := \sum_{i=1}^q \int_0^t B_1^i(s)dM^i(s)$ and $b_2(t) := \sum_{i=1}^q \int_0^t B_2^i(s)dM^i(s)$ and obtain

$$\left[\tilde{Z}_1, \tilde{Z}_2\right]_t = [b_1, b_2]_t$$

for $t \geq 0$ considering Eq. (2), because $\tilde{Z}_i$ has the same jumps and the same martingale part as b_i $(i = 1, 2)$. We know that

$$[b_1, b_2]_t = \frac{1}{2}\left([b_1 + b_2, b_1 + b_2]_t - [b_1, b_1]_t - [b_2, b_2]_t\right) \tag{7}$$

for $t \geq 0$. Using the definition in (1) yields

$$[b_1, b_1]_t = (b_1(t))^2 - 2\int_0^t b_1(s-)db_1(s)$$
$$= (b_1(t))^2 - 2\sum_{i=1}^q \int_0^t b_1(s-)B_1^i(s)dM^i(s).$$

Thus,

$$\mathbb{E}\,[b_1, b_1]_t = \mathbb{E}\left[(b_1(t))^2\right].$$

Since M^i and M^j are uncorrelated processes for $i \neq j$, we get

$$\mathbb{E}\left[(b_1(t))^2\right] = \sum_{i=1}^q \mathbb{E}\left[\left(\int_0^t B_1^i(s)dM^i(s)\right)^2\right] = \sum_{i=1}^q \int_0^t \mathbb{E}\left[\left(B_1^i(s)\right)^2\right] ds \cdot c_i$$

by applying Theorem 2.11 (ii), where $c_i := \mathbb{E}\left[\left(M^i(1)\right)^2\right]$. Hence,

$$\mathbb{E}\,[b_1, b_1]_t = \sum_{i=1}^q \int_0^t \mathbb{E}\left[\left(B_1^i(s)\right)^2\right] ds \cdot c_i.$$

Analogously, we can show that

$$\mathbb{E}\,[b_2, b_2]_t = \sum_{i=1}^q \int_0^t \mathbb{E}\left[\left(B_2^i(s)\right)^2\right] ds \cdot c_i \quad \text{and}$$
$$\mathbb{E}\,[b_1 + b_2, b_1 + b_2]_t = \sum_{i=1}^q \int_0^t \mathbb{E}\left[\left(B_1^i + B_2^i(s)\right)^2\right] ds \cdot c_i$$

hold for $t \geq 0$. Considering Eq. (7), we obtain

$$\mathbb{E}\left[\tilde{Z}_1, \tilde{Z}_2\right]_t = \mathbb{E}\,[b_1, b_2]_t = \sum_{i=1}^q \int_0^t \mathbb{E}\left[B_1^i B_2^i(s)\right] ds \cdot c_i. \tag{8}$$

At the end of this section, we refer to Sect. 4.4.3 in Applebaum [2]. There, one can find some remarks regarding the quadratic covariation of the Levy type integrals defined in that book.

3 Linear control with levy noise

Before describing balanced truncation for the stochastic case, we define observability and reachability. We introduce observability and reachability Gramians for our Levy driven system like Benner and Damm [5] do (Sect. 2.2). We additionally show that the sets of observable and reachable states are characterized by these Gramians. This is analogous to deterministic systems, where observability and reachability concepts are described in Sects. 4.2.1 and 4.2.2 in Antoulas [1]. This section extends [7] by providing more details and by considering a more general framework.

3.1 Reachability concept

Let $M_1, \ldots, M_q$ be real-valued uncorrelated and square integrable Levy processes with mean zero defined on a filtered probability space $\left(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P}\right)$.[5] In addition, we assume M_k $(k = 1, \ldots, q)$ to be $(\mathscr{F}_t)_{t\geq 0}$-adapted and the increments $M_k(t+h) - M_k(t)$ to be independent of $\mathscr{F}_t$ for $t, h \geq 0$. We consider the following equations:

$$\begin{aligned} dX(t) &= [AX(t) + Bu(t)]dt + \sum_{k=1}^{q} \Psi^k X(t-)dM_k(t), \quad t \geq 0, \\ X(0) &= x_0 \in \mathbb{R}^n, \end{aligned} \tag{9}$$

where $A, \Psi^k \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. With L^2_T we denote the space of all adapted stochastic processes v with values in $\mathbb{R}^m$, which are square integrable with respect to $\mathbb{P} \otimes dt$. The norm in L^2_T we call *energy norm*. It is given by

$$\|v\|^2_{L^2_T} := \mathbb{E} \int_0^T v^T(t)v(t)dt = \mathbb{E} \int_0^T \|v(t)\|_2^2 \, dt,$$

where we define the processes v_1 and v_2 to be equal in L^2_T if they coincide almost surely with respect to $\mathbb{P}\otimes dt$. For the case $T = \infty$, we denote the space by L^2. Further, we assume controls $u \in L^2_T$ for every $T > 0$. We start with the definition of a solution of (9).

Definition 3.1 An $\mathbb{R}^n$-valued and $(\mathscr{F}_t)_{t\geq 0}$-adapted cadlag process $(X(t))_{t\geq 0}$ is called solution of (9) if

$$X(t) = x_0 + \int_0^t [AX(s) + Bu(s)]ds + \sum_{k=1}^{q} \int_0^t \Psi^k X(s-)dM_k(s) \tag{10}$$

$\mathbb{P}$-almost surely holds for all $t \geq 0$.

Below, the solution of (9) at time $t \geq 0$ with initial condition $x_0 \in \mathbb{R}^n$ and given control u is always denoted by $X(t, x_0, u)$. For the solution of (9) in the uncontrolled case ($u \equiv 0$), we briefly write $Y_{x_0} := X(t, x_0, 0)$. Y_{x_0} is called homogeneous solution.

[5] We assume that $(\mathscr{F}_t)_{t\geq 0}$ is right continuous and that $\mathscr{F}_0$ contains all $\mathbb{P}$ null sets.

Furthermore, by $\|\cdot\|_2$ we denote the Euclidean norm. We assume the homogeneous solution to be asymptotically mean square stable, which means that

$$\mathbb{E}\left\|Y_{y_0}(t)\right\|_2^2 \to 0$$

for $t \to \infty$ and $y_0 \in \mathbb{R}^n$. This concept of stability is also used in Benner and Damm [5] and is necessary for defining (infinite) Gramians, which are introduced later.

Proposition 3.2 *Let Y_{y_0} be the solution of (9) in the uncontrolled case with any initial value $y_0 \in \mathbb{R}^n$, then $\mathbb{E}\left[Y_{y_0}(t)Y_{y_0}^T(t)\right]$ is the solution of the matrix integral equation*

$$\begin{aligned}\mathbb{Y}(t) = y_0 y_0^T &+ \int_0^t \mathbb{Y}(s)ds\ A^T + A\int_0^t \mathbb{Y}(s)ds \\ &+ \sum_{k=1}^q \Psi^k \int_0^t \mathbb{Y}(s)ds\ \left(\Psi^k\right)^T\ \mathbb{E}\left[M_k(1)^2\right]\end{aligned} \tag{11}$$

for $t \geq 0$.

Proof We determine the stochastic differential of the matrix-valued process $Y_{y_0}Y_{y_0}^T$ via using the Ito formula in Corollary 2.5. This yields

$$\begin{aligned}Y_{y_0}(t)Y_{y_0}^T(t) = y_0 y_0^T &+ \int_0^t Y_{y_0}(s-)dY_{y_0}^T(s) + \int_0^t dY_{y_0}(s)Y_{y_0}^T(s-) \\ &+ \left([e_i^T Y_{y_0}, Y_{y_0}^T e_j]_t\right)_{i,j=1,\ldots,n},\end{aligned}$$

where e_i is the i-th unit vector. We obtain

$$\begin{aligned}\int_0^t Y_{y_0}(s-)dY_{y_0}^T(s) &= \int_0^t Y_{y_0}(s-)Y_{y_0}^T(s)A^T ds \\ &\quad+ \sum_{k=1}^q \int_0^t Y_{y_0}(s-)Y_{y_0}^T(s-)(\Psi^k)^T dM_k(s) \text{ and} \\ \int_0^t dY_{y_0}(s)Y_{y_0}^T(s-) &= \int_0^t AY_{y_0}(s)Y_{y_0}^T(s-)ds \\ &\quad+ \sum_{k=1}^q \int_0^t \Psi^k Y_{y_0}(s-)Y_{y_0}^T(s-)dM_k(s)\end{aligned}$$

by inserting the stochastic differential of Y_{y_0}. Thus, by taking the expectation, we obtain

$$\mathbb{E}\left[Y_{y_0}(t)Y_{y_0}^T(t)\right] = y_0 y_0^T + \int_0^t \mathbb{E}\left[Y_{y_0}(s-)Y_{y_0}^T(s)\right] A^T ds$$
$$+ \int_0^t A\mathbb{E}\left[Y_{y_0}(s)Y_{y_0}^T(s-)\right] ds + \left(\mathbb{E}[e_i^T Y_{y_0}, Y_{y_0}^T e_j]_t\right)_{i,j=1,\dots,n}$$

applying Theorem 2.11 (i). Considering Eq. (8), we have

$$\mathbb{E}[e_i^T Y_{y_0}, Y_{y_0}^T e_j]_t = e_i^T \sum_{k=1}^q \int_0^t \mathbb{E}\left[\Psi^k Y_{y_0}(s)Y_{y_0}^T(s)\left(\Psi^k\right)^T\right] ds \cdot c_k e_j,$$

where $c_k := \mathbb{E}\left[M_k(1)^2\right]$. In addition, we use the property that a cadlag process has at most countably many jumps on a finite time interval (see Theorem 2.7.1 in Applebaum [2]), such that we can replace the left limit by the function value itself. Thus,

$$\mathbb{E}\left[Y_{y_0}(t)Y_{y_0}^T(t)\right] = y_0 y_0^T + \int_0^t \mathbb{E}\left[Y_{y_0}(s)Y_{y_0}^T(s)\right] ds\ A^T + A\int_0^t \mathbb{E}\left[Y_{y_0}(s)Y_{y_0}^T(s)\right] ds$$
$$+ \sum_{k=1}^q \Psi^k \int_0^t \mathbb{E}\left[Y_{y_0}(s)Y_{y_0}^T(s)\right] ds\ \left(\Psi^k\right)^T \cdot c_k. \tag{12}$$

□

We introduce an additional concept of stability for the homogeneous system ($u \equiv 0$) corresponding to Eq. (9). We call Y_{y_0} *exponentially mean square stable* if there exist $c, \beta > 0$ such that

$$\mathbb{E}\left\|Y_{y_0}(t)\right\|_2^2 \leq \|y_0\|_2^2\, c\, \mathrm{e}^{-\beta t}$$

for $t \geq 0$. This stability turns out to be equivalent to asymptotic mean square stability, which is stated in the next theorem.

Theorem 3.3 *The following are equivalent:*

(i) *The uncontrolled Eq. (9) is asymptotically mean square stable.*
(ii) *The uncontrolled Eq. (9) is exponentially mean square stable.*
(iii) *The eigenvalues of* $\left(I_n \otimes A + A \otimes I_n + \sum_{k=1}^q \Psi^k \otimes \Psi^k \cdot \mathbb{E}\left[M_k(1)^2\right]\right)$ *have negative real parts.*

Proof Due to the similarity of the proofs we refer to Theorem 1.5.3 in Damm [11], where these results are proven for the Wiener case. □

As in the deterministic case, there exists a fundamental solution, which we define by

$$\Phi(t) := \left[Y_{e_1}(t), Y_{e_2}(t), \dots, Y_{e_n}(t)\right]$$

for $t \geq 0$, where e_i is the i-th unit vector ($i = 1, \ldots, n$). Thus, Φ fulfills the following integral equation:

$$\Phi(t) = I_n + \int_0^t A\Phi(s)ds + \sum_{k=1}^{q} \int_0^t \Psi^k \Phi(s-)dM_k(s).$$

The columns of Φ represent a minimal generating set such that we have $Y_{y_0}(t) = \Phi(t)y_0$. With $B = [b_1, b_2, \ldots, b_m]$ one can see that

$$\Phi(t)B = [\Phi(t)b_1, \Phi(t)b_2, \ldots, \Phi(t)b_m] = \left[Y_{b_1}(t), Y_{b_2}(t) \ldots, Y_{b_m}(t)\right].$$

Hence, we have

$$\Phi(t)BB^T\Phi^T(t) = Y_{b_1}(t)Y_{b_1}^T(t) + Y_{b_2}(t)Y_{b_2}^T(t) + \ldots + Y_{b_m}(t)Y_{b_m}^T(t),$$

such that

$$\begin{aligned} \mathbb{E}\left[\Phi(t)BB^T\Phi^T(t)\right] = BB^T &+ \int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds\ A^T \\ &+ A\int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds \\ &+ \sum_{k=1}^{q} \Psi^k \int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds\ (\Psi^k)^T\ \mathbb{E}\left[M_k(1)^2\right] \end{aligned} \tag{13}$$

holds for every $t \geq 0$. Due to the assumption that the homogeneous solution Y_{y_0} is asymptotically mean square stable for an arbitrary initial value y_0, yielding $\mathbb{E}\left[Y_{y_0}^T(t)Y_{y_0}(t)\right] \to 0$ for $t \to \infty$, we obtain

$$\begin{aligned} 0 = BB^T &+ \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds\ A^T + A\int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds \\ &+ \sum_{k=1}^{q} \Psi^k \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds\ (\Psi^k)^T\ \mathbb{E}\left[M_k(1)^2\right] \end{aligned}$$

by taking the limit $t \to \infty$ in Eq. (13). Therefore, we can conclude that $P := \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds$, which exists by the asymptotic mean square stability assumption, is the solution of a generalized Lyapunov equation

$$AP + PA^T + \sum_{k=1}^{q} \Psi^k P \left(\Psi^k\right)^T\ \mathbb{E}\left[M_k(1)^2\right] = -BB^T.$$

P is the reachability Gramian of system (9), where this definition of the Gramian is also used in Benner and Damm [5] for stochastic systems driven by Wiener noise. Note that in this case $\mathbb{E}\left[M_k(1)^2\right] = 1$.

Remark The solution of the matrix equation

$$0 = BB^T + AP + PA^T + \sum_{k=1}^{q} \Psi^k P (\Psi^k)^T \cdot \mathbb{E}\left[M_k(1)^2\right] \tag{14}$$

is unique if and only if the solution of

$$-\operatorname{vec}(BB^T) = \left(I_n \otimes A + A \otimes I_n + \sum_{k=1}^{q} \Psi^k \otimes \Psi^k \cdot \mathbb{E}\left[M_k(1)^2\right]\right) \operatorname{vec}(P)$$

is unique. By the assumption of mean square asymptotic stability the eigenvalues of the matrix $I \otimes A + A \otimes I + \sum_{k=1}^{q} \Psi^k \otimes \Psi^k \cdot \mathbb{E}\left[M_k(1)^2\right]$ are non zero, hence the matrix Eq. (14) is uniquely solvable.

More general, we consider stochastic processes $(\Phi(t,\tau))_{t\geq\tau}$ with starting time $\tau \geq 0$ and initial condition $\Phi(\tau,\tau) = I_n$ satisfying

$$\Phi(t,\tau) = I_n + \int_\tau^t A\Phi(s,\tau)ds + \sum_{k=1}^{q} \int_\tau^t \Psi^k \Phi(s-,\tau) dM_k(s) \tag{15}$$

for $t \geq \tau \geq 0$. Of course, we have $\Phi(t,0) = \Phi(t)$. Analogous to Eq. (13), we can show that

$$\begin{aligned}\mathbb{E}\left[\Phi(t,\tau)BB^T\Phi^T(t,\tau)\right] = BB^T &+ \int_\tau^t \mathbb{E}\left[\Phi(s,\tau)BB^T\Phi^T(s,\tau)\right] ds\ A^T \\ &+ A \int_\tau^t \mathbb{E}\left[\Phi(s,\tau)BB^T\Phi^T(s,\tau)\right] ds \\ &+ \sum_{k=1}^{q} \Psi^k \int_\tau^t \mathbb{E}\left[\Phi(s,\tau)BB^T\Phi^T(s,\tau)\right] ds\ (\Psi^k)^T \\ &\times \mathbb{E}\left[M_k(1)^2\right].\end{aligned} \tag{16}$$

This yields that $\mathbb{E}\left[\Phi(t,\tau)BB^T\Phi^T(t,\tau)\right]$ is the solution of the differential equation

$$\dot{\mathbb{Y}}(t) = A\mathbb{Y}(t) + \mathbb{Y}(t)A^T + \sum_{k=1}^{q} \Psi^k \mathbb{Y}(t)(\Psi^k)^T\ \mathbb{E}\left[M_k(1)^2\right] \tag{17}$$

for $t \geq \tau$ with initial condition $\mathbb{Y}(\tau) = BB^T$.

Remark For $t \geq \tau \geq 0$, we have $\Phi(t, \tau) = \Phi(t)\Phi^{-1}(\tau)$, since $\Phi(t)\Phi^{-1}(\tau)$ fulfills Eq. (15).

Compared to the deterministic case ($\Psi^k = 0$) we do not have the semigroup property for the fundamental solution. So, it is not true that $\Phi(t, \tau) = \Phi(t - \tau)$ $\mathbb{P}$-almost surely holds, because the trajectories of the noise processes on $[0, t - \tau]$ and $[\tau, t]$ are different in general. We can however conclude that $\mathbb{E}\left[\Phi(t, \tau)BB^T\Phi^T(t, \tau)\right] = \mathbb{E}\left[\Phi(t - \tau)BB^T\Phi^T(t - \tau)\right]$, since both terms solve Eq. (17) as can be seen employing (13).

Now, we derive the solution representation of the system (9) via using the stochastic variation of constants method. For the Wiener case, this result is stated in Theorem 1.4.1 in Damm [11].

Proposition 3.4 *$(\Phi(t)z(t))_{t\geq 0}$ is a solution of Eq. (9), where z is given by*

$$dz(t) = \Phi^{-1}(t)Bu(t)dt, \quad z(0) = x_0.$$

Proof We want to determine the stochastic differential of $\Phi(t)z(t)$, $t \geq 0$, where its i-th component is given by $e_i^T\Phi(t)z(t)$. Applying the Ito product formula from Corollary 2.4 yields

$$e_i^T\Phi(t)z(t) = e_i^T x_0 + \int_0^t e_i^T\Phi(s-)d(z(s)) + \int_0^t z^T(s)d(\Phi^T(s)e_i).$$

Above, the quadratic covariation terms are zero, since z is a continuous semimartingale with a martingale part of zero (see Eq. (2)). Applying that $s \mapsto \Phi(\omega, s)$ and $s \mapsto \Phi(\omega, s-)$ coincide ds-almost everywhere for $\mathbb{P}$-almost all fixed $\omega \in \Omega$, we have

$$\begin{aligned} e_i^T\Phi(t)z(t) &= e_i^T x_0 + \int_0^t e_i^T\Phi(s)\Phi^{-1}(s)Bu(s)ds + \int_0^t z^T(s)\Phi^T(s)A^T e_i ds \\ &\quad + \sum_{k=1}^q \int_0^t z^T(s)\Phi^T(s-)(\Psi^k)^T e_i dM_k(s) \\ &= e_i^T x_0 + e_i^T \int_0^t Bu(s)ds + e_i^T \int_0^t A\Phi(s)z(s)ds \\ &\quad + e_i^T \sum_{k=1}^q \int_0^t \Psi^k\Phi(s-)z(s)dM_k(s). \end{aligned}$$

This yields

$$\Phi(t)z(t) = x_0 + \int_0^t A\Phi(s)z(s)ds + \sum_{k=1}^q \int_0^t \Psi^k\Phi(s-)z(s)dM_k(s) + \int_0^t Bu(s)ds.$$

□

Below, we set $P_t := \int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds$ and call P_t finite *reachability Gramian* at time $t \geq 0$. Furthermore, we define the so-called finite deterministic Gramian $P_{D,t} := \int_0^t \mathrm{e}^{As} BB^T \mathrm{e}^{A^T s} ds$. P_t and $P_{D,t}$, $t \geq 0$, coincide in the case $\Psi^k = 0$. By $X(T, 0, u)$ we denote the solution of the inhomogeneous system (9) at time T with initial condition zero for a given input u. From Proposition 3.4, we already know that

$$X(T,0,u) = \int_0^T \Phi(T)\Phi^{-1}(t)Bu(t)dt = \int_0^T \Phi(T,t)Bu(t)dt.$$

Now, we have the goal to steer the average state of the system (9) from zero to any given $x \in \mathbb{R}^n$ via the control u with minimal energy. First of all we need the following definition, which is motivated by the remarks above Theorem 2.3 in [5].

Definition 3.5 A state $x \in \mathbb{R}^n$ is called *reachable on average (from zero)* if there is a time $T > 0$ and a control function $u \in L^2_T$, such that we have

$$\mathbb{E}\left[X(T,0,u)\right] = x.$$

We say that the stochastic system is *completely reachable* if every average vector $x \in \mathbb{R}^n$ is reachable. Next, we characterize the set of all reachable average states. First of all, we need the following proposition, where we define $P := \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds$ in analogy to the deterministic case.

Proposition 3.6 *The finite reachability Gramians P_t, $t > 0$, have the same image as the infinite reachability Gramian P, i.e.,*

$$\operatorname{im} P_t = \operatorname{im} P$$

for all $t > 0$.

Proof Since P and P_t are positive semidefinite and symmetric by definition it is sufficient to show that their kernels are equal. First, we assume $v \in \ker P$. Thus,

$$0 \leq v^T P_t v \leq v^T P v = 0,$$

since $t \mapsto v^T P_t v$ is increasing such that $v \in \ker P_t$ follows. On the other hand, if $v \in \ker P_t$ we have

$$0 = v^T P_t v = \int_0^t v^T \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] v ds.$$

Hence, we can conclude that $v^T\mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] v = 0$ for almost all $s \in [0, t]$. Additionally, we know that $t \mapsto \mathbb{E}\left[\Phi(t)BB^T\Phi^T(t)\right]$ is the solution of the linear matrix differential equation

$$\dot{\mathbb{Y}}(t) = A\mathbb{Y}(t) + \mathbb{Y}(t)A^T + \sum_{k=1}^{q} \Psi^k \mathbb{Y}(t)(\Psi^k)^T \; \mathbb{E}\left[M_k(1)^2\right]$$

with initial condition $\mathbb{Y}(0) = BB^T$ for $t \geq 0$. The vectorized form $\mathrm{vec}(\mathbb{Y})$ satisfies

$$\mathrm{vec}(\dot{\mathbb{Y}}(t)) = \left(I_n \otimes A + A \otimes I_n + \sum_{k=1}^{q} \Psi^k \otimes \Psi^k \cdot \mathbb{E}\left[M_k(1)^2\right]\right)\mathrm{vec}(\mathbb{Y}(t)),$$
$$\mathrm{vec}(\mathbb{Y}(0)) = \mathrm{vec}(BB^T).$$

Thus, the entries of $\mathbb{E}\left[\Phi(t)BB^T\Phi^T(t)\right]$ are analytic functions. This implies that the function $f(t) := v^T\mathbb{E}\left[\Phi(t)BB^T\Phi^T(t)\right]v$ is analytic, such that $f \equiv 0$ on $[0, \infty)$. Thus,

$$0 = \int_0^\infty v^T\mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]vds = v^T Pv.$$

□

The next proposition shows that the reachable average states are characterized by the deterministic Gramian $P_D := \int_0^\infty \mathrm{e}^{As}\, BB^T\, \mathrm{e}^{A^T s}\, ds$, which exists due to the asymptotic stability of the matrix A, which is a necessary condition for asymptotic mean square stability of system (9).

Proposition 3.7 *An average state $x \in \mathbb{R}^n$ is reachable (from zero) if and only if $x \in \mathrm{im}\, P_D$, where $P_D := \int_0^\infty \mathrm{e}^{As}\, BB^T\, \mathrm{e}^{A^T s}\, ds$.*

Proof Provided $x \in \mathrm{im}\, P_D$, we will show that this average state can be reached with the following input function:

$$[0, T] \ni t \mapsto u(t) = B^T\, \mathrm{e}^{A^T(T-t)}\, P_{D,T}^{\#}x, \tag{18}$$

where $P_{D,T}^{\#}$ denotes the Moore-Penrose pseudoinverse of $P_{D,T}$. Thus,

$$\mathbb{E}\left[X(T, 0, u)\right] = \mathbb{E}\left[\int_0^T \Phi(T, t)BB^T\, \mathrm{e}^{A^T(T-t)}\, P_{D,T}^{\#}xdt\right]$$

by inserting the function u. Applying the expectation to both sides of Eq. (15) yields

$$\mathbb{E}\left[\Phi(t, \tau)\right] = \mathrm{e}^{A(t-\tau)}.$$

Using this fact, we obtain

$$\mathbb{E}\left[X(T, 0, u)\right] = \int_0^T \mathrm{e}^{A(T-t)}\, BB^T\, \mathrm{e}^{A^T(T-t)}\, P_{D,T}^{\#}xdt.$$

We substitute $s = T - t$ and since $x \in \operatorname{im} P_{D,T}$ by Proposition 3.6, we get

$$\mathbb{E}\left[X(T,0,u)\right] = \int_0^T \mathrm{e}^{As}\, BB^T\, \mathrm{e}^{A^T s}\, ds\, P_{D,T}^{\#} x = P_{D,T} P_{D,T}^{\#} x = x.$$

The energy of the input function $u(t) = B^T\, \mathrm{e}^{A^T(T-t)}\, P_{D,T}^{\#} x$ is

$$\|u\|_{L_T^2}^2 = x^T P_{D,T}^{\#} x < \infty.$$

On the other hand, if $x \in \mathbb{R}^n$ is reachable, then there exists an input function u and a time $t > 0$ such that

$$x = \mathbb{E}\left[X(t,0,u)\right] = \mathbb{E}\left[\int_0^t \Phi(t,s) Bu(s) ds\right] = \int_0^t \mathrm{e}^{A(t-s)}\, B\mathbb{E}\left[u(s)\right] ds$$

by definition. The last equation we get by applying the expectation to both sides of Eq. (9). We assume that $v \in \ker P_D$. Hence,

$$|\langle x, v\rangle_2| = \left|\int_0^t \left\langle \mathrm{e}^{A(t-s)}\, B\mathbb{E}\left[u(s)\right], v\right\rangle_2 ds\right| = \left|\int_0^t \left\langle \mathbb{E}\left[u(s)\right], B^T\, \mathrm{e}^{A^T(t-s)}\, v\right\rangle_2 ds\right|.$$

Employing the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\langle x, v\rangle_2| &\leq \int_0^t \left\|\mathbb{E}\left[u(s)\right]\right\|_2 \left\|B^T\, \mathrm{e}^{A^T(t-s)}\, v\right\|_2 ds \\ &\leq \int_0^t \left(\mathbb{E}\left\|u(s)\right\|_2^2\right)^{\frac{1}{2}} \left\|B^T\, \mathrm{e}^{A^T(t-s)}\, v\right\|_2 ds. \end{aligned}$$

By the Hölder inequality, we have

$$\begin{aligned} |\langle x, v\rangle_2| &\leq \|u\|_{L_t^2} \left(\int_0^t \left\|B^T\, \mathrm{e}^{A^T(t-s)}\, v\right\|_2^2 ds\right)^{\frac{1}{2}} \\ &= \|u\|_{L_t^2} \left(v^T \int_0^t \mathrm{e}^{A(t-s)}\, BB^T\, \mathrm{e}^{A^T(t-s)}\, ds\; v\right)^{\frac{1}{2}} = \|u\|_{L_t^2} \left(v^T P_{D,t} v\right)^{\frac{1}{2}}. \end{aligned}$$

Since $t \mapsto v^T P_{D,t} v$ is increasing, we obtain

$$|\langle x, v\rangle_2| \leq \|u\|_{L_t^2} \left(v^T P_D v\right)^{\frac{1}{2}} = 0.$$

Thus, $\langle x, v\rangle_2 = 0$, such that we can conclude that $x \in \operatorname{im} P_D$ due to $\operatorname{im} P_D = (\ker P_D)^{\perp}$. □

Below, we point out the relation between the reachable set and the Gramian $P := \int_0^\infty \mathbb{E}\left[\Phi(s) BB^T \Phi^T(s)\right] ds$.

Proposition 3.8 *If an average state $x \in \mathbb{R}^n$ is reachable (from zero), then $x \in \operatorname{im} P$. Consequently,* $\operatorname{im} P_D \subseteq \operatorname{im} P$ *by Proposition 3.7.*

Proof By definition, there exists an input function u and a time $t > 0$ such that

$$x = \mathbb{E}\left[X(t, 0, u)\right] = \mathbb{E}\left[\int_0^t \Phi(t, s)Bu(s)ds\right]$$

for reachable $x \in \mathbb{R}^n$. We assume that $v \in \ker P$. So, we have

$$|\langle x, v\rangle_2| = \left|\mathbb{E}\left[\int_0^t \langle \Phi(t, s)Bu(s), v\rangle_2\, ds\right]\right| = \left|\mathbb{E}\left[\int_0^t \left\langle u(s), B^T\Phi^T(t, s)v\right\rangle_2 ds\right]\right|.$$

Employing the Cauchy-Schwarz inequality, we obtain

$$|\langle x, v\rangle_2| \leq \mathbb{E}\left[\int_0^t \|u(s)\|_2 \left\|B^T\Phi^T(t, s)v\right\|_2 ds\right].$$

By the Hölder inequality, we have

$$\begin{aligned}|\langle x, v\rangle_2| &\leq \|u\|_{L_t^2}\left(\mathbb{E}\left[\int_0^t \left\|B^T\Phi^T(t, s)v\right\|_2^2 ds\right]\right)^{\frac{1}{2}}\\ &= \|u\|_{L_t^2}\left(v^T\mathbb{E}\left[\int_0^t \Phi(t, s)BB^T\Phi^T(t, s)ds\right]v\right)^{\frac{1}{2}}.\end{aligned}$$

With the remarks above Proposition 3.4, we obtain

$$\mathbb{E}\left[\Phi(t-s)BB^T\Phi^T(t-s)\right] = \mathbb{E}\left[\Phi(t, s)BB^T\Phi^T(t, s)\right],$$

such that

$$|\langle x, v\rangle_2| \leq \|u\|_{L_t^2}\left(v^T P_t v\right)^{\frac{1}{2}}.$$

Since $t \mapsto v^T P_t v$ is increasing, it follows

$$|\langle x, v\rangle_2| \leq \|u\|_{L_t^2}\left(v^T P v\right)^{\frac{1}{2}} = 0.$$

Thus, $\langle x, v\rangle_2 = 0$, such that we can conclude that $x \in \operatorname{im} P$ due to $\operatorname{im} P = (\ker P)^{\perp}$. □

Now, we state the minimal energy to steer the system to a desired average state.

Proposition 3.9 *Let $x \in \mathbb{R}^n$ be reachable, then the input function given by (18) is the one with the minimal energy to reach x at any time $T > 0$. This minimal energy is given by $x^T P_{D,T}^{\#} x$.*

Proof We use the following representation from the proof of Proposition 3.7:

$$\mathbb{E}\left[X(T,0,u)\right] = \int_0^T \mathrm{e}^{A(T-t)}\, B\mathbb{E}\left[u(t)\right] dt.$$

Let $u(t)$ be like in (18) and $\tilde{u}(t)$, $t \in [0, T]$, an additional function for which we can reach the average state x at time T, then

$$\int_0^T \mathrm{e}^{A(T-t)}\, B\left(\mathbb{E}\left[\tilde{u}(t)\right] - u(t)\right) dt = 0$$

such that

$$\mathbb{E}\left[\int_0^T u(t)^T \left(\tilde{u}(t) - u(t)\right) dt\right] = \int_0^T u(t)^T \left(\mathbb{E}\left[\tilde{u}(t)\right] - u(t)\right) dt = 0$$

follows. Hence, we have

$$\|\tilde{u}\|^2_{L^2_T} = \|u + (\tilde{u} - u)\|^2_{L^2_T} = \|u\|^2_{L^2_T} + \|\tilde{u} - u\|^2_{L^2_T} \geq \|u\|^2_{L^2_T}.$$

From the proof of Proposition 3.7, we know that the energy of u is given by $x^T P^{\#}_{D,T} x$. □

The following result shows that the finite reachability Gramian P_T provides information about the degree of reachability of an average state as well.

Proposition 3.10 *Let $x \in \mathbb{R}^n$ be reachable, then*

$$x^T P^{\#}_T x \leq x^T P^{\#}_{D,T} x$$

for every time $T > 0$.

Proof Since x is reachable, $x \in \operatorname{im} P_T$ by Proposition 3.6 and Proposition 3.8. Hence, we can write $x = P_T P^{\#}_T x$, where $P^{\#}_T$ denotes the Moore-Penrose pseudoinverse of P_T. From its definition, the finite reachability Gramian is represented by $P_T = \mathbb{E}\left[\int_0^T \Phi(T-t) B B^T \Phi^T(T-t) dt\right]$ and since

$$\mathbb{E}\left[\Phi(T-t) B B^T \Phi^T(T-t)\right] = \mathbb{E}\left[\Phi(T,t) B B^T \Phi^T(T,t)\right],$$

we have

$$x = \mathbb{E}\left[\int_0^T \Phi(T,t) B B^T \Phi^T(T,t) P^{\#}_T x dt\right].$$

Now, we choose the control $u(t) = B^T \mathrm{e}^{A^T(T-t)} P^{\#}_{D,T} x$, $t \in [0, T]$, of minimal energy to reach x, then

$$\mathbb{E}\left[\int_0^T \Phi(T,t)B\left(B^T\Phi^T(T,t)P_T^{\#}x-u(t)\right)dt\right]=0.$$

Setting $v(t)=B^T\Phi^T(T,t)P_T^{\#}x$ for $t\in[0,T]$ yields

$$\mathbb{E}\left[\int_0^T v^T(t)\left(v(t)-u(t)\right)dt\right]=0.$$

We obtain

$$\begin{aligned}x^TP_{D,T}^{\#}x&=\|u\|_{L_T^2}^2=\|v+(u-v)\|_{L_T^2}^2=\|v\|_{L_T^2}^2+\|u-v\|_{L_T^2}^2\geq\|v\|_{L_T^2}^2\\&=x^TP_T^{\#}x.\end{aligned}$$

□

Consequently, the expression $x^TP_T^{\#}x$ yields a lower bound for the energy to reach x and is the L_T^2-norm squared of the function $v(t)=B^T\Phi^T(T,t)P_T^{\#}x$, $t\in[0,T]$. With v we would also be able to steer the system to x in case it would be a valid control. Unfortunately, unavailable future information enters in v which means that it is not $(\mathscr{F}_t)_{t\in[0,T]}$- adapted. So, one can interpret the energy $x^T\left(P_{T,D}^{\#}-P_T^{\#}\right)x$ as the benefit of knowing the future until time T.

By Proposition 3.9, the minimal energy that is needed to steer the system to x is given by $\inf_{T>0}x^TP_{D,T}^{\#}x$. By definition of $P_{D,T}$ we know that it is increasing in time such that the pseudoinverse $P_{D,T}^{\#}$ is decreasing. Hence, it is clear that the minimal energy is given by $x^TP_D^{\#}x$, where $P_D^{\#}$ is the pseudoinverse of the deterministic Gramian P_D. Using the result in Proposition 3.10 provides a lower bound for the minimal energy to reach x:

$$x^TP^{\#}x\leq x^TP_D^{\#}x \tag{19}$$

with $P^{\#}$ being the pseudoinverse of the reachability Gramian P. Using inequality (19), we get only partial information about the degree of reachability of an average state x from $P^{\#}$. So, it remains an open question whether an alternative reachability concept would be more suitable to motivate the Gramian P.

Similar results are obtained by Benner and Damm [5] in Theorem 2.3 for stochastic differential equations driven by Wiener processes. For the deterministic case we refer to Sect. 4.3.1 in Antoulas [1].

3.2 Observability concept

Below, we introduce the concept of observability for the output equation

$$\mathscr{Y}(t)=CX(t) \tag{20}$$

corresponding to the stochastic linear system (9), where $C \in \mathbb{R}^{p\times n}$. Therefore, we need the following proposition.

Proposition 3.11 *Let $\hat{Q}$ be a symmetric positive semidefinite matrix and $Y_a := X(\cdot, a, 0)$, $Y_b := X(\cdot, b, 0)$ the homogeneous solutions to (9) with initial conditions $a, b \in \mathbb{R}^n$, then*

$$\mathbb{E}\left[Y_a(t)^T \hat{Q} Y_b(t)\right] = a^T \hat{Q} b + \mathbb{E}\left[\int_0^t Y_a^T(s)\hat{Q} A Y_b(s)ds\right] + \mathbb{E}\left[\int_0^t Y_a^T(s) A^T \hat{Q} Y_b(s)ds\right] + \mathbb{E}\left[\int_0^t Y_a(s)^T \sum_{k=1}^q (\Psi^k)^T \hat{Q} \Psi^k \mathbb{E}\left[M_k(1)^2\right] Y_b(s)ds\right]. \quad (21)$$

Proof By applying the Ito product formula from Corollary 2.4, we have

$$Y_a^T(t)\hat{Q}Y_b(t) = a^T\hat{Q}b + \int_0^t Y_a^T(s-)d(\hat{Q}Y_b(s)) + \int_0^t Y_b^T(s-)\hat{Q}d(Y_a(s)) + \sum_{i=1}^n [e_i^T Y_a(t), e_i^T \hat{Q} Y_b(t)]_t,$$

where e_i is the i-th unit vector ($i = 1, \ldots, n$). We get

$$\int_0^t Y_a^T(s-)d(\hat{Q}Y_b(s)) = \int_0^t Y_a^T(s-)\hat{Q}AY_b(s)ds + \sum_{k=1}^q \int_0^t Y_a^T(s-)\hat{Q}\Psi^k Y_b(s-)dM_k(s)$$

and

$$\int_0^t Y_b^T(s-)\hat{Q}d(Y_a(s)) = \int_0^t Y_b(s-)^T\hat{Q}AY_a(s)ds + \sum_{k=1}^q \int_0^t Y_b(s-)^T\hat{Q}\Psi^k Y_a(s-)dM_k(s).$$

By Eq. (8), the mean of the quadratic covariations is given by

$$\mathbb{E}[e_i^T Y_a(t), e_i^T \hat{Q} Y_b(t)]_t = \sum_{k=1}^q \mathbb{E}\int_0^t e_i^T \Psi^k Y_a(s) e_i^T \hat{Q}\Psi^k Y_b(s)ds\, \mathbb{E}\left[M_k(1)^2\right].$$

With Theorem 2.11 (i), we obtain

$$\begin{aligned}\mathbb{E}\left[Y_a(t)^T \hat{Q} Y_b(t)\right] &= a^T \hat{Q} b + \mathbb{E}\left[\int_0^t Y_a^T(s)\hat{Q} A Y_b(s)ds\right]\\ &\quad + \mathbb{E}\left[\int_0^t Y_a^T(s) A^T \hat{Q} Y_b(s)ds\right]\\ &\quad + \sum_{k=1}^{q} \mathbb{E}\left[\int_0^t Y_a(s)^T (\Psi^k)^T \hat{Q} \Psi^k Y_b(s)ds\right] \mathbb{E}\left[M_k(1)^2\right]\end{aligned}$$

using that the trajectories of Y_a and Y_b only have jumps on Lebesgue zero sets. □

If we set $a = e_i$ and $b = e_j$ in Proposition 3.11, we obtain

$$\begin{aligned}\mathbb{E}\left[e_i^T \Phi(t)^T \hat{Q}\Phi(t)e_j\right] &= e_i^T \hat{Q} e_j + \mathbb{E}\left[\int_0^t e_i^T \Phi^T(s)\hat{Q} A\Phi(s)e_j ds\right]\\ &\quad + \mathbb{E}\left[\int_0^t e_i^T \Phi^T(s) A^T \hat{Q}\Phi(s)e_j ds\right]\\ &\quad + \mathbb{E}\left[\int_0^t e_i^T \Phi(s)^T \sum_{k=1}^{q} (\Psi^k)^T \hat{Q}\Psi^k \mathbb{E}\left[M_k(1)^2\right]\Phi(s)e_j ds\right].\end{aligned}$$

This yields

$$\begin{aligned}\mathbb{E}\left[\Phi(t)^T \hat{Q}\Phi(t)\right] &= \hat{Q} + \mathbb{E}\left[\int_0^t \Phi^T(s)\hat{Q} A\Phi(s)ds\right] + \mathbb{E}\left[\int_0^t \Phi^T(s) A^T \hat{Q}\Phi(s)ds\right]\\ &\quad + \mathbb{E}\left[\int_0^t \Phi(s)^T \sum_{k=1}^{q} (\Psi^k)^T \hat{Q}\Psi^k \mathbb{E}\left[M_k(1)^2\right]\Phi(s)ds\right].\end{aligned}$$

Let Q be the solution of the generalized Lyapunov equation

$$A^T Q + QA + \sum_{k=1}^{q} (\Psi^k)^T Q\Psi^k \, \mathbb{E}\left[M_k(1)^2\right] = -C^T C. \tag{22}$$

Then,

$$\mathbb{E}\left[\Phi(t)^T Q\Phi(t)\right] = Q - \mathbb{E}\left[\int_0^t \Phi^T(s)C^T C\Phi(s)ds\right]$$

and by taking the limit $t \to \infty$, we have

$$Q = \mathbb{E}\left[\int_0^\infty \Phi^T(s)C^T C\Phi(s)ds\right] \tag{23}$$

due to the asymptotic mean square stability of the homogeneous equation ($u \equiv 0$), which provides the existence of the integral in Eq. (23) as well.

Remark The matrix Eq. (22) is uniquely solvable, since

$$L := \left(A^T \otimes I_n + I_n \otimes A^T + \sum_{k=1}^{q} (\Psi^k)^T \otimes (\Psi^k)^T \cdot \mathbb{E}\left[M_k(1)^2 \right] \right)$$

has non zero eigenvalues and hence the solution of $L \cdot \mathrm{vec}(Q) = -\mathrm{vec}(C^T C)$ is unique.

Next, we assume that the system (9) is uncontrolled, that means $u \equiv 0$. By using our knowledge concerning the homogeneous system, $X(t, x_0, 0)$ is given by $\Phi(t)x_0$, where here, $x_0 \in \mathbb{R}^n$ denotes the initial value of the system. So, we obtain $\mathscr{Y}(t) = C\Phi(t)x_0$.

We observe $\mathscr{Y}$ on a time interval $[0, \infty)$. The problem is to find x_0 from the observations we have. The energy produced by the initial value x_0 is

$$\|\mathscr{Y}\|_{L^2}^2 := \mathbb{E} \int_0^\infty \mathscr{Y}^T(t)\mathscr{Y}(t)dt = x_0^T \mathbb{E} \int_0^\infty \Phi^T(t)C^T C\Phi(t)dt \; x_0 = x_0^T Q x_0, \tag{24}$$

where we set $Q := \mathbb{E} \int_0^\infty \Phi^T(s)C^T C\Phi(s)ds$. As in Benner and Damm [5], Q takes the part of the observability Gramian of the stochastic system with output Eq. (20). We call a state x_0 unobservable if it is in the kernel of Q. Otherwise it is said to be observable. We say that a system is completely observable if the kernel of Q is trivial.

4 Balanced truncation for stochastic systems

For obtaining a reduced order model for a deterministic LTI system, balanced truncation is a method of major importance. For the procedure of balanced truncation in the deterministic case, see Antoulas [1], Benner et al. [4] and Obinata, Anderson [20]. In this section, we want to generalize this method for stochastic linear systems, which are influenced by Levy noise.

4.1 Procedure

We assume $A, \Psi^k \in \mathbb{R}^{n\times n}$ ($k = 1, \dots, q$), $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, and consider the following stochastic system:

$$\begin{aligned} dX(t) &= [AX(t) + Bu(t)]dt + \sum_{k=1}^{q} \Psi^k X(t-)dM_k(t), \quad t \geq 0, \; X(0) = x_0, \\ \mathscr{Y}(t) &= CX(t), \end{aligned} \tag{25}$$

where the noise processes M_k ($k = 1, \ldots, q$) are uncorrelated real-valued and square integrable Levy processes with mean zero. We assume the homogeneous solution Y_{y_0}, which fulfills

$$dY(t) = AY(t)dt + \sum_{k=1}^{q} \Psi^k Y(t-)dM_k(t), \quad t \geq 0, \; Y(0) = y_0,$$

to be mean square asymptotically stable. In addition, we require that the system (25) is completely reachable and observable, which is equivalent to P_D and Q being positive definite. Hence, the reachability Gramian P is also positive definite using Proposition 3.8.

Let $T \in \mathbb{R}^{n\times n}$ be a regular matrix. If we transform the states using

$$\hat{X}(t) = TX(t),$$

we obtain the following system:

$$\begin{aligned} d\hat{X}(t) &= [\tilde{A}\hat{X}(t) + \tilde{B}u(t)]dt + \sum_{k=1}^{q} \tilde{\Psi}^k \hat{X}(t-)dM_k(t), \quad \hat{X}(0) = Tx_0, \\ \mathscr{Y}(t) &= \tilde{C}\hat{X}(t), \quad t \geq 0, \end{aligned} \tag{26}$$

where $\tilde{A} = TAT^{-1}$, $\tilde{\Psi}^k = T\Psi^k T^{-1}$, $\tilde{B} = TB$ and $\tilde{C} = CT^{-1}$. For an arbitrary fixed input, the transformed system (26) has always the same output as the system (25).

The reachability Gramian $P := \int_0^\infty \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right] ds$ of system (25) fulfills

$$-BB^T = AP + PA^T + \sum_{k=1}^{q} \Psi^k P(\Psi^k)^T \cdot c_k,$$

where $c_k = \mathbb{E}\left[M_k(1)^2\right]$. By multiplying T from the left and T^T from the right hand side, we obtain

$$\begin{aligned} -\tilde{B}\tilde{B}^T &= TAPT^T + TPA^TT^T + \sum_{k=1}^{q} T\Psi^k P(\Psi^k)^T T^T \cdot c_k \\ &= \tilde{A}TPT^T + TPT^T\tilde{A}^T + \sum_{k=1}^{q} \tilde{\Psi}^k TPT^T(\tilde{\Psi}^k)^T \cdot c_k. \end{aligned}$$

Hence, the reachability Gramian of the transformed system (26) is given by $\tilde{P} = TPT^T$. For the observability Gramian of the transformed system it holds $\tilde{Q} = T^{-T}QT^{-1}$, where $Q := \int_0^\infty \mathbb{E}\left[\Phi^T(s)C^TC\Phi(s)\right] ds$ is the observability Gramian of the original system. Hence,

$$-\tilde{C}^T\tilde{C} = \tilde{A}^T\tilde{Q} + \tilde{Q}\tilde{A} + \sum_{k=1}^{q}(\tilde{\Psi}^k)^T\tilde{Q}\tilde{\Psi}^k \cdot c_k.$$

In addition, it is easy to verify that the generalized Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_n > 0$ of (25), which are the square roots of the eigenvalues of PQ, are equal to those of (26).

Like in the deterministic case (see [1] and [20]), we choose T such that $\tilde{Q}$ and $\tilde{P}$ are equal and diagonal. A system with equal and diagonal Gramians is called balanced. The corresponding balancing T is given by

$$T = \Sigma^{\frac{1}{2}}K^TU^{-1} \text{ and } T^{-1} = UK\Sigma^{-\frac{1}{2}}, \tag{27}$$

where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$, U comes from the Cholesky decomposition of $P = UU^T$ and K is an orthogonal matrix corresponding to the eigenvalue decomposition (singular value decomposition (SVD) respectively) of $U^TQU = K\Sigma^2K^T$. So, we obtain

$$\tilde{Q} = \tilde{P} = \Sigma.$$

Our aim is to truncate the average states that are difficult to observe and difficult to reach, which are those producing least observation energy and causing the most energy to reach, respectively. By equation (24), we can say that the states which are difficult to observe are contained in the space spanned by the eigenvectors corresponding to the small eigenvalues of Q. Using (19), an average state x is particularly difficult to reach if the expression $x^TP^{-1}x$ is large. Those states are contained in the space spanned by the eigenvectors corresponding to the small eigenvalues of P (or to the large eigenvalues of P^{-1}, respectively). The eigenspaces that correspond to the small eigenvalues of P contain all difficult-to-reach states if we would know the future completely, see the remarks below Proposition 3.10. In a balanced system, the dominant reachable and observable states are the same.

We consider the following partitions:

$$T = \begin{bmatrix} W^T \\ T_2^T \end{bmatrix}, \; T^{-1} = \begin{bmatrix} V & T_1 \end{bmatrix} \text{ and } \hat{X} = \begin{pmatrix} \tilde{X} \\ X_1 \end{pmatrix},$$

where $W^T \in \mathbb{R}^{r\times n}$, $V \in \mathbb{R}^{n\times r}$ and $\tilde{X}$ takes values in $\mathbb{R}^r$ $(r < n)$. Hence, we have

$$\begin{aligned}\begin{pmatrix} d\tilde{X}(t) \\ dX_1(t) \end{pmatrix} &= \left(\begin{bmatrix} W^TAV & W^TAT_1 \\ T_2^TAV & T_2^TAT_1 \end{bmatrix}\begin{pmatrix} \tilde{X}(t) \\ X_1(t) \end{pmatrix} + \begin{bmatrix} W^TB \\ T_2^TB \end{bmatrix}u(t)\right)dt \\ &\quad + \sum_{k=1}^{q}\begin{bmatrix} W^T\Psi^kV & W^T\Psi^kT_1 \\ T_2^T\Psi^kV & T_2^T\Psi^kT_1 \end{bmatrix}\begin{pmatrix} \tilde{X}(t-) \\ X_1(t-) \end{pmatrix}dM_k(t)\end{aligned} \tag{28}$$

and

$$\mathscr{Y}(t) = \begin{bmatrix} CV & CT_1 \end{bmatrix} \begin{pmatrix} \tilde{X}(t) \\ X_1(t) \end{pmatrix}.$$

By truncating the system and neglecting the X_1 terms, the approximating reduced order model is given by

$$\begin{aligned} d\tilde{X}(t) &= [W^T AV\tilde{X}(t) + W^T Bu(t)]dt + \sum_{k=1}^{q} W^T \Psi^k V\tilde{X}(t-)dM_k(t), \\ \hat{\mathscr{Y}}(t) &= CV\tilde{X}(t). \end{aligned} \tag{29}$$

We will show now that the homogeneous solution $\tilde{Y}_{y_0}$ of the reduced system (29) fulfilling

$$dY(t) = W^T AVY(t)dt + \sum_{k=1}^{q} W^T \Psi^k VY(t-)dM_k(t), \quad Y(0) = y_0, \tag{30}$$

is mean square stable which means that it is bounded in mean square.

Proposition 4.1 *Let $\tilde{Y}_{y_0}$ be the homogeneous solution satisfying Eq. (30) with initial condition $y_0 \in \mathbb{R}^r$, then*

$$\mathbb{E}\left\|\tilde{Y}_{y_0}(t)\right\|_2^2 \leq \frac{\sigma_1}{\sigma_r}\|y_0\|_2^2, \quad t \geq 0. \tag{31}$$

Proof In Eq. (28), we block-wise set

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} := \begin{bmatrix} W^T AV & W^T AT_1 \\ T_2^T AV & T_2^T AT_1 \end{bmatrix} \text{ and } \begin{bmatrix} \tilde{\Psi}_{11}^k & \tilde{\Psi}_{12}^k \\ \tilde{\Psi}_{21}^k & \tilde{\Psi}_{22}^k \end{bmatrix} := \begin{bmatrix} W^T \Psi^k V & W^T \Psi^k T_1 \\ T_2^T \Psi^k V & T_2^T \Psi^k T_1 \end{bmatrix}.$$

In the corresponding output equation, we block-wise define

$$\begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} := \begin{bmatrix} CV & CT_1 \end{bmatrix}.$$

We know

$$\begin{aligned} &\begin{bmatrix} \tilde{A}_{11}^T & \tilde{A}_{21}^T \\ \tilde{A}_{12}^T & \tilde{A}_{22}^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \\ &\quad + \sum_{k=1}^{q} \begin{bmatrix} (\tilde{\Psi}_{11}^k)^T & (\tilde{\Psi}_{21}^k)^T \\ (\tilde{\Psi}_{12}^k)^T & (\tilde{\Psi}_{22}^k)^T \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} \tilde{\Psi}_{11}^k & \tilde{\Psi}_{12}^k \\ \tilde{\Psi}_{21}^k & \tilde{\Psi}_{22}^k \end{bmatrix} \cdot c_k = - \begin{bmatrix} \tilde{C}_1^T \tilde{C}_1 & \tilde{C}_1^T \tilde{C}_2 \\ \tilde{C}_2^T \tilde{C}_1 & \tilde{C}_2^T \tilde{C}_2 \end{bmatrix}, \end{aligned}$$

where $\Sigma_1 = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$, $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n)$ and $c_k = \mathbb{E}[M_k(1)^2]$. Considering the left upper block, we obtain

$$\tilde{A}_{11}^T \Sigma_1 + \Sigma_1 \tilde{A}_{11} + \sum_{k=1}^{q} (\tilde{\Psi}_{11}^k)^T \Sigma_1 \tilde{\Psi}_{11}^k \cdot c_k = -\left(\sum_{k=1}^{q} (\tilde{\Psi}_{21}^k)^T \Sigma_2 \tilde{\Psi}_{21}^k \cdot c_k + \tilde{C}_1^T \tilde{C}_1 \right) =: L.$$

From Eq. (21), we can conclude that

$$\begin{aligned}\mathbb{E}\left[\tilde{Y}_{y_0}(t)^T \Sigma_1 \tilde{Y}_{y_0}(t)\right] = y_0^T \Sigma_1 y_0 &+ \mathbb{E}\left[\int_0^t \tilde{Y}_{y_0}^T(s) \Sigma_1 \tilde{A}_{11} \tilde{Y}_{y_0}(s) ds\right] \\ &+ \mathbb{E}\left[\int_0^t \tilde{Y}_{y_0}^T(s) \tilde{A}_{11}^T \Sigma_1 \tilde{Y}_{y_0}(s) ds\right] \\ &+ \mathbb{E}\left[\int_0^t \tilde{Y}_{y_0}(s)^T \sum_{k=1}^{q} (\tilde{\Psi}_{11}^k)^T \Sigma_1 \tilde{\Psi}_{11}^k c_k \tilde{Y}_{y_0}(s) ds\right].\end{aligned}$$

Thus,

$$\mathbb{E}\left[\tilde{Y}_{y_0}(t)^T \Sigma_1 \tilde{Y}_{y_0}(t)\right] = y_0^T \Sigma_1 y_0 + \mathbb{E}\left[\int_0^t \tilde{Y}_{y_0}^T(s) L \tilde{Y}_{y_0}(s) ds\right] \leq y_0^T \Sigma_1 y_0.$$

Using $\sigma_r v^T v \leq v^T \Sigma_1 v \leq \sigma_1 v^T v$, we obtain

$$\sigma_r \mathbb{E}\left[\tilde{Y}_{y_0}(t)^T \tilde{Y}_{y_0}(t)\right] \leq \sigma_1 y_0^T y_0.$$

□

Remark (i) One persisting problem is to find an explicit structure of the Gramians of the reduced order model. As we will see in an example below, in contrast to the deterministic case the reduced order model is not balanced, that means the Gramians are neither diagonal nor equal. In addition, the Hankel singular values are different from those of the original system.

(ii) From Proposition 4.1, we know that $\tilde{Y}_{y_0}$ is bounded. This is equivalent to

$$I_r \otimes A_{11} + A_{11} \otimes I_r + \sum_{k=1}^{q} \Psi_{11}^k \otimes \Psi_{11}^k \cdot \mathbb{E}\left[M_k(1)^2\right]$$

has just eigenvalues with non positive real parts, where $A_{11} := W^T A V$ and $\Psi_{11}^k := W^T \Psi^k V$. To prove the asymptotic mean square stability of the uncontrolled reduced order model it remains to show that the Kronecker matrix above has no eigenvalues on the imaginary axis. This was shown in [6]. Hence, we know that balanced truncation preserves asymptotic mean square stability, also in the stochastic case.

Example 4.2 We consider the case, where $q = 1$ and the noise process is a Wiener process W. So, the system we focus on is

$$\begin{aligned} dX(t) &= [AX(t) + Bu(t)]dt + \Psi X(t)dW(t), \\ \mathscr{Y}(t) &= CX(t). \end{aligned} \tag{32}$$

The following matrices (up to the digits shown) provide a balanced and asymptotically mean square stable system:

$$A = \begin{pmatrix} -4.4353 & 3.9992 & -0.3287 \\ 2.9337 & -11.0285 & -0.4319 \\ -0.0591 & -0.1303 & -11.5362 \end{pmatrix}, \quad B = \begin{pmatrix} -3.4648 & -1.9391 & -3.6790 \\ 5.7925 & 4.1379 & 2.3036 \\ -0.3258 & 1.1359 & 2.8972 \end{pmatrix},$$
$$\Psi = \begin{pmatrix} -1.4886 & 2.8510 & -0.2429 \\ 0.4720 & 0.5803 & 3.1152 \\ -1.6123 & -0.8082 & -0.0917 \end{pmatrix}, \quad C = \begin{pmatrix} -3.0588 & 0.4275 & 0.2630 \\ -4.8686 & 1.2886 & 1.0769 \\ -4.3349 & 0.6747 & -0.1734 \end{pmatrix}.$$

The Gramians are given by

$$P = Q = \Sigma = \begin{pmatrix} 8.4788 & 0 & 0 \\ 0 & 3.3232 & 0 \\ 0 & 0 & 1.4726 \end{pmatrix}.$$

The reduced order model ($r = 2$) is asymptotically mean square stable and has the following Gramians:

$$P_R = \begin{pmatrix} 7.7470 & -0.3562 \\ -0.3562 & 2.5496 \end{pmatrix} \text{ and } Q_R = \begin{pmatrix} 7.7495 & -0.2074 \\ -0.2074 & 2.8980 \end{pmatrix}.$$

The Hankel singular values of the reduced order model are 7.6633 and 2.7001.

At the end of this section, we provide a short example that shows that the reduced order model need not be completely observable and reachable even if the original system is completely observable and reachable:

Example 4.3 We consider the Eqs. (32) with the matrices

$$(A, B, \Psi, C) = \left(\begin{pmatrix} -0.25 & 1 \\ 1 & -9 \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{7} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{7} \end{pmatrix} \right)$$

and obtain a balanced and asymptotically mean square stable system being completely reachable and observable. The Hankel singular values are 2 and 1. Truncating yields a system with coefficients
$(A_{11}, B_1, \Psi_{11}, C_1) = (-0.25, 0, 0, 0)$ having Gramians $P_R = Q_R = 0$.

4.2 Error bound for balanced truncation

Let $\left(A, \Psi^k, B, C\right)$ ($k = 1, \ldots, q$) be a realization of system (25). Furthermore, we assume the initial condition of the system to be zero. We introduce the following partitions:

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \; T\Psi^k T^{-1} = \begin{bmatrix} \Psi_{11}^k & \Psi_{12}^k \\ \Psi_{21}^k & \Psi_{22}^k \end{bmatrix}, \; TB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \text{ and}$$
$$CT^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{33}$$

where T is the balancing transformation defined in (27) and $\left(A_{11}, \Psi_{11}^k, B_1, C_1\right)$ are the coefficients of the reduced order model. The output of the reduced (truncated) system is given by

$$\hat{\mathscr{Y}}(t) = C_1 \tilde{X}(t) = C_1 \int_0^t \tilde{\Phi}(t, s) B_1 u(s) ds,$$

where $\tilde{\Phi}$ is the fundamental matrix of the truncated system. In addition, we use a result from [6]. Therein it is proven that the homogeneous Eq. ($u \equiv 0$) of the reduced system is still asymptotically mean square stable. This is vital for the error bound we provide below since the existence of the Gramians of the reduced order model is ensured. Moreover, we know

$$\mathscr{Y}(t) = CX(t) = C \int_0^t \Phi(t, s) Bu(s) ds.$$

It is our goal to steer the average state via the control u and to truncate the average states that are difficult to reach for obtaining a reduced order model. Therefore, it is a meaningful criterion to consider the worst case mean error of $\hat{\mathscr{Y}}(t)$ and $\mathscr{Y}(t)$. Below, we give a bound for that kind of error:

$$\begin{aligned} \mathbb{E}\left\|\hat{\mathscr{Y}}(t) - \mathscr{Y}(t)\right\|_2 &= \mathbb{E}\left\| C \int_0^t \Phi(t, s) Bu(s) ds - C_1 \int_0^t \tilde{\Phi}(t, s) B_1 u(s) ds \right\|_2 \\ &\leq \mathbb{E} \int_0^t \left\| \left(C\Phi(t, s) B - C_1 \tilde{\Phi}(t, s) B_1 \right) u(s) \right\|_2 ds \\ &\leq \mathbb{E} \int_0^t \left\| C\Phi(t, s) B - C_1 \tilde{\Phi}(t, s) B_1 \right\|_F \|u(s)\|_2 \, ds, \end{aligned}$$

and by the Cauchy-Schwarz inequality, it holds

$$\begin{aligned} &\mathbb{E}\left\|\hat{\mathscr{Y}}(t) - \mathscr{Y}(t)\right\|_2 \\ &\quad \leq \left(\mathbb{E} \int_0^t \left\| C\Phi(t, s) B - C_1 \tilde{\Phi}(t, s) B_1 \right\|_F^2 ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^t \|u(s)\|_2^2 \, ds \right)^{\frac{1}{2}}. \end{aligned}$$

Now,

$$\begin{aligned} &\mathbb{E} \int_0^t \left\| C\Phi(t, s) B - C_1 \tilde{\Phi}(t, s) B_1 \right\|_F^2 ds \\ &\quad = \mathbb{E} \int_0^t \|C\Phi(t, s) B\|_F^2 + \left\| C_1 \tilde{\Phi}(t, s) B_1 \right\|_F^2 - 2 \left\langle C\Phi(t, s) B, C_1 \tilde{\Phi}(t, s) B_1 \right\rangle_F ds \\ &\quad = \mathbb{E} \int_0^t \operatorname{tr}\left(C\Phi(t, s) BB^T \Phi^T(t, s) C^T \right) ds \end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^t \operatorname{tr}\left(C_1\tilde{\Phi}(t,s)B_1B_1^T\tilde{\Phi}^T(t,s)C_1^T\right)ds\\
&-2\mathbb{E}\int_0^t \operatorname{tr}\left(C\Phi(t,s)BB_1^T\tilde{\Phi}^T(t,s)C_1^T\right)ds\\
=&\operatorname{tr}\left(C\int_0^t \mathbb{E}\left[\Phi(t,s)BB^T\Phi^T(t,s)\right]ds\ C^T\right)\\
&+\operatorname{tr}\left(C_1\int_0^t \mathbb{E}\left[\tilde{\Phi}(t,s)B_1B_1^T\tilde{\Phi}^T(t,s)\right]ds\ C_1^T\right)\\
&-2\ \operatorname{tr}\left(C\int_0^t \mathbb{E}\left[\Phi(t,s)BB_1^T\tilde{\Phi}^T(t,s)\right]ds\ C_1^T\right).
\end{aligned}
\tag{34}
$$

Due to the remarks before Proposition 3.4, we have

$$
\begin{aligned}
\mathbb{E}\left[\Phi(t,s)BB^T\Phi^T(t,s)\right]&=\mathbb{E}\left[\Phi(t-s)BB^T\Phi^T(t-s)\right]\text{ and}\\
\mathbb{E}\left[\tilde{\Phi}(t,s)B_1B_1^T\tilde{\Phi}^T(t,s)\right]&=\mathbb{E}\left[\tilde{\Phi}(t-s)B_1B_1^T\tilde{\Phi}^T(t-s)\right]
\end{aligned}
$$

for $0\leq s\leq t$. Furthermore, we need to analyze the term in (34). For that reason, we need the following proposition:

Proposition 4.4 *The $\mathbb{R}^{n\times r}$-valued function $\mathbb{E}\left[\Phi(t)BB_1^T\tilde{\Phi}^T(t)\right]$, $t\geq 0$, is the solution of the following differential equation:*

$$
\dot{\mathbb{Y}}(t)=\mathbb{Y}(t)A_{11}^T+A\mathbb{Y}(t)+\sum_{k=1}^q \Psi^k\mathbb{Y}(t)(\Psi_{11}^k)^T\ \mathbb{E}\left[M_k(1)^2\right],\quad \mathbb{Y}(0)=BB_1^T.
\tag{35}
$$

Proof With $B=[b_1,\ldots,b_m]$ and $B_1=\left[\tilde{b}_1,\ldots,\tilde{b}_m\right]$, we obtain

$$
\Phi(t)BB_1^T\tilde{\Phi}^T(t)=\Phi(t)b_1\tilde{b}_1^T\tilde{\Phi}^T(t)+\cdots+\Phi(t)b_m\tilde{b}_m^T\tilde{\Phi}^T(t).
\tag{36}
$$

By applying the Ito product formula from Corollary 2.5, we have

$$
\begin{aligned}
\Phi(t)b_l\tilde{b}_l^T\tilde{\Phi}^T(t)=b_l\tilde{b}_l^T+\int_0^t d(\Phi(s)b_l)\tilde{b}_l^T\tilde{\Phi}^T(s-)+\int_0^t \Phi(s-)b_l d(\tilde{b}_l^T\tilde{\Phi}^T(s))\\
+\left(\left[e_i^T\Phi b_l, e_j^T\tilde{\Phi}\tilde{b}_l\right]_t\right)_{\substack{i=1,\ldots,n\\ j=1,\ldots,r}}.
\end{aligned}
$$

From (8), we know that

$$
\mathbb{E}\left[e_i^T\Phi b_l, e_j^T\tilde{\Phi}\tilde{b}_l\right]_t=\sum_{k=1}^q\mathbb{E}\left[\int_0^t e_i^T\Psi^k\Phi(s)b_l\tilde{b}_1^T\tilde{\Phi}^T(s)(\Psi_{11}^k)^Te_j ds\right]\mathbb{E}\left[M_k(1)^2\right].
$$

With Theorem 2.11 (i), we obtain

$$\begin{aligned}\mathbb{E}\left[\Phi(t)b_l\tilde{b}_l^T\tilde{\Phi}^T(t)\right] = b_l\tilde{b}_l^T &+ \mathbb{E}\left[\int_0^t \Phi(s)b_l\tilde{b}_l^T\tilde{\Phi}^T(s)ds\right] A_{11}^T \\ &+ A\mathbb{E}\left[\int_0^t \Phi(s)b_l\tilde{b}_l^T\tilde{\Phi}^T(s)ds\right] \\ &+ \sum_{k=1}^q \Psi^k\mathbb{E}\left[\int_0^t \Phi(s)b_l\tilde{b}_l^T\tilde{\Phi}^T(s)ds\right] (\Psi_{11}^k)^T\mathbb{E}\left[M_k(1)^2\right]\end{aligned}$$

using that the trajectories of Φ and $\tilde{\Phi}$ only have jumps on Lebesgue zero sets. By Eq. (36), we have

$$\begin{aligned}\mathbb{E}\left[\Phi(t)BB_1^T\tilde{\Phi}^T(t)\right] = BB_1^T &+ \mathbb{E}\left[\int_0^t \Phi(s)BB_1^T\tilde{\Phi}^T(s)ds\right] A_{11}^T \\ &+ A\mathbb{E}\left[\int_0^t \Phi(s)BB_1^T\tilde{\Phi}^T(s)ds\right] \\ &+ \sum_{k=1}^q \Psi^k\mathbb{E}\left[\int_0^t \Phi(s)BB_1^T\tilde{\Phi}^T(s)ds\right] (\Psi_{11}^k)^T\mathbb{E}\left[M_k(1)^2\right]\end{aligned} \tag{37}$$

which proves the result. □

By Proposition 4.4, we can conclude that the function $\mathbb{E}\left[\Phi(t-\tau)BB_1^T\tilde{\Phi}^T(t-\tau)\right]$, $t \geq \tau \geq 0$, is the solution of the equation

$$\dot{\mathbb{Y}}(t) = \mathbb{Y}(t)A_{11}^T + A\mathbb{Y}(t) + \sum_{k=1}^q \Psi^k\mathbb{Y}(t)(\Psi_{11}^k)^T\ \mathbb{E}\left[M_k(1)^2\right], \quad \mathbb{Y}(\tau) = BB_1^T, \tag{38}$$

for all $t \geq \tau \geq 0$. Analogous to Proposition 4.4 we can conclude that $\mathbb{E}\left[\Phi(t,\tau)BB_1^T\tilde{\Phi}^T(t,\tau)\right]$ is also a solution of Eq. (38), which yields

$$\mathbb{E}\left[\Phi(t,\tau)BB_1^T\tilde{\Phi}^T(t,\tau)\right] = \mathbb{E}\left[\Phi(t-\tau)BB_1^T\tilde{\Phi}^T(t-\tau)\right] \tag{39}$$

for all $t \geq \tau \geq 0$. Using Eq. (39), we have

$$\begin{aligned}&\mathbb{E}\int_0^t \left\|C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1\right\|_F^2 ds \\ &= \operatorname{tr}\left(C\int_0^t \mathbb{E}\left[\Phi(t-s)BB^T\Phi^T(t-s)\right]ds\ C^T\right)\end{aligned}$$

$$+\operatorname{tr}\left(C_1\int_0^t \mathbb{E}\left[\tilde{\Phi}(t-s)B_1B_1^T\tilde{\Phi}^T(t-s)\right]ds\ C_1^T\right)$$
$$-2\ \operatorname{tr}\left(C\int_0^t \mathbb{E}\left[\Phi(t-s)BB_1^T\tilde{\Phi}^T(t-s)\right]ds\ C_1^T\right).$$

By substitution, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^t \left\|C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1\right\|_F^2 ds\\
&=\operatorname{tr}\left(C\int_0^t \mathbb{E}\left[\Phi(s)BB^T\Phi^T(s)\right]ds\ C^T\right)\\
&\quad+\operatorname{tr}\left(C_1\int_0^t \mathbb{E}\left[\tilde{\Phi}(s)B_1B_1^T\tilde{\Phi}^T(s)\right]ds\ C_1^T\right)\\
&\quad-2\ \operatorname{tr}\left(C\int_0^t \mathbb{E}\left[\Phi(s)BB_1^T\tilde{\Phi}^T(s)\right]ds\ C_1^T\right)\\
&=\mathbb{E}\int_0^t \left\|C\Phi(s)B - C_1\tilde{\Phi}(s)B_1\right\|_F^2 ds.
\end{aligned}$$

The homogeneous equation of the truncated system is still asymptotically mean square stable due to [6]. Hence, the matrices $P_R = \mathbb{E}\int_0^\infty \tilde{\Phi}(\tau)B_1B_1^T\tilde{\Phi}^T(\tau)d\tau \in \mathbb{R}^{r\times r}$ and $P_M = \mathbb{E}\int_0^\infty \Phi(\tau)BB_1^T\tilde{\Phi}^T(\tau)d\tau \in \mathbb{R}^{n\times r}$ exist. So, it holds

$$\begin{aligned}
\mathbb{E}\left\|\hat{\mathscr{Y}}(t)-\mathscr{Y}(t)\right\|_2 &\leq \left(\mathbb{E}\int_0^\infty \left\|C\Phi(s)B - C_1\tilde{\Phi}(s)B_1\right\|_F^2 ds\right)^{\frac{1}{2}}\left(\mathbb{E}\int_0^t \|u(s)\|_2^2\, ds\right)^{\frac{1}{2}}\\
&=\left(\operatorname{tr}\left(CPC^T\right)+\operatorname{tr}\left(C_1P_RC_1^T\right)-2\ \operatorname{tr}\left(CP_MC_1^T\right)\right)^{\frac{1}{2}}\|u\|_{L_t^2},
\end{aligned}$$

where $P = \mathbb{E}\int_0^\infty \Phi(\tau)BB^T\Phi^T(\tau)d\tau$ is the reachability Gramian of the original system, P_R the reachability Gramian of the approximating system and P_M a matrix that fulfills the following equation:

$$0 = BB_1^T + P_MA_{11}^T + AP_M + \sum_{k=1}^q \Psi^k P_M(\Psi_{11}^k)^T\ \mathbb{E}\left[M_k(1)^2\right], \tag{40}$$

which we get by taking the limit $t\to\infty$ on both sides of Eq. (37). We summarize these results in the following theorem:

Theorem 4.5 *Let* $\left(A, \Psi^k, B, C\right)$ *be a realization of system (25) and* $\left(A_{11}, \Psi_{11}^k, B_1, C_1\right)$ *the coefficients of the reduced order model defined in (33), then*

$$\sup_{t\in[0,T]}\mathbb{E}\left\|\hat{\mathscr{Y}}(t)-\mathscr{Y}(t)\right\|_2 \leq \left(\operatorname{tr}\left(CPC^T\right)+\operatorname{tr}\left(C_1P_RC_1^T\right)-2\ \operatorname{tr}\left(CP_MC_1^T\right)\right)^{\frac{1}{2}}\|u\|_{L_T^2} \tag{41}$$

for every $T > 0$, where $\mathscr{Y}$ and $\hat{\mathscr{Y}}$ are the outputs of the original and the reduced system, respectively. Here, P denotes the reachability Gramian of system (25), P_R denotes the reachability Gramian of the reduced system and P_M satisfies Eq. (40).

Remark If $u \in L^2$ we can replace $\|\cdot\|_{L^2_T}$ by $\|\cdot\|_{L^2}$ and $[0, T]$ by $\mathbb{R}_+$ in inequality (41).

Now, we specify the error bound from (41) in the following proposition.

Proposition 4.6 *If the realization (A, Ψ^k, B, C) is balanced, then*

$$\operatorname{tr}\left(CPC^T + C_1P_RC_1^T - 2CP_MC_1^T\right)$$
$$= \operatorname{tr}(\Sigma_2(B_2B_2^T + 2P_{M,2}A_{21}^T)) + \sum_{k=1}^{q} \operatorname{tr}(\Sigma_2(2\Psi_{22}^k P_{M,2}(\Psi_{21}^k)^T$$
$$+ 2\Psi_{21}^k P_{M,1}(\Psi_{21}^k)^T - \Psi_{21}^k P_R(\Psi_{21}^k)^T))c_k,$$

where $P_{M,1}$ are the first r and $P_{M,2}$ the last $n-r$ rows of P_M, $c_k = \mathbb{E}\left[M_k(1)^2\right]$ and $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n)$.

Proof For simplicity of notation, we prove this result just for the case $q = 1$ but of course it is easy to generalize the proof for an arbitrary q. Here, we additionally set $\Psi := \Psi^1$ and $c := c_1$. Then, we have

$$\begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
$$+ \begin{bmatrix} \Psi_{11}^T & \Psi_{21}^T \\ \Psi_{12}^T & \Psi_{22}^T \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix} c = -\begin{bmatrix} C_1^TC_1 & C_1^TC_2 \\ C_2^TC_1 & C_2^TC_2 \end{bmatrix}.$$

Hence,

$$A_{11}^T\Sigma_1 + \Sigma_1A_{11} + \Psi_{11}^T\Sigma_1\Psi_{11}c + \Psi_{21}^T\Sigma_2\Psi_{21}c = -C_1^TC_1, \tag{42}$$
$$A_{22}^T\Sigma_2 + \Sigma_2A_{22} + \Psi_{22}^T\Sigma_2\Psi_{22}c + \Psi_{12}^T\Sigma_1\Psi_{12}c = -C_2^TC_2 \tag{43}$$

and

$$A_{21}^T\Sigma_2 + \Sigma_1A_{12} + \Psi_{11}^T\Sigma_1\Psi_{12}c + \Psi_{21}^T\Sigma_2\Psi_{22}c = -C_1^TC_2. \tag{44}$$

Furthermore,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix}$$
$$+ \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} \Psi_{11}^T & \Psi_{21}^T \\ \Psi_{12}^T & \Psi_{22}^T \end{bmatrix} c = -\begin{bmatrix} B_1B_1^T & B_1B_2^T \\ B_2B_1^T & B_2B_2^T \end{bmatrix},$$

such that one can conclude

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}^T + \Psi_{11}\Sigma_1\Psi_{11}^T c + \Psi_{12}\Sigma_2\Psi_{12}^T c = -B_1 B_1^T \tag{45}$$

and

$$A_{22}\Sigma_2 + \Sigma_2 A_{22}^T + \Psi_{22}\Sigma_2\Psi_{22}^T c + \Psi_{21}\Sigma_1\Psi_{21}^T c = -B_2 B_2^T. \tag{46}$$

From

$$\begin{aligned}&\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} P_{M,1} \\ P_{M,2} \end{bmatrix} + \begin{bmatrix} P_{M,1} \\ P_{M,2} \end{bmatrix} A_{11}^T + \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}\begin{bmatrix} P_{M,1} \\ P_{M,2} \end{bmatrix}\Psi_{11}^T c \\ &= -\begin{bmatrix} B_1 B_1^T \\ B_2 B_1^T \end{bmatrix}\end{aligned}$$

we also know that

$$A_{11}P_{M,1} + A_{12}P_{M,2} + P_{M,1}A_{11}^T + \Psi_{11}P_{M,1}\Psi_{11}^T c + \Psi_{12}P_{M,2}\Psi_{11}^T c = -B_1 B_1^T. \tag{47}$$

We define $\mathscr{E} := \left(\mathrm{tr}\left(C\Sigma C^T\right) + \mathrm{tr}\left(C_1 P_R C_1^T\right) - 2\ \mathrm{tr}\left(C P_M C_1^T\right)\right)^{\frac{1}{2}}$ and obtain

$$\begin{aligned}\mathscr{E}^2 &= \mathrm{tr}\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}\begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix}\right) + \mathrm{tr}\left(C_1 P_R C_1^T\right) \\ &\quad - 2\ \mathrm{tr}\left(\begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} P_{M,1} \\ P_{M,2} \end{bmatrix} C_1^T\right) \\ &= \mathrm{tr}(C_2\Sigma_2 C_2^T + C_1\Sigma_1 C_1^T + C_1 P_R C_1^T - 2C_1 P_{M,1} C_1^T - 2C_2 P_{M,2} C_1^T).\end{aligned}$$

Using Eq. (44) yields

$$\begin{aligned}&\mathrm{tr}(-C_2 P_{M,2} C_1^T) \\ &= \mathrm{tr}(-C_1^T C_2 P_{M,2}) \\ &= \mathrm{tr}\left(A_{21}^T\Sigma_2 P_{M,2} + \ \Sigma_1 A_{12} P_{M,2} + \Psi_{11}^T\Sigma_1\Psi_{12} P_{M,2} c + \Psi_{21}^T\Sigma_2\Psi_{22} P_{M,2} c\right) \\ &= \mathrm{tr}(A_{21}^T\Sigma_2 P_{M,2} + A_{12} P_{M,2}\Sigma_1 + \Psi_{12} P_{M,2}\Psi_{11}^T\Sigma_1 c + \Psi_{21}^T\Sigma_2\Psi_{22} P_{M,2} c).\end{aligned}$$

By Eq. (47), we obtain

$$\begin{aligned}&\mathrm{tr}(-C_2 P_{M,2} C_1^T) \\ &\quad = \mathrm{tr}(A_{21}^T\Sigma_2 P_{M,2} + \Psi_{21}^T\Sigma_2\Psi_{22} P_{M,2} c) \\ &\qquad - \mathrm{tr}((B_1 B_1^T + P_{M,1} A_{11}^T + A_{11} P_{M,1} + \Psi_{11} P_{M,1}\Psi_{11}^T c)\Sigma_1).\end{aligned}$$

Using Eq. (42), we have

$$\begin{aligned}&\operatorname{tr}(P_{M,1}A_{11}^T + A_{11}P_{M,1} + \Psi_{11}P_{M,1}\Psi_{11}^T c)\Sigma_1)\\&\quad = \operatorname{tr}(A_{11}^T\Sigma_1 + \Sigma_1 A_{11} + \Psi_{11}^T\Sigma_1\Psi_{11}c)P_{M,1})\\&\quad = -\operatorname{tr}(C_1^T C_1 P_{M,1} + \Psi_{21}^T\Sigma_2\Psi_{21}P_{M,1}c),\end{aligned}$$

and hence

$$\begin{aligned}\mathscr{E}^2 =& \operatorname{tr}(C_2\Sigma_2C_2^T + C_1\Sigma_1C_1^T + C_1P_RC_1^T) + 2\operatorname{tr}(A_{21}^T\Sigma_2P_{M,2} + \Psi_{21}^T\Sigma_2\Psi_{22}P_{M,2}c)\\&- 2\operatorname{tr}(B_1B_1^T\Sigma_1) + 2\operatorname{tr}(\Psi_{21}^T\Sigma_2\Psi_{21}P_{M,1}c).\end{aligned}$$

Thus,

$$\begin{aligned}\mathscr{E}^2 =& \operatorname{tr}(\Sigma_2(C_2^TC_2 + 2P_{M,2}A_{21}^T + 2\Psi_{22}P_{M,2}\Psi_{21}^Tc + 2\Psi_{21}P_{M,1}\Psi_{21}^Tc))\\&+ \operatorname{tr}(C_1\Sigma_1C_1^T - B_1B_1^T\Sigma_1 + B_1^T(Q_R - \Sigma_1)B_1)\end{aligned}$$

using the identity $\operatorname{tr}(C_1P_RC_1^T) = \operatorname{tr}(B_1^TQ_RB_1)$. Inserting Eq. (45) provides

$$\begin{aligned}\operatorname{tr}(-B_1B_1^T\Sigma_1) &= \operatorname{tr}(A_{11}\Sigma_1\Sigma_1 + \Sigma_1A_{11}^T\Sigma_1 + \Psi_{11}\Sigma_1\Psi_{11}^T\Sigma_1c + \Psi_{12}\Sigma_2\Psi_{12}^T\Sigma_1c)\\&= \operatorname{tr}(\Sigma_1\Sigma_1A_{11} + \Sigma_1A_{11}^T\Sigma_1 + \Sigma_1\Psi_{11}^T\Sigma_1\Psi_{11}c + \Psi_{12}\Sigma_2\Psi_{12}^T\Sigma_1c)\\&= -\operatorname{tr}(\Sigma_1C_1^TC_1) - \operatorname{tr}(\Sigma_1\Psi_{21}^T\Sigma_2\Psi_{21}c) + \operatorname{tr}(\Psi_{12}\Sigma_2\Psi_{12}^T\Sigma_1c).\end{aligned}$$

So, it holds

$$\begin{aligned}\mathscr{E}^2 =& \operatorname{tr}(\Sigma_2(C_2^TC_2 + 2P_{M,2}A_{21}^T + 2\Psi_{22}P_{M,2}\Psi_{21}^Tc + 2\Psi_{21}P_{M,1}\Psi_{21}^Tc))\\&+ \operatorname{tr}(\Sigma_2(\Psi_{12}^T\Sigma_1\Psi_{12}c - \Psi_{21}\Sigma_1\Psi_{21}^Tc))\\&+ \operatorname{tr}(B_1^T(Q_R - \Sigma_1)B_1).\end{aligned}$$

From (43), it follows

$$\begin{aligned}\operatorname{tr}(\Sigma_2\Psi_{12}^T\Sigma_1\Psi_{12}c) &= \operatorname{tr}(-\Sigma_2(A_{22}^T\Sigma_2 + \Sigma_2A_{22} + \Psi_{22}^T\Sigma_2\Psi_{22}c + C_2^TC_2))\\&= \operatorname{tr}(-\Sigma_2(\Sigma_2A_{22}^T + A_{22}\Sigma_2 + \Psi_{22}\Sigma_2\Psi_{22}^Tc + C_2^TC_2)).\end{aligned}$$

Using (46) yields

$$\operatorname{tr}(\Sigma_2\Psi_{12}^T\Sigma_1\Psi_{12}c) = \operatorname{tr}(\Sigma_2(\Psi_{21}\Sigma_1\Psi_{21}^Tc + B_2B_2^T - C_2^TC_2)),$$

such that

$$\operatorname{tr}(\Sigma_2(\Psi_{12}^T\Sigma_1\Psi_{12}c - \Psi_{21}\Sigma_1\Psi_{21}^Tc)) = \operatorname{tr}(\Sigma_2(B_2B_2^T - C_2^TC_2)),$$

and hence,

$$\mathscr{E}^2 = \operatorname{tr}(\Sigma_2(B_2B_2^T + 2P_{M,2}A_{21}^T + 2\Psi_{22}P_{M,2}\Psi_{21}^T c + 2\Psi_{21}P_{M,1}\Psi_{21}^T c)) \\ + \operatorname{tr}(B_1^T(Q_R - \Sigma_1)B_1).$$

By definition, the Gramians P_R and Q_R satisfy

$$A_{11}^T Q_R + Q_R A_{11} + \Psi_{11}^T Q_R \Psi_{11} c = -C_1^T C_1$$

and

$$A_{11}P_R + P_R A_{11}^T + \Psi_{11}P_R\Psi_{11}^T c = -B_1B_1^T.$$

Thus,

$$\begin{aligned}&\operatorname{tr}(B_1B_1^T(Q_R - \Sigma_1)) \\ &= \operatorname{tr}(-(A_{11}P_R + P_R A_{11}^T + \Psi_{11}P_R\Psi_{11}^T c)(Q_R - \Sigma_1)) \\ &= \operatorname{tr}(-P_R(A_{11}^T(Q_R - \Sigma_1) + (Q_R - \Sigma_1)A_{11} + \Psi_{11}^T(Q_R - \Sigma_1)\Psi_{11}c)) \\ &= \operatorname{tr}(-P_R\Psi_{21}^T\Sigma_2\Psi_{21}c).\end{aligned}$$

Finally, we have

$$\mathscr{E}^2 = \operatorname{tr}(\Sigma_2(B_2B_2^T + 2P_{M,2}A_{21}^T + 2\Psi_{22}P_{M,2}\Psi_{21}^T c + 2\Psi_{21}P_{M,1}\Psi_{21}^T c - \Psi_{21}P_R\Psi_{21}^T c)).$$

□

The error bound we obtained in Proposition 4.6 has the same structure as the $\mathscr{H}_2$ error bound in the deterministic case, which can be found in Sect. 7.2.2 in Antoulas [1]. Furthermore, with this representation of the error bound we are able to emphasize the cases in which balanced truncation is a good approximation. In Proposition 4.6 the bound depends on Σ_2 which contains the $n - r$ smallest Hankel singular values $\sigma_{r+1}, \ldots, \sigma_n$ of the original system. In case these values are small, the reduced order model computed by balanced truncation is of good quality.

5 Applications

In order to demonstrate the use of the model reduction method introduced in Sect. 4 we apply it in the context of the numerical solution of linear controlled evolution equations with Levy noise. For that reason, we apply the Galerkin scheme to the evolution equation and end up with a sequence of ordinary stochastic differential equations. Then, we use balanced truncation for reducing the dimension of the Galerkin solution. Finally, we compute the error bounds and exact errors for the example considered here.

5.1 Finite dimensional approximations for stochastic evolution equations

In this section, we deal with an infinite dimensional system, where the noise process is denoted by M. We suppose that M is a Levy process with values in a separable Hilbert space U. Additionally, we assume that M is square integrable with zero mean. The most important properties regarding this process and the definition of an integral with respect to M can be found in the book of Peszat, Zabczyk [21].

Suppose $A : D(A) \to H$ is a densely defined linear operator being self adjoint and negative definite such that we have an orthonormal basis $(h_k)_{k\in\mathbb{N}}$ of H consisting of eigenvectors of A:

$$Ah_k = -\lambda_k h_k,$$

where $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ are the corresponding eigenvalues. Furthermore, the linear operator A generates a contraction C_0-semigroup $(S(t))_{t\geq 0}$ defined by

$$S(t)x = \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k t} \langle x, h_k\rangle\, h_k$$

for $x \in H$. It is exponentially stable for the case $0 < \lambda_1$. By $\mathscr{Q}$ we denote the covariance operator of M which is a symmetric and positive definite trace class operator that is characterized by

$$\mathbb{E}\,\langle M(t), x\rangle_U\, \langle M(s), y\rangle_U = \min\{t, s\}\, \langle \mathscr{Q}x, y\rangle_U$$

for $x, y \in U$ and $s, t \geq 0$. We can choose an orthonormal basis of U consisting of eigenvectors $(u_k)_{k\in\mathbb{N}}$ of $\mathscr{Q}$.[6] The corresponding eigenvalues we denote by $(\mu_k)_{k\in\mathbb{N}}$ such that

$$\mathscr{Q}u_k = \mu_k u_k.$$

We then consider the following stochastic differential equation:

$$\begin{aligned} dX(t) &= [AX(t) + Bu(t)]\,dt + \Psi(X(t-))dM(t), \quad X(0) = x_0 \in H,\\ Y(t) &= CX(t), \quad t \geq 0. \end{aligned} \tag{48}$$

We make the following assumptions:

- Ψ is a linear mapping on H with values in the set of all linear operators from U to H such that $\Psi(h)\mathscr{Q}^{\frac{1}{2}}$ is a Hilbert-Schmidt operator for every $h \in H$. In addition,

$$\left\| \Psi(h)\mathscr{Q}^{\frac{1}{2}} \right\|_{L_{HS}(U,H)} \leq \tilde{M}\, \|h\|_H \tag{49}$$

[6] By Theorem VI.21 in Reed, Simon [23], $\mathscr{Q}$ is a compact operator such that this property follows by the spectral theorem.

holds for some constant $\tilde{M} > 0$, where L_{HS} indicates the Hilbert-Schmidt norm.
- The process $u : \mathbb{R}_+ \times \Omega \to \mathbb{R}^m$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted with

$$\int_0^T \mathbb{E}\, \|u(s)\|_2^2 \, ds < \infty$$

for each $T > 0$.
- B is a linear and bounded operator on $\mathbb{R}^m$ with values in H and $C \in L(H, \mathbb{R}^p)$.

Definition 5.1 An adapted cadlag process $(X(t))_{t\geq 0}$ with values in H is called *mild solution* of (48) if $\mathbb{P}$-almost surely

$$X(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds + \int_0^t S(t-s)\Psi(X(s-))dM(s) \tag{50}$$

holds for all $t \geq 0$.

Remark Since the operator A generates a contraction semigroup, the stochastic convolution in Eq. (50) has a cadlag modification (Theorem 9.24 in [21]), which enables us to construct a cadlag mild solution of Eq. (48). This solution is unique for every fixed u considering Theorem 9.29 in [21].

We will now approximate the mild solution of the infinite dimensional Eq. (48). We use the Galerkin method for a finite dimensional approximation that one can find for example in Grecksch, Kloeden [12]. Therein they deal with strong solutions of stochastic evolution equations with scalar Wiener noise.

We construct a sequence $(X_n)_{n\in\mathbb{N}}$ of finite dimensional cadlag processes with values in $H_n = \operatorname{span}\{h_1, \ldots, h_n\}$ given by

$$\begin{aligned} dX_n(t) &= [A_n X_n(t) + B_n u(t)]\, dt + \Psi_n(X_n(t-))dM_n(t), \quad t \geq 0, \\ X_n(0) &= x_{0,n}, \end{aligned} \tag{51}$$

where

- $M_n(t) = \sum_{k=1}^n \langle M(t), u_k\rangle_U \, u_k$, $t \geq 0$, is a $\operatorname{span}\{u_1, \ldots, u_n\}$-valued Levy process,
- $A_n x = \sum_{k=1}^n \langle Ax, h_k\rangle_H \, h_k \in H_n$ holds for all $x \in D(A)$,
- $B_n x = \sum_{k=1}^n \langle Bx, h_k\rangle_H \, h_k \in H_n$ holds for all $x \in \mathbb{R}^m$,
- $\Psi_n(x)y = \sum_{k=1}^n \langle \Psi(x)y, h_k\rangle_H \, h_k \in H_n$ holds for all $y \in U$ and $x \in H$,
- $x_{0,n} = \sum_{k=1}^n \langle x_0, h_k\rangle_H \, h_k \in H_n$.

Since A_n is a bounded operator for every $n \in \mathbb{N}$, we know that A_n generates a C_0-semigroup on H of the form $S_n(t) = \mathrm{e}^{A_n t}, t \geq 0$. For all $x \in H_n$ it has the representation $S_n(t)x = \sum_{k=1}^n \mathrm{e}^{-\lambda_k t} \langle x, h_k\rangle_H \, h_k$ such that the mild solution of Eq. (51) is given by

$$X_n(t) = S_n(t)x_{0,n} + \int_0^t S_n(t-s)B_n u(s)ds + \int_0^t S_n(t-s)\Psi_n(X_n(s-))dM_n(s)$$

for $t \geq 0$. Furthermore, we consider the p dimensional approximating output

$$Y_n(t) = CX_n(t), \quad t \geq 0.$$

With similar arguments like in the proof of Theorem 1 in Grecksch, Kloeden [12] one can show the Theorem 5.2 below. This shows that

$$\mathbb{E}\,\|Y_n(t) - Y(t)\|_2^2 \to 0$$

is true for $n \to \infty$ and $t \geq 0$:

Theorem 5.2 *It holds*

$$\mathbb{E}\,\|X_n(t) - X(t)\|_H^2 \to 0$$

for $n \to \infty$ and $t \geq 0$.

Remark If $U = \mathbb{R}^q$, one has to replace M_n by M in Eq. (51) and Theorem 5.2 holds for this case as well.

We now determine the components of Y_n. They are given by

$$Y_n^l(t) = \langle Y_n(t), e_l\rangle_{\mathbb{R}^p} = \langle CX_n(t), e_l\rangle_{\mathbb{R}^p} = \sum_{k=1}^{n} \langle Ch_k, e_l\rangle_{\mathbb{R}^p} \langle X_n(t), h_k\rangle_H$$

for $l = 1, \ldots, p$, where e_l is the l-th unit vector in $\mathbb{R}^p$. We set

$$\mathscr{X}(t) = \left(\langle X_n(t), h_1\rangle_H, \ldots, \langle X_n(t), h_n\rangle_H\right)^T, \quad \mathscr{C} = \left(\langle Ch_k, e_l\rangle_{\mathbb{R}^p}\right)_{\substack{l=1,\ldots,p \\ k=1,\ldots,n}}$$

and obtain

$$Y_n(t) = \mathscr{C}\mathscr{X}(t), \quad t \geq 0.$$

The components of $\mathscr{X}$ fulfill the following equation:

$$\begin{aligned}\langle X_n(t), h_k\rangle_H &= \left\langle S_n(t)x_{0,n}, h_k\right\rangle_H + \int_0^t \langle S_n(t-s)B_n u(s), h_k\rangle_H \, ds \\ &\quad + \left\langle \int_0^t S_n(t-s)\Psi_n(X_n(s-))dM_n(s), h_k\right\rangle_H.\end{aligned}$$

Considering the representation $S_n(t)x = \sum_{i=1}^n \mathrm{e}^{-\lambda_i t} \langle x, h_i\rangle_H h_i$ ($x \in H_n$), we have

$$\left\langle S_n(t)x_{0,n}, h_k\right\rangle_H = \mathrm{e}^{-\lambda_k t} \left\langle x_{0,n}, h_k\right\rangle_H = \mathrm{e}^{-\lambda_k t} \langle x_0, h_k\rangle_H$$

and

$$
\begin{aligned}
&\langle S_n(t-s)B_n u(s), h_k\rangle_H \\
&\quad = \mathrm{e}^{-\lambda_k(t-s)} \langle B_n u(s), h_k\rangle_H = \sum_{l=1}^{m} \mathrm{e}^{-\lambda_k(t-s)} \langle Be_l, h_k\rangle_H \langle u(s), e_l\rangle_{\mathbb{R}^m}
\end{aligned}
$$

for $k = 1, \ldots, n$, where e_l is the l-th unit vector in $\mathbb{R}^m$. Furthermore,

$$
\begin{aligned}
&\left\langle \int_0^t S_n(t-s)\Psi_n(X_n(s-))dM_n(s), h_k \right\rangle_H \\
&\quad = \sum_{j=1}^{n} \int_0^t \left\langle S_n(t-s)\Psi_n(X_n(s-))u_j, h_k\right\rangle_H d\left\langle M(s), u_j\right\rangle_U \\
&\quad = \sum_{j=1}^{n}\sum_{i=1}^{n} \int_0^t \left\langle S_n(t-s)\Psi_n(h_i)u_j, h_k\right\rangle_H \langle X_n(s-), h_i\rangle_H \, d\left\langle M(s), u_j\right\rangle_U \\
&\quad = \sum_{j=1}^{n}\sum_{i=1}^{n} \int_0^t \mathrm{e}^{-\lambda_k(t-s)} \left\langle \Psi(h_i)u_j, h_k\right\rangle_H \langle X_n(s-), h_i\rangle_H \, d\left\langle M(s), u_j\right\rangle_U .
\end{aligned}
$$

Hence, in compact form $\mathscr{X}$ is given by

$$
\mathscr{X}(t) = \mathrm{e}^{\mathscr{A}t}\,\mathscr{X}_0 + \int_0^t \mathrm{e}^{\mathscr{A}(t-s)}\,\mathscr{B}u(s)ds + \sum_{j=1}^{n}\int_0^t \mathrm{e}^{\mathscr{A}(t-s)}\,\mathscr{N}^j\,\mathscr{X}(s-)dM^j(s), \tag{52}
$$

where

- $\mathscr{A} = \mathrm{diag}(-\lambda_1, \ldots, -\lambda_n)$, $\mathscr{B} = (\langle Be_i, h_k\rangle_H)_{\substack{k=1,\ldots,n\\ i=1,\ldots,m}}$,
  $\mathscr{N}^j = \left(\left\langle \Psi(h_i)u_j, h_k\right\rangle_H\right)_{k,i=1,\ldots,n}$,
- $\mathscr{X}_0 = (\langle x_0, h_1\rangle_H, \ldots, \langle x_0, h_n\rangle_H)^T$ and $M^j(s) = \left\langle M(s), u_j\right\rangle_U$.

The processes M^j are uncorrelated real-valued Levy processes with $\mathbb{E}\left|M^j(t)\right|^2 = t\mu_j$, $t \geq 0$, and zero mean. Below, we show that the solution of Eq. (52) fulfills the strong solution equation as well. We set

$$
f(t) := \mathscr{X}_0 + \int_0^t \mathrm{e}^{-\mathscr{A}s}\,\mathscr{B}u(s)ds + \sum_{j=1}^{n}\int_0^t \mathrm{e}^{-\mathscr{A}s}\,\mathscr{N}^j\,\mathscr{X}(s-)dM^j(s), \quad t \geq 0,
$$

and determine the stochastic differential of $\mathrm{e}^{\mathscr{A}t} f(t)$ via the Ito product formula in Corollary 2.4:

$$e_i^T \mathscr{X}(t) = e_i^T \, \mathrm{e}^{\mathscr{A}t} f(t) = e_i^T f(0) + \int_0^t d\left(e_i^T \, \mathrm{e}^{\mathscr{A}s}\right) f(s-) + \int_0^t e_i^T \, \mathrm{e}^{\mathscr{A}s} \, df(s)$$
$$= e_i^T \Bigg(\mathscr{X}_0 + \int_0^t \mathscr{A} \, \mathrm{e}^{\mathscr{A}s} f(s)ds$$
$$+ \int_0^t \mathscr{B}u(s)ds + \sum_{j=1}^n \int_0^t \mathscr{N}^j \mathscr{X}(s-)dM^j(s) \Bigg),$$

where e_i is the i-th unit vector of $\mathbb{R}^n$ and the quadratic covariation terms are zero, since $t \mapsto e_i^T \, \mathrm{e}^{\mathscr{A}t}$ is a continuous semimartingale with a martingale part of zero. Hence,

$$\mathscr{X}(t) = \mathscr{X}_0 + \int_0^t \mathscr{A}\mathscr{X}(s) + \mathscr{B}u(s)ds + \sum_{j=1}^n \int_0^t \mathscr{N}^j \mathscr{X}(s-)dM^j(s), \quad t \geq 0.$$

Example 5.3 We consider a bar of length π, which is heated on $[0, \frac{\pi}{2}]$. The temperature of the bar is described by the following stochastic partial differential equation:

$$\begin{aligned} \frac{\partial X(t,\zeta)}{\partial t} &= \frac{\partial^2}{\partial \zeta^2} X(t,\zeta) + 1_{[0,\frac{\pi}{2}]}(\zeta)u(t) + aX(t-,\zeta)\frac{\partial M(t)}{\partial t}, \\ X(t,0) &= 0 = X(t,\pi), \\ X(0,\zeta) &= x_0(\zeta) \end{aligned} \tag{53}$$

for $t \geq 0$ and $\zeta \in [0, \pi]$. Here, we assume that M is a scalar square integrable Levy process with zero mean, $H = L^2([0,\pi])$, $U = \mathbb{R}$, $m = 1$, $A = \frac{\partial^2}{\partial \zeta^2}$. Furthermore, we set $B = 1_{[0,\frac{\pi}{2}]}(\cdot)$ and $\Psi(x) = ax$ for $x \in L^2([0,\pi])$. Additionally, we assume $\mathbb{E}\left[M(1)^2\right] a^2 < 2$, which is equivalent to that the solution of the uncontrolled Eq. (53) satisfies

$$\mathbb{E}\left\| X^h(t,\cdot) \right\|_H^2 \leq c \, \mathrm{e}^{-\alpha t} \left\| x_0(\cdot) \right\|_H^2 \tag{54}$$

for $c, \alpha > 0$. This equivalence is a consequence of Theorem 3.1 in Ichikawa [14] and Theorem 5 in Haussmann [13]. For further information regarding the exponential mean square stability condition (54), see Sect. 5 in Curtain [9].[7] It is a well-known fact that here the eigenvalues of the second derivative are given by $-\lambda_k = -k^2$ and the corresponding eigenvectors which represent an orthonormal basis are $h_k = \sqrt{\frac{2}{\pi}} \sin(k \cdot)$. We are interested in the average temperature of the bar on $[\frac{\pi}{2}, \pi]$ such that the scalar output of the system is

[7] Curtain, Ichikawa and Haussmann stated these conditions for exponential mean square stability for the Wiener case, which can be easily generalized for the case of square integrable Levy process with mean zero.

$$Y(t) = \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} X(t, \zeta) d\zeta,$$

where $Cx = \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} x(\zeta) d\zeta$ for $x \in L^2([0, \pi])$. We approximate Y via

$$Y_n(t) = \mathscr{C}\mathscr{X}(t),$$

$$\mathscr{C}^T = (Ch_k)_{k=1,\dots,n} = \left(\left(\tfrac{2}{\pi}\right)^{\frac{3}{2}} \tfrac{1}{k} \left[\cos(\tfrac{k\pi}{2}) - \cos(k\pi)\right] \right)_{k=1,\dots,n}.$$

$\mathscr{X}$ is given by

$$\mathscr{X}(t) = \mathscr{X}_0 + \int_0^t \mathscr{A}\mathscr{X}(s) + \mathscr{B}u(s) ds + \int_0^t \mathscr{N}\mathscr{X}(s-) dM(s), \tag{55}$$

where

- $\mathscr{A} = \operatorname{diag}\left(-1, -4, \dots, -n^2\right)$,
- $\mathscr{N} = (\langle \Psi(h_i), h_k \rangle_H)_{k,i=1,\dots,n} = (\langle ah_i, h_k \rangle_H)_{k,i=1,\dots,n} = aI_n$,
- $\mathscr{B} = (\langle B, h_k \rangle_H)_{k=1,\dots,n} = \left(\left\langle 1_{[0,\frac{\pi}{2}]}(\cdot), h_k \right\rangle_H \right)_{k=1,\dots,n}$
  $= \left(\left(\tfrac{2}{\pi}\right)^{\frac{1}{2}} \tfrac{1}{k} \left[1 - \cos(\tfrac{k\pi}{2})\right] \right)_{k=1,\dots,n}.$

Since we now choose $x_0 \equiv 0$ for simplicity, we additionally have $\mathscr{X}_0 = 0$.

Next, we consider a more complex example with a two dimensional spatial variable:

Example 5.4 We determine the Galerkin solution of the following controlled stochastic partial differential equation:

$$\begin{aligned}
&\frac{\partial X(t,\zeta)}{\partial t} = \Delta X(t,\zeta) + 1_{[\frac{\pi}{4},\frac{3\pi}{4}]^2}(\zeta)u(t) + \mathrm{e}^{-|\zeta_1 - \frac{\pi}{2}| - \zeta_2} X(t-,\zeta)\frac{\partial M(t)}{\partial t},\\
&\qquad t \geq 0,\ \zeta \in [0,\pi]^2,\\
&\frac{\partial X(t,\zeta)}{\partial \mathbf{n}} = 0, \quad t \geq 0,\ \zeta \in \partial[0,\pi]^2,\\
&X(0,\zeta) \equiv 0.
\end{aligned} \tag{56}$$

Again, M is a scalar square integrable Levy process with zero mean, $H = L^2([0, \pi]^2)$, $U = \mathbb{R}$, $m = 1$, A is the Laplace operator, $B = 1_{[\frac{\pi}{4},\frac{3\pi}{4}]^2}(\cdot)$, and $\Psi(x) = \mathrm{e}^{-|\cdot - \frac{\pi}{2}| - \cdot}\, x$ for $x \in L^2([0, \pi]^2)$. The eigenvalues of the Laplacian on $[0, \pi]^2$ are given by $-\lambda_{ij} = -(i^2 + j^2)$ and the corresponding eigenvectors which represent an orthonormal basis are $h_{ij} = \frac{f_{ij}}{\|f_{ij}\|_H}$, where $f_{ij} = \cos(i\cdot)\cos(j\cdot)$. For simplicity we write $-\lambda_k$ for the k-th largest eigenvalue, and the corresponding eigenvector we denote by h_k. The scalar output of the system is

$$Y(t) = \frac{4}{3\pi^2} \int_{[0,\pi]^2 \setminus [\frac{\pi}{4},\frac{3\pi}{4}]^2} X(t,\zeta) d\zeta,$$

where $Cx = \frac{4}{3\pi^2}\int_{[0,\pi]^2\setminus[\frac{\pi}{4},\frac{3\pi}{4}]^2} x(\zeta)d\zeta$ for $x \in L^2([0,\pi]^2)$. The output of the Galerkin system is

$$Y_n(t) = \mathscr{C}\mathscr{X}(t)$$

with $\mathscr{C}^T = (Ch_k)_{k=1,\dots,n}$. The Galerkin solutions $\mathscr{X}$ satisfies

$$\mathscr{X}(t) = \int_0^t \mathscr{A}\mathscr{X}(s) + \mathscr{B}u(s)ds + \int_0^t \mathscr{N}\mathscr{X}(s-)dM(s), \tag{57}$$

where

$$\mathscr{A} = \operatorname{diag}(0,-1,-1,\dots),\ \mathscr{N} = \left(\left\langle \mathrm{e}^{-\left|\cdot-\frac{\pi}{2}\right|-\cdot} h_i, h_k\right\rangle_H\right)_{k,i=1,\dots,n},$$
$$\mathscr{B} = \left(\left\langle 1_{[\frac{\pi}{4},\frac{3\pi}{4}]^2}(\cdot), h_k\right\rangle_H\right)_{k=1,\dots,n}.$$

5.2 Error bounds for the examples

We consider the system from Example 5.3. Using Theorem 3.3, the uncontrolled Eq. (55) is asymptotically mean square stable if and only if the Kronecker matrix

$$I_n\otimes\mathscr{A}+\mathscr{A}\otimes I_n+\mathscr{N}\otimes\mathscr{N}\cdot\mathbb{E}\left[M(1)^2\right]=I_n\otimes\mathscr{A}+(\mathscr{A}+\mathbb{E}\left[M(1)^2\right]a^2I_n)\otimes I_n$$

is Hurwitz. From Sect. 2.6 in Steeb [25] we can conclude that the largest eigenvalue of the Kronecker matrix is $-2+\mathbb{E}\left[M(1)^2\right]a^2$. Thus, the solution of the uncontrolled system (55) is asymptotically mean square stable if and only if $\mathbb{E}\left[M(1)^2\right]a^2 < 2$, which is fulfilled by (54).

We want to obtain a reduced order model via balanced truncation. We choose $a = \mathbb{E}\left[M(1)^2\right] = 1$ and additionally let $n = 1000$. It turns out that the system is neither completely observable nor completely reachable since the Gramians do not have full rank. So, we need an alternative method to determine the reduced order model. We use a method for non minimal systems that is known from the deterministic case and which is for example described in Sect. 1.4.2 in Benner et al. [4]. In this algorithm we do not compute the full transformation matrix T. So, we obtain the matrices of the reduced order model by

$$\tilde{\mathscr{A}} = W^T\operatorname{diag}(-1,\dots,-n^2)V,\quad \tilde{\mathscr{N}} = W^TI_nV = I_r,\quad \tilde{\mathscr{B}} = W^T\mathscr{B},\quad \tilde{\mathscr{C}} = \mathscr{C}V.$$

Above, we set

$$W^T = \Sigma_1^{-\frac{1}{2}}V_1^TR \text{ and } V = S^TU_1\Sigma_1^{-\frac{1}{2}},$$

where V_1 and U_1 are obtained from the SVD of SR^T:

$$SR^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix},$$

where $Q = R^T R$ and $P = S^T S$. Reducing the model yields the following error bounds:

Dimension of the reduced order model	$\left(\operatorname{tr}\left(\mathscr{C} P \mathscr{C}^T\right) + \operatorname{tr}\left(\tilde{\mathscr{C}} P_R \tilde{\mathscr{C}}^T\right) - 2 \operatorname{tr}\left(\mathscr{C} P_M \tilde{\mathscr{C}}^T\right)\right)^{\frac{1}{2}}$
8	$4.5514 \cdot 10^{-6}$
4	$2.3130 \cdot 10^{-4}$
2	$1.7691 \cdot 10^{-3}$
1	0.0879

Below, we reduce the Galerkin solution of Example 5.4 with dimension $n = 1000$ and $\mathbb{E}\left[M(1)^2\right] = 1$. Here, the matrix $\mathscr{A} = \operatorname{diag}(0, -1, -1, -2, \ldots)$ is not stable, such that we need to stabilize system (57) before using balanced truncation. Inserting the feedback control $u(t) = -2e_1^T \mathscr{X}(t)$, $t \geq 0$, where e_1 is the first unit vector in $\mathbb{R}^n$, yields a asymptotically mean square stable system, since the following sufficient condition holds (see Corollary 3.6.3 in [11] and Theorem 5 in [13]): $\mathscr{A}_S = \mathscr{A} - 2\mathscr{B} e_1^T$ is stable and

$$\left\| \int_0^\infty \mathrm{e}^{\mathscr{A}_S^T t}\, \mathscr{N}^T \mathscr{N}\, \mathrm{e}^{\mathscr{A}_S t}\, dt \right\| = 0.0658 < 1.$$

We repeat the procedure from above and obtain

Dimension of the reduced order model	$\left(\operatorname{tr}\left(\mathscr{C} P \mathscr{C}^T\right) + \operatorname{tr}\left(\tilde{\mathscr{C}} P_R \tilde{\mathscr{C}}^T\right) - 2 \operatorname{tr}\left(\mathscr{C} P_M \tilde{\mathscr{C}}^T\right)\right)^{\frac{1}{2}}$
8	$3.7545 \cdot 10^{-6}$
4	$6.4323 \cdot 10^{-4}$
2	$3.1416 \cdot 10^{-3}$
1	0.0333

for the the stabilized system (57) meaning that we replaced $\mathscr{A}$ by $\mathscr{A}_S$.

5.3 Comparison between exact error and error bound

Since Eqs. (55) and (57) do not have an explicit solution in general, we need to discretize in time for estimating the exact error of the estimation given here. For simplicity, we assume that $n = 80$ and M is a scalar Wiener process and use the Euler-Maruyama scheme[8] for approximating the original system, yielding

$$\mathscr{X}_{k+1} = \mathscr{X}_k + (\mathscr{A}\mathscr{X}_k + \mathscr{B}u(t_k))\,h + \mathscr{N}\mathscr{X}_k \Delta M_k$$

and the reduced order model:

$$\tilde{\mathscr{X}}_{k+1} = \tilde{\mathscr{X}}_k + \left(\tilde{\mathscr{A}}\tilde{\mathscr{X}}_k + \tilde{\mathscr{B}}u(t_k)\right) h + \tilde{\mathscr{N}}\tilde{\mathscr{X}}_k \Delta M_k,$$

where we consider these equations on the time interval $[0, \pi]$. Furthermore, we choose $\mathscr{X}_0 = 0$, $h = \frac{\pi}{10000}$ and $t_k = kh$ for $k = 0, 1, \ldots, 10000$, $\Delta M_k = M(t_{k+1}) - M(t_k)$.

For system (55) we insert the normalized control functions $u_1(t) = \frac{\sqrt{2}}{\pi} M(t)$, $u_2(t) = \sqrt{\frac{2}{\pi}}\cos(t)$, $u_3(t) = \sqrt{\frac{2}{1-\mathrm{e}^{-2\pi}}}\,\mathrm{e}^{-t}$, $t \in [0, \pi]$ and obtain $\mathscr{D} := \max_{k=1,\ldots,10000} \mathbb{E}\left|\mathscr{C}X_k - \tilde{\mathscr{C}}\tilde{X}_k\right|$ for different dimensions of the reduced order model (ROM) and different inputs:

Dimension of the ROM	$\mathscr{D}$ with $u = u_1$	$\mathscr{D}$ with $u = u_2$	$\mathscr{D}$ with $u = u_3$	$\mathscr{E}\mathscr{B}$
8	9.0615×10^{-9}	7.8832×10^{-8}	1.3987×10^{-7}	1.4813×10^{-6}
4	3.8702×10^{-6}	6.4204×10^{-6}	1.1353×10^{-5}	2.2706×10^{-4}
2	6.8932×10^{-5}	1.1195×10^{-4}	1.9549×10^{-4}	1.7671×10^{-3}
1	0.0141	0.0243	0.0354	0.0879

Here, $\mathscr{E}\mathscr{B} := \left(\operatorname{tr}\left(\mathscr{C}P\mathscr{C}^T\right) + \operatorname{tr}\left(\tilde{\mathscr{C}}P_R\tilde{\mathscr{C}}^T\right) - 2\ \operatorname{tr}\left(\mathscr{C}P_M\tilde{\mathscr{C}}^T\right)\right)^{\frac{1}{2}}$ is the balanced truncation error bound derived in Sect. 4.

For system (57) we use the inputs $\tilde{u}_i(t) = -2e_1^T\mathscr{X}(t) + u_i(t)$, $t \geq 0$, $i = 1, 2, 3$ and obtain

Dimension of the ROM	$\mathscr{D}$ with $u = \tilde{u}_1$	$\mathscr{D}$ with $u = \tilde{u}_2$	$\mathscr{D}$ with $u = \tilde{u}_3$	$\mathscr{E}\mathscr{B}$
8	5.6162×10^{-7}	5.5374×10^{-7}	6.5699×10^{-7}	3.5376×10^{-6}
4	4.7245×10^{-5}	5.2722×10^{-5}	6.8758×10^{-5}	3.1487×10^{-4}
2	5.1270×10^{-4}	4.6627×10^{-4}	6.2103×10^{-4}	2.4164×10^{-3}
1	3.7520×10^{-3}	0.0118	9.9629×10^{-3}	0.0327

These results show that the balanced truncation error bound, which is a worst case bound holding for all feasible input functions, also provides a good prediction of the true time domain error. In particular, it quite well predicts the decrease of the true error for increased dimension of the reduced order model.

[8] The theory regarding this method can be found in Kloeden and Platen [16].

6 Conclusions

We generalized balanced truncation for stochastic system with noise processes having jumps. In particular, we focused on a linear controlled state equation driven by uncorrelated Levy processes which is asymptotically mean square stable and equipped with an output equation. We showed that the Gramians we defined are solutions of generalized Lyapunov equations. Furthermore, we proved that the observable states and the corresponding energy are characterized by the observability Gramian Q and that the reachability Gramian P provides partial information about the reachabilty of an average state. We showed that the reduced order model (ROM) is mean square stable, not balanced, the Hankel singular values (HV) of the ROM are not a subset of the HVs of the original system and one can lose complete observability and reachability. Furthermore, we provided an error bound for balanced truncation of the Levy driven system. Finally, we demonstrated the use of balanced truncation for stochastic systems. We applied it in the context of the numerical solution of linear controlled evolution equations with Levy noise and computed the error bounds and exact errors for the example considered here.

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