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Nawab Hussain · Hind Alamri · Saud Alsulami

Fixed point approximation for a class of generalized nonexpansive multi-valued mappings in Banach spaces

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Abstract In this paper, we propose a new iteration process, called multi-valued F-iteration process, for the approximation of fixed points. We introduce a new class of multi-valued generalized nonexpansive mappings satisfying a $B_{\gamma,\mu}$ property. Moreover, we establish certain weak and strong convergence theorems in uniformly convex Banach spaces. We also discuss the stability of the modified F-iteration process. Furthermore, a numerical example is presented to illustrate the superiority of our results.

1 Introduction

Fixed point theory provides essential tools for solving various types of nonlinear problems. Fixed point theory for different types of single-valued and multi-valued mappings has attracted the attention of many researchers. Many types of iterative processes have been utilized to approximate the fixed points of multi-valued mappings in Banach spaces, (see, e.g., [1,2,6,8,16,17]). Let U be a Banach space with norm $\|.\|$ and $\Re$ a nonempty subset of U. A mapping $\mathcal{S} : \Re \to \Re$ is contraction if and only if there is a real number $\alpha \in (0, 1)$ such that

$$\|\mathcal{S}\ell - \mathcal{S}\eta\| \leq \alpha\|\ell - \eta\|, \tag{1}$$

for all $\ell, \eta \in \Re$. The mapping $\mathcal{S}$ is said to be nonexpansive if $\alpha = 1$ in (1). The set of all fixed points of $\mathcal{S}$ denote by $Fix(\mathcal{S}) := \{\ell \in \Re : \mathcal{S}\ell = \ell\}$. It is well known that if $\Re$ is a closed, bounded, and convex subset of a uniformly convex Banach space $\Re$, then $Fix(\mathcal{S})$ is nonempty for a nonexpansive mapping [5]. Many authors have years, several extensions and generalizations of nonexpansive mappings in recent years due to their diverse applications. Suzuki [15] introduced an interesting generalization of single-valued nonexpansive mappings and obtained some existence and convergence results. Such mappings are known as mappings satisfying condition (C). A mapping $\mathcal{S} : \Re \to \Re$ is said to be satisfy condition (C) if

$$\|\ell - \mathcal{S}\ell\| \leq \|\ell - \eta\| \Rightarrow \|\mathcal{S}\ell - \mathcal{S}\eta\| \leq \|\ell - \eta\|,$$

All authors contributed equally to this work.

N. Hussain (✉) · H. Alamri · S. Alsulami
Department of Mathematics, King Abdulaziz University, 80203, Jeddah 21589, Saudi Arabia
E-mail: nhusain@kau.edu.sa

S. Alsulami
E-mail: alsulami@kau.edu.com

H. Alamri
Department of Mathematics, College of Science, Taif University, 11099, Taif 21944, Saudi Arabia
E-mail: h.amry@tu.edu.sa

for all $\ell, \eta \in \Re$. Recently, in 2018, Patir et al. [11] suggested two parametric conditions, which they called Condition $B_{\gamma,\mu}$. They proved that Condition $B_{\gamma,\mu}$ is weaker than the corresponding condition (C). A self-mapping $\mathcal{S}$ of a subset $\Re$ of a metric space is said to satisfy Condition $B_{\gamma,\mu}$ (or called Patir map) if there are some $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ with $2\mu \leq \gamma$ such that for all $\ell, \eta \in \Re$,

$$\gamma\|\ell - \mathcal{S}\ell\| \leq \|\ell - \eta\| + \mu\|\eta - \mathcal{S}\eta\| \Rightarrow \|\mathcal{S}\ell - \mathcal{S}\eta\| \leq (1-\gamma)\|\ell - \eta\| + \mu(\|\ell - \mathcal{S}\eta\| + \|\eta - \mathcal{S}\ell\|).$$

In 2011, Abkar and Eslamian [1] extended the notion of condition (C) to the multi-valued mappings. To avoid the endpoint condition, Shahzad and Zegeye [14] introduced another Ishikawa iterative scheme using $\mathcal{P}_{\mathcal{S}}(\ell) = \{\eta \in \mathcal{S}\ell : \|\ell - \eta\| = d(\ell, \mathcal{S}\ell)\}$, where $\mathcal{S}$ is a given multi-valued mapping. Very recently, in 2020, Ali and Ali [3] introduced a new iteration process, called the F-iterative scheme for generalized contractions as follows:

$$\begin{cases} \ell_1 \in \Re \\ \ell_{k+1} = \mathcal{S}\eta_k \\ \eta_k = \mathcal{S}\hbar_k \\ \hbar_k = \mathcal{S}((1-\delta_k)\ell_k + \delta_k\mathcal{S}\ell_k), \end{cases} \tag{2}$$

where $\delta_k \in (0, 1)$ and for all $k \in \mathbb{N}$. The authors showed that the sequence $\{\ell_k\}$ defined by iterative process (2) is stable and has a better rate of convergence when compared with the other iterations in the setting of generalized contractions.

Following are some basic definitions and results needed in the sequel.

Let $(U, \|.\|)$ be a Banach space and $\Re$ be a nonempty subset of U. The set $\Re$ is said to be a proximinal if there exists some η in $\Re$ such that $d(\ell, \eta) = d(\ell, \Re)$, where $d(\ell, \Re) = \inf\{d(\ell, \eta) : \eta \in \Re\}$, for each $\ell \in U$. From now on, the notations $\mathcal{P}_{px}(\Re)$, $\mathcal{P}_{cb}(\Re)$ and $\mathcal{P}(\Re)$ denotes the families of nonempty proximinal subsets, closed bounded subsets and all possible subsets of $\Re$ respectively. A point $\ell \in \Re$ is called an endpoint of $\mathcal{S}$ if $\{\ell\} = \mathcal{S}(\ell)$. A multi-valued mapping $\mathcal{S}$ is said to satisfy the endpoint condition, if $\{\ell\} = \mathcal{S}(\ell)$ for all $\ell \in Fix(\mathcal{S})$. The Pompeiu–Hausdorff metric [4] on the set $\mathcal{P}_{cb}(\Re)$ is defined by

$$\mathcal{H}(\mathcal{M}, \mathcal{N}) = \max\{\sup_{\ell \in \mathcal{M}} d(\ell, \mathcal{N}), \sup_{\eta \in \mathcal{N}} d(\eta, \mathcal{M})\},$$

for all $\mathcal{M}, \mathcal{N} \in \mathcal{P}_{cb}(\Re)$.

Let $\Re$ be a subset of a Banach space and a multi-valued mapping $\mathcal{S} : \Re \to \mathcal{P}(\Re)$ is said to be:

(i) a contraction mapping if there exists an $\alpha \in [0, 1)$ such that

$$\mathcal{H}(\mathcal{S}\ell, \mathcal{S}\eta) \leq \alpha\|\ell - \eta\|,$$

for all $\ell, \eta \in \Re$.

(ii) a nonexpansive mapping if

$$\mathcal{H}(\mathcal{S}\ell, \mathcal{S}\eta) \leq \|\ell - \eta\|,$$

for all $\ell, \eta \in \Re$.

(iii) a quasi-nonexpansive mapping if $Fix(\mathcal{S}) \neq \phi$ and

$$\mathcal{H}(\mathcal{S}\ell, \mathcal{S}q) \leq \|\ell - q\|,$$

for every $q \in Fix(\mathcal{S})$.

A multi-valued mapping $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ is said to satisfy condition (C) if for all $\ell, \eta \in \Re$ the following condition holds:

$$d(\ell, \mathcal{S}\ell) \leq \|\ell - \eta\| \Rightarrow \mathcal{H}(\mathcal{S}\ell, \mathcal{S}\eta) \leq \|\ell - \eta\|.$$

Every multi-valued nonexpansive mapping also satisfies condition (C).

Definition 1.1 A Banach space U is said to have Opial's condition if and only if for each weakly convergent sequence $\{\ell_k\} \subset U$ with a weak limit ℓ in U, we have

$$\limsup_{k\to\infty} \|\ell_k - \ell\| < \limsup_{k\to\infty} \|\ell_k - \eta\|,$$

for each η in U and $\ell \neq \eta$.

Lemma 1.2 [14] *Let* $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ *and* $\mathcal{P}_{\mathcal{S}}(\ell) = \{\eta \in \mathcal{S}\ell : \|\ell - \eta\| = d(\ell, \mathcal{S}\ell)\}$. *Then the following conditions are equivalent:*

(i) $q \in Fix(\mathcal{S})$.
(ii) $\mathcal{P}_{\mathcal{S}}(q) = \{q\}$.
(iii) $q \in Fix(\mathcal{P}_{\mathcal{S}}(q))$.

Moreover, $Fix(\mathcal{S}) = Fix(\mathcal{P}_{\mathcal{S}})$.

Definition 1.3 Let $\{\ell_k\}$ be a bounded sequence in U and $\Re$ be a subset of U. Then,

(i) The asymptotic radius of $\{\ell_k\}$ at a point ℓ in U is defined as

$$r(\ell, \{\ell_k\}) = \limsup_{k\to\infty} \|\ell_k - \ell\|.$$

(ii) The asymptotic radius of $\{\ell_k\}$ with respect to $\Re$ is defined as

$$r(\ell, \Re) = \inf\{r(\ell, \{\ell_k\}) : \ell \in \Re\}.$$

(iii) The asymptotic center of $\{\ell_k\}$ with respect to $\Re$ is defined as

$$\mathcal{A}(\Re, \{\ell_k\}) = \{\ell \in \Re; r(\ell, \{\ell_k\}) = r(\Re, \ell_k)\}.$$

Definition 1.4 Let $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$. A sequence $\{\ell_k\} \in \Re$ is said to be an approximate fixed point sequence (or AFPS) for $\mathcal{S}$ provided that $d(\ell_k, \mathcal{S}\ell_k) \to 0$ as $k \to \infty$.

Definition 1.5 A multi-valued mapping $\mathcal{S} : \Re \to \mathcal{P}(\Re)$ is called demiclosed at $\eta \in \Re$ if for any sequence $\{\ell_k\}$ in $\Re$ weakly convergent to t in $\Re$ and $\eta_k \in \mathcal{S}(\ell_k)$ strongly convergent to η, we have $\eta \in \mathcal{S}(t)$.

In 1974, Senter and Dotson [13] provided the multi-valued version of condition (I).

Definition 1.6 [13] A multi-valued mapping $\mathcal{S} : \Re \to \mathcal{P}(\Re)$ is said to satisfy Condition (I) if there exists a continuous nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(s) > 0$ for all $s \in (0, \infty)$ such that $d(\ell, \mathcal{S}\ell) \geq f(d(\ell, Fix(\mathcal{S}))$ for all $\ell \in \Re$.

Lemma 1.7 [12] *Suppose that* U *is a uniformly convex Banach space. Assume that* $\{\alpha_k\}$ *is any sequence of real numbers such that* $0 < \beta \leq \{\alpha_k\} \leq \delta < 1$ *for all* $k \geq 1$. *If* $\{\ell_k\}$ *and* $\{\eta_k\}$ *are any two sequences in* U *such that* $\limsup_{k\to\infty} \|\ell_k\| \leq j$, $\limsup_{k\to\infty} \|\eta_k\| \leq j$ *and* $\lim_{k\to\infty} \|(1-\alpha_k)\ell_k + \alpha_k\eta_k\| = j$ *hold for some* $j \geq 0$. *Then* $\lim_{k\to\infty} \|\ell_k - \eta_k\| = 0$.

The purpose of this paper is to study a new class of multi-valued mappings generalized nonexpansive mappings satisfies Condition $B_{\gamma,\mu}$ and present a fixed point result. We establish weak and strong convergence results for mapping which satisfies the Condition $B_{\gamma,\mu}$, using the multi-valued version of F-iteration process in uniformly convex Banach spaces. Furthermore, we provide a stability of the modified iteration process and an interesting example to illustrate the results.

2 Main results

We define multi-valued mapping satisfying Condition $B_{\gamma,\mu}$ as follows:

Definition 2.1 Let $\Re$ be a nonempty subset of a Banach space U. A mapping $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ is called a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$ if there are some $\gamma \in [0, 1]$ and $\mu \in [0, \frac{1}{2}]$ with $2\mu \leq \gamma$ such that for all $\ell, \eta \in \Re$,

$$\gamma d(\ell - \mathcal{S}\ell) \leq \|\ell - \eta\| + \mu d(\eta - \mathcal{S}\eta) \Rightarrow \mathcal{H}(\mathcal{S}\ell, \mathcal{S}\eta) \leq (1-\gamma)\|\ell - \eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)).$$

Lemma 2.2 *Let* $\Re$ *be a nonempty subset of a Banach space* U *and consider a multi-valued mapping* $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$. *If* $\mathcal{S}$ *is mapping satisfying Condition* $B_{\gamma,\mu}$ *with a fixed point* $q \in Fix(\mathcal{S})$ *and satisfies the endpoint condition, then* $\mathcal{S}$ *is quasi-nonexpansive.*

Proof Assume that $q \in Fix(\mathcal{S})$. Then

$$\gamma d(q, \mathcal{S}q) = 0 \leq \|q - \eta\| + \mu d(\eta, \mathcal{S}\eta),$$

for any $\eta \in \Re$, so

$$\begin{aligned}\mathcal{H}(\mathcal{S}q, \mathcal{S}\eta) &\leq (1-\gamma)\|q - \eta\| + \mu(d(q, \mathcal{S}\eta) + d(\eta, \mathcal{S}q)) \\ &\leq (1-\gamma)\|q - \eta\| + \mu(\mathcal{H}(\mathcal{S}q, \mathcal{S}\eta) + \|\eta - q\|).\end{aligned}$$

This yields

$$(1-\mu)\mathcal{H}(\mathcal{S}q, \mathcal{S}\eta) \leq (1-\gamma+\mu)\|q - \eta\|.$$

Since $2\mu \leq \gamma$, we obtain the result. □

Lemma 2.3 *Let $\Re$ be a nonempty subset of a Banach space U and $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ be a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$. For any $\ell, \eta \in \Re$, $\nu \in \mathcal{S}\ell$ and $h \in [0, 1]$. Then the following results hold:*

(i) $d(\nu, \mathcal{S}\nu) \leq \|\ell - \nu\|$.
(ii) *at least one of the following ((a) and (b)) holds:*
  (a) $\frac{h}{2} d(\ell, \mathcal{S}\ell) \leq \|\ell - \eta\|$,
  (b) $\frac{h}{2} d(\nu, \mathcal{S}\nu) \leq \|\nu - \eta\|$.

Proof (i) Since $\gamma d(\ell, \mathcal{S}\ell) \leq \|\ell - \eta\| + \mu d(\eta, \mathcal{S}\eta)$ for any $\nu \in \mathcal{S}\ell$, we obtain

$$\begin{aligned}\mathcal{H}(\mathcal{S}\ell, \mathcal{S}\nu) &\leq (1-\gamma)\|\ell - \nu\| + \mu(d(\ell, \mathcal{S}\nu) + d(\nu, \mathcal{S}\ell)) \\ &= (1-\gamma)\|\ell - \nu\| + \mu d(\ell, \mathcal{S}\nu) \\ &\leq (1-\gamma)\|\ell - \nu\| + \mu(\|\ell - \nu\| + d(\nu, \mathcal{S}\nu)) \\ &= (1-\gamma+\mu)\|\ell - \nu\| + \mu d(\nu, \mathcal{S}\nu).\end{aligned}$$

Since $\nu \in \mathcal{S}\ell$, $2\mu \leq \gamma$, and $1 - \mu > 0$, we can write

$$d(\nu, \mathcal{S}\nu) \leq (1-\gamma+\mu)\|\ell - \nu\| + \mu d(\nu, \mathcal{S}\nu).$$

which implies

$$\begin{aligned}d(\nu, \mathcal{S}\nu) &\leq \left(\frac{1-\gamma+\mu}{1-\mu}\right)\|\ell - \nu\| \\ &\leq \|\ell - \nu\|,\end{aligned}$$

for any $\nu \in \mathcal{S}\ell$.

(ii) In contrast, assume that for any $\ell, \eta \in \Re$, $\nu \in \mathcal{S}\ell$, and $h \in [0, 1]$, we have

$$\begin{aligned}&\frac{h}{2} d(\ell, \mathcal{S}\ell) > \|\ell - \eta\|, \\ &\frac{h}{2} d(\nu, \mathcal{S}\nu) > \|\nu - \eta\|.\end{aligned} \tag{3}$$

It follows from (3) and (i) that

$$\begin{aligned}\|\ell - \nu\| &\leq \|\ell - \eta\| + \|\eta - \nu\| \\ &< \frac{h}{2} d(\ell, \mathcal{S}\ell) + \frac{h}{2} d(\nu, \mathcal{S}\nu) \\ &\leq \frac{h}{2}\|\ell - \nu\| + \frac{h}{2}\|\ell - \nu\|,\end{aligned}$$

which is a contradiction. Hence, the result follows. □

Lemma 2.4 *Let $\Re$ be a nonempty subset of a Banach space U and $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ be a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$ .Then,*

$$d(\ell, \mathcal{S}\eta) \leq \left(\frac{3+\mu}{1-\mu}\right) d(\ell, \mathcal{S}\ell) + \|\ell - \eta\|, \tag{4}$$

for $\ell, \eta \in \Re$.

Proof By Lemma 2.3, we have the following two cases: *Case 1.* If $\gamma d(\ell, \mathcal{S}\ell) \leq \|\ell - \eta\|$ (for $\gamma = \frac{h}{2}$, $h \in [0, 1]$), we obtain

$$\begin{aligned} d(\ell, \mathcal{S}\eta) &\leq d(\ell, \mathcal{S}\ell) + \mathcal{H}(\mathcal{S}\ell, \mathcal{S}\eta) \\ &\leq d(\ell, \mathcal{S}\ell) + (1-\gamma)\|\ell - \eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) \\ &\leq d(\ell, \mathcal{S}\ell) + (1-\gamma)\|\ell - \eta\| + \mu(d(\ell, \mathcal{S}\eta) + \|\ell - \eta\| + d(\ell, \mathcal{S}\ell)) \\ &= (1+\mu)d(\ell, \mathcal{S}\ell) + (1-\gamma+\mu)\|\ell - \eta\| + \mu d(\ell, \mathcal{S}\eta). \end{aligned}$$

From the previous inequalities, we obtain

$$(1-\mu)d(\ell, \mathcal{S}\eta) \leq (1+\mu)d(\ell, \mathcal{S}\ell) + (1+\mu-\gamma)\|\ell - \eta\|,$$

since $2\mu \leq \gamma$, we have

$$d(\ell, \mathcal{S}\eta) \leq \left(\frac{1+\mu}{1-\mu}\right) d(\ell, \mathcal{S}\ell) + \|\ell - \eta\|.$$

The required result is proven.

Case 2. Let $\gamma d(\nu, \mathcal{S}\nu) \leq \|\ell - \nu\|$ (for $\gamma = \frac{h}{2}$, $h \in [0, 1]$). Then, we have

$$\begin{aligned} d(\ell, \mathcal{S}\eta) &\leq \|\ell - \nu\| + d(\nu, \mathcal{S}\nu) + \mathcal{H}(\mathcal{S}\nu, \mathcal{S}\eta) \\ &\leq 2\|\ell - \nu\| + (1-\gamma)\|\nu - \eta\| + \mu(d(\nu, \mathcal{S}\eta) + d(\eta, \mathcal{S}\nu)) \\ &\leq 2\|\ell - \nu\| + (1-\gamma)\|\nu - \eta\| + \mu\|\nu - \ell\| + \mu(d(\ell, \mathcal{S}\eta) + \|\eta - \nu\| + d(\nu, \mathcal{S}\nu)). \end{aligned}$$

Therefore, using Lemma 2.3, we have

$$\begin{aligned} (1-\mu)d(\ell, \mathcal{S}\eta) &\leq (2+2\mu)\|\ell - \nu\| + (1+\mu-\gamma)\|\nu - \eta\| \\ &\leq (2+2\mu)\|\ell - \nu\| + (1+\mu-\gamma)(\|\nu - \ell\| + \|\ell - \eta\|) \\ &= (3+3\mu-\gamma)\|\ell - \nu\| + (1+\mu-\gamma)\|\ell - \eta\|, \end{aligned}$$

which implies

$$d(\ell, \mathcal{S}\eta) \leq \left(\frac{3+3\mu-\gamma}{1-\mu}\right)\|\ell - \nu\| + \frac{1+\mu-\gamma}{1-\mu}\|\ell - \eta\|,$$

since $2\mu \leq \gamma$, $\nu \in \mathcal{S}\ell$ and $\gamma d(\ell, \mathcal{S}\ell) \leq \|\ell - \eta\|$, we obtain

$$d(\ell, \mathcal{S}\eta) \leq \left(\frac{3+\mu}{1-\mu}\right) d(\ell, \mathcal{S}\ell) + \|\ell - \eta\|.$$

□

Hence, in both cases the result is proven.

Lemma 2.5 *Let U be a Banach space and $\Re$ be a nonempty closed convex and bounded subset of U. Let $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ be a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$. Let $\{\ell_k\}$ be a bounded approximate fixed point sequence for $\mathcal{S}$ in $\Re$ and $k \in \mathbb{N}$. Then, $\mathcal{A}(\Re, \{\ell_k\})$ is $\mathcal{S}$-invariant.*

Proof Let $\ell \in \mathcal{A}(\Re, \{\ell_k\})$. Since the mapping $\mathcal{S}$ satisfies (4), we have

$$d(\ell_k, \mathcal{S}\ell) \leq \left(\frac{3+\mu}{1-\mu}\right) d(\ell_k, \mathcal{S}\ell_k) + \|\ell_k - \ell\|.$$

Then,

$$\begin{aligned} r(\mathcal{S}\ell, \{\ell_k\}) &= \limsup_{k\to\infty} d(\ell_k, \mathcal{S}\ell) \\ &\leq \left(\frac{3+\mu}{1-\mu}\right) \limsup_{k\to\infty} d(\ell_k, \mathcal{S}\ell_k) + \limsup_{k\to\infty} \|\ell_k - \ell\| \\ &= \limsup_{k\to\infty} \|\ell_k - \ell\| \\ &= r(\ell, \{\ell_k\}). \end{aligned}$$

We have, $\mathcal{S}\ell \in \mathcal{A}(\Re, \{\ell_k\})$ by the definition of the asymptotic center. Hence, $\mathcal{A}(\Re, \{\ell_k\})$ is $\mathcal{S}$-invariant. □

Lemma 2.6 *Let U be a Banach space and $\Re$ be a nonempty closed convex and bounded subset of U. Let $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ be a multi-valued mapping Satisfying Condition $B_{\gamma,\mu}$. Suppose $\{\ell_k\}$ is an approximate fixed point sequence for $\mathcal{S}$. Then*

$$\limsup_{k\to\infty} d(\ell_k, \mathcal{S}\ell) \leq \limsup_{k\to\infty} \|\ell_k - \ell\|,$$

for each $\ell \in \Re$ and $k \in \mathbb{N}$.

Proof Since $\mathcal{S}$ satisfies (4), for any $\ell \in \Re$, we obtain

$$d(\ell_k, \mathcal{S}\ell) \leq \left(\frac{3+\mu}{1-\mu}\right) d(\ell_k, \mathcal{S}\ell_k) + \|\ell_k - \ell\|.$$

Since $\{\ell_k\}$ is an approximately fixed point sequence in $\Re$, we obtain

$$\limsup_{k\to\infty} d(\ell_k, \mathcal{S}\ell) \leq \left(\frac{3+\mu}{1-\mu}\right) \limsup_{k\to\infty} [d(\ell_k, \mathcal{S}\ell_k) + \|\ell_k - \ell\|],$$

which implies

$$\limsup_{k\to\infty} d(\ell_k, \mathcal{S}\ell) \leq \limsup_{k\to\infty} \|\ell_k - \ell\|,$$

□

for $\ell \in \Re$.

We conclude next theorem with the property of demiclosedness.

Theorem 2.7 (Demiclosed principle) *Let $\Re$ be a nonempty closed convex subset of a uniformly convex Banach space U with Opial's property. $\mathcal{S} : \Re \to \mathcal{P}_{cb}(\Re)$ a multi-valued mapping satisfying the Condition $B_{\gamma,\mu}$ and $\{\ell_k\}$ be a sequence in U. If $\{\ell_k\}$ converges weakly to some point $q \in \Re$ and $\limsup_{k\to\infty} d(\ell_k, \mathcal{S}\ell_k) = 0$, then $q \in \mathcal{S}q$, i.e., $(I - \mathcal{S})$ is demiclosed at zero.*

Proof Since $q \in \Re$ and $\mathcal{S}q$ is closed and bounded, for each $k \in \mathbb{N}$ there exist $\ell_k \in \mathcal{S}q$ such that $\|q_k - \ell_k\| = d(q_k, \mathcal{S}q)$. Then by Lemma 2.4,

$$\begin{aligned} \|q_k - \ell_k\| = d(q_k, \mathcal{S}q) &\leq d(q_k, \mathcal{S}q_k) + H(\mathcal{S}q_k, \mathcal{S}q) \\ &\leq d(q_k, \mathcal{S}q_k) + \left(\frac{3+\mu}{1-\mu}\right) d(q_k, \mathcal{S}q_k) + \|q_k - q\|. \end{aligned}$$

Taking limsup on both sides and using $\limsup_{k\to\infty} d(q_k, \mathcal{S}q_k) = 0$, we obtain

$$\limsup_{k\to\infty} \|q_k - \ell_k\| \leq \limsup_{k\to\infty} \|q_k - q\|, \qquad \text{for all } k \in \mathbb{N}. \tag{5}$$

As the sequence $\{q_k\}$ converges weakly to q and $\Re$ possesses Opail's property, for any $K \in \mathbb{N}$ if $\ell_k \neq q$ then it follows that

$$\limsup_{k\to\infty} \|q_k - q\| < \limsup_{k\to\infty} \|q_k - \ell_k\|,$$

which contradicts (5), therefore we can infer $\ell_k = q$ for all $k \in \mathbb{N}$. As a consequence of $\ell_k \in \mathcal{S}q$ we have $q \in \mathcal{S}q$, i.e., $(I - \mathcal{S})$ is demiclosed at zero. □

Now, we prove the existence of a fixed point of a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$.

Theorem 2.8 *Let U be a Banach space and $\Re$ be a nonempty closed convex and bounded subset of U. Let $\mathcal{S}:\Re\to\mathcal{P}_{cb}(\Re)$ be a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$. Suppose $\{\ell_k\}$ is an approximate fixed point sequence $\{\ell_k\}\in\Re$ for $\mathcal{S}$, the asymptotic center $\mathcal{A}(\Re,\{\ell_k\})$ is nonempty and compact. Then, $\mathcal{S}$ has a fixed point.*

Proof Let $\{\ell_k\}$ be an approximate fixed point sequence in the asymptotic center $\mathcal{A}(\Re,\{\ell_k\})$. Since this center is compact, there exists a subsequence $\{\ell_{k_j}\}$ of $\{\ell_k\}$ such that

$$\{\ell_{k_j}\}\to q\in\mathcal{A}(\Re,\{\ell_k\}).$$

As Lemma 2.5 the asymptotic center is $\mathcal{S}$-invariant, $\mathcal{S}q\in\mathcal{A}(\Re,\{\ell_k\})$. Additionally, by Lemma 2.6, we obtain

$$\limsup_{k\to\infty} d(\ell_{k_j},\mathcal{S}q)\le\limsup_{k\to\infty}\|\ell_{k_j}-q\|,$$

□

which implies that $q\in\mathcal{S}q$.

3 Convergence results

We now define an F-iterative process as follow. Let $\Re$ be a nonempty closed and convex subset of a Banach space U and $\mathcal{S}:\Re\to\mathcal{P}(\Re)$ be a multi-valued mapping. Let $\{\ell_k\}$ be a sequence in $\Re$ defined by

$$\begin{cases}\ell_{k+1}=p'''_k,\\ \eta_k=p''_k,\\ \hbar_k=p'_k,\\ \xi_k=(1-\delta_k)\ell_k+\delta_k p_k,\end{cases}\tag{6}$$

where $p_k\in\mathcal{P}_{\mathcal{S}}(\ell_k)$, $p'_k\in\mathcal{P}_{\mathcal{S}}(\xi_k)$, $p''_k\in\mathcal{P}_{\mathcal{S}}(\hbar_k)$, $p'''_k\in\mathcal{P}_{\mathcal{S}}(\eta_k)$, and $\delta_k\in(0,1)$ We start with the following lemmas:

Lemma 3.1 *Let U be a uniformly convex Banach space and $\Re$ a nonempty closed convex subset of U. Let $\mathcal{S}:\Re\to\mathcal{P}_{px}(\Re)$ be a multi-valued mapping satisfying Condition $B_{\gamma,\mu}$ such that $Fix(\mathcal{S})\neq\phi$. Furthermore, assume that $\mathcal{P}_{\mathcal{S}}$ is a mapping satisfying Condition $B_{\gamma,\mu}$. Let $\{\ell_k\}$ be the sequence defined by (6). Then, $\lim_{k\to\infty}\|\ell_k-q\|$ exists for all $q\in Fix(\mathcal{S})$ and $\lim_{k\to\infty}d(\ell_k,\mathcal{P}_{\mathcal{S}}(\ell_k))=0$.*

Proof Suppose we have the sequence $\{\ell_k\}$ generated by (6), and that $q\in Fix(\mathcal{S})$. Using Lemma 2.2 and (6), we have

$$\begin{aligned}\|\xi_k-q\|&\le(1-\delta_k)\|\ell_k-q\|+\delta_k\|p_k-q\|\\&\le(1-\delta_n)\|\ell_k-q\|+\delta_k\mathcal{H}(\mathcal{P}_{\mathcal{S}}\ell_k,\mathcal{P}_{\mathcal{S}}q)\\&\le(1-\delta_k)\|\ell_k-q\|+\delta_k\|\ell_k-q\|\\&\le\|\ell_k-q\|,\end{aligned}\tag{7}$$

for all $k\in\mathbb{N}$, and

$$\begin{aligned}\|\hbar_k-q\|=\|p'_k-q\|&\le\mathcal{H}(\mathcal{P}_{\mathcal{S}}(\xi_k),\mathcal{P}_{\mathcal{S}}(q))\\&\le\|\xi_k-q\|.\end{aligned}$$

Furthermore

$$\begin{aligned}\|\eta_k-q\|=\|p''_k-q\|&\le\mathcal{H}(\mathcal{P}_{\mathcal{S}}(\hbar_k),\mathcal{P}_{\mathcal{S}}(q))\\&\le\|\hbar_k-q\|.\end{aligned}$$

These imply that

$$\begin{aligned}\|\ell_{k+1}-q\| &= \|p_k'''-q\| \le \mathcal{H}(\mathcal{P_S}(\eta_k),\mathcal{P_S}(q))\| \\ &= \|\eta_k-q\| \le \mathcal{H}(\mathcal{P_S}(\hbar_k),\mathcal{P_S}(q))\| \\ &= \|\hbar_k-q\| \le \mathcal{H}(\mathcal{P_S}(\xi_k),\mathcal{P_S}(q))\| \\ &\le \|\xi_k-q\| \\ &\le \|\ell_k-q\|. \end{aligned} \tag{8}$$

Thus, $\|\ell_k - q\|$ is non-increasing and bounded, implying that $\lim_{k\to\infty}\|\ell_k - q\|$ exists for all $q \in Fix(\mathcal{S})$. Then, we prove that

$$\lim_{k\to\infty}\|\ell_k - p_k\| = 0.$$

Assume that

$$\lim_{k\to\infty}\|\ell_k - q\| = c \quad \text{where} \quad q \in Fix(\mathcal{S}).$$

If $c = 0$, then the proof is trivial, we consider $c > 0$, from (7), we have

$$\begin{aligned}&\|\xi_k - q\| \le \|\ell_k - q\| \\ &\Rightarrow \limsup_{k\to\infty}\|\xi_k - q\| \le \limsup_{k\to\infty}\|\ell_k - q\| \le c.\end{aligned} \tag{9}$$

Since $q \in \mathcal{P_S}(q)$ and $\|p_k - q\| = d(p_k, \mathcal{P_S}(q))$, by Lemma 2.2, we have

$$\|p_k - q\| \le d(p_k, \mathcal{P_S}(q)) \le \mathcal{H}(\mathcal{P_S}(\ell_k), \mathcal{P_S}(q)) \le \|\ell_k - q\|.$$

Taking limsup on both sides of the above inequality, we obtain

$$\limsup_{k\to\infty}\|p_k - q\| \le \limsup_{k\to\infty}\|\ell_k - q\| = c. \tag{10}$$

Again from (8), we have

$$\begin{aligned}&\|\ell_{k+1} - q\| \le \|\xi_k - q\| \\ &\Rightarrow c = \liminf_{k\to\infty}\|\ell_{k+1} - q\| \le \liminf_{k\to\infty}\|\xi_k - q\|.\end{aligned} \tag{11}$$

Using (11) and (9)

$$c \le \liminf_{k\to\infty}\|\xi_k - q\| \le \limsup_{k\to\infty}\|\ell_k - q\| \le c.$$

Thus

$$\lim_{k\to\infty}\|\xi_k - q\| = c$$

which implies that

$$c = \lim_{k\to\infty}\|\xi_k - q\| = \lim_{k\to\infty}\|(1-\delta_k)(\ell_k - q) + \delta_k(p_k - q)\|.$$

By Lemma 1.7, we obtain

$$\lim_{k\to\infty}\|\ell_k - p_k\| = 0,$$

which yields

$$\lim_{k\to\infty} d(\ell_k, \mathcal{P_S}(\ell_k)) = 0.$$

□

We now prove a strong convergence result for $\{\ell_k\}$ generated by (6) for multi-valued mapping satisfying Condition $B_{\gamma,\mu}$

Theorem 3.2 *Let U be a uniformly convex Banach space and $\Re$ be a nonempty compact convex subset of U. Let $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ be such that $\mathcal{P}_{\mathcal{S}}$ is satisfying Condition $B_{\gamma,\mu}$ and $Fix(\mathcal{S}) \neq \phi$. Then $\{\ell_k\}$ generated by (6) converges strongly to a fixed point $\mathcal{S}$.*

Proof By Lemma 3.1, $\lim_{k\to\infty} d(\ell_k, \mathcal{P}_{\mathcal{S}}(\ell_k)) = 0$. Due to the compactness of $\Re$ we can find a subsequence $\{\ell_{k_i}\}$ of $\{\ell_k\}$ such that $\{\ell_{k_i}\}$ converges to some $q \in \Re$. In the view of Lemma 2.4, we have

$$\begin{aligned} d(q, \mathcal{P}_{\mathcal{S}}(q)) &\leq \|q - \ell_{k_i}\| + d(\ell_{k_i}, \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \|q - \ell_{k_i}\| + \left(\frac{3+\mu}{1-\mu}\right) d(\ell_{k_i}, \mathcal{P}_{\mathcal{S}}(\ell_{k_i})) + \|\ell_{k_i} - q\| \\ &= 2\|\ell_{k_i} - q\| + \left(\frac{3+\mu}{1-\mu}\right) d(\ell_{k_i}, \mathcal{P}_{\mathcal{S}}(\ell_{k_i})) \to 0. \end{aligned}$$

Hence, $q \in \mathcal{P}_{\mathcal{S}}(q)$. By Lemma 1.2, $q \in Fix(\mathcal{P}_{\mathcal{S}}) = Fix(\mathcal{S})$. By Lemma 3.1, $\lim_{k\to\infty} \|\ell_k - q\|$ exists. Hence, q is the strong limit of $\{\ell_q\}$. □

The proof of the following result is elementary;

Theorem 3.3 *Let U be a uniformly convex Banach space and $\Re$ be a nonempty closed convex subset of U. Let $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ be such that $\mathcal{P}_{\mathcal{S}}$ satisfies Condition $B_{\gamma,\mu}$. If $Fix(\mathcal{S}) \neq \phi$. Let $\{\ell_k\}$ be the sequence defined by (6), and let $\liminf_{k\to\infty} d(\ell_k, Fix(\mathcal{S})) = 0$. Then $\{\ell_k\}$ converges strongly to a fixed point of $\mathcal{S}$.*

We use condition (I) to prove another strong convergence theorem.

Theorem 3.4 *Let U be a uniformly convex Banach space and $\Re$ be a nonempty closed convex subset of U. Let $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ be a multi-valued mapping with $Fix(\mathcal{S}) \neq \phi$. If $\mathcal{P}_{\mathcal{S}}$ satisfies condition $B_{\gamma,\mu}$. Then $\{\ell_k\}$ generated by (6) converges strongly to a fixed point of $\mathcal{S}$ provided that $\mathcal{S}$ satisfies the condition (I).*

Proof By Lemma 3.1, $\lim_{k\to\infty} \|\ell_k - q\|$ exists for all $q \in Fix(\mathcal{S})$. Set $c = \lim_{k\to\infty} \|\ell_k - q\|$ for some $c \geq 0$. If $c = 0$ then the result is trivial. Moreover, suppose that $c > 0$. Then,

$$\begin{aligned} \|\ell_{k+1} - q\| &\leq \|\ell_k - q\| \\ \liminf_{k\to\infty} \|\ell_{k+1} - q\| &\leq \liminf_{k\to\infty} \|\ell_k - q\| \\ d(\ell_{k+1}, Fix(\mathcal{S})) &\leq d(\ell_k, Fix(\mathcal{S})). \end{aligned}$$

Hence $\lim_{k\to\infty} d(\ell_k, Fix(\mathcal{S}))$ exists. We show that $\lim_{k\to\infty} d(\ell_k, Fix(\mathcal{S})) = 0$. From Lemma 3.1, it follows that $\lim_{k\to\infty} d(\ell_k, \mathcal{P}_{\mathcal{S}}(\ell_k)) = 0$. Additionally, from Lemma 2.2, $Fix(\mathcal{S}) = Fix(\mathcal{P}_{\mathcal{S}})$. Using these facts and condition (I), we have

$$\lim_{k\to\infty} f(d(\ell_k, Fix(\mathcal{S})) = 0.$$

Since f is nondecreasing and $f(0) = 0$. We obtain

$$\lim_{k\to\infty} d(\ell_k, Fix(\mathcal{S})) = 0.$$

□

By Theorem 3.3, we obtain the required conclusions.

Finally, we prove a weak convergence of the sequence $\{\ell_k\}$.

Theorem 3.5 *Let U be a uniformly convex Banach space satisfying Opial's condition and $\Re$ be a nonempty closed convex subset of U. Assume $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ is a multi-valued mapping with $Fix(\mathcal{S}) \neq \phi$. If $\mathcal{P}_{\mathcal{S}}$ satisfies Condition $B_{\gamma,\mu}$ and $I - \mathcal{P}_{\mathcal{S}}$ is demiclosed with respect to zero. Suppose $\{\ell_k\}$ is a sequence generated by (6). Then $\{\ell_k\}$ converges weakly to a fixed point of $\mathcal{S}$.*

Proof By the proof of Lemma 3.1 $\{\ell_k\}$ is bounded. Since U is uniformly convex, so U is reflexive by Milman-Pettis's Theorem. By Eberlin's Theorem, every bounded sequence in U has a weakly convergent subsequence. Thus, we can find a weakly convergent subsequence $\{\ell_{k_i}\}$ of $\{\ell_k\}$ with weak limit say q_1 in $\Re$. By the demi-closeness of $I - \mathcal{P}_{\mathcal{S}}$ at 0, $q_1 \in Fix(\mathcal{P}_{\mathcal{S}}) = Fix(\mathcal{S})$. We prove that q_1 is the unique weak limit of $\{\ell_k\}$. Let us find another weakly convergent subsequence $\{\ell_k\}$ of $\{\ell_k\}$ with a weak limit, say $q_2 \in \Re$ and $q_2 \neq q_1$. Again, $q_2 \in Fix(\mathcal{P}_{\mathcal{S}}) = Fix(\mathcal{S})$. By Opial property and Lemma 3.1, we have

$$\begin{aligned}\lim_{k\to\infty} \|\ell_k - q_1\| &= \lim_{i\to\infty} \|\ell_{k_i} - q_1\| \\ &< \lim_{i\to\infty} \|\ell_{k_i} - q_2\| \\ &= \lim_{k\to\infty} \|\ell_k - q_2\| \\ &< \lim_{j\to\infty} \|\ell_{k_j} - q_2\| \\ &< \lim_{j\to\infty} \|\ell_{k_j} - q_1\| \\ &= \lim_{k\to\infty} \|\ell_k - q_1\|.\end{aligned}$$

□

This is a contradiction. Hence, $\{\ell_k\}$ converges weakly to q_1.

4 Stability analysis

This section concerns with the convergence and stability of the iteration process (6) for a multi-valued contraction mapping.

Theorem 4.1 *Let $\Re$ be a nonempty, closed, and convex subset of a Banach space U. Let $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ be a multi-valued mapping and $\mathcal{P}_{\mathcal{S}}$ a multi-valued contraction with $\vartheta \in [0, 1)$. If $\{\ell_k\}$ is a sequence defined in (6) with $\delta_k \in (0, 1)$ and $\sum_{k=0}^{\infty} \delta_k = \infty$, then $\{\ell_k\}$ converges to a fixed point of $\mathcal{S}$.*

Proof By Nadler contraction principle $\mathcal{P}_{\mathcal{S}}$ has a fixed point. Now, we will show that $\{\ell_k\}$ converges to a fixed point q. From Lemma 1.2, we have $q \in \mathrm{Fix}(\mathcal{P}_{\mathcal{S}})$. We use (6), to obtain

$$\begin{aligned}\|\xi_k - q\| &\leq (1-\delta_k)\|\ell_k - q\| + \delta_k\|p_k - q\| \\ &\leq (1-\delta_k)\|\ell_k - q\| + \delta_k d(p_k, \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq (1-\delta_k)\|\ell_k - q\| + \delta_k \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell_k), \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq (1-\delta_k)\|\ell_k - q\| + \delta_k \vartheta \|\ell_k - q\| \\ &\leq (1-\delta_k(1-\vartheta))\|\ell_k - q\|. \end{aligned} \tag{12}$$

Furthermore,

$$\begin{aligned}\|\hbar_k - q\| \leq \|p' - q\| &\leq d(p', \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\xi_k), \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \vartheta\|\xi_k - z\|. \end{aligned} \tag{13}$$

Similarly

$$\begin{aligned}\|\eta_n - q\| \leq \|p'' - q\| &\leq d(p'', \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\hbar_k), \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \vartheta\|\hbar_k - q\| \\ &\leq \vartheta^2\|\xi_k - q\|. \end{aligned} \tag{14}$$

From (12), (13), (14), and using the fact that $(1-\delta_k(1-\vartheta)) < 1$, for $\vartheta \in (0,1)$ and $\{\alpha_n\} \in (0,1)$, we obtain that

$$\begin{aligned}\|\ell_{k+1}-q\| &= \|p'''-q\| \leq \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\eta_k), \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \vartheta\|\eta_k - q\| \leq \vartheta\mathcal{H}(\mathcal{P}_{\mathcal{S}}(\hbar_k), \mathcal{P}_{\mathcal{S}}(q)) \\ &\leq \vartheta^2\|\hbar_k - q\| \\ &\leq \vartheta^3\|\xi_k - q\| \\ &\leq \vartheta^3(1-\delta_k(1-\vartheta))\|\ell_k - q\|. \end{aligned} \tag{15}$$

From (15), we have

$$\begin{aligned}\|\ell_{k+1}-q\| &\leq \vartheta^3(1-\delta_k(1-\vartheta))\|\ell_k - q\| \\ &\leq \vartheta^3(1-\delta_{k-1}(1-\vartheta))\|\ell_{k-1} - q\| \\ &\leq . \\ &\quad . \\ &\quad . \\ &\quad . \\ &\leq \vartheta^3(1-\delta_0(1-\vartheta))\|\ell_0 - q\|. \end{aligned} \tag{16}$$

By (16), we obtain

$$\|\ell_{k+1}-q\| \leq \|\ell_0 - q\| \left(\vartheta^3\right)^{k+1} \prod_{i=0}^{k}(1-\delta_i(1-\vartheta)).$$

Since δ_k and $\vartheta \in (0,1)$, we have $1-\delta_i(1-\vartheta) < 1$, for all $k \in \mathbb{N}$. We know that $1-\ell \leq e^{-\ell}$ for $0 \leq u \leq 1$. It follows that,

$$\|\ell_{k+1}-q\| \leq \|\ell_0 - q\| \left(\vartheta^3\right)^{k+1} e^{-(1-\vartheta)\sum_{i=0}^{k}\delta_i} \to \infty \tag{17}$$

If we take the limit in both sides of (17), we obtain $\lim \|\ell_k - q\| = 0$, which implies that $\{\ell_k\}$ converges to q. Since $q \in Fix(\mathcal{P}_{\mathcal{S}})$, from Lemma 1.2, we have $q \in Fix(\mathcal{S})$, and hence $\{\ell_k\}$ converges strongly to $q \in Fix(\mathcal{S})$. □

Next, we give the definition of $\mathcal{S}$-stable iteration process.

Definition 4.2 [7] Let $\{x_k\}$ be any arbitrary sequence in U. Then, an iteration procedure $x_{k+1} = f(\mathcal{S}, x_k)$, converging to fixed point q, is said to be $\mathcal{S}$-stable or stable with respect to $\mathcal{S}$, if for $\varepsilon_k = \|x_{k+1} - f(\mathcal{S}, x_k)\|$, for all $k \in \mathbb{N}$, we have

$$\lim_{k\to\infty} \varepsilon_k = 0 \Leftrightarrow \lim_{k\to\infty} x_k = q.$$

Lemma 4.3 [18] *Let $\{t_k\}$ and $\{\varepsilon_k\}$ be two nonnegative real sequences satisfying the following inequality:*

$$t_{k+1} \leq (1-\varpi_k)t_k + \varepsilon_k,$$

where $\varpi_k \in (0,1)$ for all $k \in \mathbb{N}$, $\sum_{k=0}^{\infty}\varpi_k = \infty$ and $\lim_{k\to\infty}\frac{\varepsilon_k}{\varpi_k} = 0$; then, $\lim_{k\to\infty} t_k = 0$.

Theorem 4.4 [18] *Let $\Re$ be a nonempty, closed and convex subset of a Banach space U, let $\mathcal{S} : \Re \to \mathcal{P}_{px}(\Re)$ and $\mathcal{P}_{\mathcal{S}}$ be a multi-valued contractions. If $\{\ell_k\}$ is a sequence given by (6) with $\delta_k \in (0,1)$ and $\sum_{k=0}^{\infty}\delta_k = \infty$; then, the iteration process (6) is $\mathcal{S}$-stable.*

Proof Let $\{\ell_k\} \subset \Re$ be any arbitrary sequence in U and suppose that the sequence generated by (6) is $\ell_{k+1} = f(\mathcal{S}, \ell_k)$ converging to a unique fixed point q and that $\varepsilon_k = \|\ell_{k+1} - f(\mathcal{S}, \ell_k)\|$. To establish that $\mathcal{S}$ is stable, we need to prove that $\lim_{k\to\infty} \varepsilon_k = 0 \iff \lim_{k\to\infty} \ell_k = q$.

Suppose that $\lim_{k\to\infty} \varepsilon_k = 0$. Using triangular inequality and (15), we have that

$$\begin{aligned}\|\ell_{k+1} - q\| &\le \|\ell_{k+1} - f(\mathcal{S}, \ell_k)\| + \|f(\mathcal{S}, \ell_k) - q\| \\ &\le \varepsilon_k + \vartheta^3(1 - \delta_k(1 - \vartheta))\|\ell_k - q\|.\end{aligned}$$

If $d_k = \|\ell_k - q\|$, and $\varpi_k = \delta_k(1 - \vartheta)$, then we have

$$d_{k+1} \le (1 - \varpi_k)d_k + \varepsilon_k.$$

As $\sum_{k=0}^{\infty} \varpi_k = \infty$ and $\lim_{k\to\infty} \varepsilon_k = 0$, $\lim_{k\to\infty} \frac{\varepsilon_k}{\varpi_k} = 0$, by Lemma 4.3 we have that $\lim_{k\to\infty} \ell_k = q$.

Consequently, suppose that $\lim_{k\to\infty} \ell_k = q$. We have that

$$\begin{aligned}\varepsilon_k &= \|\ell_{k+1} - f(\mathcal{S}, \ell_k)\| \\ &\le \|\ell_{k+1} - q\| + \|f(\mathcal{S}, \ell_k) - q\| \\ &\le \|\ell_{k+1} - q\| + \vartheta^3(1 - \delta_k(1 - \vartheta))\|\ell_k - q\| \to 0, \quad as \;\; k \to \infty.\end{aligned}$$

Using our hypothesis that $\lim_{k\to\infty} \ell_k = q$, we then have that $\lim_{k\to\infty} \varepsilon_k = 0$. Hence, the iteration process (6) is stable with respect to $\mathcal{S}$. □

5 Example

In this section we provide an example of multi-valued mapping for which best approximate operator $\mathcal{P}_{\mathcal{S}}$ is a generalized nonexpansive mapping satisfying Condition $B_{\gamma,\mu}$.

Example 5.1 Let $\Re = [0, 1] \subset \mathbb{R}$ be endowed with usual norm. Define $\mathcal{S} : \Re \to \mathcal{P}(\Re)$ by

$$\mathcal{S}\ell = \begin{cases} 0 & \ell \in [0, \frac{1}{2}) \setminus \{\frac{1}{4}\} \\ [0, \frac{\ell}{4}] & \ell \in [\frac{1}{2}, 1] \\ [0, \frac{1}{12}] & \ell = \{\frac{1}{4}\}. \end{cases}$$

If $\ell \in [0, \frac{1}{2}]\setminus\{\frac{1}{4}\}$, then $\mathcal{P}_{\mathcal{S}}(\ell) = \{0\}$. For $\ell \in [\frac{1}{2}, 1]$, then we have $\mathcal{P}_{\mathcal{S}}(\ell) = \{\frac{1}{4}\}$. If $\ell = \{\frac{1}{4}\}$, then $\mathcal{P}_{\mathcal{S}}(\ell) = \{\frac{1}{12}\}$. We show that $\mathcal{P}_{\mathcal{S}}$ is mapping satisfies Condition $B_{\gamma,\mu}$. For this choose $\gamma = 1$, and $\mu = \frac{1}{2}$. We shall consider the following three cases:

Case (1) For $\ell, \eta \in [0, \frac{1}{2}] \setminus \{\frac{1}{4}\}$, we get

$$\begin{aligned}(1 - \gamma)\|\ell - \eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) &= \frac{1}{2}(\|\ell\| + \|\eta\|) \\ &\ge 0 = \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell), \mathcal{P}_{\mathcal{S}}(\eta)).\end{aligned}$$

Case(2) For $\ell, \eta \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned}(1 - \gamma)\|\ell - \eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) &= \frac{1}{2}\left(\left\|\ell - \frac{\eta}{4}\right\| + \left\|\eta - \frac{\ell}{4}\right\|\right) \\ &= \frac{1}{2}\left(\left\|\ell - \frac{\eta}{4}\right\| + \left\|\frac{\ell}{4} - \eta\right\|\right) \\ &\ge \frac{1}{2}\left\|\frac{5\ell}{4} - \frac{5\eta}{4}\right\| \\ &= \frac{5}{8}\|\ell - \eta\| > \frac{2}{8}\|\ell - \eta\| \\ &= \frac{1}{4}\|\ell - \eta\| = \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell), \mathcal{P}_{\mathcal{S}}(\eta)).\end{aligned}$$

Case(3) For $\ell \in [0, \frac{1}{2}) \setminus \{\frac{1}{4}\}$ and $\eta = \{\frac{1}{4}\}$, we obtain

$$\begin{aligned}(1-\gamma)\|\ell-\eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) &= \frac{1}{2}\left(\|\ell - \frac{1}{12}\| + \|\frac{1}{4}\|\right) \\ &\geq \frac{1}{2}\left\|\ell - \frac{1}{12}\right\| + \frac{1}{8} \\ &\geq \frac{1}{12} = \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell), \mathcal{P}_{\mathcal{S}}(\eta)).\end{aligned}$$

Case(4) For $\ell \in [\frac{1}{2}, 1]$, $\eta = \{\frac{1}{4}\}$, we get

$$\begin{aligned}(1-\gamma)\|\ell-\eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) &= \frac{1}{2}\left(\|\ell - \frac{1}{12}\| + \|\frac{1}{4} - \frac{\ell}{4}\|\right) \\ &\geq \frac{1}{2}\left\|\ell - \frac{1}{12} + \frac{1}{4} - \frac{\ell}{4}\right\| \\ &= \frac{1}{2}\left\|\frac{3\ell}{4} + \frac{3}{12}\right\| = \frac{1}{4}\left\|\frac{3\ell}{2} + \frac{1}{2}\right\| \\ &\geq \frac{1}{4}\left\|\frac{\ell}{2} - \frac{1}{12}\right\| = \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell), \mathcal{P}_{\mathcal{S}}(\eta))\end{aligned}$$

Case(5) For $\ell \in [0, \frac{1}{2}) \setminus \{\frac{1}{4}\}$, $\eta \in [\frac{1}{2}, 1]$, we obtain

$$(1-\gamma)\|\ell-\eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) = \frac{1}{2}\left(\|\ell - \frac{\eta}{4}\| + \|\eta\|\right). \tag{18}$$

Then, we have two cases:

$$\|\ell - \frac{\eta}{4}\| = \begin{cases} \ell - \frac{\eta}{4} & \text{if } \ell > \frac{\eta}{4} \\ \frac{\eta}{4} - \ell & \text{if } \ell \leq \frac{\eta}{4}. \end{cases}$$

For the first case, $\ell > \frac{\eta}{4}$ and (18), imply

$$\begin{aligned}(1-\gamma)\|\ell-\eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) &= \frac{1}{2}\left(\ell - \frac{\eta}{4} + \eta\right) \\ &= \frac{1}{2}\left(\ell + \frac{3\eta}{4}\right) \\ &\geq \frac{3\eta}{8} \geq \frac{1}{4}\eta = \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell), \mathcal{P}_{\mathcal{S}}(\eta)).\end{aligned}$$

Then, from the second case, $\ell \leq \frac{\eta}{4}$, and using (18) we obtain

$$(1-\gamma)\|\ell-\eta\| + \mu(d(\ell, \mathcal{S}\eta) + d(\eta, \mathcal{S}\ell)) \geq \frac{1}{2}\eta \geq \frac{1}{4}\eta = \mathcal{H}(\mathcal{P}_{\mathcal{S}}(\ell), \mathcal{P}_{\mathcal{S}}(\eta)).$$

Thus, the mapping $\mathcal{P}_S$ satisfies Condition $B_{\gamma,\mu}$, and $q = 0$ is fixed point in $\Re$. Let δ_k be sequence such that $\delta_k = \frac{1}{2k}$ for all $k \in \mathbb{N}$, for any given $\ell_1 \in [0, 1]$, we can assume that $\ell_1 = \frac{3}{4}$. Using the F-iteration defined by (6), we have $\mathcal{S}(\ell_1) = \mathcal{S}(\frac{3}{4}) = [0, \frac{3}{16}]$. Then

$$\begin{aligned}\xi_1 &= (1-\delta_1)\ell_1 + \delta_1 p_1 \\ &= \left(1 - \frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{16}\right) \\ &= \frac{15}{32}\end{aligned}$$

implies $\hbar_1 = 0$, then we get $\eta_1 = \ell_2 = 0$. Hence the sequence $\{\ell_n\}$ converges to fixed point which is zero. If we take the initial point $\ell_1 = \frac{1}{4}$, then $\mathcal{S}(\ell_1) = \mathcal{S}(\frac{1}{4}) = [0, \frac{1}{12}]$, again we have

$$\begin{aligned}\xi_1 &= (1-\delta_1)\ell_1 + \delta_1 p_1 \\ &= \left(1-\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{12}\right) \\ &= \frac{1}{6}\end{aligned}$$

we have $\hbar_1 = 0$ and $\eta_1 = \ell_2 = 0$. Continuing in this manner, $\ell_n = 0$, for all $k \geq 2$ and hence the sequence $\{\ell_n\}$ generated by (6) converges to a fixed point.

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