Arab. J. Math. (2024) 13:63–78
https://doi.org/10.1007/s40065-023-00444-x

Arabian Journal of Mathematics

Kheireddine Biroud

A nonlocal type problem involving a mixed local and nonlocal operator

Received: 25 April 2023 / Accepted: 30 August 2023 / Published online: 22 September 2023

Abstract In this paper, we consider the nonlocal elliptic problem involving a mixed local and nonlocal operator,

$$(P)\begin{cases} \left(\int\limits_{\Omega} f(x,u)dx\right)^{\beta} \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u) = f^{\alpha}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $\mathfrak{L}_{\mathfrak{p},\mathfrak{s}} \equiv -\Delta_p + (-\Delta)^s_p$, $0 < s < 1 < p < N$, $\alpha,\ \beta \in \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonnegative function which is defined almost everywhere with respect to the variable x. Using Schauder and Tychonoff fixed point theorems, we get two existence theorems of weak positive solutions under some hypothesis on α, β and f.

Mathematics Subject Classification 35R11 · 35A01 · 35B09

1 Introduction

In this work, we deal with the following nonlocal elliptic problem involving a mixed local and nonlocal operator,

$$\begin{cases} \left(\int\limits_{\Omega} f(x,u)dx\right)^{\beta} \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u) = f^{\alpha}(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.0}$$

where $\Omega \subset \mathbb{R}^N$ be a bounded regular domain with $0 < s < 1 < p < N$, $(\alpha,\ \beta) \in \mathbb{R}^2$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a nonnegative function.

Here $\mathfrak{L}_{\mathfrak{p},\mathfrak{s}} \equiv -\Delta_p + (-\Delta)^s_p$ where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is standard $p-$Laplace operator and $(-\Delta)^s_p$ denotes the so-called *fractional* p*−Laplacian* operator, is defined as,

$$(-\Delta)^s_p u(x) := \text{P.V} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}}\, dy,$$

K. Biroud (✉)
Laboratoire d'Analyse Nonlinéaire et Mathématiques Appliquées, École Supérieure de Management, No. 01, Rue Barka Ahmed Bouhannak Imama, 13000 Tlemcen, Algeria
E-mail: kh_biroud@yahoo.fr

where P.V. stands for Cauchy principal value of the integral.

In this paper, we assume two sets of assumptions.

• First set,

$(\mathcal{H}_1)$ $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a function such that $t\longmapsto f(x,t)$ is continuous for a.e in Ω, $x\longmapsto f(x,t)$ is measurable for all $t\in\mathbb{R}$ and for some nonnegative function $g\in L^1(\Omega)$, we have

$$0<f(x,t)\le g(x)\quad \text{for all } t\in\mathbb{R} \text{ and a.e } x\in\Omega,$$

and $\alpha,\ \beta$ satisfy one of the two following assumptions

$(\mathcal{H}_2)$ $0\le\alpha\le\dfrac{1}{p'}$ where $p'=\dfrac{p}{p-1}$, $(p'>1)$ and $\beta\le\alpha$;

$(\mathcal{H}_3)$ $\alpha>\dfrac{1}{p'}$, $\beta\le\dfrac{1}{p'}$ with $g(x)\le 1$ a.e $x\in\Omega$.

• In the second set, we assume that

$(\mathcal{G}_1)$ $x\longmapsto f(x,t)$ is measurable for all $t\in\mathbb{R}$ and

$$f(x,t)>0\quad \text{for all } t\in\mathbb{R} \text{ and a.e } x\in\Omega,$$

.

$(\mathcal{G}_2)$ $f(.,0)\in L^1(\Omega)$, $|f(x,t_1)-f(x,t_2)|\le h_\delta(x)|t_1-t_2|^\delta$ for all $t_1,t_2\in\mathbb{R}$ and a.e $x\in\Omega$ such that

$(\mathcal{G}_3)$ $0<\delta\le p$, $h_\delta\in L^{\frac{p}{p-\delta}}(\Omega)$ for $\delta\in(0,p)$, $h_p\in L^\infty(\Omega)$, $0<\beta\le\alpha\le\dfrac{1}{p'}$ with $p'>1$.

In recent years, the problems driven by operators like $\mathcal{L}_{p,s}$ have attracted a huge amount of attention, both for the mathematical complications that the combination of two so different operators imply and for a wide range of applications, for example in [25], these operators in the case $p=2$, describe a biological species whose individuals diffuse either by a random walk or by a jump process, according to prescribed probabilities. For other applications in the biological field generated by mixed operators, we refer readers [24,38,39]. See also [11] for further applications.

Moreover, several studies have been carried out on mixed local and nonlocal operators from different points of view, like the regularity theory, existence and non-existence results, calculus of variations, shape of optimization and eigenvalue problems see for instance [1,5–10,12,15,20,21,23,32,41,42] and the references therein.

Before stating our main results, we begin by recalling some well-known results related to our problem.

– *Local case:* $s=1$. In this case, we consider the following problem,

$$\begin{cases} -\Big(\displaystyle\int_\Omega f_1(x,u)dx\Big)^\beta \Delta_p u=\rho^\gamma(x)f_2^\alpha(x,u) \text{ in } \Omega,\\ \qquad\qquad u>0 \qquad\qquad \text{in } \Omega,\\ \qquad\qquad u=0 \qquad\qquad \text{in } \partial\Omega,\end{cases}\tag{1.1}$$

where f_1, f_2, ρ are measurable functions, a β and γ are real parameters. Several works are devoted to classes of problems (1.1).

In the case of semilinear problem corresponding to $p=2,\gamma=0$ and $f_1(x,t)\equiv f_2(x,t)\equiv f(t)$ depends only on the variable of t, has been treated in [31], where the authors have used a fixed-point theorem in cone, to show that problem (1.1) has at least one positive solution, in which f is a positive continuous, nondecreasing function and satisfies a boundedness condition. In [18], the authors proved the existence and uniqueness results of problem (1.1) for $p=2,\gamma=1$ and under some conditions on the functions f_1, f_2, ρ and the constants α,β. In the case, $p=2,\gamma=\alpha=1$, $f_1(x,u)=f_1(u)$, $f_2(x,u)=f_2(u)$ and $\rho(x)=\sigma=const>0$, problem (1.1) was treated in [3], where the authors have showed the existence of positive solution using Bolzano theorem. Recently in [45], the author gave sufficient conditions on the function f and the real parameters α,β

in order to have an existence of positive solutions via Schauder and Tychonoff fixed point theorems for the following problem

$$\begin{cases} -\Big(\int\limits_{\Omega} f(x,u)dx\Big)^{\beta} \operatorname{div}(a(x)\nabla u) = f^{\alpha}(x,u) \text{ in } \Omega, \\ \qquad\qquad u > 0 \qquad \text{in } \Omega, \\ \qquad\qquad u = 0 \qquad \text{in } \partial\Omega, \end{cases} \tag{1.2}$$

where $a(x)$ be an $n \times n$ matrix function defined a.e in Ω.

For other classes of nonlocal elliptic problems namely for $p \in (1,\infty)$ see for instance [14,16,17,33] and the references therein.

Non-local case: $0 < s < 1$. In this case, the Kirchhoff-type problems involving the non-local integro-differential operator has been widely studied in recent years, we refer to [4,13,27,35,36,40,43,44] and the references therein.

Needless to say, the references mentioned above do not exhaust the rich literature on the subject.

Mixed local and non-local case. Unlike the two cases above, Kirchhoff-type problems with mixed local and non-local operators are even less known to our knowledge.

Inspired by the work [45], we will try to reproduce the same reasoning applied to a class nonlocal elliptic problem (1.2) and adapt it to our problem with mixed local and nonlocal operator. As far as we are aware, our main results are new even in the semilinear case $p = 2$.

Notice that, the two sets of assumptions mentioned above $(\mathcal{H}_1)-(\mathcal{H}_3)$ and $(\mathcal{G}_1)-(\mathcal{G}_3)$ are purely technical, so they allowed us to apply the fixed-point theorems of Tychonoff and Schauder respectively, to obtain existence results (see sect. 3).

The article is organized as follows. In Sect. 2 , we collect some preliminaries dealing with the functional setting associated to our problem, like the concepts of solutions and useful lemmas are included that will be needed along of the paper. In the last Section, by using the Schauder and Tychonoff fixed point theorems, we prove the main existence results of this work under two sets of hypothesis on α, β, f and g.

2 The functional setting and tools

In this section, we collect some well-known results on Sobolev spaces and give some tools as they are needed to prove our main results.

Let $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ and $0 < s < 1 < p < \infty$ be the real numbers. The fractional Sobolev space is defined by,

$$W^{s,p}(\Omega) = \Big\{u \in L^p(\Omega) : \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy < +\infty\Big\}.$$

Notice that $W^{s,p}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} = \Big(||u||_{L^p(\Omega)} + \int_{\Omega}\int_{\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy\Big)^{\frac{1}{p}}.$$

We denote by $W_0^{s,p}(\Omega)$ the following space,

$$W_0^{s,p}(\Omega) = \Big\{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega\Big\},$$

endowed with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \Big(\iint_{D_\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy\Big)^{1/p},$$

where

$$D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega) = (\Omega \times \mathbb{R}^N) \cup (\mathcal{C}\Omega \times \Omega),$$

is reflexive Banach space we refer to [2,22,37] for more details and properties of the fractional Sobolev spaces.

Now we define, for $1 < p < \infty$, the Sobolev space

$$W^{1,p}(\Omega) = \Big\{u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega)\Big\},$$

endowed with the classical norm,

$$||u||_{W^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||\nabla u||_{L^p(\Omega)}.$$

Notice that $(W^{1,p}(\Omega), ||.||_{W^{1,p}(\Omega)})$ is reflexive Banach space.

The space $W_0^{1,p}(\Omega)$ is defined as the closure of the space $C_0^\infty(\Omega)$ of smooth functions with compact support in the norm of the Sobolev space $W^{1,p}(\Omega)$.

The following result asserts that Sobolev space $W^{1,p}(\Omega)$ is continuously embedded in the fractional Sobolev space, see [22] for more details.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be bounded Lipschitz domain and $0 < s < 1 < p < \infty$, then, there exists a constant $C = C(N, p, s)$ such that*

$$||u||_{W^{s,p}(\Omega)} \leq C||u||_{W^{1,p}(\Omega)}, \quad \forall u \in W^{1,p}(\Omega). \tag{2.3}$$

We need also the following result, where the proof can be found in [12],

Lemma 2.2 *Under the same hypothesis of the previous lemma, there exists a constant $C = C(N, p, s, \Omega)$ such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy \leq C\int_\Omega |\nabla u|^p dx, \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.4}$$

Remarks 2.3 It is clear that from the previous lemma, the following norm on the space $W_0^{1,p}(\Omega)$ defined by

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}} \tag{2.5}$$

is equivalent to

$$||u||_{W_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}. \tag{2.6}$$

For the following Sobolev embedding, see for example [26,34].

Lemma 2.4 *The following embedding operators*

$$W_0^{1,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & \text{for } q \in [1, p^*], \text{ if } p \in (1, N) \\ L^q(\Omega), & \text{for } q \in [1, \infty), \text{ if } p = N \\ L^\infty(\Omega), & \text{if } q > N, \end{cases}$$

are continuous. Also, the above embeddings are compact, except for $q = p^* = \frac{pN}{N-p}$ if $p \in (1, N)$.

Now, we define the notion of zero of Dirichlet boundary condition as follows,

Definition 2.5 We say that $u \leq 0$ on $\partial\Omega$, if $u = 0$ in $\mathbb{R}^N\backslash\Omega$ and for every $\varepsilon > 0$, we have

$$(u-\varepsilon)_+ \in W_0^{1,p}(\Omega).$$

We say that $u = 0$ on $\partial\Omega$, if u in nonnegative and $u \leq 0$ on $\partial\Omega$.

Now, we need to render precise the sense of the weak solution for the problem (1.0).

Definition 2.6 Assume that $u \in W_0^{1,p}(\Omega)$. We say that u is a weak solution to problem (1.0), if $u > 0$ in Ω, $u = 0$ on $\partial\Omega$ in the sense of Definition 2.5 and for every $\phi \in W_0^{1,p}(\Omega)$, we have

$$\int_\Omega \phi \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u)dx = \frac{\int_\Omega f^\alpha(x,u)\phi dx}{\left(\int_\Omega f(x,u)dx\right)^\beta}. \tag{2.7}$$

To close this section, we recall the following elementary algebraic inequality (see [19]) that will be used in some arguments.

Lemma 2.7 *Let $p > 1$. Then for every $a,\ b \in \mathbb{R}^N$, there exists a positive constant $C = C(p) > 0$ such that*

$$\left\langle |a-b|^{p-2}(a-b), a-b \right\rangle \geq C \frac{|a-b|^2}{(|a|+|b|)^{2-p}}. \tag{2.8}$$

3 Existence results

In this section, we focus to prove two existing results of nontrivial solutions to (1.0) by using, respectively, Tychonoff and Schauder fixed-point theorems under two sets of hypothesis on α, β and f.

We begin with the following result.

Theorem 3.1 *Let $s \in (0,1)$ and $1 < p < N$. Assume that $s \in (0,1)$, $(\mathcal{H}_1)$ and one of $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ hold. Then, problem (1.0) has a positive weak solution $u \in W_0^{1,p}(\Omega)$ in the sense of Definition 2.6.*

Proof The proof will be given in several steps.
Step 1: For $v \in W_0^{1,p}(\Omega)$, we consider the following problem,

$$\begin{cases} \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u) = \dfrac{f^\alpha(x,v)}{\left(\int_\Omega f(x,v)dx\right)^\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.9}$$

To show that the previous problem has a solution $u \in W_0^{1,p}(\Omega)$, we need to show that

$$h(v) := \frac{f^\alpha(x,v)}{\left(\int_\Omega f(x,v)dx\right)^\beta} \in (W_0^{1,p}(\Omega))'. \tag{3.10}$$

To do this, using $\phi \in W_0^{1,p}(\Omega)$ as a test function in problem (3.9), we get

$$|\langle h(v), \phi \rangle| \leq \frac{\int_\Omega f^\alpha(x,v)|\phi|dx}{\left(\int_\Omega f(x,v)dx\right)^\beta},$$

Using Hölder and Poincaré inequalities, it holds that

$$
\begin{aligned}
|\langle h(v), \phi\rangle| &\leq \frac{\left(\int\limits_{\Omega} f^{\alpha p'}(x,v)dx\right)^{\frac{1}{p}'}\left(\int\limits_{\Omega} |\phi|^p dx\right)^{\frac{1}{p}}}{\left(\int\limits_{\Omega} f(x,v)dx\right)^{\beta}} \\
&\leq \frac{C||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_{\Omega} f^{\alpha p'}(x,v)dx\right)^{\frac{1}{p}'}}{\left(\int\limits_{\Omega} f(x,v)dx\right)^{\beta}}.
\end{aligned}
$$

First, if we assume that $(\mathcal{H}_2)$ holds, it follows that

- if $\alpha \in (0, \frac{1}{p'})$ and $\alpha \leq \beta$, we have

$$
\begin{aligned}
|\langle h(v), \phi\rangle| &\leq \frac{C|\Omega|^{\frac{1-\alpha p'}{p'}}||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_{\Omega} f(x,v)dx\right)^{\alpha}}{\left(\int\limits_{\Omega} f(x,v)dx\right)^{\beta}} \\
&\leq C|\Omega|^{\frac{1-\alpha p'}{p'}}||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_{\Omega} f(x,v)dx\right)^{\alpha-\beta} \\
&\leq C|\Omega|^{\frac{1-\alpha p'}{p'}}||g||_{L^1(\Omega)}^{\alpha-\beta}||\phi||_{W_0^{1,p}(\Omega)};
\end{aligned}
\tag{3.11}
$$

- if $\alpha = \frac{1}{p'}$ and $\beta \leq \alpha$, we get

$$
\begin{aligned}
|\langle h(v), \phi\rangle| &\leq \frac{C||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_{\Omega} f(x,v)dx\right)^{\frac{1}{p'}}}{\left(\int\limits_{\Omega} f(x,v)dx\right)^{\beta}} \\
&\leq C||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_{\Omega} f(x,v)dx\right)^{\frac{1}{p'}-\beta} \\
&\leq C||g||_{L^1(\Omega)}^{\frac{1}{p'}-\beta}||\phi||_{W_0^{1,p}(\Omega)};
\end{aligned}
\tag{3.12}
$$

- if $\beta \leq \alpha = 0$, we obtain

$$
\begin{aligned}
|\langle h(v), \phi\rangle| &\leq C|\Omega|^{1-\frac{1}{p}}||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_{\Omega} f(x,v)dx\right)^{-\beta} \\
&\leq C|\Omega|^{1-\frac{1}{p}}||g||_{L^1(\Omega)}^{-\beta}||\phi||_{W_0^{1,p}(\Omega)}.
\end{aligned}
\tag{3.13}
$$

Now, if suppose that $(\mathcal{H}_3)$ holds, we get

$$
\begin{aligned}
|\langle h(v), \phi\rangle| &\leq \frac{\left(\int\limits_\Omega f^{\alpha p'}(x,v)dx\right)^{\frac{1}{p}'}\left(\int\limits_\Omega |\phi|^p dx\right)^{\frac{1}{p}}}{\left(\int\limits_\Omega f(x,u)dx\right)^{\beta}} \\
&\leq C||\phi||_{W_0^{1,p}(\Omega)}\left(\int\limits_\Omega f(x,v)dx\right)^{\frac{1}{p}'-\beta} \\
&\leq C||g||_{L^1(\Omega)}^{\frac{1}{p}'-\beta}||\phi||_{W_0^{1,p}(\Omega)}.
\end{aligned} \tag{3.14}
$$

Hence, combining with (3.11), (3.12), (3.14), (3.10) follows.

Now, using Theorem 1.1 of [29], this problem possess a unique solution $u \in W_0^{1,p}(\Omega)$.

Obviously, from $(\mathcal{H}_1)$, we derive that $u = 0$ is not solution to problem (3.9). On the other hand, to show that u is nonnegative solution, we choose u^- as a test function in (3.9), it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{N+ps}}dxdy \\
&+\int\limits_\Omega |\nabla u^-|^p dx = \frac{\int\limits_\Omega f^{\alpha}(x,v)u^- dx}{\left(\int\limits_\Omega f(x,v)dx\right)^{\beta}} \leq 0.
\end{aligned} \tag{3.15}
$$

Following the same arguments from the proof of Lemma 3.1 in [30], we get

$$
|u(x)-u(y)|^{p-2}(u(x)-u(y))(u^-(x)-u^-(y)) \geq 0, \quad \text{for a.e } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{3.16}
$$

Using (3.15) and (3.16), it follows that

$$
\int\limits_\Omega |\nabla u^-|^p dx \leq 0
$$

Hence, $u \geq 0$ in Ω. Since $u = 0$ in $\mathbb{R}^N\backslash\Omega$, we derive that the function u is nonnegative in $\mathbb{R}^N$. As $g(x,t) > 0$ for a.e $x \in \Omega$ and for all $t \in \mathbb{R}$. Then, by using theorem 8.3 of [28], we have that for every $\omega \Subset \Omega$, there exists a constant $C(\omega) > 0$ such that $u \geq C(\omega) > 0$ in Ω. Hence $u > 0$ in Ω.

Therefore, the operator

$$
\begin{aligned}
T : &W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega) \\
&v \mapsto T(v) = u.
\end{aligned}
$$

is well defined.

Step 2: Let us show $T(\overline{B}_R(0)) \subset \overline{B}_R(0)$ where $\overline{B}_R(0)$ is closed ball in $W_0^{1,p}(\Omega)$ with center 0 and radius R.

For this purpose, it is sufficient to prove that if u solves Problem (3.9), then

$$
||u||_{W_0^{1,p}(\Omega)} \leq R,
$$

where R will be precise later.

To do this, we use u as a test function in (3.9), we get

$$\int_\Omega |\nabla u|^p dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy = \frac{\int_\Omega f^\alpha(x,v)udx}{\left(\int_\Omega f(x,v)dx\right)^\beta}. \tag{3.17}$$

As in the first step, we assume that α, β satisfy $(\mathcal{H}_2)$ and we distinguish three cases. Therefore, by using Hölder and Poincaré inequalities, we get from (3.17),

- if $\alpha \in (0, \frac{1}{p'})$ and $\alpha \leq \beta$, we have

$$\begin{aligned}||u||^p_{W_0^{1,p}(\Omega)} &\leq \frac{C|\Omega|^{\frac{1-\alpha p'}{p'}}||u||_{W_0^{1,p}(\Omega)}\left(\int_\Omega f(x,v)dx\right)^\alpha}{\left(\int_\Omega f(x,v)dx\right)^\beta}\\ &\leq C|\Omega|^{\frac{1-\alpha p'}{p'}}||u||_{W_0^{1,p}(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{\alpha-\beta}\\ &\leq C|\Omega|^{\frac{1-\alpha p'}{p'}}||g||^{\alpha-\beta}_{L^1(\Omega)}||u||_{W_0^{1,p}(\Omega)}.\end{aligned}$$

Therefore,

$$||u||_{W_0^{1,p}(\Omega)} \leq \left(C|\Omega|^{\frac{1-\alpha p'}{p'}}||g||^{\alpha-\beta}_{L^1(\Omega)}\right)^{\frac{1}{p-1}}; \tag{3.18}$$

- if $\alpha = \frac{1}{p'}$ and $\beta \leq \alpha$, we get

$$\begin{aligned}||u||^p_{W_0^{1,p}(\Omega)} &\leq \frac{C||u||_{W_0^{1,p}(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{\frac{1}{p'}}}{\left(\int_\Omega f(x,v)dx\right)^\beta}\\ &\leq C||u||_{W_0^{1,p}(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{\frac{1}{p'}-\beta}\\ &\leq C||g||^{\frac{1}{p'}-\beta}_{L^1(\Omega)}||u||_{W_0^{1,p}(\Omega)}.\end{aligned}$$

Thus

$$||u||_{W_0^{1,p}(\Omega)} \leq \left(C||g||^{\frac{1}{p'}-\beta}_{L^1(\Omega)}\right)^{\frac{1}{p-1}}. \tag{3.19}$$

- if $\beta \leq \alpha = 0$, we obtain

$$\begin{aligned}||u||^p_{W_0^{1,p}(\Omega)} &\leq C|\Omega|^{1-\frac{1}{p}}||u||_{W_0^{1,p}(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{-\beta}\\ &\leq C|\Omega|^{1-\frac{1}{p}}||g||^{-\beta}_{L^1(\Omega)}||u||_{W_0^{1,p}(\Omega)}.\end{aligned}$$

Therefore

$$||u||_{W_0^{1,p}(\Omega)} \le \left(C|\Omega|^{1-\frac{1}{p}} ||g||_{L^1(\Omega)}^{-\beta} \right)^{\frac{1}{p-1}}. \tag{3.20}$$

Now, by using $(\mathcal{H}_3)$, and, Hölder and Poincaré inequalities, it follows that

$$\begin{aligned} ||u||^p_{W_0^{1,p}(\Omega)} &\le \frac{\left(\int\limits_\Omega f^{\alpha p'}(x,v)dx \right)^{\frac{1}{p'}} \left(\int\limits_\Omega |u|^p dx \right)^{\frac{1}{p}}}{\left(\int\limits_\Omega f(x,u)dx \right)^{\beta}} \\ &\le C||u||_{W_0^{1,p}(\Omega)} \left(\int\limits_\Omega f(x,v)dx \right)^{\frac{1}{p'}-\beta} \\ &\le C||g||_{L^1(\Omega)}^{\frac{1}{p'}-\beta} ||u||_{W_0^{1,p}(\Omega)}. \end{aligned}$$

Thus, we deduce that

$$||u||_{W_0^{1,p}(\Omega)} \le \left(C||g||_{L^1(\Omega)}^{\frac{1}{p}'-\beta} \right)^{\frac{1}{p-1}}. \tag{3.21}$$

Hence, from (3.18)-(3.21), there exists a positive constant $R := R(\alpha, \beta, \Omega, p, ||g||_{L^1(\Omega)}) > 0$ such that $T(\overline{B}_R(0)) \subset \overline{B}_R(0)$.

Step 3: Now, we prove that $T : \overline{B}_R(0) \to \overline{B}_R(0)$ is weakly continuous.

In order to show this, let $\{v_i\}_{i\in I}$ be a generalized sequence in $\overline{B}_R(0)$ such that $v_i \rightharpoonup v$ weakly in $W_0^{1,p}(\Omega)$, and, define $u_i = T(v_i)$ and $u = T(v)$.

Since $\overline{B}_R(0)$ is weakly compact in $W_0^{1,p}(\Omega)$ and to prove that $\{u_i\}_{i\in I}$ converge weakly in $\overline{B}_R(0)$, it suffices to show that $\{u_i\}_{i\in I}$ has u as unique weak limit for that topology. To do this, we assume by contradiction that

$$u_i \rightharpoonup \overline{u} \quad \text{weakly in } W_0^{1,p}(\Omega), \tag{3.22}$$

On the other hand, from Lemma 2.4, we have $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, and, therefore we get the existence of a subsequence of $\{v_i\}_{i\in I}$ such that

$$v_i(x) \to v(x) \quad \text{a.e in } \Omega. \tag{3.23}$$

Since u_i and u are two positive solutions to (3.9), then, by taking $\phi = (u_i - u)$ as a test function in (3.9), we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}dxdy \\ &+ \int\limits_\Omega |\nabla u_i|^{p-2}\nabla u_i \nabla\phi(x)dx = \frac{\int\limits_\Omega f^\alpha(x,v_i)\phi(x)dx}{\left(\int\limits_\Omega f(x,v_i)dx \right)^{\beta}}, \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}dxdy \\ &+ \int\limits_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi(x)dx = \frac{\int\limits_\Omega f^\alpha(x,v)\phi(x)dx}{\left(\int\limits_\Omega f(x,v)dx \right)^{\beta}}. \end{aligned} \tag{3.25}$$

Subtracting (3.24) with (3.25), we get

$$\begin{aligned}\Big\langle \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u) - \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u), \phi \Big\rangle &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}dxdy \\ &\quad - \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+ps}}dxdy \\ &\quad + \int_\Omega \left(|\nabla u_i|^{p-2}\nabla u_i - |\nabla u|^{p-2}\nabla u\right)\nabla\phi(x)dx \\ &= \frac{\int_\Omega f^\alpha(x,v_i)\phi(x)dx}{\left(\int_\Omega f(x,v_i)dx\right)^\beta} - \frac{\int_\Omega f^\alpha(x,v)\phi(x)dx}{\left(\int_\Omega f(x,v)dx\right)^\beta}. \end{aligned} \tag{3.26}$$

Hence, by using Lemma 2.7, for $p \geq 2$ it holds that

$$\Big\langle \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u) - \mathfrak{L}_{\mathfrak{p},\mathfrak{s}}(u), u_i - u \Big\rangle \geq c||u_i-u||^p_{W_0^{1,p}(\Omega)}, \tag{3.27}$$

Combining with (3.26) and (3.27), we reach that

$$c||u_i-u||^p_{W_0^{1,p}(\Omega)} \leq \frac{\int_\Omega f^\alpha(x,v_i)\phi(x)dx}{\left(\int_\Omega f(x,v_i)dx\right)^\beta} - \frac{\int_\Omega f^\alpha(x,v)\phi(x)dx}{\left(\int_\Omega f(x,v)dx\right)^\beta}. \tag{3.28}$$

So,

$$\begin{aligned} c||u_i-u||^p_{W_0^{1,p}(\Omega)} &\leq \frac{\int_\Omega f^\alpha(x,v_i)\phi(x)dx}{\left(\int_\Omega f(x,v_i)dx\right)^\beta} - \frac{\int_\Omega f^\alpha(x,v)\phi(x)dx}{\left(\int_\Omega f(x,v)dx\right)^\beta} \\ &\leq \frac{1}{\left(\int_\Omega f(x,v_i)dx\right)^\beta}\int_\Omega \left(f^\alpha(x,v_i)-f^\alpha(x,v)\right)(u_i-u)dx \\ &\quad + \left[\frac{1}{\left(\int_\Omega f(x,v_i)dx\right)^\beta} - \frac{1}{\left(\int_\Omega f(x,v)dx\right)^\beta}\right]\int_\Omega f^\alpha(x,v)(u_i-u)dx \end{aligned} \tag{3.29}$$

On the other hand, by using $(\mathcal{H}_1)$, we get

$$\int_\Omega f(x,v_i)dx > 0 \quad \text{and} \int_\Omega f(x,v)dx > 0, \quad \text{for every } k \in I\!N.$$

So, from $(\mathcal{H}_1)$, (3.23) and by applying the dominated convergence theorem, we reach that

$$f^\alpha(x,v_i) \to f^\alpha(x,v) \quad \text{strongly in } L^{p'}(\Omega), \tag{3.30}$$

and

$$\left(\int_\Omega f(x, v_i)\right)^\beta \to \left(\int_\Omega f(x, v)\right)^\beta \quad \text{strongly in } \mathbb{R}. \tag{3.31}$$

Notice that $u_i \in \overline{B}_R(0)$, then, by using (3.30), (3.31) and letting $i \to +\infty$ in (3.29), it follows that

$$u_i \to u \quad \text{strongly in } W_0^{1,p}(\Omega).$$

Going back to (3.22), we derive that $u = \overline{u}$. Hence, u is the unique weak limit point of $\{u_i\}_{i\in I}$, and then

$$u_i = T(v_i) \rightharpoonup u = T(v) \quad \text{weakly in } W_0^{1,p}(\Omega).$$

Therefore, by applying Tychonoff fixed point theorem, the operator T admits at least one fixed point $u \in \overline{B}_R(0)$ such that $T(u) = u$, which is a weak solution to (1.0).

The case $1 < p < 2$ is made using similar arguments, and we will omit its proof. □

Now, we are going to establish the second result.

Theorem 3.2 *Let $s \in (0, 1)$ and $1 < p < N$. Assume that $(\mathcal{G}_1)-(\mathcal{G}_3)$ hold. Then, problem (1.0) has a positive weak solution $u \in W_0^{1,p}(\Omega)$ in sense of Definition 2.6.*

Proof The proof will be done in several steps.
Step 1: For $v \in L^p(\Omega)$, as in the proof of the previous result, we want to show that

$$h(v) := \frac{f^\alpha(x, v)}{\left(\int_\Omega f(x, v)dx\right)^\beta} \in (W_0^{1,p}(\Omega))'.$$

Observe that,

- if $0 < \delta < p$ and $h_\delta \in L^{\frac{p}{p-\delta}}(\Omega)$, so, by $(\mathcal{G}_2)$-$(\mathcal{G}_3)$, and by applying Hölder and Poincaré inequalities, we derive that

$$f(., v) \in L^1(\Omega) \quad \text{and } f^\alpha(., v) \in L^{p'}(\Omega).$$

  Hence, by the same computations as in first step in the proof of theorem 3.1 we get $h(v) \in (W_0^{1,p}(\Omega))'$.
- Similarly, if we take $\delta = p$ and $h_\delta \in L^\infty(\Omega)$, we get also $h(v) \in (W_0^{1,p}(\Omega))'$.

Therefore, by applying theorem 1.1 in [29], there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ to (3.9).

Notice that $f(x, t) > 0$ for all $t \in \mathbb{R}$ and a.e $x \in \Omega$, then, by the same reasoning as in first step in the proof of theorem 3.1, we get $u > 0$ in Ω.

Hence, the operator,

$$\begin{aligned} H :& L^p(\Omega) \to W_0^{1,p}(\Omega) \\ & v \mapsto H(v) = u. \end{aligned}$$

is well defined. Moreover, if u is a fixed point of H, then u is a weak solution to Problem (1.0). Thus, we just have to show that H has a fixed point.
Step 2: In this step, we will prove the existence $R > 0$ such that $H(\overline{B}_R(0)) \subset \overline{B}_R(0)$ where $\overline{B}_R(0)$ is the closed ball of center 0 and radius R.

Taking u as a test function in Problem (3.9) and using Poincaré inequality, it follows that

$$||u||^p_{L^p(\Omega)} \leq C \frac{\int_\Omega f^\alpha(x, v)|u|dx}{\left(\int_\Omega f(x, v)dx\right)^\beta}. \tag{3.32}$$

Therefore, by taking into consideration $(\mathcal{G}_1)$-$(\mathcal{G}_3)$ and by using Hölder's inequality, we get

- if $0 < \beta < \alpha < \frac{1}{p'}$, we have

$$||u||^p_{L^p(\Omega)} \le \frac{C|\Omega|^{\frac{1-\alpha p'}{p'}}||u||_{L^p(\Omega)}\left(\int_\Omega f(x,v)dx\right)^\alpha}{\left(\int_\Omega f(x,v)dx\right)^\beta}$$

$$\le C|\Omega|^{\frac{1-\alpha p'}{p'}}||u||_{L^p(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{\alpha-\beta}$$

$$\le C|\Omega|^{\frac{1-\alpha p'}{p'}}||u||_{L^p(\Omega)}\left(\int_\Omega (h_\delta(x)|v|^\delta + f(x,0))dx\right)^{\alpha-\beta}.$$

Thus

$$||u||^{p-1}_{L^p(\Omega)} \le \begin{cases} C|\Omega|^{\frac{1-\alpha p'}{p'}}\left(||h_\delta||_{L^{\frac{p}{p-\delta}}(\Omega)}||v||^\delta_{L^p(\Omega)} + ||f(\cdot,0)||_{L^1(\Omega)}\right)^{\alpha-\beta} & \text{if } \delta \in (0,p), \\ C|\Omega|^{\frac{1-\alpha p'}{p'}}\left(||h_\delta||_{L^\infty(\Omega)}||v||^p_{L^p(\Omega)} + ||f(\cdot,0)||_{L^1(\Omega)}\right)^{\alpha-\beta} & \text{if } \delta = p, \end{cases}$$

which implies

$$||u||_{L^p(\Omega)} \le \begin{cases} C|\Omega|^{\frac{1-\alpha p'}{p}}\left(||h_\delta||^{\frac{\alpha-\beta}{p-1}}_{L^{\frac{p}{p-\delta}}(\Omega)}||v||^{\frac{\delta(\alpha-\beta)}{p-1}}_{L^p(\Omega)} + ||f(\cdot,0)||^{\frac{\alpha-\beta}{p-1}}_{L^1(\Omega)}\right) & \text{if } \delta \in (0,p), \\ C|\Omega|^{\frac{1-\alpha p'}{p}}\left(||h_\delta||^{\frac{\alpha-\beta}{p-1}}_{L^\infty(\Omega)}||v||^{p'(\alpha-\beta)}_{L^p(\Omega)} + ||f(\cdot,0)||^{\frac{\alpha-\beta}{p-1}}_{L^1(\Omega)}\right) & \text{if } \delta = p. \end{cases}$$

Using now the fact $0 < \frac{\delta(\alpha-\beta)}{p-1} < 1$, then, we can find a positive constant R_1 depends only of the data such that

$$||u||_{L^p(\Omega)} \le R_1; \tag{3.33}$$

- if $0 < \beta < \alpha = \frac{1}{p'}$, we get

$$||u||^p_{L^p(\Omega)} \le \frac{C||u||_{L^p(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{\frac{1}{p'}}}{\left(\int_\Omega f(x,v)dx\right)^\beta}$$

$$\le C||u||_{L^p(\Omega)}\left(\int_\Omega f(x,v)dx\right)^{\frac{1}{p'}-\beta}$$

$$\le C||u||_{L^p(\Omega)}\left(\int_\Omega (h_\delta(x)|v|^\delta + f(x,0))dx\right)^{\frac{1}{p'}-\beta},$$

which leads to

$$||u||^{p-1}_{L^p(\Omega)} \le \begin{cases} C\left(||h_\delta||_{L^{\frac{p}{p-\delta}}(\Omega)}||v||^\delta_{L^p(\Omega)} + ||f(\cdot,0)||_{L^1(\Omega)}\right)^{\frac{1}{p'}-\beta} & \text{if } \delta \in (0,p), \\ C\left(||h_\delta||_{L^\infty(\Omega)}||v||^p_{L^p(\Omega)} + ||f(\cdot,0)||_{L^1(\Omega)}\right)^{\frac{1}{p'}-\beta} & \text{if } \delta = p, \end{cases}$$

thus

$$||u||_{L^p(\Omega)} \leq \begin{cases} C\left(||h_\delta||_{L^{\frac{p}{p-\delta}}(\Omega)}^{\frac{1-\beta p'}{p}}||v||_{L^p(\Omega)}^{\frac{\delta(1-\beta p')}{p}} + ||f(\cdot,0)||_{L^1(\Omega)}^{\frac{1-\beta p'}{p}}\right) & \text{if } \delta \in (0,p),\\ C\left(||h_\delta||_{L^\infty(\Omega)}^{\frac{1-\beta p'}{p}}||v||_{L^p(\Omega)}^{1-\beta p'} + ||f(\cdot,0)||_{L^1(\Omega)}^{\frac{1-\beta p'}{p}}\right) & \text{if } \delta = p, \end{cases}$$

Since, $0 < \frac{\delta(1-\beta p')}{p} < 1$ for every $\delta \in (0,p]$, therefore, we get $R_2 > 0$ depends only of the data such that

$$||u||_{L^p(\Omega)} \leq R_2; \tag{3.34}$$

- finally, if $0 < \beta = \alpha \leq \frac{1}{p'}$, then, by using the same arguments as above, we get

$$||u||_{L^p(\Omega)} \leq R_3, \tag{3.35}$$

where $R_3 := C|\Omega|^{\frac{1-\alpha p'}{p}}$.

So, combining (3.33)-(3.35), we get the existence a positive constant R such that

$$||u||_{L^p(\Omega)} \leq R, \tag{3.36}$$

as desired.

Step 3: Now, we show that H is a continuous and compact operator on $\overline{B}_R(0)$.

Let us begin by proving the continuity of H. Consider $\{v_n\}_n \subset \overline{B}_R(0)$ and $v \in \overline{B}_R(0)$ such that

$$v_n \to v \quad \text{strongly in } L^p(\Omega).$$

Since u_n and u are two positive solutions to (3.9), then, by taking $(u_n - u)$ as a test function in both equations of u_n and u, and, by using the same computations as in the proof of theorem 3.1 for $p \geq 2$, we get

$$\begin{aligned} c||u_n - u||_{L^p(\Omega)}^p &\leq \frac{\int_\Omega f^\alpha(x,v_n)(u_n-u)(x)dx}{\left(\int_\Omega f(x,v_n)dx\right)^\beta} - \frac{\int_\Omega f^\alpha(x,v)(u_n-u)(x)dx}{\left(\int_\Omega f(x,v)dx\right)^\beta} \\ &\leq \frac{1}{\left(\int_\Omega f(x,v_n)dx\right)^\beta}\int_\Omega \big(f^\alpha(x,v_n) - f^\alpha(x,v)\big)(u_n-u)dx \\ &\quad + \left[\frac{1}{\left(\int_\Omega f(x,v_n)dx\right)^\beta} - \frac{1}{\left(\int_\Omega f(x,v)dx\right)^\beta}\right]\int_\Omega f^\alpha(x,v)(u_n-u)dx, \end{aligned} \tag{3.37}$$

where in the last inequality we have used Poincaré inequality.

Since $\alpha \leq \frac{1}{p'} < 1$, so, by Hölder's inequality, we get

$$\begin{aligned} \left|\int_\Omega \big(f^\alpha(x,v_n) - f^\alpha(x,v)\big)(u_n-u)dx\right| &\leq \int_\Omega \big|f^\alpha(x,v_n) - f^\alpha(x,v)\big||u_n-u|dx \\ &\leq \int_\Omega \big|f(x,v_n) - f(x,v)\big|^\alpha |u_n-u|dx \\ &\leq \left(\int_\Omega \big|f(x,v_n) - f(x,v)\big|^{p'\alpha}dx\right)^{\frac{1}{p'}}\left(\int_\Omega |u_n-u|^p\right)^{\frac{1}{p}} \end{aligned} \tag{3.38}$$

So, by $(\mathcal{G}_2)$, $(\mathcal{G}_3)$ and from (3.38), we distinguish two cases,

- if $0 < \alpha < \frac{1}{p'}$, it follows that

$$\begin{aligned}&\left|\int_\Omega \big(f^\alpha(x, v_n) - f^\alpha(x, v)\big)(u_n - u)dx\right|\\&\leq |\Omega|^{\frac{1-\alpha p'}{p'}} ||u_n - u||_{L^p(\Omega)}\bigg(\int_\Omega |f(x, v_n) - f(x, v)\big|dx\bigg)^\alpha\\&\leq |\Omega|^{\frac{1-\alpha p'}{p'}} ||u_n - u||_{L^p(\Omega)}\bigg(\int_\Omega |h_\delta(x)|v_n - v|^\delta dx\bigg)^\alpha.\end{aligned}$$

Therefore, by using Hölder's inequality, it holds that

$$\left|\int_\Omega \big(f^\alpha(x, v_n) - f^\alpha(x, v)\big)(u_n - u)dx\right| \leq C \begin{cases} ||h_\delta||^\alpha_{L^{\frac{p}{p-\delta}}(\Omega)} ||v_n - v||^{\alpha\delta}_{L^p(\Omega)}||u_n - u||_{L^p(\Omega)} \text{ if } \delta \in (0, p),\\ ||h_\delta||^\alpha_{L^\infty(\Omega)}||v_n - v||^{\alpha p}_{L^p(\Omega)}||u_n - u||_{L^p(\Omega)} \text{ if } \delta = p.\end{cases} \tag{3.39}$$

- If $\alpha = \frac{1}{p'}$, then, we have

$$\begin{aligned}\left|\int_\Omega \big(f^{\frac{1}{p'}}(x, v_n) - f^{\frac{1}{p'}}(x, v)\big)(u_n - u)dx\right| &\leq C||u_n - u||_{L^p(\Omega)}\bigg(\int_\Omega |f(x, v_n) - f(x, v)\big|dx\bigg)^{\frac{1}{p'}}\\&\leq C||u_n - u||_{L^p(\Omega)}\bigg(\int_\Omega |h_\delta(x)|v_n - v|^\delta dx\bigg)^{\frac{1}{p'}}.\end{aligned}$$

So,

$$\left|\int_\Omega \big(f^{\frac{1}{p'}}(x, v_n) - f^{\frac{1}{p'}}(x, v)\big)(u_n - u)dx\right| \leq \begin{cases} C||h_\delta||^{\frac{1}{p'}}_{L^{\frac{p}{p-\delta}}(\Omega)} ||v_n - v||^{\frac{\delta}{p'}}_{L^p(\Omega)}||u_n - u||_{L^p(\Omega)} \text{ if } \delta \in (0, p),\\ C||h_\delta||^{\frac{1}{p'}}_{L^\infty(\Omega)}||v_n - v||^{\frac{p}{p'}}_{L^p(\Omega)}||u_n - u||_{L^p(\Omega)} \text{ if } \delta = p.\end{cases} \tag{3.40}$$

On the other hand, by using $(\mathcal{G}_1) - (\mathcal{G}_2)$ and Hölder's inequality, we reach that

$$\left|\int_\Omega \big(f(x, v_n) - f(x, v)\big)dx\right| \leq \begin{cases} ||h_\delta||_{L^{\frac{p}{p-\delta}}(\Omega)} ||v_n - v||^\delta_{L^p(\Omega)} \text{ if } \delta \in (0, p),\\ ||h_\delta||_{L^\infty(\Omega)}||v_n - v||^p_{L^p(\Omega)} \text{ if } \delta = p.\end{cases} \tag{3.41}$$

Now, combining (3.38)-(3.41), we obtain from (3.37) by letting $n \to +\infty$,

$$u_n \to u \quad \text{strongly in } L^p(\Omega),$$

and the continuity of H follows.

By using the same reasoning as above with lemma 2.7, we get also the continuity of H in the case $1 < p < 2$, and we will omit its proof.

Now, we prove that the operator H is compact. Let $v \in \overline{B}_R(0)$ such that $u = H(v)$.

Thus, by using the same arguments as in the second step, we get the existence a positive constant C depends only of the data such that

$$||u||_{W_0^{1,p}(\Omega)} \leq C.$$

Since, by Lemma 2.4, we have that, the set $H(\overline{B}_R(0))$ is relatively compact in $L^p(\Omega)$, hence, the compactness of H follows.

To conclude the proof, we apply Schauder fixed-point theorem. Hence, we get the existence of $u \in W_0^{1,p}(\Omega)$ which is solution to Problem (1.0). □

Remarks 3.3 (A) Under the hypothesis $(\mathcal{G}_1) - (\mathcal{G}_2)$ and one of

(1) $\delta \in (0, p]$, $\alpha \in [0, \frac{1}{p'}]$, $\beta \leq 0$ and $0 < \frac{\delta(\alpha-\beta)}{p-1} < 1$,

(2) $\delta \in (0, p)$, $\beta = -\frac{p-1}{\delta}$, $\alpha = 0$ and $C|\Omega|^{\frac{1}{p}}||h_\delta||^{\frac{1}{\delta}}_{L^{\frac{p}{p-\delta}}(\Omega)} < 1$,

(3) $\delta \in (0, p)$, $\alpha \in (0, \frac{1}{p'}]$, $\beta \leq 0$, $\delta(\alpha - \beta) = p - 1$ and $C|\Omega|^{\frac{1-\alpha p'}{p}}|h_\delta||^{\frac{\alpha-\beta}{p-1}}_{L^{\frac{p}{p-\delta}}(\Omega)} < 1$,

(4) $\delta = p$, $\alpha = 0$, $\beta = -\frac{1}{p'}$ and $C|\Omega|^{\frac{1}{p}}||h_\delta||^{\frac{1}{p}}_{L^\infty(\Omega)} < 1$,

(5) $\delta = p$, $\alpha \in (0, \frac{1}{p'}]$, $\beta \leq 0$, $\alpha - \beta = \frac{1}{p'}$ and $C|\Omega|^{\frac{1-\alpha p'}{p}}|h_\delta||^{\frac{\alpha-\beta}{p-1}}_{L^\infty(\Omega)} < 1$

hold, then, Problem (1.0) has a positive weak solution $u \in W_0^{1,p}(\Omega)$. Moreover by using [9, Theorem 3.1] (see also [30, Lemma 3.1]), we get, $u \in L^\infty(\Omega)$ in the cases (2) and (4).

(B) Notice that, all the existence results obtained in (A), can be proved, by using the same stratigy as in the proof of theorem 3.2.

Acknowledgements This research was partially supported by a research project from DGRSDT, C00L03UN380120220001, Algeria.

Funding Not applicable.

Data availability statement Not applicable.

Declarations

Conflict of interest This work does not have any conflicts of interest.

Informed consent statement Not applicable.

References

1. Abatangelo, N.; Cozzi, M.: An elliptic boundary value problem with fractional nonlinearity. SIAM J. Math. Anal. **53**(3), 3577–3601 (2021)
2. Adams, R.A.: Sobolev spaces. Academic Press, New York (1975)
3. Arcoya, D.; Leonori, T.; Primo, A.: Existence of solutions for semilinear nonlocal elliptic problems via a Bolzano theorem. Acta Appl. Math. **127**, 87–104 (2013)
4. Autuori, G.; Fiscella, A.; Pucci, P.: Stationary Kirchhoff problems involving a fractional operator and a critical nonlinearity. Nonlinear Anal. **125**, 699–714 (2015)
5. Biagi, S.; Dipierro, S.; Valdinoci, E.; Vecchi, E.: A Brezis-Nirenberg type result for mixed local and nonlocal operators. preprint (2022).
6. Biagi, S.; Dipierro, S.; Valdinoci, E.; Vecchi, E.: Mixed local and nonlocal elliptic operators: regularity and maximum principles. Comm. Partial Differ. Equ. **47**(3), 585–629 (2022)
7. Biagi, S.; Dipierro, S.; Valdinoci, E.; Vecchi, E.: A Faber-Krahn inequality for mixed local and nonlocal operators. JAMA (2023). https://doi.org/10.1007/s11854-023-0272-5
8. Biagi, S.; Vecchi, E.; Dipierro, S.; Valdinoci, E.: Semilinear elliptic equations involving mixed local and nonlocal operators. Proc. Roy. Soc. Edinburgh Sect. A **151**(5), 1611–1641 (2021)
9. Biroud, K.: Mixed local and nonlocal equation with singular nonlinearity having variable exponent. J. Pseudo-Differ. Oper. Appl. **14**, 13 (2023). https://doi.org/10.1007/s11868-023-00509-7
10. Biswas, I.H.; Jakobsen, E.R.; Karlsen, K.H.: Viscosity solutions for a system of integro-PDEs and connections to optimal switching and control of jump-diffusion processes. Appl. Math. Optim. **62**(1), 47–80 (2010)
11. Blazevski, D.; del-Castillo-Negrete, D..: Local and nonlocal anisotropic transport in reversed shear magnetic fields: shearless Cantori and nondiffusive transport. Phys. Rev. **87**, 063106 (2013)

12. Buccheri, S.; da Silva, J.V.; de Miranda, L.H.: A system of local/nonlocal $p-$ Laplacians: the eigenvalue problem and its asymptotic limit as $p \to \infty$. Asymptot. Anal. **128**(2), 149–181 (2022)
13. Caponi, M.; Pucci, P.: Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations. Ann. Mat. Pura Appl. **195**, 2099–2129 (2016)
14. Chen, C.; Huang, J.; Liu, L.: Multiple solutions to the nonhomogeneous p-Kirchhoff elliptic equation with concave-convex nonlinearities. Appl. Math. Lett. **26**(7), 754–759 (2013)
15. Chen, Z.-Q.; Kim, P.; Song, R.; Vondracek, Z.: Boundary Harnack principle for $\Delta + \Delta^{\frac{\alpha}{2}}$. Trans. Am. Math. Soc. **364**(8), 4169–4205 (2012)
16. Corrêa, F.J.S.A.; Figueiredo, G.M.: On an elliptic equation of p-Kirchhoff-type via variational methods. Bull. Austral. Math. Soc. **77**, 263–277 (2006)
17. Corrêa, F.J.S.A.; Figueiredo, G.M.: On a p-Kirchhoff equation via Krasnoselskiiś genus. Appl. Math. Lett. **22**, 819–822 (2009)
18. Corrêa, F.J.S.A.; de Morais Filho, D.C.: On a class of nonlocal elliptic problems via Galerkin method. J. Math. Anal. Appl. **310**(1), 177–187 (2005)
19. Damascelli, L.: Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. Ann. Inst. H. Poincaré. Analyse Nonlinéaire **15**(4), 493–516 (1998)
20. Del Pezzo, L.M.; Ferreira, R.; Rossi, J.D.: Eigenvalues for a combination between local and nonlocal $p-$ Laplacians. Fract. Calc. Appl. Anal. **22**(5), 1414–1436 (2019)
21. del Teso, F.; Endal, J.; Jakobsen, E.R.: On distributional solutions of local and nonlocal problems of porous medium type. CR Math. Acad. Sci. Paris **355**(11), 1154–1160 (2017)
22. Di Nezza, E.; Palatucci, G.; Valdinoci, E.: Hitchhiker's guide to the fractional Sobolev spaces. Bull. Sci. math. **136**(5), 521–573 (2012)
23. Dipierro, S.; Lippi, E.P.; Valdinoci, E.: Linear theory for a mixed operator with Neumann conditions. Asymptot. Anal. **128**(4), 571–594 (2022)
24. Dipierro, S.; Proietti Lippi, E.; Valdinoci, E.: (Non)local logistic equations with Neumann conditions. Ann. Inst. H Poincaré Anal. NonLinéaire (2022). https://doi.org/10.4171/AIHPC/57
25. Dipierro, S.; Valdinoci, E.: Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and L'evy processes. Phys. A **575**, 20 (2021)
26. Evans, LC: Partial differential equations, vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI (1998)
27. Fiscella, A.; Valdinoci, E.: A critical Kirchhoff type problem involving a nonlocal operator. Nonliear Anal. **94**, 156–170 (2014)
28. Garain, Prashanta; Kinnunen, Juha: On the regularity theory for mixed local and nonlocal qasilinear elliptic equations. Trans. Am. Math. Soc. **375**(8), 5393–5423 (2022)
29. Garain, P.; Lindgren, E.: Higher Hölder regularity for mixed local and nonlocal degenerate elliptic equations. Calc. Var. Partial Differ. Equ **62**(2), 67 (2023)
30. Garain, P.; Ukhlov, A.: Mixed local and nonlocal Sobolev inequalities with extremal and associated quasilinear singular elliptic problems. Nonlinear Anal. **223**, 113022 (2022)
31. Guo, D.; Lakshmikantham, V.: Nonlinear problems in abstract cones. Academic Press, Boston (1988)
32. Lamao, C.; Huang, S.; Tian, Q.; Huang, C.: Regularity results of solutions to elliptic equations involving mixed local and nonlocal operators. AIMS Math. **7**, 4199–4210 (2021)
33. Liu, D.C.: On a p-Kirhhoff equation via fountain theorem and dual fountain theorem. Nonlinear Anal. **72**, 208–302 (2010)
34. Mazýa, Vladimir: Sobolev spaces with applications to elliptic partial differential equations, volume 342 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, augmented edition, (2011)
35. Mingqi, X.; Molica Bisci, G.; Tian, G.; Zhang, B.: Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian. Nonlinearity **29**, 357–374 (2016)
36. Mingqi, X.; Radulescu, V.; Zhang, B.: Fractional Kirchhoff problems with critical Trudinger-Moser nonlinearity. Calc. Var. Partial Differ. Equ. **58**, 1–7 (2019)
37. Molica Bisci, G.; Radulescu, V.D.; Servadei, R.: Variational methods for nonlocal fractional problems. Cambridge University Press, (2016)
38. Montefusco, Eugenio; Pellacci, Benedetta; Verzini, Gianmaria: Fractional diffusion with Neumann boundary conditions: the logistic equation. Discrete Contin. Dyn. Syst. Ser. B **18**(8), 2175–2202 (2013)
39. Pellacci, Benedetta; Verzini, Gianmaria: Best dispersal strategies in spatially heterogeneous environments: optimization of the principal eigenvalue for indefinite fractional Neumann problems. J. Math. Biol. **76**(6), 1357–1386 (2018)
40. Pucci, P.; Xiang, M.; Zhang, B.: Existence and multiplicity of entire solutions for fractional p-Kirchhoff equations. Adv. Nonlinear Anal. **5**, 27–55 (2016)
41. Salort, A.; Vecchi, E.: On the mixed local-nonlocal Hénon equation. Differ. Integr. Equ. **35**(11–12), 795–818 (2022)
42. Su, X.; Valdinocci, E.; Wei, Y.; Zhang, J.: Regularity results for solutions of mixed local and nonlocal elliptic equations. Math. Z. **302**, 1855–1878 (2022)
43. Xiang, M.Q.; Zhang, B.L.; Ferrara, M.: Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian. J. Math. Anal. Appl. **424**, 1021–1041 (2015)
44. Xiang, M.; Zhang, B.; Zhang, X.: A nonhomogeneous fractional p-Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$. Adv. Nonlinear Stud. **17**, 611–640 (2017)
45. Zaouche, E.: Nontrivial weak solutions for nonlocal nonhomogeneous elliptic problems. Appl. Anal. **101**(4), 1261–1270 (2020)