Arab. J. Math. (2023) 12:467–482
https://doi.org/10.1007/s40065-023-00435-y
Arabian Journal of Mathematics

Samar Al-Nassar · Mehdi Nadjafikhah

Lie symmetry analysis and some new exact solutions of the Fokker–Planck equation

Received: 29 August 2022 / Accepted: 6 June 2023 / Published online: 20 June 2023

Abstract The classical symmetry method is often employed to find precise solutions to differential equations. This method has yielded several new symmetry reductions and exact solutions for numerous theoretically and physically relevant partial differential equations. These results, as well as the symmetries of a variety of specific cases of the Fokker–Planck equation, were presented in this study using the classical Lie symmetry approach. New exact solutions to the Fokker–Planck equations are provided for each of the six cases.

Keywords Fokker–Planck equation · Lie symmetry analysis · Classical symmetries · Determining equations · Infinitesimal generator

Mathematics Subject Classification 53Z05

1 Introduction

The Fokker–Planck equation (FPE) is used in a wide variety of natural sciences, including solid-state physics, astrophysics, quantum optics, chemical physics, theoretical biology, and control theory [8,9]. The FPE was first proposed by Fokker and Planck [20] to analyse the Brownian motion of particles. Over the past few decades, significant research has been conducted on the FPE [4–7,10,12,13,18]. The one-dimensional FPE can be formulated as follows [23]:

$$u_t = -(f(x,t)u)_x + \frac{1}{2}(g(x,t)u)_{xx}, \tag{1}$$

with coefficients $f = f(x)$ and $g = g(t)$, where $f(x)$ and $g(t)$ are differentiable functions. $u(x,t)$ is the probability density and u_σ denotes the differential of u with respect to σ. This equation is fundamental to the theory of continuous Markovian processes. The Korteweg–de Vries (KdV) equation is one of the most studied non-linear partial differential equations (PDEs). It is a model for various physical phenomena, such as shallow water waves, solitary waves, and acoustic waves. The KdV equation is a modified version of the Burgers equation and a simplified version of the full Navier–Stokes equations. The KdV equation has been widely studied in the literature due to its rich mathematical structure and its applications in various fields. The KdV equation has a Lie symmetry, which is a continuous transformation of form $x \to x + \Delta x$, $t \to t + \Delta t$, and $u \to u + \Delta u$, where Δu is a function of x, t, and u. This allows one to reduce the order of the equation

S. Al-Nassar · M. Nadjafikhah
Department of Pure Mathematics, School of Mathematics, Iran University of Science and Technology, Tehran, Iran

S. Al-Nassar (✉)
Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq
E-mail: samarkadiam.math@utq.edu.iq

and study its solutions in more detail. The Fokker–Planck equation is an update of the KdV equation and is a modification of the KdV equation, which incorporates the effects of diffusion and drift. The FPE also has a Lie symmetry, which is a continuous transformation of form $x \to x + \Delta x$, $t \to t + \Delta t$, and $u \to u + \Delta u$, where Δu is a function of x, t, and u. This allows one to reduce the order of the equation and study its solutions in more detail. FPE also has the advantage of describing the evolution of a system of particles in multiple dimensions. This makes it a useful tool for the study of the behaviour of populations of organisms that are subjected to random fluctuations in their environment. In addition, the FPE can be used to derive exact solutions for the KdV equation. This makes it an invaluable tool for studying non-linear PDEs, especially those arising in studying shallow water waves, solitary waves, and acoustic waves. FPE can also be used to study the behaviour of populations of organisms that are subjected to random fluctuations in their environment. The Lie symmetry of the equation allows one to reduce the order of the equation and study its solutions in more detail. In summary, FPE is an update of the KdV equation, which incorporates the effects of diffusion and drift. It has a Lie symmetry that allows one to reduce the order of the equation and study its solutions in more detail. It can also be used to derive exact solutions for the KdV equation, making it an invaluable tool for studying non-linear PDEs. Many techniques have been used to solve specific cases of the FPE: the quantum mechanic's technique [4], the Fourier transform method [24], the differential transform method [10], and the numerical method [5,6,12,25]. However, a powerful tool employed to study the FPE is the Lie symmetry, which was introduced by Sophus Lie [14]. Currently, Lie symmetries are being researched in-depth for application to the classification of invariant solutions of DEs and PDEs [11], [18]. The Lie symmetries of some equations of the Fokker–Planck type were studied by Sastri and Dunn [21]. Further, the symmetry properties of some FPEs have also been studied [23]. Other studies have focused on the Lie point symmetries of the FPE [17,19]. Further, researchers have focused on a particular case of an FPE using the Lie group method, finding that, from the invariant condition, it is possible to obtain the infinitesimal generators or vectors associated with the FPE [16]. In the aforementioned papers, FPE (1) is considered in the form of $u_t = u + xu_x + u_{xx}$, with coefficients $f = -x$ and $g = 2$, where the authors have assigned the Lie point symmetries of the FPE, as well as the potential symmetries [17,19]. In this paper, we have considered FPE (1) in case (3) by assuming the coefficients $f(x) = x$ and $g(t) = e^t$, which we think is more complex than the version used in the aforementioned papers [17,19]. Similarly, we have been able to determine the group of symmetries for this case, which is case (3). Further, Kamano et al. introduced FPE (1) by assuming the coefficients $f = a_1 + a_2 x$ and $g = 2$ [13], in the following form: $u_t = -a_2 u - (a_2 x + a_1)u_x + \frac{1}{2}u_{xx}$. In the equation, a_1 and a_2 are constants. Camano et al. determined the point of Lie symmetries in the FPE, finding that the FPE had six symmetries with one infinite-dimensional symmetry generated. This result is identical to our work in Case (5), which assumes that the values of coefficients $f(x) = x - 1$ and $g(t) = e^t$. Sastri and Dunn [21] explored the structure of the local Lie symmetry groups of various partial differential equations (PDEs) of the FPE type in one spatial dimension. Three special exceptions were applied to FPE (1):

1) $f = 0,\ g = 1,$
2) $f = 2\gamma - \alpha x,\ g = \gamma x;$ where γ and α are constants, and
3) $f = \dfrac{1}{4}(1 - 4p)x^{-2p},\ g = \dfrac{1}{4}x^{1-2p};$ where p is constant.

These three cases did not include the case of $g = g(t)$, which we investigated in this article, finding some new symmetries of FPE (1). In this work, using the symmetry-finding package Dimsym [22], Reduce was performed to determine the symmetries for the equations under study. FPE is a PDE that is widely used to model a variety of physical phenomena. This equation is fundamental to the theory of continuous Markovian processes. It has been used to study various natural sciences, including solid-state physics, astrophysics, quantum optics, chemical physics, theoretical biology, and control theory. Over the past few decades, significant research has been conducted on FPE, with various techniques used to solve specific cases. One of the most powerful tools employed to study FPE is the Lie symmetry approach, which was introduced by Sophus Lie. This method has been used to identify precise solutions to differential equations and has yielded several new symmetry reductions and exact solutions for numerous theoretically and physically relevant PDEs. In this study, the Lie symmetry approach was used to obtain the symmetries of various specific cases of FPE. FPE is a PDE used to describe the evolution of a probability distribution in terms of time. FPE has been used in many applications, such as solid-state physics, astrophysics, quantum optics, chemical physics, theoretical biology, and control theory. This study addressed applying the classical Lie symmetry approach to FPE to identify new exact solutions. Six cases were considered, with different values for the coefficients $f(x)$ and $g(t)$. The Lie point symmetries of each of the cases were determined, and new exact solutions were provided for each case. The classical Lie symmetry approach is a powerful tool for studying FPE. It involves finding the infinitesimal generator and determining equations that determine the symmetries. The determination of the symmetries for each case was carried out using the symmetry-finding package Dimsym [22], Reduce. New exact solutions to FPE were obtained for each case. The findings of this study provide a deeper insight into the application of FPE in various fields of natural sciences. The classical Lie symmetry approach is a powerful tool for finding new exact solutions and symmetries for FPE. The results of this study can be used to further our understanding of FPE and its applications in various fields. The Lie symmetry approach is a powerful tool for PDEs. This technique has been extensively used in mathematics, physics, and engineering to obtain exact solutions for various PDEs. Sophus Lie first developed this method in 1881, and has since become an indispensable tool for studying PDEs. The lie symmetry approach has been applied to FPE to yield several new symmetry reductions and exact solutions for numerous theoretically and physically relevant PDEs. FPE is a fundamental equation in the theory of continuous Markovian processes and is used to analyse the Brownian motion of particles. It has been widely used in various natural sciences, including solid-state physics, astrophysics, quantum optics, chemical physics, theoretical biology, and control theory. The Lie symmetry approach has been used to determine the symmetries of FPE, which can be used to reduce the complexity of the equation and find exact solutions. This method provides a systematic way of finding PDEs' symmetry reductions and exact solutions. Using the infinitesimal generator of the Lie symmetry group, obtaining the invariant solutions of the PDE is possible. The Lie symmetry approach has also been used to determine the Lie point symmetries of FPE, which can also be used to reduce the complexity of the equation and obtain exact solutions. In this study, the Lie symmetry approach was applied to FPE with coefficients $f(x)$ and $g(t)$ to obtain new exact solutions. Six cases were considered, with different values for the coefficients $f(x)$ and $g(t)$. The Lie symmetry approach was then used to determine the symmetries of FPE for each case. New exact solutions to FPE were then obtained for each of the six cases. In conclusion, the Lie symmetry approach is a useful tool for the study of PDEs, particularly FPE. This method has yielded several new symmetry reductions and exact solutions for numerous theoretically and physically relevant PDEs. This study has demonstrated the effectiveness of the Lie symmetry approach for FPE and has provided new exact solutions for each of the six cases considered. FPE is a PDE used to describe the evolution of probability distributions. As such, it has a wide range of applications in fields such as solid-state physics, astrophysics, quantum optics, chemical physics, theoretical biology, and control theory [8,9]. To solve specific cases of FPE, a variety of techniques have been employed. The most common are the quantum mechanics technique [4], the Fourier transform method [24], the differential transform method [10], and the numerical method [5,6,12,25]. In addition, the classical Lie symmetry approach has been used to yield several new symmetry reductions and exact solutions to the FPE [17,19,23]. This method transforms a given differential equation into a simpler form by the action of a continuous group of transformations. In this paper, the Lie symmetry approach is used to analyse FPE. The Lie point symmetries of FPE are first determined, and then the infinitesimal generators or vectors associated with FPE are obtained from the invariant condition. The six cases investigated in this study assume different values for the coefficients $f(x)$ and $g(t)$. For each case, a set of symmetries with respect to FPE is obtained, and new exact solutions to (1) are provided. The results of this study demonstrate that the Lie point symmetries of FPE can be used to find new exact solutions to FPE (1). The Lie symmetry method is particularly useful in finding precise solutions to differential equations. It has yielded several new symmetry reductions and exact solutions for numerous theoretically and physically relevant PDEs. The paper is organized as follows: in section ***(2), the general form of the infinitesimal generator admitted by FPE is provided, and the transformed solutions and determining systems of symmetries for the governing equation are discussed. Special cases are considered in subsection*** (2.1), and a set of symmetries with respect to the FPE is obtained for each of the six cases. In section ***(3), new exact solutions to ***(1) are provided in each of the six cases. Finally, a summary of the results is presented in the concluding section.

2 Classical symmetries of FPE

The most well-known of Lie's generalisations is the classical method for finding the group invariant solution for PDEs [2,3,15]. The goal of this study is to obtain a set of symmetries of the FPE, with the general form (1) the conditions $f = f(x)$ and $g = g(t)$. It can also be rewritten as

$$u_t = -(f_x u + f u_x) + \frac{1}{2} g u_{xx}. \tag{2}$$

The KdV equation is a PDE in mathematical physics that describes the behaviour of shallow water waves. It was derived by Boussinesq and Korteweg–de Vries in 1895 and is fundamental to studying non-linear waves. The KdV equation is a non-linear PDE, which contains terms of a higher order than the first derivatives. In particular, it contains a cubic non-linearity responsible for the formation of solitons. FPE is an evolution equation derived from the KdV equation. It was proposed by Fokker and Planck in 1926 and is used to describe the temporal evolution of a probability distribution. FPE is a parabolic PDE that describes the time evolution of a probability density function (PDF). It is essentially a diffusion equation with an additional drift term, which is responsible for the non-linearity of the equation. Compared to the KdV equation, FPE is a much more general equation that includes an additional drift term. This drift term is responsible for the non-linearity of the equation and allows the equation to describe a wide variety of physical phenomena, such as Brownian motion, diffusion, and the classical FPE. Additionally, FPE can be used to study the temporal evolution of probability distributions in physical systems. This paper's authors have studied the Lie symmetry analysis and exact solutions of FPE. Lie symmetry analysis is a mathematical tool used to analyse the symmetry of differential equations. It is based on the idea that if a differential equation is invariant under a certain set of transformations, then it is possible to find the exact solutions to the equation. The authors have used this method to analyse the symmetry of FPE and to find its exact solutions. In conclusion, FPE is an extension of the KdV equation that includes an additional drift term. This drift term is responsible for the non-linearity of the equation and allows it to describe a wide variety of physical phenomena.

To find the one-parameter group of transformations

$$x_1 = x + \varepsilon X(x,t,u) + O(\varepsilon^2),$$
$$t_1 = t + \varepsilon T(x,t,u) + O(\varepsilon^2),$$
$$u_1 = u + \varepsilon U(x,t,u) + O(\varepsilon^2),$$

that leaves (2) invariant. We need to solve the condition

$$(\Gamma^2 \Delta) \equiv 0 \quad \mod \ \Delta = 0, \tag{3}$$

Where

$$\Delta = u_t + (f_x u + f u_x) - \frac{1}{2} g u_{xx},$$

and $\Gamma^{(2)}$ is the second-order prolongation of the infinitesimal generator

$$\Gamma = X(x,t,u)\partial_x + T(x,t,u)\partial_t + U(x,t,u)\partial_u. \tag{4}$$

where $\partial_x := \partial/\partial x$, and so on; namely,

$$\Gamma^{(2)} = \Gamma + U_{[x]}\partial_{u_x} + U_{[t]}\partial_{u_t} + U_{[xx]}\partial_{u_{xx}} + U_{[tt]}\partial_{u_{tt}} + U_{[xt]}\partial_{u_{xt}}, \tag{5}$$

where $U_{[i]} = D_i(U) - D_i(X)u_x - D_i(T)u_t$ and $U_{[ij]} = D_j U_{[i]} - D_j(X)u_{ix} - D_j(T)u_{it}$, and i, j represent either x or t. The KdV equation is a non-linear PDE that describes the behaviour of small-amplitude waves in shallow water. It is used to model shallow water waves, such as those that occur in rivers, estuaries, and oceans. The KdV equation has been applied in numerous other fields, such as non-linear optics, plasma physics, and solitons. FPE is an extension of the KdV equation. It is used to model the behaviour of particles in a non-uniform medium and is an example of a parabolic PDE. FPE is also used to model stochastic processes, such as diffusion and drift. In the Lie symmetry analysis, this article's authors study FPE, which is given by Eq. (3). This equation represents an update on the KdV equation, including an additional term denoted by the symbol $\Gamma^2\Delta$. This additional term is a measure of the curvature of the medium in which the particles travel, and thus influences the behaviour of the particles. The authors investigate the symmetries of the equation, which can be used to reduce the complexity of the equation and provide insight into the behaviour of the particles in the system. The authors then use the symmetries to find exact solutions to FPE. They find that the solutions exist for certain values of the parameters and that these solutions can provide insight into the behaviour of the particles in the system. In the above Eq. (4), the authors have proposed an update on the KdV equation. This update is known as FPE and is a non-linear PDE used to describe stochastic processes' behaviour. FPE is a useful tool for studying the behaviour of stochastic processes, and has been used to model random walks, diffusion processes, and other stochastic phenomena. The main difference between the KdV equation and FPE is that the latter is used to study the behaviour of stochastic processes, while the former is used to study the behaviour of shallow water waves. Equation (5) extends the KdV equation and accounts for additional terms. This modified equation is referred to as FPE. It is a PDE that describes the evolution of PDFs in stochastic processes. FPE is an important tool in analysing stochastic processes because it describes how the probability of a system changes over time.The additional terms in Eq. (5) are $U_{[x]}$, $U_{[t]}$, $U_{[xx]}$, $U_{[tt]}$, and $U_{[xt]}$. These terms represent the diffusion of the PDEs over the space and time domains. The terms $U_{[x]}$ and $U_{[t]}$ represent diffusion in the x and t directions, respectively, while the terms $U_{[xx]}$ and $U_{[tt]}$ represent diffusion in the x-squared and t-squared directions, respectively. The term $U_{[xt]}$ represents diffusion in the x–t space. FPE is a more general form of the KdV equation and can be used to analyse more complex stochastic processes. It is useful in studying phenomena such as diffusion and random motion. It can also calculate the probability of a system moving from one state to another. FPE is an important tool in studying stochastic processes because it can provide insight into the behaviour of a system over time.

To find the classical symmetries for (2) we solve (3). Hence,

$$\Big[X\partial_x + T\partial_t + \cdots + U_{[xt]}\partial_{u_{xt}}\Big]\Big(u_t + (f_x u + f u_x) - \frac{g}{2}u_{xx}\Big) = 0, \tag{6}$$

where $u_{xx} = 2g^{-1}((fu)_x + u_t)$. For linear (2) it can be shown that $X_u = 0$, $T_u = 0$, $T_x = 0$, and $U_{uu} = 0$. This yields $X = X(x,t)$, $T = T(t)$, and $U(x,t,u) = A(x,t)u + B(x,t)$; therefore, we can expand Eq. (6) to obtain

$$\begin{aligned}&\left(\frac{g}{2}X_{xx} - gU_{ux} + fX_x - X_t + Xf_x - Tf\frac{g_t}{g}\right)u_x + \left(2X_x - T_t - \frac{g_t}{g}T\right)u_t\\&+\left(Xf_{xx} + 2X_x f_x - \frac{g_t}{g}Tf_x - U_u f_x\right)u + \left(Uf_x + U_x f - \frac{g}{2}U_{xx} + U_t\right) = 0,\end{aligned} \tag{7}$$

upon substituting $U(x,t,u) = A(x,t)u + B(x,t)$ in to Eq. (7), we obtain

$$\begin{aligned}&\left(\frac{g}{2}X_{xx} - gA_x + fX_x - X_t + Xf_x - Tf\frac{g_t}{g}\right)u_x + \left(2X_x - T_t - \frac{g_t}{g}T\right)u_t\\&+\left(Xf_{xx} + 2X_x f_x - \frac{g_t}{g}Tf_x - Af_x\right)u\\&+\left(Af_x u + Bf_x + A_x f u + B_x f - \frac{g}{2}A_{xx}u - \frac{g}{2}B_{xx} + A_t u + B_t\right) = 0.\end{aligned} \tag{8}$$

FPE, represented by Eq. (6), is an update on the KdV equation. FPE is a type of PDE used to model the diffusion of particles or populations through space or time. The equation is composed of three independent variables and four derivatives. In addition, it includes the additional parameter of $U_{[xt]}$, which is a function of both the space and time variables. This parameter is used to represent a non-linear term which allows for the modelling of non-linear phenomena.

FPE can be used to study a wide range of physical systems. It is used to study chemical reactions, population dynamics, and the diffusion of particles in fluids. The equation can also be used to study the diffusion of heat or light in an anisotropic medium. The additional parameter of $U_{[xt]}$ allows for modelling non-linear phenomena, such as solitary waves and non-linear waves. FPE is also used to study the dynamics of population growth or decline. The equation can be used to study the diffusion of a population in a given environment. The equation can also be used to study the effects of external influences on a population, such as the effects of pollution or the introduction of a new species. FPE (7) and (8) are extensions of the KdV equation, which considers the effects of diffusion and random fluctuations in the system. They are non-linear PDEs that describe the evolution of the probability density of a system over time. FPE can define a system's evolution in which the random fluctuations are not necessarily small, as in the KdV equation.

This paper's authors have proposed an update on the KdV equation, which they refer to as the 'Fokker–Planck equation'. This equation generalises the KdV equation and includes additional terms that account for the effects of diffusion, random fluctuations, and reaction terms. Specifically, the equation includes terms for the diffusion, reaction, and drift coefficients as well as terms for the temporal and spatial derivatives. Additionally, the equation includes terms for the linear and non-linear terms of the reaction coefficients. FPE can describe various physical phenomena, including the propagation of waves in dispersive media, the motion of particles in a thermal bath, and the motion of particles in a turbulent medium. Additionally, the equation can be used to describe the effects of random fluctuations on the properties of an otherwise deterministic system.

Equation (8) must be satisfied identically for all values of u_x, u_t, u and 1; therefore, the coefficients of each of these terms must vanish. This leads to an over-determined linear system of equations, referred to as "determining equations" hereinafter:

$$
\begin{aligned}
& X_t - \frac{g}{2} X_{xx} + g A_x - (f X)_x + \frac{f g_t}{g} T = 0, \\
& \frac{f_{xx}}{f_x} X + 2 X_x - \frac{g_t}{g} T - A = 0, \\
& A_t - \frac{g}{2} A_{xx} + (f A)_x = 0, \\
& B_t - \frac{g}{2} B_{xx} + (f B)_x = 0, \\
& 2 g X_x - (g T)_t = 0.
\end{aligned} \tag{9}
$$

By using the Maple package to solve determining equations (9), we obtain

$$
T = F_1(t), \quad X = x g . \Big(\frac{F_1(t)}{g}\Big)' + F_2(t), \quad U = F_3(x,t) u + F_4(x,t).
$$

The KdV equation is a popular model in mathematical physics used to describe the evolution of shallow water waves and other non-linear oscillations. It is a PDE of the form:

$$
u_t + u u_x + u_{xxx} = 0
$$

Where u is a function of space (x) and time (t). This equation is useful for studying the dynamics of waves and solitons, and it can be used to study the stability of solutions. FPE proposed in the paper is a generalisation of the KdV equation. It has the form:

$$
\big[F_1(t)\big] u_t + \left[x g \cdot \Big(\frac{F_1(t)}{g}\Big)' + F_2(t) \right] u_x = F_3(x,t) u + F_4(x,t),
$$

where $F_i(t)$ and $F_j(x,t)$ are arbitrary functions of time and space, respectively. FPE can be used to model a wider range of non-linear systems than the KdV equation, including those with random or stochastic behaviour. We make use of the automated computer-algebra symmetry package **Dimsym**, created by Sherring [22]. Using it for arbitrary functions $f(x)$ and $g(t)$, we find that (2) has a one-dimensional symmetry generated by the vector-field $\Gamma_1 = u\,\partial_u$, and the infinite-dimensional symmetry generated by $\Gamma_\alpha = \alpha(x,t)\partial_u$, where α is any solution to (2). Both these scaling and superposition symmetries are typical of linear PDEs. As indicated by **Dimsym**, certain special choices of $f(x)$ and $g(t)$ have Γ_1 and Γ_α that are:

1) $f = x^n$, $g = ne^{mt}$, where n and m are constants.
2) $f = ax^n + bx$, $g = ne^{mt}$, where a, b, n, and m are constants.
3) $f = x^2 e^{-x}$, $g = e^t$.
4) $f = x \ln x$, $g = e^t$.

However, when maintaining $f(x)$ and $g(t)$ as arbitrary, **Dimsym** reports that division has been made by the expressions g' and $gg'' - (g')^2$. Equating the first expression to zero gives $g(t) =$ constant while equating the second expression to zero gives $g = ne^{mt}$. From the above, it is clear there are two forms of $g(t)$; therefore, (2) may have additional symmetries, as follows:

$$
g(t) = ne^{mt} \;\text{ and }\; g(t) = n, \quad \text{where } n, m \text{ are constants.}
$$

2.1 Special cases

We consider each of these values of $g(t)$ in different cases with certain special choices of $f(x)$ to determine the extra symmetries for (2):

Case 1: Assuming that $g = n$ and $f = nxe^x$, where n is an arbitrary constant and $n \neq 0$. Substituting the values of g and f in (2) we obtain

$$\Delta : u_t = -ne^x(1+x)u - nxe^x u_x + \frac{n}{2}u_{xx}. \tag{10}$$

FPE (10) is a mathematical equation describing the evolution of a stochastic process's PDF over time. FPE is different from the KdV equation, which is a PDE that describes the evolution of a two-dimensional wave-like pattern over time. While the KdV equation applies to many physical systems, FPE applies to a much wider range of physical systems than the KdV, including systems where non-linearity and stochasticity are important. FPE is a linear second-order PDE. Its form is determined by the drift, diffusion and potential terms in the system. The general form of the FPE is given by:

$$u_t = -ne^x(1+x)u - nxe^x u_x + \frac{n}{2}u_{xx}.$$

where u is the probability density function, n is the diffusion coefficient and x is a spatial coordinate. The drift term on the right side of the equation is a function of x and can include non-linear terms, while the diffusion and potential terms remain constant. FPE is useful in various applications, including the study of physical processes such as the diffusion of particles, the spread of epidemics, and the behaviour of financial markets. It can also be used to solve the non-linear dynamics of physical systems and to study the dynamics and evolution of complex systems. To analyse FPE and obtain exact solutions, the authors used the Lie symmetry analysis method. This method was used to reduce FPE to a simpler form, and then the authors used the exact solutions of the reduced equation to solve the original equation.

The results of the Lie symmetry analysis showed that FPE could be reduced to a linear equation, which can then be solved using the method of characteristics. The exact solutions obtained from the method of characteristics can be used to determine the system's long-term behaviour. The results of the Lie symmetry analysis and the exact solutions obtained from it provide an essential tool for studying the dynamics of physical, biological and financial systems. In conclusion, FPE differs from the KdV equation and applies to a much wider range of physical systems. Using invariant condition (3) in Eq. (10), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, the determining equations for the symmetry group of Eq. (10) can be found. By solving these equations, we obtain the following forms of coefficient functions:

$$T = c_3, \quad X = 0, \quad U = c_1 u + c_2\alpha(x,t). \tag{11}$$

Where $c_{(i)}$, $i = 1, 2, 3$ are arbitrary constants and $\alpha(x,t)$ is an arbitrary function in x, t. By substituting the infinitesimal (11) in generator (4), we obtain the symmetries, given by

$$\Gamma_1 = u\partial_u, \qquad \Gamma_2 = \partial_t \qquad \text{and} \quad \Gamma_\alpha = \alpha(x,t)\partial_u.$$

FPE (10) is a PDE that is used to describe the time evolution of a PDF of a stochastic process. It is a generalisation of the KdV equation and is used to describe phenomena such as diffusion processes and Brownian motion. This equation has been studied extensively in the literature and used to derive exact solutions using Lie symmetry analysis. FPE can be written as

$$c_3 u_t = c_1 u + c_2\alpha(x,t)$$

Where u is the probability density function, T is a diffusion coefficient, c_1, c_2, and c_3 are constants, and $\alpha(x,t)$ is a function which depends on the spatial and temporal coordinates. The Lie symmetry analysis of FPE is based on the idea that if a system of equations has a certain symmetry, then it can be transformed into a simpler form by making certain coordinate transformations. In this case, the Lie symmetry analysis can be used to determine the form of the exact solution of FPE. Now, we use one of the symmetries to find a solution to Eq. (10). Equation (10) has symmetry with a generator $\Gamma_2 = \partial_t$, where the infinitesimals are $X = 0$, $T = 1$, and $U = 0$. By substituting the infinitesimals in the invariant surface condition (ISc), the following can be obtained:

$$Xu_x + Tu_t = U. \tag{12}$$

By solving the corresponding invariant surface condition $u_t = 0$, we obtain $u = \varphi(\zeta)$ and $\zeta = x$. By substituting this functional form into governing Eq. (10), we find that φ needs to satisfy $\varphi'' - 2\zeta e^{\zeta}\varphi' - 2e^{\zeta}(1+\zeta)\varphi = 0$. By solving the above equation, we obtain $\varphi(\zeta) = c\exp(2(\zeta-1)e^{\zeta})$ where c is an arbitrary constant. Hence, the solution to Eq. (10) is

$$u = c\exp(2(x-1)e^{x}).$$

FPE (10) is an important mathematical tool used to study various physical phenomena. It is a PDE that describes the evolution of a PDF of a stochastic process over time. It is closely related to the KdV equation, an example of an integrable system that describes the evolution of a wave in a dispersive medium. FPE can be written as $Xu_x + Tu_t = U$, where X and T are coefficients that describe the drift and diffusion terms of the equation, respectively. This equation can describe a wide range of phenomena, such as the diffusion of particles in a fluid medium, the behaviour of a population under the influence of natural selection, and the motion of particles in a gravitational field. The authors of this paper have proposed a new version of FPE, which includes a Lie symmetry analysis and exact solutions. This new version, the $Xu_x + Tu_t = U$ equation, is derived from the KdV equation by introducing an additional term. This additional term reduces the number of independent variables in FPE and thus simplifies the analysis. The authors have also presented exact solutions to this new version of FPE, which are obtained by applying the Lie symmetry analysis. These solutions can be used to describe the dynamics of the system and to obtain information about the behaviour of the PDF. Furthermore, these exact solutions can also be used to study the time evolution of the PDF and to analyse the stability of the system. In conclusion, the authors have proposed an update to the KdV equation, which is a PDE used to describe the evolution of a PDF over time.

Case 2: Assuming that $g = ne^{mt}$ and $f = x$, where n, m are arbitrary constants and $m, n \neq 0$. By substituting the values of g and f in Eq. (2) we obtain

$$\Delta\ :\ u_t = -u - xu_x + \frac{n}{2}e^{mt}u_{xx}, \tag{13}$$

The KdV equation is a non-linear PDE that describes the evolution of non-linear shallow water waves. This equation has been widely studied in the literature and used to model numerous physical phenomena, such as fluid flow and wave propagation. However, the KdV equation cannot accurately describe all physical phenomena, such as those with dissipation and diffusion effects. Equation (13), an update on the KdV equation, is FPE, which describes the evolution of a PDF of particles. This equation incorporates dissipation and diffusion effects, which are not accounted for in the KdV equation. The addition of the term $[\frac{n}{2}e^{mt}u_{xx}]$ to the KdV equation introduces a diffusion coefficient, allowing for the description of diffusion effects. Additionally, the term $(-xu_x)$ in Eq. (13) describes the dissipation of the wave, which is not present in the KdV equation.

Using invariant condition (3) in Eq. (13), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, the determining equations for the symmetry group of Eq. (13) can be found. By solving these equations, the following forms of coefficient functions are obtained

$$\begin{aligned}
T &= 2c_1 + c_2e^{-(m-2)t} + c_3e^{(m-2)t},\\
X &= (m-1)c_3e^{(m-2)t}x + c_1mx + c_4e^{t} + c_5e^{(m-1)t} + c_2xe^{-(m-2)t},\\
U &= \frac{-1}{2n}\Big(c_3e^{-2t}m^2x^2u - 4c_3me^{-2t}x^2u + c_3\,n\,m\,e^{(m-2)t}u + 4c_3e^{-2t}x^2u\\
&\quad + 2c_5me^{-t}xu + 2c_2nue^{-(m-2)t} - 4c_5e^{-t}xu - 2c_6nu - 2n\,\alpha(x,t)\Big).
\end{aligned} \tag{14}$$

Where $c_{(i)}$, $i = 1, 2, \ldots, 6$ are arbitrary constants and α is an arbitrary function in x, t. By substituting infinitesimal (14) into generator (4), the symmetries can be obtained as follows:

$$\Gamma_1 = mx\partial_x + 2\partial_t, \qquad \Gamma_4 = e^t\partial_x,$$

$$\Gamma_2 = e^{-(m-2)t}\Big(x\partial_x + \partial_t - u\partial_u\Big), \qquad \Gamma_5 = e^{(m-1)t}\partial_x - \frac{m-2}{ne^t}xu\,\partial_u,$$

$$\Gamma_3 = (m-1)xe^{(m-2)t}\partial_x + e^{(m-2)t}\partial_t - \frac{e^{-2t}}{2n}\Big(n\,me^{mt} + (m-2)^2x^2\Big)u\,\partial_u, \qquad \Gamma_6 = u\partial_u, \qquad \Gamma_\alpha = \alpha(x,t)\partial_u.$$

Now, we can find the solution to Eq. (13) by using one of the above mentioned symmetries. Equation (13) has symmetry with generator $\Gamma_1 = mx\partial_x + 2\partial_t$, where the infinitesimals are $X = mx$, $T = 2$ and $U = 0$. By substituting the infinitesimals into invariant surface condition (12), we obtain the corresponding invariant surface condition $mxu_x + 2u_t = 0$ with its solution, and we get $u = \varphi(\zeta)$ where $\zeta = xe^{-mt/2}$. Substituting this functional form into Eq. (13), we find that φ needs to satisfy $\varphi'' + \zeta(m-2)\varphi' - 2\varphi = 0$. By solving the above mentioned equation, we obtain

$$\varphi(\zeta) = c_1 \exp\Big(\frac{2-m}{n}\zeta^2\Big)\mathrm{M}\Big(\frac{m-1}{m-2}, \frac{3}{2}, \frac{1}{2}\frac{(m-2)\zeta^2}{n}\Big)\zeta + c_2 \exp\Big(\frac{2-m}{n}\zeta^2\Big)\mathrm{U}\Big(\frac{m-1}{m-2}, \frac{3}{2}, \frac{1}{2}\frac{(m-2)\zeta^2}{n}\Big)\zeta,$$

where c_1 and c_2 are arbitrary constants, and M, U are the Kummer-M and Kummer-U, functions respectively (see e.g. [1]). Hence, the solution to Eq. (13), is

$$u(x,t) = c_1 \exp\Big(\frac{2-m}{2n}x^2e^{-mt}\Big)\mathrm{M}\Big(\frac{m-1}{m-2}, \frac{3}{2}, \frac{1}{2}\frac{(m-2)x^2e^{-mt}}{n}\Big)xe^{-mt/2} + c_2 \exp\Big(\frac{2-m}{2n}x^2e^{-mt}\Big)\mathrm{U}\Big(\frac{m-1}{m-2}, \frac{3}{2}, \frac{1}{2}\frac{(m-2)x^2e^{-mt}}{n}\Big)xe^{-mt/2}.$$

The KdV equation is a PDE used to model the behaviour of shallow water waves. It is a non-linear dispersive wave equation that describes the propagation of long waves in an inviscid medium. The equation is named after the Dutch mathematician Diederik Korteweg and the physicist Gustav de Vries, who derived it in 1895. The KdV equation is a good approximation for a wide range of physical phenomena, such as surface waves in shallow water, wave-structure interactions, internal wave-structure interactions, and non-linear optics. The Lie symmetry analysis and exact solutions of FPE presented in Eq. (13) are an update on the KdV equation. This update incorporates FPE to more accurately model the behaviour of shallow water waves. FPE is a non-linear integro-differential equation that describes the time evolution of a PDF. This equation is used to model the behaviour of a system where particles, such as water molecules, are subject to random forces. This equation is used to model the diffusion of particles and is a good approximation for a wide range of physical phenomena, such as turbulence, optical beams, and chemical reactions. The main difference between the KdV equation and the updated FPE is that the KdV equation is a dispersive wave equation, while FPE is an integro-differential equation. The KdV equation describes the propagation of shallow water waves, while FPE describes the diffusion of particles in a system. The KdV equation is a non-linear equation, while FPE is a linear equation. The KdV equation is used to model wave-structure interactions, while FPE is used to model non-linear optics, turbulence, and chemical reactions. In conclusion, the Lie symmetry analysis and exact solutions of FPE presented in Eq. (13) is an update on the KdV equation. This update incorporates FPE to model the behaviour of shallow water waves more accurately.

Case 3: Assuming that $g = e^t$ and $f = x$, by substituting the values of g and f into (2), we obtain

$$\Delta\ :\ u_t = -u - xu_x + \frac{e^t}{2}u_{xx}. \tag{15}$$

Using invariant condition (3) in Eq. (15), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, we can find the determining equations for the symmetry group of Eq. (15). By solving these equations, we obtain the following forms of the coefficient functions:

$$T = 2c_1 + 4c_2e^t + 4c_3e^{-t}, \qquad X = (4c_2e^t + c_1)x + c_4 + c_5e^t,$$

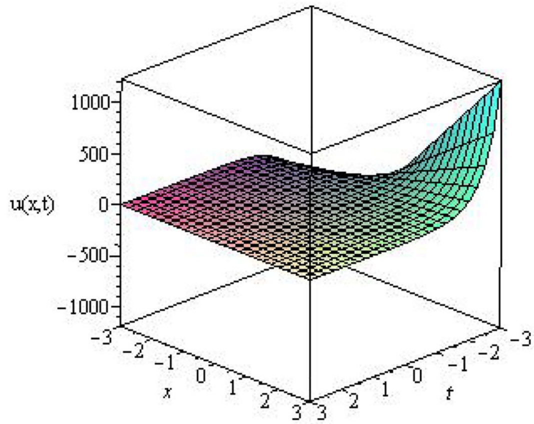

Fig. 1 Plot of the exact solution of (16) obtained by the classical generator of Γ_2 for the values of $c_1 = 1,\ c_2 = 1$; spaces $x = -3\ldots3, t = -3\ldots3$ in 3D

$$U = -\big(c_2e^t + c_3e^{-t}\big)x^2u - \big((c_3e^{-t} - c_2e^t)x - c_4\big)xue^{-t} - \big(4c_2e^t + 2c_3e^{-t} - c_6\big)u + \alpha(x, t).$$

where $c_{(i)}$, $i = 1, 2, \ldots, 6$ are arbitrary constants and α is an arbitrary function in x, t. By substituting the infinitesimal X, T, and U into generator (4), we obtain the symmetries given by

$$\Gamma_1 = x\partial_x + 2\partial_t, \qquad \Gamma_4 = \partial_x + xe^{-t}u\partial_u,$$
$$\Gamma_2 = xe^t\partial_x + e^t\partial_t - e^tu\partial_u, \qquad \Gamma_5 = e^t\partial_x,$$
$$\Gamma_3 = 2e^{-t}\partial_t - e^{-2t}\big(x^2 + e^t\big)u\,\partial_u, \qquad \Gamma_6 = u\partial_u,$$
$$\Gamma_\alpha = \alpha(x, t)\partial_u.$$

The KdV equation is a non-linear PDE frequently used to describe the dynamics of shallow water waves in a uniform depth channel. It has been used to model various physical phenomena, including the propagation of solitary waves and small-amplitude waves in homogeneous media. FPE (15) is a generalisation of the KdV equation that considers the effects of diffusion and randomness on the propagation of waves. FPE contains additional terms in comparison to the KdV equation, which is designed to capture the effects of diffusion and randomness. These terms include the diffusion coefficient ($\frac{1}{2}e^t$), as well as the drift term, ($-x\,u_x$), which accounts for the effect of random external forces on the system. By including these terms, FPE can describe the effects of randomness, diffusion, and non-linearity on the propagation of waves in a homogeneous medium. We can use one of these symmetries to find the solution to Eq. (15). Equation (15) has symmetry with generator $\Gamma_2 = xe^t\partial x + e^t\partial t - e^t\partial u$, where the infinitesimals are $X = xe^t,\ T = e^t$ and $U = -e^tu$. By substituting the infinitesimals in ISc (12) and solving the corresponding ISc $xe^tu_x + e^tu_t = -e^tu$, we obtain: $u = \varphi(\zeta)/x$, where $\zeta = xe^{-t}$. By substituting this functional form into governing Eq. (15), we find that φ needs to satisfy $\zeta^2\varphi'' - 2\zeta\varphi' + 2\varphi = 0$. By solving this equation, we get $\varphi(\zeta) = c_1\zeta + c_2\zeta^2$, where c_1 and c_2 are arbitrary constants. Hence, the solution to Eq. (15) is

$$u = c_1\,e^{-t} + c_2(xe^{-2t}). \tag{16}$$

FPE is a generalisation of the KdV equation and describes the evolution of a probability density in a system of particles. It is derived from the KdV equation by including a diffusion term, which accounts for the random motion of the particles. This diffusion term is represented in the equation by a second-order PDE, which is solved using a technique known as Lie symmetry analysis. The Lie symmetry analysis allows for the exact solution of the equation, solving the form of a linear combination of exponential functions. FPE describes the evolution of a probability density in a system of particles, represented by the equation [$u = c_1e^{-t} + c_2(xe^{-2t})$]. This equation updates the KdV equation because it considers the particles' random motion by adding a diffusion term. This diffusion term is represented in the equation by a second-order PDE, which is solved using Lie symmetry analysis. The Lie symmetry analysis is a powerful tool that can be used to determine the exact solution of FPE. This technique allows the exact solution of the equation to be obtained in the form of a linear combination of exponential functions. This form of solution is easier to interpret and understand than the original equation as it is written in terms of simple mathematical functions. It also allows for the analysis of the behaviour of the system with respect to different initial conditions. FPE provides a more sophisticated method of analyzing a system of particles than the KdV equation. The addition of the diffusion term makes the equation more complex but also more accurate in its prediction of the behaviour of the system. Additionally, the Lie symmetry analysis provides an exact solution to the equation, making it easier to interpret and understand. This makes FPE a powerful tool for studying the behaviour of a system of particles.

Case 4: Assuming that $g = be^t$ and $f = x+a$, where a, b are arbitrary constants and $a, b \neq 0$. By substituting the values of g and f into (2), we obtain

$$\Delta \,:\, u_t = -u - (x+a)u_x + \frac{1}{2}be^t u_{xx}. \tag{17}$$

By using invariant condition (3) in Eq. (17), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, we can find the determining equations for the symmetry group of Eq. (17). By solving these equations, we obtain the following forms of coefficient functions:

$$\begin{aligned}
&T = 2c_1 + c_2e^{-t} + c_3e^t, \qquad X = \big(c_1 + c_3e^t\big)x + c_4 + c_5e^t,\\
&U = \frac{1}{2b}\Big(- \big(c_2e^{-t} + c_3e^t\big)e^{-t}x^2u\\
&\quad + \Big(\big(2a + x\big)\big(- c_2e^{-t} + c_3e^t\big) - 2\big(2c_1 + c_2e^{-t} + c_3e^t\big)a + 2c_4\Big)xe^{-t}u\\
&\quad + \Big(2\big[2bc_6 - \big(2c_3e^t + (2a^2c_1 + bc_2 - 2ac_4)e^{-t} + c_2a^2e^{-2t}\big)\big]u + \alpha(x,t)\Big)\Big),
\end{aligned}$$

where $c_{(i)}$, $i = 1, 2, \ldots, 6$ are arbitrary constants and α is an arbitrary function in x, t. By substituting the infinitesimal X, T, and U into generator (4), we obtain the symmetries given by

$$\begin{aligned}
&\Gamma_1 = x\partial_x + 2\partial_t - \frac{a(x+a)}{be^t}u\,\partial_u, &&\Gamma_4 = \partial_x + \frac{x+a}{be^t}u\partial_u,\\
&\Gamma_2 = e^{-t}\partial_t - \frac{1}{2be^t}\big((x+a)^2e^{-t} + b\big)u\partial_u, &&\Gamma_5 = e^t\partial_x,\\
& &&\Gamma_6 = u\partial_u,\\
&\Gamma_3 = xe^t\partial_x + e^t\partial_t - e^tu\partial_u, &&\Gamma_\alpha = \alpha(x,t)\partial_u.
\end{aligned}$$

The KdV equation is a non-linear PDE that describes the evolution of a unidirectional wave in a dispersive medium. It is used to model various phenomena, including shallow water waves, ion-acoustic waves in plasmas, and ion-acoustic shocks in semiconductors. FPE (17) is a generalisation of the KdV equation that considers the diffusion of particles in the medium. In this equation, variable (u) represents the density of particles in the medium and variable (x) is the spatial coordinate. Variable (a) is a constant that describes the strength of non-linearity in the medium. Variable (be^t) is the diffusion coefficient, which increases with time. Therefore, this equation describes the evolution of particles in a non-linear, dispersive medium with diffusion. The Lie symmetry analysis of FPE can be used to determine the exact solutions of the equation. The Lie symmetry analysis is an approach to solving non-linear PDEs that involve finding continuous transformations of the independent and dependent variables that leave the equation invariant. This approach can reduce the number of independent variables or reduce the equation to a simpler form. It can also be used to find the exact solutions to the equation. In the case of FPE (17), the Lie symmetry analysis reveals several possible exact solutions, depending on the initial conditions. For example, there is an exact solution for a free particle, which is valid for any initial conditions. There is also an exact solution for a particle in a potential, valid in the absence of diffusion. Finally, there is an exact solution for a particle in a potential with diffusion, valid for any initial conditions. Overall, FPE (17) updates the KdV equation. This equation considers the diffusion of particles in the medium, which is not present in the KdV equation. The Lie symmetry analysis can be used to determine the exact solutions of the equation, which can help to understand the behaviour of particles in a non-linear, dispersive medium with diffusion. Now, one of the symmetries is used to find a solution to Eq. (17). Equation (17) has symmetry with generator Γ_4, where the infinitesimals are

$$X = 1, \qquad T = 0 \quad \text{and} \quad U = (x+a)u/(be^t).$$

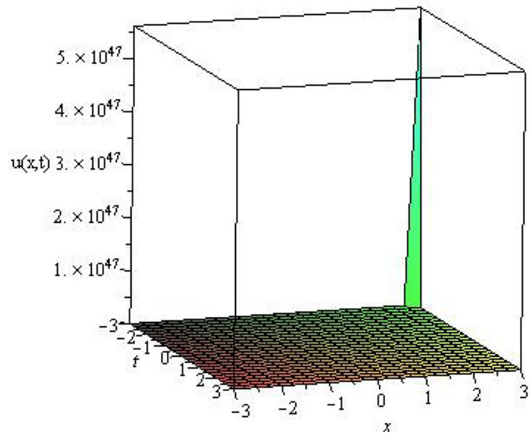

Fig. 2 Plot of the exact solution of (18) obtained by the classical generator of Γ_4 for $c_1 = 1$, $a = 2$ and $b = 2$; spaces $x = -3\ldots3$, $t = -3\ldots3$ in 3D

By substituting the infinitesimals into invariant surface condition (12) and solving the corresponding invariant surface condition $u_x = (x+a)u/(be^t)$, we obtain

$$u = \varphi(\zeta)\exp\left(\frac{x(x+2a)}{2be^t}\right) \text{ where } \zeta = t.$$

By substituting this functional form into Eq. (17), we find that φ needs to satisfy

$$\varphi' + \left(\frac{a^2+b}{2b}\right)\varphi = 0.$$

By solving the aforementioned equation, we get

$$\varphi(\zeta) = c_1 \exp\left(\frac{-1}{2b}(a^2+b)\zeta\right),$$

where c_1 is an arbitrary constant. Hence, the solution to Eq. (17) is

$$u(x,t) = c_1 \exp\left(\frac{1}{2b}\left[x(x+2a)e^{-t} - (a^2+b)t\right]\right), \tag{18}$$

The KdV equation is used to model a variety of wave phenomena, such as shallow water waves or solitons. It has been studied extensively since its introduction in 1895 and has been found to possess various exact solutions. On the other hand, FPE is a PDE, which models the evolution of a probability density. It was first proposed by Fokker and Planck in 1920 and is used to describe the time evolution of a system with random fluctuations, such as diffusion and heat conduction. Equation (18) is an exact solution to the FPE, and as such, it has some similarities and some differences with the KdV equation. Firstly, FPE does not possess the same non-linear terms as the KdV equation. Thus, it does not possess the same kind of non-linear wave-like solutions. Secondly, FPE is a first-order PDE, while the KdV equation is a third-order equation.

Case 5: Assuming that $g = e^t$ and $f = x - 1$, and by substituting the values of g and f in (2), we obtain

$$\Delta \;:\; u_t = -u - (x-1)u_x + \frac{e^t}{2}u_{xx}. \tag{19}$$

Using invariant condition (3) in (19), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, we can find the determining equations for the symmetry group of (19). By solving these equations, we obtain the following forms of the coefficient functions

$$T = 2c_1 + 2c_2e^{-t} + 2c_3e^t, \qquad X = \left(c_1 + 2c_3e^t\right)x + c_4 + c_5e^t,$$
$$U = \frac{-x^2u}{2e^t}\left(c_2e^{-t} + c_3e^t\right) + \frac{xu}{e^t}\left(\frac{x}{2}\left[c_3e^t - c_2e^{-t}\right] + c_1 + 2c_2e^{-t} + c_4\right)$$

$$- u\big(2c_3e^t + (c_1 + c_2 + c_4)e^{-t} + c_2e^{-2t} - c_6\big) + \alpha(x, t),$$

where $c_{(i)}$, $i = 1, 2, \ldots, 6$ are arbitrary constants and α is an arbitrary function in x, t. By substituting the infinitesimal X, T, and U in generator (4), we obtain the symmetries given by

$$\begin{aligned} &\Gamma_1 = x\,\partial_x + 2\partial_t + e^{-t}(x-1)\,u\,\partial_u, && \Gamma_4 = \partial_x + e^{-t}(x-1)\,u\,\partial_u,\\ &\Gamma_2 = 2e^{-t}\partial_t - e^{-2t}\big((x-1)^2 + e^t\big)\,u\,\partial_u, && \Gamma_5 = e^t\,\partial_x,\\ &\Gamma_3 = xe^t\,\partial_x + e^t\,\partial_t - e^t u\,\partial_u, && \Gamma_6 = u\,\partial_u,\\ & && \Gamma_\alpha = \alpha(x,t)\,\partial_u. \end{aligned}$$

Now, we use one of the symmetries to find a solution to (19). Equation (19) has symmetry with generator $\Gamma_3 = xe^t\partial x + e^t\partial t - e^t u\,\partial u$, where the infinitesimals are $X = xe^t$, $T = e^t$, and $U = -e^t u$. By substituting the infinitesimals in ISc (12) and solving the corresponding invariant surface condition $xe^t u_x + e^t u_t = -e^t u$ we obtain $u = e^{-t}\,\varphi(\zeta)$ where $\zeta = xe^{-t}$. By substituting this functional form into Eq. (19), we find that φ needs to satisfy $\varphi'' + 2\varphi' = 0$. By solving the aforementioned equation, we obtain $\varphi(\zeta) = c_1 + c_2e^{-2\zeta}$, where c_1 and c_2 are arbitrary constants. Hence, the solution to (19) is

$$u(x,t) = e^{-t}\Big(c_1 + c_2\exp\big(-2xe^{-t}\big)\Big).$$

The KdV equation is a non-linear PDE that describes the behaviour of a wide range of physical phenomena, including waves in shallow water, small-amplitude gravity waves, and the propagation of solitons in non-linear media. The KdV equation is a mathematical model used to describe waves in various systems, including shallow water waves, plasma waves, and acoustic waves. Equation (19) is an update on the KdV equation. This equation is a version of FPE, which is a linear PDE used to describe the evolution of a probability distribution over time. FPE can be used to describe the behaviour of a variety of physical systems, including thermodynamic systems, chemical reactions, biological systems, and financial markets. The main difference between the KdV equation and Eq. (19) is that the KdV equation is non-linear, while Eq. (19) is linear. FPE is a linear equation because it is derived from a system of linear differential equations, which means that FPE is more amenable to analytical solutions than the KdV equation. Additionally, FPE can be used to model the evolution of probability distributions, while the KdV equation is primarily used to model wave behaviour. The specific form of Eq. (19) also differs from the KdV equation in several important ways. The equation includes a temporal term, a spatial term, and a diffusion term, which are not present in the KdV equation. Additionally, the equation includes a coefficient for the spatial term of $(x - 1)$, which is not present in the KdV equation. Finally, the equation includes a coefficient for the diffusion term of $(\frac{1}{2}e^t)$ which is also not present in the KdV equation.

Case 6: Assuming that $g = a$ and $f = b$, where a, b are arbitrary constants and $a, b \neq 0$. Let $a = 1$ and $b = 1$. Substituting g and f into Eq. (2), we obtain

$$\Delta\ :\ u_t = -u_x + \frac{1}{2}u_{xx}, \tag{20}$$

Using invariant condition (3) in Eq. (20), and equating the coefficients of the various monomials in partial derivatives with respect to x and various powers of u, we can obtain the determining equations for the symmetry group of Eq. (20). By solving these equations, we obtain the following forms of the coefficient functions:

$$\begin{aligned} T &= 2c_1t^2 + 2c_2t + c_3,\\ X &= \big((1+2x)c_1 + c_2 + c_5\big)t + c_2x + c_4,\\ U &= -u\big(c_1(x-t)^2 + (c_5 + c_1)x - c_5t - c_6\big) + \alpha(x,t). \end{aligned}$$

Where $c_{(i)}$, $i = 1, 2, \ldots, 6$ are arbitrary constants and α is an arbitrary function in x, t. By substituting the infinitesimal X, T and U into the generator (4), we obtain the symmetries given by

$$\begin{aligned} &\Gamma_1 = t(2x+1)\partial_x + 2t^2\partial_t - \big((x-t)^2 + x\big)u\partial_u, &&\\ &\Gamma_2 = (x+t)\partial_x + 2t\partial_t, && \Gamma_3 = \partial_t,\\ &\Gamma_4 = \partial_x, && \Gamma_5 = t\partial_x + \big(t - x\big)u\partial_u,\\ &\Gamma_6 = u\partial_u, && \Gamma_\alpha = \alpha(x,t)\partial_u. \end{aligned}$$

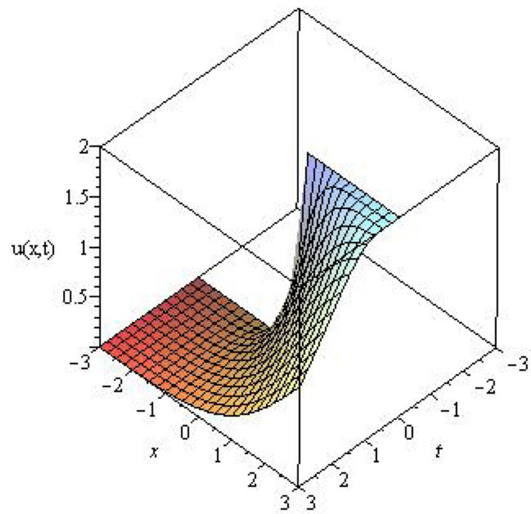

Fig. 3 Plot of the exact solution of (21) obtained by the classical generator of Γ_2 for $c_1 = 1, c_2 = 1$; spaces $x = -3\ldots3$, $t = -3\ldots3$ in 3D

FPE, as described in Eq. (20), is an update on the KdV equation but includes a diffusion term. FPE is a non-linear PDE used in studying systems with random fluctuations and is a type of diffusion equation. Specifically, FPE has the form $[u_t = -u_x + \frac{1}{2}u_{xx}]$, which is similar to the KdV equation, but with the addition of the second-order diffusion term on the right-hand side. The KdV equation is a non-linear dispersive PDE that describes one-dimensional wave propagation in shallow waters. It is used to study problems in fluid dynamics, such as the transmission of waves in shallow water and the formation of shock waves. The KdV equation is linear, meaning the solutions can be found by solving a linear system. In contrast, FPE is a non-linear equation, and the solutions cannot be found by solving a linear system. Now, it is possible to find the solution to Eq. (20) by using one of the symmetries above. Equation (20) has symmetry with generator $\Gamma_2 = (x + t)\,\partial x + 2t\,\partial t$ where the infinitesimals are $X = x + t$, $T = 2t$ and $U = 0$. By substituting the infinitesimals into ISc (12) and solving the corresponding invariant surface condition $(x + t)u_x + 2tu_t = 0$, we obtain $u = \varphi(\zeta)$ where $\zeta = (x-t)/\sqrt{t}$. By substituting this functional form into Eq. (20), we find that φ needs to satisfy $\varphi'' + \zeta\,\varphi' = 0$, such that $\varphi = c_1 + c_2\,\mathrm{erf}(\zeta/\sqrt{2})$, where c_1 and c_2 are arbitrary constants, and $\mathrm{erf}\,z \equiv \frac{2}{\sqrt{\pi}}\int_0^z e^{-\xi}d\xi$ is the error function (also called the probability integral) [1]. Hence, the solution to Eq. (20) is

$$u(x, t) = c_1 + c_2\ \mathrm{erf}\left(\frac{x - t}{\sqrt{2t}}\right). \tag{21}$$

The KdV equation is a canonical non-linear PDE that describes the evolution of shallow water waves and other physical phenomena. It has been used to model many physical phenomena, such as water plasma, and other dispersive waves. However, the KdV equation is limited in the behaviour of more complex systems. In particular, it does not accurately model the effects of random noise on the system. To address this, the authors of this paper have proposed an update on the KdV equation in the form of FPE, a type of stochastic differential equation. FPE is similar to the KdV equation as it is also a non-linear PDE that describes the evolution of a system. However, it is different as it includes a random noise term, which is responsible for introducing random fluctuations in the system. The authors have also used the Lie symmetry analysis to find exact solutions to FPE. They found that the exact solution to FPE is $u(x, t) = c_1 + c_2\ \mathrm{erf}\left(\frac{x-t}{\sqrt{2t}}\right)$.

3 Conclusion

In this paper, the Lie symmetry group method is applied to study the Fokker–Planck equation, and determining systems of symmetries for the FPE are discussed. By applying the classical symmetry method to several special cases of the FPE, we found that (1) has six symmetries, with one infinite-dimensional symmetry generated in each of the six cases. Some 3D plots for solution Eq. (1) are shown in Figs. 1, 2 and 3 that use suitable parameter values. Finally, in each of these cases, new exact solutions to the FPE are provided.

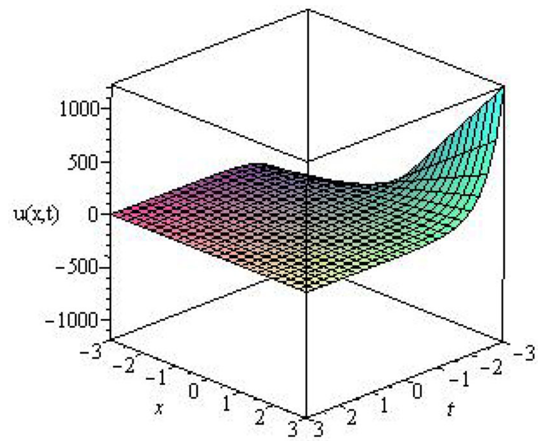

Fig. 4 PLOT CASE3

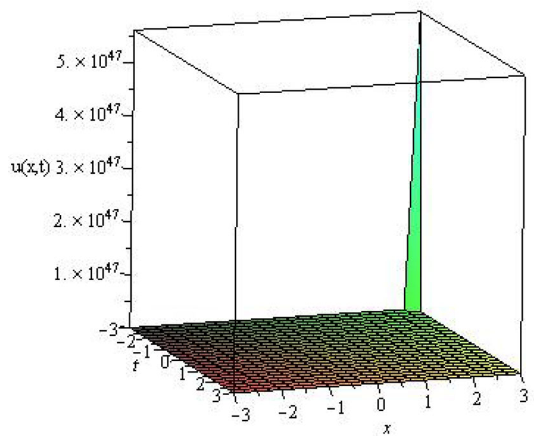

Fig. 5 PLOT CASE4

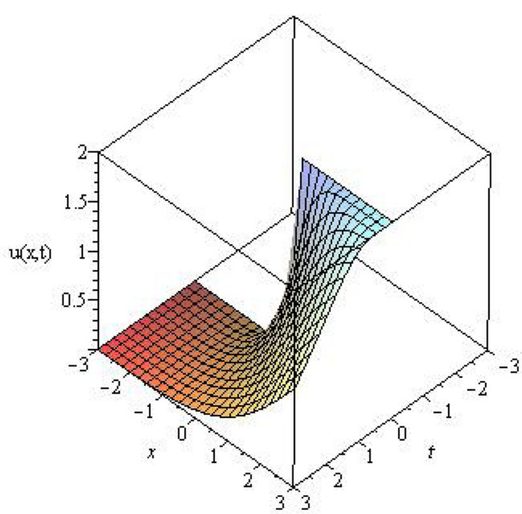

Fig. 6 PLOT CASE6

Funding This work was not supported by any specific funding.

Availability of data and materials No datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Author contributions The authors made equal contributions to this work. SA found the analysis of classical symmetries and the exact solution and wrote the article. MN read,corrected, revised and supervised the article.

References

1. Abramowitz, M.; Stegun, I.A.: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover Publications Inc, New York (1970)
2. Bluman, G.W.; Cole, J.D.: Similarity Methods for Differential Equations, vol. 13. Springer, Berlin (2012)
3. Bluman, G.W.; Kumei, S.: Symmetries and Differential Equations, vol. 81. Springer, Berlin (2013)
4. Brics, M.; Kaupuzs, J.; Mahnke, R.: How to solve Fokker–Planck equation treating mixed eigenvalue spectrum?. arXiv:1303.5211 (arXiv preprint) (2013)
5. Carrillo, J.A., Cordier, S., Mancini, S.: One dimensional Fokker–Planck reduced dynamics of decision making models in computational neuroscience. arXiv:1112.3794 (arXiv preprint) (2011).
6. Carrillo, J.A.; Cordier, S.; Mancini, S.: A decision-making Fokker-Planck model in computational neuroscience. J. Math. Biol. **635**, 801–830 (2011)
7. Elham, D.; RezaHejazi, S.: New solutions for Fokker–Plank equation of special stochastic process via lie point symmetries. Comput. Methods Differ. Equ. **51**, 30–42 (2017)
8. Frank, T.D.: Nonlinear Fokker–Planck Equations: Fundamentals and Applications. Springer, Berlin (2005)
9. Gardiner, C. W.: Handbook of Stochastic Methods, 4th ed (2009)
10. Hesam, S.; Nazemi, A.R.; Haghbin, A.: Analytical solution for the Fokker–Planck equation by differential transform method. Sci. Iran. **19**(4), 1140–1145 (2012)
11. Hill, J.M.: Solution of differential equations by means of one-parameter groups. Res. Notes Math. **20**, 20 (1982)
12. Hottovy, S.: The Fokker–Planck Equation. University of Wisconsin, Department of Mathematics, Madison (2011)
13. Kamano, F.D., Manga, B., Tossa, J.: Point and potential symmetries of the Fokker–Planck equation. arXiv:1503.02209 (arXiv preprint) (2015)
14. Lie, S.: Über die Integration durch bestimmte Integrale von einer Classe linearer partieller Differentialgleichungen. Cammermeyer **20**, 20 (1880)
15. Olver, P.J.: Applications of Lie Groups to Differential Equations, vol. 107. Springer, Berlin (2012)
16. Ortiz-Álvarez, H.H.; FrancyNelly, J.-G.; AbelEnrique, P.-A.: Some exact solutions for a unidimensional Fokker–Planck equation by using Lie Symmetries. Rev. Mat. Teoría Apl. **22**(1), 1–20 (2015)
17. Ouhadan, A.; et al.: Point and potential symmetries of the Fokker–Planck equation. Afr. J. Math. Phys. **5**, 33–41 (2007)
18. Ovsiannikov, L.V.: Group Analysis of Differential Equations. Academic press, New York (2014)
19. Pucci, E.; Saccomandi, G.: Potential symmetries and solutions by reduction of partial differential equations. J. Phys. A Math. Gen. **26**(3), 681 (1993)
20. Risken, H.; Eberly, J.H.: The Fokker–Planck equation, methods of solution and applications. J. Opt. Soc. Am. B Opt. Phys. **23**, 508 (1985)
21. Sastri, C.C.A.; Dunn, K.A.: Lie symmetries of some equations of the Fokker–Planck type. J. Math. Phys. **26**(12), 3042–3047 (1985)
22. Sherring, J.; Prince, G.: DIMSYM-Symmetry Determination and Linear Partial Differential Equations Package. Dept. of Mathematics Preprint, LaTrobe Univ, Australia (1992)
23. Stohny, V.: Symmetry properties and exact solutions of the Fokker–Planck equation. J. Nonlinear Math. Phys. **4**(1–2), 132–136 (1997)
24. Tanski, I.A.: Fundamental solution of Fokker–Planck equation. arXiv:nlin/0407007 (arXiv preprint) (2004)
25. Zorzano, M.P.; Helmut, M.; Luiz, V.: Numerical solution for Fokker–Planck equations in accelerators. Phys. Sect. D **1132**, 379–381 (1998)