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The eigenvalue problem for Kirchhoff-type operators in Musielak–Orlicz spaces

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Abstract Given a Musielak–Orlicz function $\varphi(x,s) : \Omega \times [0,\infty) \to \mathbb{R}$ on a bounded regular domain $\Omega \subset \mathbb{R}^n$ and a continuous function $M : [0,\infty) \to (0,\infty)$, we show that the eigenvalue problem for the elliptic Kirchhoff's equation $-M\left(\int\limits_\Omega \varphi(x, |\nabla u(x)|)\mathrm{d}x\right) \operatorname{div}\left(\frac{\partial\varphi}{\partial s}(x, |\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|}\right) = \lambda \frac{\partial\varphi}{\partial s}(x, |u(x)|)\frac{u(x)}{|u(x)|}$ has infinitely many solutions in the Sobolev space $W_0^{1,\varphi}(\Omega)$. No conditions on φ are required beyond those that guarantee the compactness of the Sobolev embedding theorem.

Keywords Variable exponent p-Laplacian · Sobolev embedding · Modular spaces · Musielak–Orlicz spaces · Variable exponent spaces · Kirchhoff equations · Nonlinear wave equation

Mathematics Subject Classification Primary 35A01; Secondary 46A80

1 Introduction

In 1883, G. Kirchhoff [13] noted that the vibration of an elastic, variable-length string is modeled by means of the following variant of the classical wave equation:

$$\frac{\partial^2 u}{\partial t^2} = M\left(\int_0^1 \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial x^2}, \tag{1.1}$$

where $M : [0,\infty) \longrightarrow [0,\infty)$ is a suitable increasing function. Since then, a vast amount of literature was devoted to studying the solvability of various Kirchhoff-type equations [2].

In higher dimensions, (1.1) takes up the form

$$\frac{\partial^2 u}{\partial t^2} = M\left(\int\limits_\Omega |\nabla u(y)|^2 dy\right)\Delta u. \tag{1.2}$$

The stationary problem

$$\begin{cases} M\left(\int\limits_\Omega |\nabla u(y)|^2 dy\right)\Delta u = f(x,u) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.3}$$

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has been extensively studied under different assumptions on M and f, see for example [1,19,20,26,27] and the references therein.

Of particular interest is the extension of (1.2) to equations involving the p-Laplacian [10,18]: If $1 < p < \infty$ is a real number and $M \geq 0$ is a continuous function, the p-Kirchhoff operator is defined as

$$K_p : W_0^{1,p}(\Omega) \longrightarrow \left(W_0^{1,p}(\Omega)\right)^* K_p(u) = -M\left(\int\limits_\Omega |\nabla u(y)|^p dy\right) \operatorname{div}\left(|\nabla u(x)|^{p-2}\nabla u(x)\right), \quad (1.4)$$

which clearly generalizes the right-hand side of (1.1). In (1.4), $W_0^{1,p}(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in the usual Sobolev space $W^{1,p(\cdot)}(\Omega)$. The reader is referred to Sect. 3 for the precise terminology to be used in this work.

From the physical point of view, this operator arose from the need of finding a mathematical model for the motion of a vibrating string under less stringent assumptions than those assumed for the classical derivation of the linear wave equation. Specifically, the linear wave equation is obtained under the assumption that the length of the vibrating string remains constant during the motion. By removing this assumption, the nonlinear operator K_p comes into the play.

Various boundary value problems associated to the p-Kirchhoff operator (1.4) have been studied for example in [10,18,19]. The emergence of the variable exponent Lebesgue spaces and the subsequent realization of their role in applications [5] sparked interest in the study of boundary value problems of the type

$$\begin{cases} K_p(u) = f(x,u) \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1.5)$$

for a variable exponent $p = p(x)$.

The variability of the exponent opens a new class of highly non-trivial difficulties, mainly related to its modular nature, that is to its direct relation to the functional

$$u \longrightarrow \int_\Omega |u(x)|^{p(x)} \mathrm{d}x$$

rather than with the norm

$$u \longrightarrow \|u\|_{p(x)},$$

as discussed in [2,21]. A vast amount of literature exists on boundary value problems of the type (1.5), under the assumption of the variability of the exponent $p(x)$. We refer the reader to some of the most significant from the point of view of the present work, such as [2–4,6–8,21,23].

In this article we observe that the treatment of a wide class of eigenvalue problem for Kirchhoff-type operators, including, but not limited to the variable-exponent case, can be unified by the consideration of Musielak–Orlicz spaces. With this objective in mind, we study the eigenvalue problem for a general Kirchhoff equation in this framework. In fact, given a suitable Musielak–Orlicz (MO)-function φ and an appropriate function M (we refer the reader to the next section for a detailed account of the notation and terminology), the generalized Kirchhoff operator is naturally given by

$$K(u) = -M\left(\rho_\varphi(|\nabla u|)\right) \operatorname{div}\left(\frac{\partial\varphi}{\partial s}(x, |\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|}\right). \quad (1.6)$$

We provide a characterization of the first eigenvalue for the operator (1.6) via a Musielak–Orlicz Sobolev embedding theorem that has been obtained in [16, Theorem 5.1].

The present work is organized as follows. In the next section we introduce the notation and terminology to be used in the exposition and present a brief survey on the literature. In Section 3 the definition and basic properties of the Musielak–Orlicz spaces needed in the sequel are given. Section 4 is a brief survey on the Sobolev embedding theorems in the context of Musielak–Orlicz spaces. In Section 5 we delve into some natural properties of the Musielak–Orlicz operators to be considered later and the functional analytic stage is set for the treatment of the eigenvalue problems developed in detail in Section 7. Section 8 contains applications to the variable exponent case, i.e., to the case $\varphi(x,t) = t^{p(x)}$.

2 Known results

In the sequel, $\Omega \subset \mathbb{R}^n$ will denote a bounded domain with a regular boundary (the cone condition will do) and $\mathcal{M}(\Omega)$ will stand for the vector space of all real-valued, Borel-measurable functions defined on Ω. The subset of $\mathcal{M}$ consisting of functions

$$p : \Omega \longrightarrow [1, \infty)$$

will be denoted by $\mathcal{P}(\Omega)$. The Lebesgue measure of a subset $A \subset \mathbb{R}^n$ will be denoted by $|A|$.

For $p \in \mathcal{P}(\Omega)$, the following notation will be used throughout this work:

$$p_- := \operatorname{essinf}_{\Omega} p, \quad p_+ := \operatorname{esssup}_{\Omega} p.$$

For $p \in \mathcal{P}(\Omega)$, the eigenvalue problem

$$\begin{cases} K_p(u) = \lambda |u|^{q-2}u \ \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

was studied in [2] for M subject to

$$m_1 t^{\alpha-1} \le M(t) \le m_2 t^{\beta-1}, \tag{2.2}$$

for $\beta \ge \alpha > 1, m_2 \ge m_1 > 0$ and variable exponents $p, q \in C(\overline{\Omega})$ satisfying either [2, Theorem 3.1, Theorem 3.4, Theorem 3.6]

$$\beta p_+ < q_- \le q_+ < p^*, \tag{2.3}$$

$$1 < q_- \le q_+ < \alpha p_-, \tag{2.4}$$

or

$$1 < q(x) < p(x) < p^*(x) \tag{2.5}$$

in Ω. Here

$$p^*(x) = \frac{np(x)}{n - p(x)} \mathbf{1}_{(1,n)}(p(x)) + \infty \mathbf{1}_{[n,\infty)}(p(x))$$

and $\mathbf{1}_A$ stands for the characteristic function of the set A.

An anisotropic variant of (2.1) was considered in [23], whereas [4] deals with the following weighted version of (2.1):

$$\begin{cases} K_p(u) = \lambda V |u|^{q-2}u \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.6}$$

for $0 \le V \in L^\infty(\Omega)$, $p_+ < n$ and M subject to (2.2) [4, Theorem 1.4]. The generalized version of (1.5) given by

$$\begin{cases} K_p(u) = B\left(\int_\Omega \int_0^{|u|} f(x,s)ds\right) f(x,u) \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2.7}$$

is studied in [12]. Specifically, problem (2.7) is shown to have a solution in $W_0^{1,p(\cdot)}(\Omega)$ under the following assumptions [12, Theorem 3.1]:

(i) $\int_0^t M(s)ds \ge mt^{\alpha_1}$, $m > 0$, for sufficiently large t,

(ii) for some positive constants c_1, c_2 the Carathéodory function

$$f : \Omega \times \mathbb{R} \to \mathbb{R}$$

satisfies the bound $|f(x,t)| \le c_1 + c_2|t|^{q(x)}$ for $q \in C(\overline{\Omega})$, $1 < q(x) < p^*(x)$,

(iii) for some positive constants A_1, A_2,

$$\int_0^t B(s)ds \le A_1 + A_2 t^{\beta_1},$$

(iv)

$$\beta_1 q_+ < \alpha_1 p_-.$$

For $M(t) = a + b\gamma t^{\gamma-1}$, the study of the solvability of a hyperbolic equation related to the operator K_p can be found in [3]. A polyharmonic version of (1.5) was studied in [6].

In [8], a discussion of problem 1.5 is presented for a linear function $M(t) = a+bt$ [8, Theorem 1.1], whereas in [7] the existence of a solution of (1.5) is proved provided, among other conditions, that $M(t) \ge m_0 > 0$ for $t > 0$ [7, Theorems 3.1−3.4].

Associated to every Musielak–Orlicz function φ, the so-called Matuszewska index of φ (see [17]) generalizes the role of the exponent p in the classical Lebesgue spaces; in particular the exponent p is easily verified to be the Matuszewska index of the MO function given by

$$(x,t) \longrightarrow t^{p(x)}.$$

As is observed in [16], sharp conditions (trivially satisfied by the exponent p for the Sobolev embedding stated in [11,15]) on the Matuszewska index of the MO function φ guarantee the compactness of the Sobolev embedding

$$W_0^{1,\varphi}(\Omega) \hookrightarrow L^{\varphi}(\Omega) \tag{2.8}$$

for a bounded domain $\Omega \subset \mathbb{R}^n$. Via the compactness of the Sobolev embedding, a natural characterization of the first eigenvalue of the Kirchhoff's operator can be given, and the results outlined above can be regarded as particular cases of our more general approach, which allows for less stringent conditions than the ones stated in the first part of this Section.

3 Musielak–Orlicz spaces

Throughout this paper $\Omega \subset \mathbb{R}^n$, $n \ge 1$ will stand for a bounded, Lipschitz domain. A convex, left-continuous function

$$\varphi : [0,\infty) \longrightarrow [0,\infty)$$

with $\varphi(0) = 0$, $\lim_{x\to\infty} \varphi(x) = \infty$ and $\lim_{x\to 0^+} \varphi(x) = 0$ will be said to be an Orlicz function. In particular, any Orlicz function is non-decreasing. The term generalized Orlicz function or Musielak–Orlicz (MO) function will refer to a function

$$\varphi : \Omega \times [0,\infty) \to [0,\infty)$$

such that

$$\varphi(x,\cdot) : [0,\infty) \to [0,\infty)$$

is an Orlicz function for each fixed $x \in \Omega$ and

$$\varphi(\cdot,y) : \Omega \to [0,\infty)$$

is Lebesgue measurable for each fixed $y \in \mathbb{R}$.

The Musielak–Orlicz space $L^{\varphi}(\Omega)$, [24,25], is the real-vector space X_{φ} of all extended-real valued, Borel-measurable functions u on Ω for which

$$\int\limits_{\Omega} \varphi(x, \lambda|u(x)|)\, \mathrm{d}x < \infty \quad \text{for some } \lambda > 0,$$

furnished with the norm

$$\|u\|_{\varphi} = \inf\left\{\lambda > 0 : \int_{\Omega} \varphi\left(x, \frac{|u(x)|}{\lambda}\right) \leq 1\right\}.$$

The functional

$$\rho_{\varphi}(u) = \int\limits_{\Omega} \varphi(x, |u(x)|)\, \mathrm{d}x \tag{3.1}$$

is a convex, left-continuous modular on X_{φ} [9,11,24]. It is well known [9] that $L^{\varphi}(\Omega)$ is a Banach space. Since it will be needed in the sequel, we define the complementary function φ^* of φ as

$$\varphi^* : \Omega \times [0, \infty) \longrightarrow [0, \infty) \tag{3.2}$$

$$\varphi^*(x, t) = \sup_{u \geq 0} (tu - \varphi(x, u)). \tag{3.3}$$

The complementary function φ^* is itself a MO-function (see [9]) and Hölder's inequality holds, namely for $f \in L^{\varphi}(\Omega)$ and $g \in L^{\varphi^*}(\Omega)$,

$$\int\limits_{\Omega} f(x)g(x)\, \mathrm{d}x \leq 2\|f\|_{\varphi}\|g\|_{\varphi^*}. \tag{3.4}$$

If in addition

$$\int\limits_{K} \varphi(x, t)\, \mathrm{d}x < \infty \tag{3.5}$$

for any $K \subset \Omega$ with Lebesgue measure $|K| < \infty$ and

$$\inf_{x \in \Omega} \varphi(x, 1) > 0, \tag{3.6}$$

the Musielak–Orlicz Sobolev space $W^{1,\varphi}(\Omega)$ consisting of all functions in $L^{\varphi}(\Omega)$ whose distributional derivatives are in $L^{\varphi}(\Omega)$, is a Banach space when furnished with the norm

$$\|u\|_{1,\varphi} = \|u\|_{\varphi} + \||\nabla u|\|_{\varphi},$$

where ∇ stands for the gradient operator and $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. The Sobolev space $W_0^{1,\varphi}(\Omega)$ is defined to be the closure of $C_0^{\infty}(\Omega)$ in $W^{1,\varphi}(\Omega)$.

4 Sobolev-type embeddings

The central idea of this Section is the Sobolev Embedding Theorem 4.7. In order to facilitate the flow of ideas we present a few definitions.

The Matuszewska index of an Orlicz function φ was introduced by Matuszewska and Orlicz in [17].

Definition 4.1 For φ as above and each $x \in \Omega$, set

$$M(x, t) = \limsup_{u \to \infty} \frac{\varphi(x, tu)}{\varphi(x, u)}. \tag{4.1}$$

The *Matuszewska index* of φ is defined to be

$$m(x) = \lim_{t\to\infty} \frac{\ln M(x,t)}{\ln t} = \inf_{t>1} \frac{\ln M(x,t)}{\ln t}. \tag{4.2}$$

Definition 4.2 The limit (4.1) is said to be *uniform* if for each $\delta > 0$ there exist $s_0 > 1$ and $T > 1$ such that, for all $(x,t) \in \Omega \times [T, \infty)$, one as

$$M(x,t) - \delta < \frac{\varphi(x, ts_0)}{\varphi(x, s_0)} < M(x,t) + \delta. \tag{4.3}$$

The following examples illustrate the above definition for some well known MO functions:

Example 4.3 Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain and

$$p : \Omega \longrightarrow (0, \infty)$$

be Borel-measurable. The MO function

$$\begin{aligned} \varphi : \Omega \times [0,\infty) &\longrightarrow [0,\infty) \\ \varphi(x,t) &= t^{p(x)} \end{aligned} \tag{4.4}$$

has Matuszewska index equal to $p(x)$. In this case, the convergence (4.2) is trivially uniform on Ω and the limit (4.2) is clearly uniform.

Lemma 4.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and φ an MO function as described above. If the Matuszewska index m is the restriction to Ω of a continuous function $\tilde{m}$ on the closure of Ω, i.e.,*

$$\tilde{m} : \overline{\Omega} \longrightarrow \mathbb{R}, \tag{4.5}$$

and the convergence to the limits (4.1) and (4.2) is uniform, then there exist $C > 1$, $T_0 > 1$ and $S_0 > 1$ such that uniformly in Ω it holds

$$\varphi(x, sT_0) \le C\varphi(x,s) \tag{4.6}$$

for any $s \ge S_0$.

Condition (4.6) will be referred to as the Δ condition.

Proof Fix $\delta > 0$, then for some $T_0 > 1$ one has for any $t \ge T_0$, by virtue of (4.2)

$$t^{m(x)-\delta} < M(x,t) < t^{m(x)+\delta} \tag{4.7}$$

uniformly in Ω. By definition of $M(x,t)$ and on account of the uniformity assumption of the infimum (4.1), there exists a positive number N for which, uniformly for $t \ge T_0$ and $x \in \Omega$, it holds that

$$\sup_{s\ge N} \frac{\varphi(x, st)}{\varphi(x,s)} < t^{m(x)+\delta}. \tag{4.8}$$

In particular, for all $s \ge N$:

$$\varphi(x, sT_0) \le T_0^{\sup_\Omega m(x)+\delta} \varphi(x,s).$$

□

Corollary 4.5 *There exists $S_0 > 1$ and a constant $C > 1$ such that*

$$\varphi(x, 2s) \le C\varphi(x,s) \tag{4.9}$$

for any $x \in \Omega$, $s \ge S_0$.

Lemma 4.6 *If the statement of corollary 4.5 holds, then ρ-convergence is equivalent to norm-convergence in $L^{\varphi}(\Omega)$.*

Proof It suffices to show that if (u_j) ρ_φ-converges to 0 and converges a.e. to 0. then it converges to 0 in the topology of the norm. This will be automatically implied by the validity of the equality

$$\lim_{j\to\infty} \rho_\varphi(\lambda x_n) = 0 \tag{4.10}$$

for any $\lambda > 0$. It is obviously necessary to show (4.10) only for $\lambda > 1$. Let $N = [\log_2 \lambda] + 1 \geq 1$. A simple argument shows that for C, S_0 as in Corollary 4.5

$$\varphi(x, \lambda|u_n(x)|) \leq \sum_{1}^{N-1} \varphi(x, 2S_0) + C^N \varphi(x, (\lambda/2^N)|u_n(x)|).$$

Since the second term in the right-hand side tends a.e. to 0 as $n \to \infty$, it follows that

$$\rho_\varphi(\lambda u_n) = \int_\Omega \varphi(x, \lambda|u_n(x)|)\mathrm{d}x \to 0 \text{ as } n \to \infty. \tag{4.11}$$

□

We refer the reader to [16] for the proof of the following theorem.

Theorem 4.7 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and*

$$\varphi : \Omega \times [0, \infty) \longrightarrow \mathbb{R}$$

a Musielak–Orlicz function that satisfies condition (3.6) and for which the limits in (4.1) and (4.2) are uniform; assume that the Matuszewska index m is the restriction to Ω of a continuous function $\tilde{m}$ defined on the closure of Ω, that

$$1 < m_- =: \inf_\Omega m,$$

and that there exists a function

$$\beta : (0, \infty) \longrightarrow (0, \infty)$$

such that, uniformly in Ω and for $t > 0$:

$$\varphi(x, t) \leq \beta(t). \tag{4.12}$$

Then the embedding

$$W_0^{1,\varphi}(\Omega) \hookrightarrow L^\varphi(\Omega) \tag{4.13}$$

is compact.

Corollary 4.8 *For φ satisfying the conditions of Theorem 4.7, there exists a positive constant C depending only on n, Ω, φ such that for any $u \in W_0^{1,\varphi}(\Omega)$*

$$\|u\|_\varphi \leq C\||\nabla u|\|_\varphi. \tag{4.14}$$

Proof See [16]. □

5 Δ_2-type Musielak–Orlicz functions

From now on we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain satisfying the cone condition and φ is a MO-function satisfying the conditions of Theorem 4.7. Denote the conjugate of φ by φ^*.

Theorem 5.1 *Let φ be a MO function on Ω; assume that φ satisfies the conditions of Theorem 4.7; in particular, φ satisfies the Δ condition 4.6, i.e., for some $K > 0$, $S_0 > 1$ it holds that*

$$\varphi(x, 2s) \leq K\varphi(x, s) \ \text{for all} \ s \geq S_0, \ x \in \Omega. \tag{5.1}$$

Then,

$$\sup\left\{\rho_\varphi(u) : \rho_\varphi(|\nabla u|) \leq r\right\} < \infty. \tag{5.2}$$

Proof It follows from (5.1) that for arbitrary $v \in L^\varphi(\Omega)$ (recall that φ is nonnegative and nondecreasing)

$$\begin{aligned}
\rho_\varphi(2v) &= \int\limits_\Omega \varphi(x, 2|v(x)|)\mathrm{d}x \\
&= \left(\int\limits_{|v|<S_0} + \int\limits_{|v|\geq S_0}\right)\varphi(x, 2|v(x)|)\mathrm{d}x \\
&\leq \left(\int\limits_{|v|<S_0} \varphi(x, 2S_0)\mathrm{d}x + K\rho_\varphi(v)\right) \\
&\leq \left(\int\limits_\Omega \varphi(x, 2S_0)\mathrm{d}x + K\rho_\varphi(v)\right).
\end{aligned} \tag{5.3}$$

If $r \geq 1$ and $u \in W_0^{1,\varphi}(\Omega)$ with

$$\rho_\varphi(|\nabla u|) \leq r, \tag{5.4}$$

it is a simple matter to verify that if $\||\nabla u|\|_\varphi \geq 1$

$$\begin{aligned}
1 = \rho_\varphi\left(\frac{|\nabla u|}{\||\nabla u|\|_\varphi}\right) &\leq \frac{1}{\||\nabla u|\|_\varphi}\rho_\varphi(|\nabla u|) \\
&\leq \frac{1}{\||\nabla u|\|_\varphi} r;
\end{aligned}$$

it is thus clear that if (5.4) holds, then:

$$\||\nabla u|\|_\varphi \leq r.$$

Therefore,

$$\||\nabla u|\|_\varphi \leq \max\{1, r\} = b.$$

It follows from the preceding reasoning in conjunction with Poincaré inequality that if $\rho_\varphi(|\nabla u|) \leq r$, then, for some $C > 0$,

$$\|u\|_\varphi \leq C\||\nabla u|\|_\varphi \leq Cb.$$

Hence,

$$\rho_\varphi\left(\frac{u}{Cb}\right) \leq 1. \tag{5.5}$$

If $Cb < 2$, (5.5) implies

$$\begin{aligned}\rho_\varphi(u) &= \rho_\varphi\left(Cb\frac{u}{Cb}\right) \leq \rho_\varphi\left(2\frac{u}{Cb}\right)\\ &\leq \int_\Omega \varphi(x, 2S_0)\mathrm{d}x + K\rho_\varphi\left(\frac{u}{Cb}\right)\\ &\leq \int_\Omega \varphi(x, 2S_0)\mathrm{d}x + K. \end{aligned}\tag{5.6}$$

Otherwise, on account of the iteration of inequality (5.3)

$$\begin{aligned}\rho_\varphi(u) &= \rho_\varphi\left(Cb\frac{u}{Cb}\right)\\ &\leq \rho_\varphi\left(2^{[\log_2 Cb]+1}\frac{u}{Cb}\right)\\ &\leq \left(1 + K + ...K^{[\log_2 Cb]}\right)\int_\Omega \varphi(x, 2S_0)\mathrm{d}x + K^{[\log_2 Cb]+1}. \end{aligned}\tag{5.7}$$

In all, (5.6) and (5.7) yield (5.2). □

An immediate consequence of the preceding theorem is the following functional-analytic result:

Lemma 5.2 *For φ as in Theorem 4.7 and $r > 0$, the modular ball*

$$B_r := \left\{u \in W_0^{1,\varphi}(\Omega) : \rho_\varphi(|\nabla u|) \leq r\right\}$$

is weakly closed.

Proof B_r is clearly convex. It suffices to show that it is also norm-closed. If (u_j) norm-converges to $u \in W_0^{1,\varphi}(\Omega)$, then $\rho_\varphi(|\nabla(u_j - u)|) \to 0$ as $n \to \infty$ and there is no loss of generality in assuming that $\nabla u_j \to \nabla u$ a.e. in Ω. Theorem 4.7 guarantees that (u_j) can be chosen so that $u_j \to u$ a.e. in Ω. Since

$$\rho_\varphi(|\nabla u|) = \int_\Omega \varphi(x, \lim_{n\to\infty} |\nabla u_n(x)|)\mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega \varphi(x, |\nabla u_n(x)|)\mathrm{d}x \leq r, \tag{5.8}$$

it follows that $u \in B_r$, i.e., B_r is norm-closed (and convex) and hence weakly closed. □

6 Differentiability properties

Aiming at a full description of the Fréchet derivative of the functionals to be introduced momentarily, a further assumption is imposed unto the MO function φ at this point, namely, it is from now on required that φ be an N function. More precisely:

Definition 6.1 An MO function is said to be an *N-function* iff it satisfies the condition

$$\lim_{t\to 0} \frac{\varphi(x, t)}{t} = 0 \;\; a.e.. \tag{6.1}$$

It is well known [9] that if φ is an N-function, it can be written as

$$\varphi(x, t) = \int_0^t \phi(x, s)\, ds, \tag{6.2}$$

where $\phi(x, \cdot)$ is the right t-derivative of φ. On the other hand, the conjugate function φ^* can be written as

$$\varphi^*(x, t) = \int_0^t \phi^{-1}(x, s)\, ds. \tag{6.3}$$

The proof of the following theorem can be found in [14,22]; we include it here in the interest of completeness.

Theorem 6.2 *Let φ be an N-function; assume that*

$$\varphi_t(x,t) = \frac{\partial \varphi}{\partial t}(x,t)$$

is continuous a.e. $x \in \Omega$. Define the operator T_{φ_t} as

$$T_{\varphi_t} : \mathcal{M}(\Omega) \longrightarrow \mathcal{M}(\Omega)$$
$$T_{\varphi_t}(u) = \varphi_t(x, |u(x)|) = \frac{\partial \varphi}{\partial t}(x, |u(x)|).$$

Then, from the assumption

$$T_{\varphi_t}(L^{\varphi}(\Omega)) \subseteq L^{\varphi^*}(\Omega) \tag{6.4}$$

it follows that the operator

$$T_{\varphi_t} : L^{\varphi}(\Omega) \longrightarrow L^{\varphi^*}(\Omega)$$

is continuous and bounded.

Proof Assume $(u_n) \subseteq L^{\varphi}(\Omega)$ converges to $u \in L^{\varphi}(\Omega)$. On $\Omega \times [0,\infty)$ define

$$w(x,t) = \varphi_t(x, |u(x)+t|) - \varphi_t(x, |u(x)|);$$

then on account of the assumption on φ_t, w is a Carathéodory function and $w(x,0) = 0$. If

$$T_w : L^{\varphi}(\Omega) \longrightarrow L^{\varphi^*}(\Omega)$$

is continuous at 0, then $T_w(u_n - u) \longrightarrow 0$ in $L^{\varphi^*}(\Omega)$ as $n \to \infty$. If φ^* satisfies the Δ_2 condition, the latter is equivalent to

$$\rho_{\varphi^*}(T_w(u_n - u)) = \int_{\Omega} \varphi^*(x, |\varphi_t(x, |u_n(x)|) - \varphi_t(x, |u(x)|)|) \to 0 \ \text{ as } \ n \to \infty,$$

that is

$$T_{\varphi_t}(u_n) \longrightarrow T_{\varphi_t}(u) \ \text{ in } L^{\varphi^*}(\Omega) \text{ as } \ n \to \infty.$$

Therefore, it is enough to show that T_{φ_t} is continuous at 0 under the assumption that $\varphi_t(x,0) = 0$ *a.e.* in Ω. Assume that T_{φ_t} is not continuous at 0; let $r > 0$ and let (u_n) be a sequence that converges to 0 in $L^{\varphi}(\Omega)$, for which

$$\|T_{\varphi_t}(u_n)\|_{\varphi^*} \geq r \ \text{ for any } n \in \mathbb{N}.$$

Since norm convergence implies modular convergence, one can, without loss of generality assume that

$$\max\left\{\rho_{\varphi}(u_n), \|u_n\|_{\varphi}\right\} < \frac{1}{2^n},$$

and hence that $\sum_{n=1}^{\infty} \int_{\Omega} \varphi(x, |u_n(x)|)\,dx < \infty$. Due to the validity of the Δ_2 condition for φ^*, norm convergence and modular convergence are equivalent on $L^{\varphi^*}(\Omega)$. It follows that there exists $\epsilon(r) > 0$ such that $\rho_{\varphi^*}\left(T_{\varphi_t}(u_n)\right) \geq \epsilon(r)$ for any $n \in \mathbb{N}$. We next claim the existence of a sequence of real numbers (ϵ_k), a sequence (Ω_k) of subsets of Ω and a subsequence (u_{n_k}) of (u_n) satisfying the following conditions:

(i) $\epsilon_{k+1} < \frac{1}{2}\epsilon_k$,
(ii) $|\Omega_k| \leq \epsilon_k$,
(iii) $\int_{\Omega_k} \varphi^*(x, |T_{\varphi_t}(u_{n_k})|)\,dx > \frac{2}{3}\epsilon(r)$.

(iv) If $E \subseteq \Omega$ is measurable and $|E| < 2\epsilon_{k+1}$, then

$$\int_E \varphi^*(x, |T_{\varphi_t}(u_{n_k})|)\,\mathrm{d}x < \frac{\epsilon(r)}{3}.$$

Set $\Omega_1 = \Omega$, $\epsilon_1 = |\Omega|$, $n_1 = 1$. We assume that ϵ_k, n_k and Ω_k are given, then, by assumption, $\varphi_t(\cdot, |u_{n_k}(\cdot)|) \in L^{\varphi^*}(\Omega)$ and on account of the Δ_2 condition one has

$$\int_\Omega \varphi^*(x, |\varphi_t(x, |u_{n_k}(x)|)|)\,\mathrm{d}x < \infty.$$

Since the measure

$$A \longrightarrow \mu(A) = \int_A \varphi^*(x, |\varphi_t(x, |u_{n_k}(x)|)|)\,\mathrm{d}x$$

defined on the Borel σ algebra $\mathcal{B}$ of subsets of Ω is absolutely continuous with respect to the Lebesgue measure, one can find ϵ_{k+1} such that any $X \in \mathcal{B}$ with $|X| < 2\epsilon_{k+1}$ satisfies $\int_X \varphi^*(x, |\varphi_t(x, |u_{n_k}(x)|)|)\,\mathrm{d}x < \frac{\epsilon(r)}{3}$. The assumption $\epsilon_k \leq 2\epsilon_{k+1}$ would contradict (iii). We now proceed to the construction of Ω_{k+1} and n_{k+1}.

It is well known [14] that the strong convergence of (u_n) in $L^\varphi(\Omega)$ implies the convergence in measure of both $(T_{\varphi_t}(u_n))$ and $(T_{\varphi^*}(T_{\varphi_t}(u_n)))$. Consequently, there exists $n_{k+1} \in \mathbb{N}$ such that

$$\left|\left\{x \in \Omega : \left|T_{\varphi^*}(T_{\varphi_t}(u_{n_{k+1}}))\right| > \frac{\epsilon(r)}{3|\Omega|}\right\}\right| < \epsilon_{k+1} < \frac{\epsilon_k}{2} < \epsilon_k.$$

Define

$$\Omega_{k+1} = \left\{x \in \Omega : \left|T_{\varphi^*}(T_{\varphi_t}(u_{n_{k+1}}))\right| > \frac{\epsilon(r)}{3|\Omega|}\right\}.$$

Next,

$$\begin{aligned}\int_{\Omega_{k+1}} \varphi^*(x, |T_{\varphi_t}(u_{n_{k+1}})|)\,\mathrm{d}x &= \left(\int_\Omega - \int_{\Omega\setminus\Omega_{k+1}}\right) \varphi^*(x, |T_{\varphi_t}(u_{n_{k+1}})|)\,\mathrm{d}x \\ &> \epsilon(r) - \frac{\epsilon(r)}{3} = \frac{2\epsilon(r)}{3}.\end{aligned}$$

By construction

$$\left|\bigcup_{j=k+1}^{\infty} \Omega_j\right| \leq \sum_{j=k+1}^{\infty} \epsilon_j < 2\epsilon_{k+1}. \tag{6.5}$$

Set

$$v(x) = \begin{cases} u_{n_k}(x) & \text{if } x \in \Omega_k \setminus \bigcup_{j=k+1}^{\infty} \Omega_j \\ 0 & \text{otherwise.}\end{cases}$$

It is clear that $v \in L^\varphi(\Omega)$. Next, observe that:

$$\int_\Omega \varphi^*(x, T_{\varphi_t}(v(x)))\,\mathrm{d}x \geq \sum_{k=1}^{\infty} \int_{\Omega_k \setminus \cup_{j=k+1}^{\infty}\Omega_j} \varphi^*(x, T_{\varphi_t}(u_{n_k}(x)))\,\mathrm{d}x$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}\left(\int\limits_{\Omega_k} - \int\limits_{\cup_{j=k+1}^{\infty}\Omega_j}\right)\varphi^*(x, T_{\varphi_t}(u_{n_k}(x)))\,\mathrm{d}x\\
&\geq \sum_{k=1}^{\infty}\left(\frac{2}{3}\epsilon(r) - \frac{1}{3}\epsilon(r)\right)\\
&= \infty,
\end{aligned}
$$

which follows by (6.5) and condition (iv). This contradicts assumption (6.4). Hence, T_{φ_t} is continuous at 0. □

We next prepare the ground for the next lemma, which deals with the differentiability properties needed in the sequel. Let $M : \mathbb{R} \longrightarrow [0, \infty)$ be continuous and write

$$
\hat{M}(s) = \left(\int_0^s M(t)\,dt\right). \tag{6.6}
$$

Consider the maps

$$
\begin{aligned}
F &: L^{\varphi}(\Omega) \longrightarrow [0, \infty)\\
F(u) &= \rho_{\varphi}(u)
\end{aligned} \tag{6.7}
$$

and

$$
\begin{aligned}
H &: W_0^{1,\varphi}(\Omega) \longrightarrow [0, \infty)\\
H(u) &= \hat{M}\left(\rho_{\varphi}(|\nabla u|)\right).
\end{aligned} \tag{6.8}
$$

Recall that a MO function φ is said to be locally integrable if for any $t > 0$ and any subset $\mathcal{W} \subseteq \Omega$ with $\mu(\mathcal{W}) < \infty$ one has

$$
\int\limits_{\mathcal{W}} \varphi(x, t)\,\mathrm{d}x < \infty.
$$

Lemma 6.3 *In the terminology of the preceding paragraph, let φ be an N function; suppose that $\frac{\partial}{\partial t}\varphi(x, \cdot)$ is continuous a.e. x. Assume that the complementary function φ^* of φ satisfies the Δ condition and is locally integrable. If the maps*

$$
D_1 : L^{\varphi}(\Omega) \longrightarrow L^{\varphi^*}(\Omega) \tag{6.9}
$$
$$
D_1(u) = \tfrac{\partial}{\partial t}\varphi(\cdot, |u(\cdot)|) \tag{6.10}
$$

and

$$
D_2 : W_0^{1,\varphi}(\Omega) \longrightarrow L^{\varphi^*}(\Omega) \tag{6.11}
$$
$$
D_2(u) = \tfrac{\partial}{\partial t}\varphi(\cdot, |\nabla u(\cdot)|) \tag{6.12}
$$

are well defined, then the functionals (6.7) and (6.8) are Fréchet differentiable for $u \neq 0$. In this case, the derivatives of F and H at $u \neq 0$ are given, respectively by

$$
\langle F^{'}(u), h\rangle = \int\limits_{\Omega} \frac{\partial\varphi}{\partial s}(x, |u(x)|)\frac{u(x)}{|u(x)|}h(x)\,\mathrm{d}x, \tag{6.13}
$$

and by

$$
\langle H^{'}(u), h\rangle = \langle -M\left(\rho_{\varphi}(|\nabla u)\right) div\left(\frac{\partial\varphi}{\partial s}(x, |\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|}, h\rangle\right), \tag{6.14}
$$

with the understanding that $\frac{x}{|x|} = 0$ if $x = 0$.

Proof It suffices to show that under the stipulated conditions (6.13) holds; a straightforward application of the chain rule will yield the full statement of the Lemma.

Observe that for $u(x) \neq 0$, $h \in L^{\varphi}(\Omega)$ one has

$$\frac{|u(x)+th(x)|-|u(x)|}{t} = \frac{2u(x)h(x)+t(h(x))^2}{|u(x)+th(x)|+|u(x)|} \longrightarrow \frac{u(x)h(x)}{|u(x)|} \text{ as } t \longrightarrow 0.$$

Therefore, for some $\theta \in (|u(x)|, |u(x)+th(x)|)$,

$$\frac{\varphi(x,|u(x)+th(x)|)-\varphi(x,|u(x)|)}{t} = \frac{|u(x)+th(x)|-|u(x)|}{t}\frac{\partial\varphi}{\partial t}(x,\theta) \longrightarrow \frac{u(x)h(x)}{|u(x)|}\frac{\partial\varphi}{\partial t}(x,|u(x)|) \text{ as } t \longrightarrow 0.$$

By hypothesis,

$$\int_{\Omega}\left|h(x)\frac{\partial\varphi}{\partial t}(x,|u(x)|)\right| \mathrm{d}x < \infty, \tag{6.15}$$

which in conjunction with the above limit yields:

$$\lim_{t\longrightarrow 0}\int_{\{x:u(x)\neq 0\}}\frac{|u(x)+th(x)|-|u(x)|}{t}\frac{\partial\varphi}{\partial t}(x,\theta(x))\mathrm{d}x = \int_{\{x:u(x)\neq 0\}}\frac{u(x)h(x)}{|u(x)|}\frac{\partial\varphi}{\partial t}(x,|u(x)|)\mathrm{d}x. \tag{6.16}$$

On the other hand, one has by assumption, one has a.e. $x \in \Omega$:

$$\frac{\varphi(x,|th(x)|)}{t} \longrightarrow 0 \text{ as } t \longrightarrow 0$$

and by convexity, for $0 < t < 1$,

$$\frac{\varphi(x,|th(x)|)}{t} \leq \varphi(x,|h(x)|).$$

Since $h \in L^{\varphi}(\Omega)$, $\int_{\Omega}\varphi(x,|h(x)|)\,\mathrm{d}x < \infty$, it is easily derived by way of application of Lebesgue's theorem that

$$\int_{\{x:u(x)=0\}}\left|\frac{\varphi(x,|u(x)+th(x)|-\varphi(x,|u(x)|)}{t}\right| \mathrm{d}x = \int_{\{x:u(x)=0\}}\varphi(x,t|h(x)|)/t\,\mathrm{d}x \to 0 \text{ as } t \to 0. \tag{6.17}$$

In all,

$$\frac{F(u+th)-F(u)}{t} = \left(\int_{\{x:u(x)\neq 0\}} + \int_{\{x:u(x)=0\}}\right)\frac{\varphi(x,|u(x)+th(x)|)-\varphi(x,|u(x)|)}{t}\mathrm{d}x \to \int_{\{x:u(x)\neq 0\}}\frac{u(x)h(x)}{|u(x)|}\frac{\partial\varphi}{\partial t}(x,|u(x)|)\mathrm{d}x \text{ as } t \longrightarrow 0.$$

We conclude that F is Gâteaux differentiable and that its derivative is equal to the right-hand side of (6.13). The proof of the Gâteaux differentiability of G follows along the same lines. Hence, it suffices to prove that under the additional assumptions (6.9) and (6.11), the operators

$$\mathcal{L}_F \quad : L^{\varphi}(\Omega) \longrightarrow (L^{\varphi}(\Omega))^*$$
$$\mathcal{L}_f(u) = F^{'}(u) \tag{6.18}$$
$$\mathcal{L}_G : W_0^{1,\varphi}(\Omega) \longrightarrow \left(W_0^{1,\varphi}(\Omega)\right)^*$$
$$\mathcal{L}(u) = G^{'}(u) \tag{6.19}$$

are continuous at $u \neq 0$. A standard functional-analytic result guarantees that in this case F and G are Fréchet differentiable and that the Fréchet and the Gâteaux derivatives coincide. To this end, consider a convergent sequence $(u_j) \subset L^{\varphi}(\Omega)$, say $u_j \longrightarrow u$ in $L^{\varphi}(\Omega)$ as $j \longrightarrow \infty$: it is well known that there is no loss of generality by assuming that u_j converges to u almost everywhere in Ω (see [9,11]).

Since $\frac{\partial\varphi}{\partial t}(x, |u(x)|) \in L^{\varphi^*}(\Omega)$ there exists $\lambda_0 > 0$ for which

$$\int_{\Omega} \varphi^* \left(x, \lambda_0 \left|\frac{\partial\varphi}{\partial t}(x, |u(x)|)\right|\right) \mathrm{d}x < \infty. \tag{6.20}$$

Notice that for $\|h\|_{\varphi} \leq 1$ one has

$$\int_{\{x: u_j(x)\neq 0, u(x)\neq 0\}} \left(\tfrac{u_j(x)}{|u_j(x)|}\tfrac{\partial\varphi}{\partial t}(x, |u_j(x)|) - \tfrac{u(x)}{|u(x)|}\tfrac{\partial\varphi}{\partial t}(x, |u(x)|)\right) h(x)\,\mathrm{d}x \tag{6.21}$$
$$= \int_{\{x: u_j(x)\neq 0, u(x)\neq 0\}} \tfrac{u_j(x)}{|u_j(x)|}\left(\tfrac{\partial\varphi}{\partial t}(x, |u_j(x)|) - \tfrac{\partial\varphi}{\partial t}(x, |u(x)|)\right) h(x)\,\mathrm{d}x$$
$$+ \int_{\{x: u_j(x)\neq 0, u(x)\neq 0\}} \left(\tfrac{u_j(x)}{|u_j(x)|} - \tfrac{u(x)}{|u(x)|}\right)\tfrac{\partial\varphi}{\partial t}(x, |u(x)|) h(x)\,\mathrm{d}x. \tag{6.22}$$

Theorem 6.2 guarantees the continuity of the map (6.9); it is apparent from this fact in conjunction with Hölder's inequality (3.4) that the integral in (6.21) tends to 0 as j tends to infinity. As to the remaining integral, set, for $j \in \mathbb{N}$:

$$\left|\frac{\partial\varphi}{\partial t}(x, |u(x)|)\right| = f(x)$$
$$r_j(x) = \left(\frac{u_j(x)}{|uj(x)|} - \frac{u(x)}{|u(x)|}\right) f(x).$$

Recall that in the terminology of Lemma 4.4,

$$\varphi^*(x, sT_0) \leq C\varphi^*(x, s), \tag{6.23}$$

for any $s \geq S_0$. For λ_0 as in (6.20) and any $\lambda > 2\lambda_0$ let k be defined by the inequalities

$$2^{k-2} < \lambda/\lambda_0 \leq 2^{k-1}.$$

Notice that $|r_j| \leq 2$. For S_0 as in Lemma 4.4, one readily obtains, for any positive integer m, using the monotonicity of φ^* in the second variable

$$\varphi^*\left(x, 2^m\lambda_0 f(x)\right)$$
$$= \varphi^*\left(x, 2^m\lambda_0 f(x)\right)\left(I_{[S_0,\infty)}\left(2^{m-1}\lambda_0 f(x)\right) + I_{[0,S_0)}\left(2^{m-1}\lambda_0 f(x)\right)\right)$$
$$\leq C\varphi^*(x, 2^{m-1}\lambda_0 f(x)) + \varphi^*(x, 2S_0). \tag{6.24}$$

The iteration of the preceding inequality yields

$$\begin{aligned}\varphi^*\left(x,\lambda r_j\right) &\le \varphi^*\left(x,2^k\lambda_0 f(x)\right)\\ &=\varphi^*\left(x,2^k\lambda_0 f(x)\right)\left(I_{[S_0,\infty)}\left(2^{k-1}\lambda_0 f(x)\right)+I_{[0,S_0)}\left(2^{k-1}\lambda_0 f(x)\right)\right)\\ &\le\left(\sum_{i=0}^{k-1}C^i\right)\varphi^*(x,2S_0)+C^k\varphi^*(x,\lambda_0 f(x)).\end{aligned}$$

A routine application of Lebesgue's dominated convergence quickly shows that

$$\int_{\{x:u_j(x)\neq 0,u(x)\neq 0\}}\varphi^*\left(x,\lambda\left(\frac{u_j(x)}{|u_j(x)|}-\frac{u(x)}{|u(x)|}\right)\frac{\partial\varphi}{\partial t}(x,|u(x)|)\right)\mathrm{d}x\longrightarrow 0\ \text{ as }\ j\longrightarrow\infty. \tag{6.25}$$

A similar argument shows that (6.25) holds for $0<\lambda\le 2\lambda_0$. It follows from the arbitrariness of λ that

$$\left\|\left(\frac{u_j}{|u_j|}-\frac{u}{|u|}\right)\frac{\partial\varphi}{\partial t}(\cdot,|u|)\right\|_{L^{\varphi^*}(\Omega)}\to 0\ \text{ as }\ j\to\infty.$$

On account of Hölders inequality (3.4), the integral (6.22) is bounded by

$$2\left\|\left(\frac{u_j}{|u_j|}-\frac{u}{|u|}\right)\frac{\partial\varphi}{\partial t}(\cdot,|u|)\right\|_{L^{\varphi^*}(\Omega)}\longrightarrow 0\ \text{ as }\ j\longrightarrow\infty. \tag{6.26}$$

The proof of the continuity of $\mathcal{L}_G$ follows along the same lines and will be skipped.
This concludes the continuity argument and hence F and G are Fréchet differentiable. □

Lemma 6.4 *If*

$$\hat{M}:[0,\infty)\longrightarrow[0,\infty)$$

is strictly increasing, the modular $\hat{M}$*- ball*

$$B_r=\left\{u\in W_0^{1,\varphi}(\Omega):\hat{M}\rho_\varphi(|\nabla u|)\le r\right\} \tag{6.27}$$

is weakly closed in $W_0^{1,\varphi}(\Omega)$.

Proof For any $r>0$ set $s_r=\hat{M}^{-1}(r)$; it is then clear from the above assumptions that

$$B_r=V_{s_r}=\left\{u\in W_0^{1,\varphi}(\Omega):\rho_\varphi(|\nabla u|)\le s_r\right\} \tag{6.28}$$

and the latter set is weakly closed (Lemma 5.2). □

7 Kirchhoff-type eigenvalue problem

We are now ready to prove the main result of this work.

Theorem 7.1 *Let* $M\in C((0,\infty),(0,\infty))$. *Set* $\hat{M}(t)=\int_0^t M(s)ds$. *Then, for any* $r>0$, *there exists a solution* $(u,\lambda)\in W_0^{1,\varphi}(\Omega)\times(0,\infty)$ *to the equation*

$$-M\left(\rho_\varphi(|\nabla u|)\right)div\left(\frac{\partial\varphi}{\partial s}(x,|\nabla u(x)|)\frac{\nabla u(x)}{|\nabla u(x)|}\right)=\lambda\frac{\partial\varphi}{\partial s}(x,|u(x)|)\frac{u(x)}{|u(x)|}, \tag{7.1}$$

satisfying

$$\hat{M}(\rho_\varphi(|\nabla u|))=r \tag{7.2}$$

Proof Theorem 5.1 guarantees that for any $r > 0$,

$$0 < \sup\left\{\rho_\varphi(u) : u \in B_r\right\} = S_r < \infty. \tag{7.3}$$

We next observe that Theorem 4.7 implies the existence of a sequence $(u_j) \subset V_{s_r}$ with $u_j \rightharpoonup u_0$ in $W_0^{1,\varphi}(\Omega)$ and $u_j \longrightarrow u_0$ in $L^\varphi(\Omega)$ such that

$$\rho_\varphi(u) \longrightarrow S_r = \rho_\varphi(u_0). \tag{7.4}$$

To see this, we notice that $a.e.$ in Ω,

$$\varphi(x, |u_n(x)|) \longrightarrow \varphi(x, |u(x)|)$$

and that on account of convexity, for any $n \in \mathbb{N}$:

$$\varphi(x, |u_n(x)|) \leq \frac{1}{2}\varphi(x, 2|u_n(x) - u(x)|) + \frac{1}{2}\varphi(x, 2|u(x)|). \tag{7.5}$$

Select n large enough so that $2\|u - u_n\|_\varphi < 1$; for such n, it holds, by way of the convexity of $\varphi(x, \cdot)$,

$$\begin{aligned}\int_\Omega \frac{1}{2}\varphi(x, 2|u_n(x) - u(x)|)\mathrm{d}x &= \int_\Omega \frac{1}{2}\varphi\left(x, \frac{2|u_n(x) - u(x)|2\|u - u_n\|_\varphi}{2\|u - u_n\|_\varphi}\right)\mathrm{d}x \\ &\leq \|u - u_n\|_\varphi \int_\Omega \varphi\left(x, \frac{|u_n(x) - u(x)|}{\|u - u_n\|_\varphi}\right)\mathrm{d}x \end{aligned} \tag{7.6}$$

Denote the left-hand side and the right-hand side of (7.5) by v_n and w_n respectively. Then the following conditions hold:

(i) $v_n(x) \to v(x) = \varphi(x, |u(x)|) \in L^1(\Omega)$ $a.e.$ in Ω
(ii) $w_n(x) \to w(x) = \frac{1}{2}\varphi(x, 2|u(x)|) \in L^1(\Omega)$ $a.e.$ in Ω
(iii) $v_n, w_n \in L^1(\Omega)$ for any $n \in \mathbb{N}$
(iv) $\int_\Omega w_n dx \to \int_\Omega \frac{1}{2}\varphi(x, 2|u|)\mathrm{d}x = \frac{1}{2}\rho_\varphi(2u)$.

Since $w - v \geq 0$ $a.e$ in Ω, Fatou's Lemma leads to:

$$\begin{aligned}\int_\Omega (w - v)\mathrm{d}x &\leq \int_\Omega w\,\mathrm{d}x + \liminf_n \int_\Omega (-v_n)\,\mathrm{d}x \\ &= \int_\Omega w\,\mathrm{d}x - \limsup_n \int_\Omega v_n\,\mathrm{d}x\end{aligned}$$

and

$$\int_\Omega (w + v)\mathrm{d}x \leq \int_\Omega w\,\mathrm{d}x + \liminf_n \int_\Omega v_n\,\mathrm{d}x.$$

The two last statements yield

$$\lim_{n\to\infty} \int_\Omega \varphi(x, |u_n(x)|)\mathrm{d}x = \int_\Omega \varphi(x, |u(x)|)\mathrm{d}x$$

or, equivalently

$$\rho_\varphi(u_n) \longrightarrow \rho_\varphi(u) \text{ as } n \to \infty. \tag{7.7}$$

By construction $\rho_\varphi(u_n) \longrightarrow S_r$; (7.7) is therefore the desired result.

Lemma 6.4 yields $\hat{M}\left(\rho_\varphi(|\nabla u_0|)\right) \leq r$. Furthermore, the continuity and monotonicity of the modular ρ_φ immediately yield

$$\hat{M}(\rho_\varphi(|\nabla u_0|)) = r.$$

As is apparent from the above, u_0 is a solution of the constrained maximization problem of the type

$$\max F(v) \quad with \quad G(v) = r. \tag{7.8}$$

It is a routine matter to show, via the Implicit Function Theorem, that the preceding statement implies that

$$\ker G'(u_0) = \ker F'(u_0), \tag{7.9}$$

from which it is clear (since neither functional is null) that u_0 satisfies equation (7.1) for some $\lambda > 0$. This concludes the proof of the claim. □

8 Applications

A particular instance of Theorem 7.1 deserves to be stressed, namely its implication in the consideration of variable exponent Lebesgue spaces.

More precisely, if $\Omega \subset \mathbb{R}^n$ is bounded then

$$\varphi : \Omega \times [0, \infty) \longrightarrow [0, \infty) \tag{8.1}$$

$$\varphi(x, t) = \frac{t^{p(x)}}{p(x)} \tag{8.2}$$

satisfies the conditions of Theorem 4.7 iff p is the restriction to Ω of a function $\tilde{p} \in C(\mathbb{R}^n, \mathbb{R})$ and

$$1 < p_- = \inf_{x\in\Omega} p(x) \leq p_+ = \sup_{x\in\Omega} p(x) < \infty. \tag{8.3}$$

The conjugate function is clearly given by

$$\varphi^*(x, t) = \frac{p(x) - 1}{p(x)} t^{\frac{p(x)}{p(x)-1}}. \tag{8.4}$$

It is straightforward to verify the conditions of Lemma 6.3 for this case. It is customary to write, in this case

$$\rho_p(u) = \int_\Omega \frac{|u(x)|^{p(x)}}{p(x)} \mathrm{d}x. \tag{8.5}$$

Therefore, Theorem 7.1 yields the following result:

Theorem 8.1 *If $\Omega \subset \mathbb{R}^n$ is bounded, M satisfies the conditions of Theorem 7.1 and*

$$p : \Omega \longrightarrow (1, \infty)$$

is a variable exponent satisfying the assumptions (8.3) then for each $r > 0$ there exists a solution (u_0, λ) to the eigenvalue problem

$$M\left(\rho_p(|\nabla u|)\right) div \left(|\nabla u|^{p(x)-2} \nabla u\right) = \lambda |u|^{p(x)-2} u \tag{8.6}$$

with $M\left(\rho_p(|\nabla u_0|)\right) = r$.

With the aid of the following compactness theorem, the techniques used in Sections 6 and 7, one can derive Theorems 8.3 and 8.4, particular cases of which were obtained in [2] and [4], respectively, via different methods.

Theorem 8.2 [9,15] *Let $\Omega \subset \mathbb{R}^n$, $n > 1$ be a bounded domain, $p \in C(\overline{\Omega})$ with*

$$1 < p_- \leq p_+ < n. \tag{8.7}$$

For $0 < \varepsilon < \frac{1}{n-1}$ and $q \in \mathcal{P}(\Omega)$ such that

$$q(x) < \frac{np(x)}{n - p(x)} - \epsilon, \tag{8.8}$$

the embedding

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega) \tag{8.9}$$

is compact.

Theorem 8.3 *Given a function $M \in C((0,\infty),[0,\infty))$, $M(t) > 0$ for $t > 0$. Under the hypotheses of Theorem 8.2, for any $r > 0$, there exists a solution $(u,\lambda) \in W_0^{1,p(\cdot)}(\Omega) \times (0,\infty)$ to the equation*

$$-M\left(\rho_p\left(|\nabla u|\right)\right) div\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{q-2}u, \tag{8.10}$$

satisfying

$$M\left(\rho_p\left(|\nabla u|\right)\right) = r. \tag{8.11}$$

Proof The proof follows from Theorem 7.1 and Theorem 8.3 by considering $\varphi(x,t) = \frac{t^{p(x)}}{p(x)}$ and by way of the bound

$$\sup\left\{\int_\Omega \frac{|u(x)|}{q(x)} : \rho_p(|\nabla u|) \leq r\right\} < \infty, \tag{8.12}$$

which is easy derived as in Theorem 5.1 via the compactness result of Theorem 8.2. □

Theorem 8.4 *Under the assumptions of Theorem 8.2, for any $0 \leq V \in L^\infty(\Omega)$ and $r > 0$, there exists a solution $(u,\lambda) \in W_0^{1,p(\cdot)}(\Omega) \times (0,\infty)$ to the equation*

$$-M\left(\rho_p\left(\frac{|\nabla u|}{p}\right)\right) div\left(|\nabla u|^{p-2}\nabla u\right) = \lambda V|u|^{q-2}u, \tag{8.13}$$

satisfying

$$\hat{M}\left(\rho_p\left(\frac{|\nabla u|}{p}\right)\right) = r. \tag{8.14}$$

Proof The proof follows along the same lines as those of Theorem 7.1 by observing that the functional

$$L^{p(\cdot)}(\Omega) \ni u \to T(u) = \int_\Omega \frac{V(x)}{q(x)}|u(x)|^{q(x)}\mathrm{d}x$$

is Fréchet differentiable for $u \neq 0$ and that, for $h \in C_0^\infty(\Omega)$,

$$\langle T'(u), h\rangle = \int_\Omega V(x)|u(x)|^{q(x)-2}u(x)h(x)\mathrm{d}x.$$

If B_r, $r > 0$, is defined as in Lemma 6.4, it follows as in the proof of Theorem 7.1 that there exists $u_0 \in B_r$ with $\hat{M}\left(\int_\Omega \left|\frac{\nabla u_0}{p(x)}\right|^{p(x)}\mathrm{d}x\right) = r$ such that

$$\int_\Omega \frac{V(x)}{q(x)}|u_0(x)|^{q(x)}\mathrm{d}x = \max\left\{\int_\Omega \frac{V(x)}{q(x)}|u(x)|^{q(x)}\mathrm{d}x, u \in B_r\right\}.$$

Reasoning *mutatis mutandis* as in the proof of Theorem 7.1 it can be shown that u_0 is in fact a sought-for solution to Problem 8.13. □

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References

1. Alves, G.; Corrêa, F.: Positive solutions for a quasilinear elliptic equation of Kirchhoffs type. Comput. Math. Appl. **49**, 85–93 (2005)
2. Arouzi, G.; Mirzapour, M.: Eigenvalue problems for $p(x)$-Kirchhoff-type equations. EJDE **2013**(253), 1–10 (2013)
3. Autuori, G.; Pucci, P.; Salvatori, M.C.: Global nonexistence for nonlinear-Kirchhoff systems. Arch. Rat. Mech. Anal. **196**, 489–516 (2010)
4. Chung, N.: On some $p(x)$-Kirchhoff type equations with weights. J. Appl. Math. Inform. **32**(1–2), 113–128 (2014)
5. Chen, Y.; Levine, S.; Rao, M.: Variable exponent, linear growth functionals in image restoration. SIAM J. Appl. Math. **66**(4), 1383–1406 (2006) **(electronic).**
6. Colasuonno, F.; Pucci, P.: Multiplicity of solutions for $p(x)$-polyharmonic Kirchhoff equations. Nonlinear Anal. **74**, 5962–5974 (2011)
7. Dai, G.; Hao, R.: Existence of solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. **79**, 275–284 (2009)
8. Dai, G.; Liu, D.: Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation. J. Math. Anal. Appl. **359**, 704–710 (2009)
9. Diening, L.; Harjulehto, P.; Hästö, P.; Růžička, M.: Lebesgue and Sobolev spaces with variable exponents, vol. 2017. Lecture notes in Mathematics. Springer, Heidelberg (2011)
10. Dreher, M.: The Kirchhoff equation for the p-Laplacian. Rend. Semin. Mat. Univ. Polit. Torino **64**(2), 217–238 (2006)
11. D.E. Edmunds, J. Lang, O. Méndez, Differential operators on spaces of variable integrability, World Scientific (2015)
12. Fan, X.: On non-local $p(x)$-Laplacian Dirichlet problems. Nonlinear Anal. Theor. **72**, 3314–3323 (2010)
13. G. Kirchoff Mechaniks, Teubner, Leipzig (1883)
14. Krasnoselskii, M.: Topological Methods in the Theory of Nonlinear Integral Equations. Pergamon, Elsford (1964)
15. O. Kováčik, J. Rákosník, On spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, Czechoslovak Math. J. 41 (1991), 592–618.
16. Lang, J.; Méndez, O.: Sharp conditions for the compactness of the Sobolev embedding theorem on Musielak–Orlicz spaces. Math. Nachr. **292**(2), 377–388 (2019)
17. Matuszewska, W.; Orlicz, W.: On certain properties of φ-functions. Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **8**, 430–443 (1960)
18. Li, H.: Blow-up of solutions to a p-Kirchhoff-type parabolic equation with general nonlinearity. J. Dyn Control Syst **26**, 383–392 (2020)
19. Li, L.; Sun, J.: Existence and multiplicity solutions for a non local equation of Kirchhoff type. Differ. Equ. Appl. **10**(4), 369–386 (2018)
20. Liang, Z.; Li, F.; Shi, J.: Positive solutions to Kirchhofs type equations with nonlinearity having prescribed asymptotic behavior. Ann. Inst. H. Poincaré Anal. Non Linéaire **31**(1), 155–167 (2014)
21. O. Méndez, On the eigenvalue problem for a class of Kirchhoff-type equations. J. Math. Anal. Appl. 494(2) (2021).
22. O. Méndez, J. Lang, Analysis on function spaces of Musielak–Orlicz type. Chapman and Hall/CRC Monographs and Research Notes in Mathematics (2018)
23. Mihailescu, M.; Radulescu, V.; Pucci, P.: Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. J. Math. Anal. Appl. **340**(1), 687–698 (2008)
24. J. Musielak, Orlicz Spaces and Modular Spaces, Lect. Notes in Math. 1034, 1983.
25. Nakano, H.: Modulared Semi-ordered Linear Spaces. Maruzen, Tokyo (1950)
26. Sun, J.; Tang, V.: Existence and multiplicity of solutions for Kirchhoff type equations. Nonlinear Anal. theo. **74**(4), 1212–1222 (2011)
27. Sun, J.; Wu, T.: Existence and multiplicity of solutions for an indefinite Kirchhoff-type equation in bounded domains. Proc. R. Soc. Edinburgh: Sect. A Math. **146**(2), 435–448 (2016)