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On controllability of driftless control systems on symmetric spaces

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Abstract Symmetric spaces arise in wide variety of problems in Mathematics and Physics. They are mostly studied in Representation theory, Harmonic analysis and Differential geometry. As many physical systems have symmetric spaces as their configuration spaces, the study of controllability on symmetric space is quite interesting. In this paper, a driftless control system of type $\dot{x} = \sum_{i=1}^{m} u_i f_i(x)$ is considered on a symmetric space. For this we have established global controllability condition which is illustrated by few examples of exponential submanifolds of $SE(3)$ and random matrix ensembles.

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1 Introduction

Lie theory was first introduced by Brockett [8,9] in motion control problems. He studied various aspects of controllability and observability of systems involving Lie groups. The controllability properties of the systems on Lie groups were investigated in [22,23,41]. Later, studies were done on these type of control systems from a geometric point of view [1,21,38]. Recent times have witnessed a lot of activities where by control theory (more particularly driftless control system) on Lie groups could play a lead role in explaining many physical, chemical and engineering problems in the area of quantum control [33], space altitude control [44], robotic controls [43], chemical reaction control [16,17] etc. which could not be easily tackled by control theory defined on a conventional state space. With the advent of new technological challenges in control theory such as motion patterns of many kinesiological/robot mechanical systems or control on ensembles, it is being felt that a modified theory should be developed for tackling such type of situation, similar but not exact, to the case where Lie group replaced the ordinary state space in control theory.

The motion pattern of human shoulder complex, human knee joint, mechanical device such as omni-wrist are associated with spaces which can be approximated or shown to be some types of symmetric spaces. Similarly considering a dynamical system on random matrix ensembles such as Circular ensembles, Gaussian ensembles [Orthogonal (GOE), Sympletic (GSE), Unitary (GUE)] [24] it becomes necessary to devise a theory of control system on symmetric space which is the main aim of our paper. We have confined ourselves to the study of controllability of non linear driftless control system on these type of spaces, as most of the physical

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system can be modelled on such background. However, one can extend such types of studies to bilinear system or affine system etc. which we have not ventured into, in this paper.

Progresses have been made to some extent in this regard by various authors [25,40] in which they have discussed optimal control problems in Lie groups where control belong to symmetric space. A class of optimal control problems defined on certain kinds of symmetric spaces is discussed by Bloch et al. [6,7]. Similarly a lot of progress has been made to study control of complex networks [5,28] and control of ensembles through structural controllability [12,13] and by other authors [2,3,26,27,39], but to our knowledge nobody has discussed about these in the context of random matrix ensembles like GOE, GSE, GUE which actually correspond to Riemannian symmetric spaces classified by Cartan [10]. These random matrix ensembles which actually correspond to Riemannian symmetric spaces, play an important role in explaining quantum transport in a disordered wire, carbon nano tube etc. Thus we hope our theory can play an important role in quantum control of these phenomena in quantum materials [11]. Taking these viewpoints into consideration we were motivated to extend the study of control theory on Lie groups (Lie algebras) to symmetric spaces (Lie triple systems).

Our task becomes simpler as Lie triple systems and symmetric spaces are related by exponential mapping similar to Lie algebras and Lie groups. As a result many of the theories on Lie groups are extended very easily to the case of symmetric spaces.

In the next section we gave an introduction to the concepts and definitions of symmetric spaces and Lie triple system along with their relations. In the third section we define a nonlinear control system (driftless) on a symmetric space and obtain the condition under which a system is globally controllable. In the fourth section we apply these theories to a few simple but important examples. The last section contains few concluding remarks.

2 Symmetric space and lie triple system

2.1 Symmetric space

A symmetric space [18,20,29,32] is a manifold S with a differential symmetric product $\cdot$ that obeys the following conditions:

(i) $a \cdot a = a$
(ii) $a \cdot (a \cdot b) = b$
(iii) $a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c)$,
and moreover
(iv) every a has a neighbourhood U such that $\forall b \in U, a \cdot b = b \Rightarrow b = a$.

In other words, for $x \in S$, there exists an element s_x in the isometry group of S which satisfies the following properties

$$s_x(x) = x, \text{ and } (ds_x)_x = -Id.$$

This isometry s_x is known as symmetry at x. Hence, a symmetric space S is a homogeneous space with a symmetry s_a at some point $a \in S$. A pointed symmetric space (S, o) is a pair which consists of a symmetric space S and a point o known as the base point. The following are some examples of symmetric spaces.

1. The subsets of a group G which are closed under the composition $a \cdot b = ab^{-1}a$, where ab is the usual multiplication in G.
2. The set of all symmetric positive definite matrices forms a symmetric space with the product defined by $a \cdot b = ab^{-1}a$.
3. Let $S = S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ with the standard scalar product. The symmetry at any $x \in S^n$ is the reflection at the line $\mathbb{R}x$ in $\mathbb{R}^{n+1}$, i.e., $s_x(y) = -y + 2\langle y, x\rangle x$. In this case, the symmetries generate the full isometry group which is the orthogonal group $O(n+1)$.

2.2 Lie triple system

Lie triple system [18,20,29,32,45] is a vector space L over a field $\mathbb{K}$ with a trilinear map $\mu : L \times L \times L \longrightarrow L$ satisfying(if we write $\mu(A_1 \otimes A_2 \otimes A_3) = [A_1, A_2, A_3]$):

1. $[A_1, A_1, A_2] = 0$
2. $[A_1, A_2, A_3] + [A_2, A_3, A_1] + [A_3, A_1, A_2,] = 0$
3. $[A_1, A_2[D, E, F]] = [[A_1, A_2, D], E, F] + [D, [A_1, A_2, E], F] + [D, E, [A_1, A_2, F]]$, for $A_1, A_2, D, E, F \in L$, $[\cdot\, ,\ \cdot,\ \cdot]$ is called ternary operation of the Lie triple system.

Generally, any subset of a Lie algebra $\mathfrak{g}$ that is closed under the operator

$$T_A(\cdot) = ad_A^2(\cdot) = [A, [A, \cdot]]$$

is a Lie triple system.

Some examples of Lie triple system is given below.

(i) The set of all $n \times n$ matrices is a Lie triple system with Lie double bracket defined by $[A_1, A_2, A_3] = [[A_1, A_2], A_3]$, for any $n \times n$ matrices A_1, A_2, A_3 and $[A_1, A_2] = A_1A_2 - A_2A_1$.
(ii) The set of all $n \times n$ symmetric matrices is a Lie triple system with Lie double bracket defined by $[A_1, A_2, A_3] = [[A_1, A_2], A_3]$, for A_1, A_2, A_3 any $n \times n$ symmetric matrices and $[A_1, A_2] = A_1A_2 - A_2A_1$.
(iii) Let T be a Lie triple system and S be any nonempty set. Then the set T^S of all functions f from S to T is a Lie triple system with respect to a ternary operation given by $[f_1, f_2, f_3](x) = [f_1(x), f_2(x), f_3(x)]$.

2.3 Lie double bracket of vector fields on a symmetric space

Let X_1 and X_2 be smooth vector fields on a n-manifold M, then X_1 and X_2 are first order differential operators on smooth functions $M \longrightarrow \mathbb{R}$. Hence for a smooth function $f : M \longrightarrow \mathbb{R}$, $X_1 f$ and $X_2 f$ are also smooth functions $M \longrightarrow \mathbb{R}$.

Now we first define a differential operator $[X_1, X_2]$ termed as Lie bracket of X_1 and X_2 as $[X_1, X_2] := X_1X_2 - X_2X_1$, i.e., for smooth functions $f : M \longrightarrow \mathbb{R}$,

$$[X_1, X_2](f) := X_1X_2(f) - X_2X_1(f).$$

So by definition, $[X_1, X_2]$ is a second order differential operator. But it appears that $[X_1, X_2]$ is indeed a first order differential operator and in fact, a vector field.

For any smooth vector fields X_1 and X_2, the Lie bracket $[X_1, X_2]$ is again a smooth vector field on M. Extending this, it can be shown that $[[X_1, X_2], X_3]$ is again a first order differential operator where

$$[[X_1, X_2], X_3]\,(f) = X_1X_2X_3(f) - X_3X_1X_2(f) - X_2X_1X_3(f) + X_3X_2X_1(f).$$

Before proceeding further, let us extend the concepts of left invariant vector fields on symmetric spaces. Let $\mathbf{T}$ be a Lie triple system (LTS) with its ternary operation $\mu(A_1 \otimes A_2 \otimes A_3) = [A_1, A_2, A_3]$ and G be a Lie group. Then G is said to act on $\mathbf{T}$ from left if there exist a function

$$\varphi : G \times \mathbf{T} \to \mathbf{T}$$
$$(g, A) \mapsto \varphi(g, A) = gA$$

which satisfies the following properties

1. $ex = x,\ \forall x \in \mathbf{T}$, where $e \in G$ is group identity,
2. $g_1(g_2x) = (g_1g_2)x, \forall g_1, g_2 \in G$ and $\forall x \in \mathbf{T}$,
3. for every $g \in G$, the left translation $\varphi_g = \varphi(g) : \mathbf{T} \to \mathbf{T},\ A \mapsto gA$ is a linear map,
4. $\forall g \in G$ and $A_1, A_2, A_3 \in \mathbf{T}$; $\mu(gA_1, gA_2, gA_3) = g\mu(A_1, A_2, A_3) = g[A_1, A_2, A_3]$.

The above action is denoted by $(G, \mathbf{T})$. The Lie triple system with an action of a group G is called $G-LTS$.

Further we see that the group G acts on the space of points G/K and maps it onto itself. In particular the coset representations $c \in G/K \subset G$ map the origin (base point) of G/K on to the point $c \in G/K$. If c is any point in G/K, it can be mapped into any other point c' in G/K by some group operation: $(c'c^{-1})c \to c',\ c'c^{-1} = g \in G$. Moreover, the only group operation which leaves every point c fixed is the identity, $gc = c,\ \forall c \in G/K \Rightarrow g = Id$.

The vector fields of the form $U(X) = XA,\ V(X) = XB,\ W(X) = XC,\ \ X \in S,\ \ A, B, C \in \mathbf{T}$ (tangent space at the base point) are called left invariant vector fields if they satisfy the following property

$$[[U, V], W](X) = [XA, XB, XC] = X[A, B, C].$$

Suppose now we have three left invariant smooth vector fields U, V, W on a symmetric space S and $V_F(S)$ denotes the space of all smooth vector fields on S. The Lie double bracket of the fields U, V, W is the vector field, $[[U, V], W]] \in V_F(S)$. It can be shown that

$$[[U, V], W](X) = \frac{d}{dt}\bigg|_{t=0} \gamma(\sqrt[3]{t}), \qquad X \in S,$$

where the curve γ is defined as

$$\gamma(t) = e^{-tW} \circ e^{-tU} \circ e^{-tV} \circ e^{tU} \circ e^{tV} \circ e^{tW} \circ e^{-tV} \circ e^{-tU} \circ e^{tV} \circ e^{tU}(X). \tag{1}$$

Here e^{tU} denotes the flow of the vector field U

$$\frac{d}{dt}e^{tU}(X) = U\left(e^{tU}(X)\right), \quad e^{tU}\bigg|_{t=0}(X) = X.$$

The left invariant vector fields now have the Lie double bracket defined as

$$\begin{aligned}
[[U, V], W](X) &= [[XA, XB], XC] \\
&= X[[A, B], C] \\
&= X((AB - BA)C + C(BA - AB)) \\
&= X(ABC - BAC + CBA - CAB).
\end{aligned}$$

The matrix exponential gives the flows of the left invariant vector fields, so

$$e^{tU}(X) = Xe^{(tA)}, \quad e^{tV}(X) = Xe^{(tB)}, \quad e^{tW}(X) = Xe^{(tC)}.$$

Computing the lower order terms of the curve γ from (1); we obtain

$$\begin{aligned}
\gamma(t) &= Xe^{(tA)}e^{(tB)}e^{(-tA)}e^{(-tB)}e^{(tC)}e^{(tB)}e^{(tA)}e^{(-tB)}e^{(-tA)}e^{(-tC)} \\
&= X\left(Id + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)\left(Id + tB + \frac{t^2}{2!}B^2 + \frac{t^3}{3!}B^3 + \cdots\right) \\
&\quad \left(Id - tA + \frac{t^2}{2!}A^2 - \frac{t^3}{3!}A^3 + \cdots\right)\left(Id - tB + \frac{t^2}{2!}B^2 - \frac{t^3}{3!}B^3 + \cdots\right) \\
&\quad \left(Id + tC + \frac{t^2}{2!}C^2 + \frac{t^3}{3!}C^3 + \cdots\right)\left(Id + tB + \frac{t^2}{2!}B^2 + \frac{t^3}{3!}B^3 + \cdots\right) \\
&\quad \left(Id + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\right)\left(Id - tB + \frac{t^2}{2!}B^2 - \frac{t^3}{3!}B^3 + \cdots\right) \\
&\quad \left(Id - tA + \frac{t^2}{2!}A^2 - \frac{t^3}{3!}A^3 + \cdots\right)\left(Id - tC + \frac{t^2}{2!}C^2 - \frac{t^3}{3!}C^3 + \cdots\right) \\
&= X\left(Id + t^2(AB - BA) + \frac{t^3}{2!}(A^2B + BA^2 + 2BAB - 2ABA - AB^2 - B^2A)\right) \\
&\quad \left(Id + tC + \frac{t^2}{2!}C^2 + \frac{t^3}{3!}C^3 + \cdots\right)\left(Id + t^2(BA - AB) + \frac{t^3}{2!}(AB^2 + B^2A\right. \\
&\quad \left. + 2ABA - 2BAB - A^2B - BA^2)\right)\left(Id - tC + \frac{t^2}{2!}C^2 - \frac{t^3}{3!}C^3 + \ldots\right) \\
&= X\left(Id + tC + \frac{t^2}{2!}\left(C^2 + 2AB - 2BA\right) + \frac{t^3}{3!}(C^3 + 6ABC - 6BAC + 3A^2B + 3BA^2\right. \\
&\quad \left. + 6BAB - 6ABA - 3AB^2 - 3B^2A)\right)\left(Id - tC + \frac{t^2}{2!}\left(C^2 + 2BA - 2AB\right)\right. \\
&\quad \left. - \frac{t^3}{3!}\left(C^3 + 6BAC - 6ABC - 3A^2B - 3B^2A + 6BAB - 6ABA + 3A^2B + 3BA^2\right)\right)
\end{aligned}$$

$$= X\left(Id + t^3(AB - BA)C - C(AB - BA)\right)$$
$$= X\left(Id + t^3[[A, B], C] + \cdots\right).$$

Hence,

$$\gamma(\sqrt[3]{t}) = X\left(Id + t[[A, B]C] + \cdots\right),$$

is a smooth curve at $t = 0$, and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(\sqrt[3]{t}) = [[A, B], C].$$

Thus, we see that these types of vector fields form a tangent space to the symmetric space at the base point and is called Lie triple system.

Special types of symmetric spaces can also be constructed as follows:
A symmetric space is associated to an involutive automorphism of a given Lie algebra $\mathfrak{g}$. To be more specific, if σ is an automorphism then it preserves the multiplication: $[\sigma(x), \sigma(y)] = \sigma([x, y])$. Let the linear automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ be such that $\sigma^2 = 1$, however σ is not the identity, which implies σ has eigen values ± 1 and it splits the algebra $\mathfrak{g}$ into orthogonal eigen spaces corresponding to these eigen values. Such type of mapping is known as an involutive automorphism.

Let σ be an involutive automorphism of a compact simple Lie algebra $\mathfrak{g}$ and $\mathfrak{g} = k \bigoplus p$ where

$$\sigma(y) = y, \quad \forall y \in k,$$
$$\sigma(y) = -y, \quad \forall y \in p.$$

It is simple to verify that k is a subalgebra whereas p is not. Moreover, the commutation relation

$$[k, k] \subset k, \quad [k, p] \subset k, \quad [p, p] \subset k \tag{2}$$

holds. A subalgebra 'k' satisfying (2) is called a symmetric subalgebra. When the elements of p are multiplied by i (weyl unitary trick), a new non compact algebra $\mathfrak{g}^* = k + p^*$ is obtained. This is called Cartan decomposition and 'k' is the maximal compact subalgebra of $\mathfrak{g}^*$. The coset spaces $P \simeq e^p \simeq G/K$, where $G \simeq e^{\mathfrak{g}}$, is a symmetric space. Similarly G^*/K is also a symmetric space. The corresponding Lie triple systems are denoted by $\mathfrak{g}/k$ and $\mathfrak{g}^*/k$ respectively. For example

$$G/K = SU(n, \mathbb{C})/SO(n, \mathbb{R});$$
$$G^*/K = SL(n, \mathbb{R})/SO(n, \mathbb{R});$$

are symmetric spaces of compact and non compact type respectively. These types of symmetric spaces has been classified by Cartan [10] and they corresponds to various random matrix ensembles.

The Lie triple systems associated with the above two types of symmetric spaces are denoted by

$$\mathfrak{g}/k = su(n, \mathbb{C})/so(n, \mathbb{R});$$
$$\mathfrak{g}^*/k = sl(n, \mathbb{R})/so(n, \mathbb{R});$$

respectively.

3 Driftless control system on a symmetric space and its controllability condition

To begin with, we introduce some basic concepts and definitions related with control system following standard literature on control theory.

Let S be a smooth n-dimensional manifold with tangent space at x denoted by T_xS. A general control system takes the form

$$\dot{x} = f(x, u)$$

where $x \in S$ denotes the state, $u \in \mathbb{R}$ is the control, and $f(\cdot, u)$, is a vector field on S for all u.

Suppose that f is locally Lipschitz relative to the second variable, then $\forall x \in S$ and $\forall u \in L^2([0, t_f], \mathbb{R}^m)$, the cauchy problem

$$\dot{x}(t) = f(x(t), u(t)), t \in [0, t_f],\ x(0) = x$$

has a unique solution

$$x(\cdot) = x(\cdot, x, u) : \left[0, t'_f\right] \longrightarrow S \text{ with } t'_f \leq t_f.$$

Mainly there are three types of controllability: global controllability, local controllability at an equilibrium point and local controllability along a reference trajectory. In this paper we extend the study of global controllability to symmetric spaces.

A control system of the form $\dot{x} = f(x, u)$ defined as above is said to be globally controllable on S if for any $x_1, x_2 \in S$ and $t_f > 0$, there exist a control $u \in L^2([0, t_f], \mathbb{R}^m)$ such that the solution of the cauchy problem starting at x_1 satisfy $x(t_f) = x_2$.

Now, given a family $\mathcal{F}$ of smooth vector fields on symmetric space S, denote $LTS\{\mathcal{F}\}$ as the Lie triple system generated by $\mathcal{F}$. It is the smallest vector subspace $\mathcal{V}$ of smooth vector fields containing $\mathcal{F}$, which satisfies

$$[[A, B], C] \in \mathcal{V},\ \forall A, B \in \mathcal{F},\ \forall C \in \mathcal{V}.$$

The sufficient condition of global controllability for a driftless control system on Lie group is given and proved independently by P. Rashevsky [35] and W. L. Chow [14]. More recently simpler proofs given by F. Jean [21] and L. Rifford [37] separately. In this paper we have extended Rashevsky and Chow's theorem to symmetric spaces following Jean closely, which we call extended Rashevsky and Chow's theorem for symmetric spaces, which is explained below:

Extended Rashevsky-Chow's Theorem. Let S be a smooth connected n-manifold of symmetric space and $\{X_1, X_2, \ldots, X_m\}$ be m smooth vector fields on S. Assume that

$$LTS\{X_1, X_2, \ldots, X_m\}(x) = T_xS, \quad \forall x \in S.$$

Then the control system

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x),$$

is globally controllable on S.

The controllability of a driftless control system on a symmetric space is characterized by the properties of the Lie triple system generated by the vector fields $\{X_1, X_2, \ldots, X_m\}$.

Let $V_F(S)$ denote the set of all smooth vector fields on S and Θ^1 be the linear subspace of $V_F(S)$ generated by vectors fields $\{X_1, \ldots, X_m\}$,

$$\text{i.e.,} \Theta^1 = span\{X_1, \ldots, X_m\}.$$

For $p \geq 1$, define

$$\Theta^{p+1} = \Theta^p + \left[\left[\Theta^1, \Theta^p\right], \Theta^1\right]$$

where

$$\left[\left[\Theta^1, \Theta^p\right], \Theta^1\right] = span\left\{[[A, B], C]] : A, C \in \Theta^1, B \in \Theta^p\right\}.$$

The vector fields $\{X_1, \ldots, X_m\}$ generates a Lie triple system, which is defined as

$$LTS\{X_1, \ldots, X_m\} = \bigcup_{p\geq 1} \Theta^p.$$

From the Jacobi identity property of Lie triple system, $LTS\{X_1, \ldots, X_m\}$ is the smallest linear subspace of $V_F(S)$ which contains $\{X_1, \ldots, X_m\}$ and is invariant under Lie double brackets.

Let $\mathcal{L}$ be the free Lie algebra generated by the elements $\{1, \ldots, m\}$. Here, $\mathcal{L}$ is the $\mathbb{R}$ vector space generated by $\{1, \ldots, m\}$ and their bracket $[\cdot, \cdot]$ combined with relation of skew symmetry and the Jacobi identity.

The length of an element I of $\mathcal{L}$ is denoted by $|I|$ and is defined inductively by $|I| = 1$ for $I = 1, \ldots, m$, $|I| = |I_1| + |I_2|$ for $I = [I_1, I_2]$ and $|I| = |I_1| + |I_2| + |I_3|$ for $I = [[I_1, I_2], I_3]$. With every element $I \in \mathcal{L}$ associate the vector field $X_I \in LTS\{X_1, \ldots, X_m\}$ obtained by plugging in X_i, $i = 1, 2, \ldots, m$ for the corresponding letter i in I. For example $X_{[1,2,3]} = [[X_1, X_2], X_3]$.

Due to Jacobi identity property of Lie triple system

$$\Theta^p = span\{X_I : |I| \leqslant p\}.$$

For $x \in S$, we set $LTS\{X_1, \ldots, X_s\}(x) = \{X(x) : X \in LTS\{X_1, \ldots, X_m\}\}$ and for $p \geqslant 1$, $\Theta^p(x) = \{X(x) : X \in \Theta^p\}$. By definition, these sets are linear subspaces of $T_x S$.

We say that the control system

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x), \tag{3}$$

and the vector fields $\{X_1, \ldots, X_m\}$ satisfies Chow's condition if $LTS\{X_1, \ldots, X_m\}(x) = T_x S$, $\forall x \in S$.

Equivalently, for any $x \in S$ $\exists$ an integer $r = r(x)$ such that $dim \Theta^r(x) = n$.

If (3) satisfies Chow's condition, then $\forall x \in S$, the reachable set R_x is a neighbourhood of x.

Now we confine ourselves to a small neighbourhood of x, i.e., $U_x \subset S$ which can be identified with a neighbourhood of 0 in $\mathbb{R}^n$.

Let $\Psi_t^i = exp(tX_i)$, $i = 1, 2, \ldots, m$ be the flow of the vector field X_i. Each curve $t \longmapsto \Psi_t^i(x_1)$ is a trajectory of (3) and we have

$$\Psi_t^i = id + tX_i + O(t).$$

For every element $I \in \mathcal{L}$, we define the local diffeomorphism Ψ_t^I on U_x by induction on the length $|I|$ of I : if $I = [I_1, I_2], I_3]$ then

$$\begin{aligned} \Psi_t^I &= \left[\left[\Psi_t^{I_1}, \Psi_t^{I_2}\right], \Psi_t^{I_3}\right] \\ &= \Psi_{-t}^{I_3} \circ \Psi_{-t}^{I_1} \circ \Psi_{-t}^{I_2} \circ \Psi_t^{I_1} \circ \Psi_t^{I_2} \circ \Psi_t^{I_3} \circ \Psi_{-t}^{I_2} \circ \Psi_{-t}^{I_1} \circ \Psi_t^{I_2} \circ \Psi_t^{I_1}. \end{aligned}$$

By construction, Ψ_t^I may be expanded as a composition of flows of the vector fields X_i, $i = 1, 2, \ldots, m$. As a result $\Psi_t^I(x_1)$ is the end point of a trajectory of (3) induced from x_1. Further, on a neighbourhood of x there holds

$$\Psi_t^I = id + t^{|I|} X_I + O\left(t^{|I|}\right). \tag{4}$$

A diffeomorphism now can be defined whose derivative with respect to the time is exactly X_I and is given by

$$\psi_t^I = id + tX_I + O(t) \tag{5}$$

and $\psi_t^I(x_1)$ is the endpoint of a trajectory of (3) starting from x_1.

Now we choose double commutators $X_{I_1}, X_{I_2}, \dots, X_{I_n}$ whose values at x span $T_x S$. This is feasible through Chow's condition. Let ϕ be a map defined on a small neighbourhood U_0 of 0 in $\mathbb{R}^n$ by

$$\phi(t_1, \dots, t_n) = \psi^{I_n}_{t_n} \circ \cdots \circ \psi^{I_1}_{t_1}(x_1) \in S.$$

We infer from (5) that this map is C^1 near 0 and has an invertible derivative at 0, which implies that it is a local C^1-diffeomorphism. Hence $\phi(U_0)$ contains a neighbourhood of x.

Now, $\forall t \in U_0,\ \phi(t)$ is the endpoint of a concatenation of trajectories of (3), the first one starting from x. It is then the endpoint of a trajectory starting from x. Hence, $\phi(U_0) \subset R_x$, which implies that R_x is a neighbourhood of x.

Let $x_1 \in S$ and if $x_2 \in R_{x_1}$, then $x_1 \in R_{x_2}$. As a result, $R_{x_1} = R_{x_2}$ for any $x_2 \in S$ and by the above result, R_{x_1} is an open set. Therefore the union of the sets R_{x_1} which are pairwise disjointed covers the manifold S. Since S is a connected manifold, there can only be one open set which can cover S, i.e., R_{x_1}. Hence, any two points in S can be connected by trajectories of (3).

Thus we see that as (3) satisfies the Chow's condition, $\forall x \in S$, the reachable set R_x is a neighbourhood of x. So now if S is connected and if (3) satisfies the Rashevsky-Chow's condition then any two points of S can be connected by a trajectory of (3) and we obtain the extended Chow's theorem.

4 Applications

Immediate examples of symmetric spaces which are applied directly to physical applications are symmetric submanifolds of special Euclidean group $SE(3)$ which are related to various kinesiological and mechanical systems and accordingly, have numerous potential applications in robot kinematics [42,43]. Similarly, as discussed earlier the random matrix ensembles like GOE, GUE and circular ensembles have a lot application in quantum transport problems, etc. So in this section we have studied the controllability aspect of some symmetric spaces. But to start with, we have simplest example of symmetric space $SO(3)/SO(2) \simeq S^2$. Each element of $SO(3)/SO(2)$ can be fully characterized by three real parameters such that their moduli sum to 1 and then there is a one-to-one correspondence between each element of $SO(3)/SO(2)$ and a set of Cartesian coordinates for S^2.

4.1 Controllability on $SO(3)/SO(2)$

The infinitesimal generators of the Lie algebra $so(3)$ of Lie group $SO(3)$, corresponds to the derivative of rotation around the each of the standard axes, evaluated at the identity, which are

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The infinitesimal generators of the Lie algebra $so(2)$ of Lie group $SO(2)$ correspond to the derivative of rotation evaluated at the identity, which is

$$X = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

This result for infinitesimal generator for rotation about the z- axis i.e. X_3 is essentially identical to those of $SO(2)$. So the generators of $so(3)/so(2)$ are $\{X_1, X_2\}$ and

$$[X_1, X_2] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X_3.$$

Also,

$$[[X_1, X_2], X_1] = [X_3, X_1] = X_2$$

and

$$[[X_1, X_2], X_2] = [X_3, X_2] = -X_1.$$

We can verify that $\{X_1, X_2\}$ form a Lie triple system.

Now we consider a driftless control system of the following type in classical notation [38]

$$\dot{x} = x \sum_{i=1}^{2} X_i u_i, \; x \in S \tag{6}$$

where $x \in SO(3)/SO(2)$, u_i are controls on it and X_i are smooth vector fields on the given symmetric space.

Let $\Theta^1 = span\{X_1, X_2\}$, for $p > 1$, $\Theta^{p+1} = \Theta^p + \left[\Theta^1, \left[\Theta^1, \Theta^p\right]\right]$, so $\Theta^2 = \Theta^1 + \left[\Theta^1, \left[\Theta^1, \Theta^1\right]\right]$. Let $A, B, C \in \Theta^1$ be arbitrary, then we can write $A = \alpha_1 X_1 + \beta_1 X_2$, $\quad B = \alpha_2 X_1 + \beta_2 X_2$, $\quad C = \alpha_3 X_1 + \beta_3 X_2$, $\quad \alpha_i, \beta_i, i \in \{1, 2, 3\}$ are scalars.
Now, $\left[\Theta^1, \left[\Theta^1, \Theta^1\right]\right]$

$$\begin{aligned}
&= [\alpha_1 X_1 + \beta_1 X_2, [\alpha_2 X_1 + \beta_2 X_2, \alpha_3 X_1 + \beta_3 X_2]] \\
&= [\alpha_1 X_1 + \beta_1 X_2, ([\alpha_2 X_1, \alpha_3 X_1] + [\alpha_2 X_1, \beta_3 X_2] + [\beta_2 X_2, \alpha_3 X_1] + [\beta_2 X_2, \beta_3 X_2])] \\
&= [\alpha_1 X_1 + \beta_1 X_2, \alpha_2 \beta_3 [X_1, X_2] + \beta_2 \alpha_3 [X_2, X_1]] \\
&= [\alpha_1 X_1 + \beta_1 X_2, (\alpha_2 \beta_3 - \beta_2 \alpha_3) X_3] \\
&= \left[\alpha_1 X_1 + \beta_1 X_2, \gamma X_3\right], \text{ where } \gamma = \alpha_2 \beta_3 - \beta_2 \alpha_3 \\
&= \alpha_1 \gamma [X_1, X_3] + \beta_1 \gamma [X_2, X_3] \\
&= \alpha_1 \gamma X_1 + \beta_1 \gamma X_2 \in \Theta^1 \\
\text{So, } \Theta^2 &= \Theta^1 + \left[\Theta^1, \left[\Theta^1, \Theta^1\right]\right] \\
&= \Theta^1
\end{aligned}$$

Hence, number of generators of $\Theta^2 = 2$. This implies dim $\Theta^2 = 2$.
And hence $so(3)/so(2)$ satisfies the Chow's condition as dim $\Theta^2 = 2$ i.e., dim Θ^2 is same as the dimension of $SO(3)/SO(2)$. So we conclude that the control system defined in (6) is controllable by Chow–Rashevsky's theorem. This is a trivial case, however we have done it in detail and the result coincides with the paper by Brockett for controllability on S^n sphere [9,30,38].

4.2 Controllability on symmetric submanifolds of SE(3)

The special Euclidean group $SE(3)$ admits an inversion symmetry through any of its elements hence it is a symmetric space. The Lie algebra of $SE(3)$ is $se(3)$ whose basis elements are these 4×4 matrices, each of which corresponds to either infinitesimal rotations or infinitesimal translations along each axis:

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$e_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, e_6 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There is one to one correspondence between Lie triple systems of $SE(3)$ and the symmetric submanifolds of $SE(3)$. Therefore, the classification of symmetric submanifolds of $SE(3)$ upto conjugation is equivalent to that of the conjugacy class of Lie triple system of $SE(3)$. There are seven conjugacy classes of symmetric submanifolds of $SE(3)$, all of which can be locally represented by $exp\ M$ with M being a Lie triple (sub)system of $se(3)$. As discussed earlier, the Lie triple system are vector subspace of $se(3)$ closed under double Lie brackets. The Lie triple systems of $se(3)$ [43] are $\{e_3, e_4\}$, $\{e_3, e_4 + \mathsf{p}e_1\}$, $\{e_4, e_5\}$, $\{e_1, e_3, e_4\}$, $\{e_3, e_4, e_5\}$, $\{e_1, e_2, e_4, e_5\}$ and $\{e_1, e_2, e_3, e_4, e_5\}$ here p takes an arbitrary finite value.

From control theory point of view, a system is controllable on a space with all available controls in hand. But generally we are interested in controlling a system with fewer number of controls. In this case we see that the following two systems are controllable on the given symmetric spaces.

1. A left invariant driftless control system defined on the symmetric submanifold of $SE(3)$, represented by $M_1 = exp\,\{e_1, e_2, e_4, e_5\}$ is given by

$$\dot{x} = x\,(e_1u_1 + e_2u_2 + e_4u_4 + e_5u_5)\,, \quad x \in M_1,\ u_i, i = 1, 2, 4, 5 \in \mathbb{R}. \tag{7}$$

   Since, $[[e_2, e_4], e_5] = e_1$, so the three generators $\{e_2, e_4, e_5\}$ upon double bracket commutation generates the full set of basis elements of $LTS\,\{e_1, e_2, e_4, e_5\}$. So the system (7) can be controlled with fewer number of controls by Rashvesky-Chow's theorem and is represented as

$$\dot{x} = x\,(e_2u_2 + e_4u_4 + e_5u_5)\,, \quad x \in M_1,\ u_i, i = 2, 4, 5 \in \mathbb{R}. \tag{8}$$

2. Similarly, a left invariant driftless control system defined on the symmetric submanifold of $SE(3)$, represented by $M_2 = exp\,\{e_1, e_2, e_3, e_4, e_5\}$ is given by

$$\dot{x} = x\,(e_1u_1 + e_2u_2 + e_3u_3 + e_4u_4 + e_5u_5)\,, \quad x \in M_2,\ u_i, i = 1, \ldots, 5 \in \mathbb{R}. \tag{9}$$

   Since, $[[e_1, e_5], e_4] = e_2$, so, the four generators $\{e_1, e_3, e_4, e_5\}$ upon double bracket commutation generates the full set of basis elements of $LTS\,\{e_1, e_2, e_3, e_4, e_5\}$. So the system (9) can be rewritten as

$$\dot{x} = x\,(e_1u_1 + e_3u_3 + e_4u_4 + e_5u_5)\,, \quad x \in M_2,\ u_i, i = 1, 3, 4, 5 \in \mathbb{R} \tag{10}$$

   and it is controllable by Rashvesky–Chow's theorem.

4.3 Controllability on symmetric spaces associated with random matrices

Random matrix theory deals with the study of matrix ensembles, i.e., matrices with a probability measure. The classical non-compact matrix Gaussian ensembles are GOE, GUE and GSE. These ensembles arises when a Gaussian-like probability measure is introduced in each set of a family of non-compact matrices. Each set of matrices, along with the probability measure on it, is invariant under the action of the orthogonal, unitary or symplectic group, respectively. The ensembles above have compact counterparts the Circular Orthogonal Ensemble (COE), the Circular Unitary Ensemble (CUE) and the Circular Symplectic Ensemble (CSE), leaving the probability measure invariant. The members of all the six ensembles discussed are submanifolds of Euclidean space. In fact, they are (essentially) Riemannian globally symmetric spaces. The integration manifolds of the random matrices, the distribution of eigenvalues and the Dyson characterizing the ensembles [19] are strictly in correspondence with symmetric spaces. This identification can lead to a number of critical outcomes.

A new classification of random matrix ensembles arises from the Cartan classification of triplets of symmetric spaces with positive, zero and negative curvature. Now we discuss controllability on two examples of GOE and COE respectively.

4.3.1 Gaussian Orthogonal Ensemble (GOE)

For each n, the set of $n \times n$ real symmetric matrices, whose entries are independent normal random variables, form the ensemble GOE i.e.,

$$S_{GOE}(n) = \left\{A = (a_{ij})_{n\times n} = (a_{ji})\right\} \tag{11}$$

with the probability measure

$$d\mu(A) \propto exp\left(-tr\frac{A^2}{2}\right)dA \tag{12}$$

where $dA = \prod_{i\leqslant j} da_{ij}$ is the (additive) Haar measure on S_{GOE}. The nonsingular matrices in (11) form an open set of full measure, one of whose components S consists of all the real symmetric positive definite $n \times n$ matrices. It is known the symmetric positive definite matrices [31] are symmetric spaces, with symmetric product $a \cdot b = ab^{-1}a$ and its Lie triple system are the set of symmetric matrices. The Lie triple system of the 3×3 symmetric positive random matrix, are the set of 3×3 symmetric matrices, whose basis elements are

$$a_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$a_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, a_4 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$a_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, a_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

So a driftless control system on the symmetric space S of symmetric positive definite matrices with six number of controls is given by

$$\dot{x} = x\sum_{i=1}^{6} a_i u_i,\ x \in S,\ u_i \in \mathbb{R}. \tag{13}$$

Since, $[[a_1, a_4], a_4] = -2a_2, [[a_2, a_4], a_4] = a_6$ and $[[a_2, a_6], a_4] = a_5$. The three basis element $\{a_1, a_3, a_4\}$ upon double bracket commutation generates the full set of basis elements of the 3×3 symmetric matrices. So now the control system with fewer number of controls is given by

$$\dot{x} = x\,(a_1u_1 + a_3u_3 + a_4u_4)\,,\quad x \in S,\ u_i, i = 1, 3, 4 \in \mathbb{R}. \tag{14}$$

It is controllable on given GOE by Rashvesky-Chow's theorem. The maximum number of controls is 6. We can construct a number of control systems with different combinations with fewer number of controls, but not all such systems are controllable. In this case, control systems with two controls, will not be controllable. In particular, if we take the control system (14) as above then we see that only with these three number of controls, we can control the whole system defined in (13).

4.3.2 Circular Orthogonal Ensemble (COE)

In $SU(n)$, the subgroup $SO(n)$ is the fixed-point set of the involution

$$a \mapsto a^\sigma := \left(a^{-1}\right)^t.$$

The space $SU(n)/SO(n)$ can be realized as the set of matrices

$$S(n) = \left\{A \in SU(n)|\ A \text{ is symmetric, i.e., } A = A^T\right\}. \tag{15}$$

The ensemble $S(n)$ endowed with its Haar probability measure is Dyson's Circular Orthogonal Ensemble (COE) [11] i.e., the integration manifold of the COE is $SU(n)/SO(n)$. If we have consider the case of $n = 3$, the group $SU(3)$ is characterized by 3×3 unitary matrices with determinant 1. The generators are Hermitian matrices 3×3 with zero trace. Such general Hermitian matrix can be parametrized with eight real numbers $a, \ldots, h$

$$\begin{bmatrix} a & c - id & e - if \\ c + id & b & g - ih \\ e + if & g + ih & -a - b \end{bmatrix}.$$

The Lie algebra $su(3)$ of $SU(3)$ is defined as the collection of 3×3 anti-Hermitian square matrices having trace zero. The following Gell-Mann matrices, are the generators of $su(3)$

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_2 = i \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$A_5 = i \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, A_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
$$A_7 = i \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, A_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

The generators $\{A_2, A_5, \text{ and } A_7\}$ of $su(3)$ are essentially identical to the generators of $so(3)$. So, the generators of $su(3)/so(3)$ are $\{A_1, A_3, A_4, A_6 \text{ and } A_8\}$. So a driftless control system on the symmetric space $SU(3)/SO(3)$ is given by

$$\dot{x} = x\,(A_1u_1 + A_3u_3 + A_4u_4 + A_6u_6 + A_8u_8)\,, \quad x \in SU(3)/SO(3),\ u_i, i = 1, 3, 4, 6, 8 \in \mathbb{R}. \tag{16}$$

Since, $[[A_1, A_3], A_4] = 2A_6$, so, the four generators $\{A_1, A_3, A_4, A_8\}$ upon double bracket commutation generates the full set of basis elements of $su(3)/so(3)$. So the system (16) can be controlled with fewer number of controls and the system can be given by

$$\dot{x} = x\,(A_1u_1 + A_3u_3 + A_4u_4 + A_8u_8)\,, \quad x \in SU(3)/SO(3),\ u_i, i = 1, 3, 4, 8 \in \mathbb{R}. \tag{17}$$

In this case, control systems with three controls, will not be controllable. But, if we take the control system (17) as above then we see that only with these four number of controls, we can control the whole system defined in (16).

5 Concluding remarks

In this article, we have extended Rashvesky-Chow's theorem for global controllability condition on Lie groups to that of symmetric spaces and shown that how it can be readily and directly implemented for driftless nonlinear control systems on various physical system of symmetric submanifold of $SE(3)$ and random matrix ensembles. The extension of Rashvesky-Chow's theorem from Lie groups to symmetric space is not straight forward. Lie group is a group but symmetric space is not. The tangent space to the group at identity element is a Lie algebra (with a bilinear Lie bracket), but the tangent space to the symmetric space at the base point is a Lie triple system (with a trilinear Lie double bracket). For Lie groups, the Lie algebra are obtained through four operations of flows on the manifold, while in case of symmetric spaces it involves ten operations of flow on the manifold. Now it is quite interesting to extend such type of studies for Hamiltonian formalism [6] (using Lie double brackets) and using various analytical and numerical integrators [15,36] to understand the system dynamics. These type of studies can also be used to understand the controllability of different other physical systems such as quantum systems, molecular systems, etc. [4,33,34].

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