Arab. J. Math. (2022) 11:585–599
https://doi.org/10.1007/s40065-022-00380-2
Arabian Journal of Mathematics

Saima Yaseen · Fiza Zafar

A new sixth-order Jarratt-type iterative method for systems of nonlinear equations

Received: 7 October 2021 / Accepted: 5 June 2022 / Published online: 11 July 2022

Abstract Many real-life problems using mathematical modeling can be reduced to scalar and system of nonlinear equations. In this paper, we develop a family of three-step sixth-order method for solving nonlinear equations by employing weight functions in the second and third step of the scheme. Furthermore, we extend this family to the multidimensional case preserving the same order of convergence. Moreover, we have made numerical comparisons with the efficient methods of this domain to verify the suitability of our method.

Mathematics Subject Classification Primary 65H10 · 47H17; Secondary 65G99 · 49M15

1 Introduction

It is well-known fact that a wide class of problems which arises in various branches of pure and applied sciences can be viewed in the general framework of the nonlinear equations and systems of nonlinear equations. Due to their importance, several iterative methods have been suggested and analyzed under certain conditions. Therefore, solving nonlinear equations and nonlinear systems efficiently and reliably has gained paramount importance in physics, engineering, operational research, and many other disciplines. This importance led to the development of many numerical techniques. However, most of them are iterative in nature, because analytic methods for such problems are almost unavailable. We can see several examples that show the applicability of these to real world problems; see [5,9]. Narang et al. (2016) in [10] proposed fourth- and sixth-order methods for the nonlinear systems which were the extensions of earlier univariate schemes. Recently, researchers have proposed sixth-order iterative methods using weight functions and parameters (for example, see [1–3,6–8,13]). By getting motivation from the recent activities in this direction, we aim to propose a sixth-order family of Jarratt-type methods for solving scalar equations. Along with the perseverance of order of convergence, we then extend this family for the multidimensional case. The outline of the paper is as follows. In Sect. 2, a sixth-order scheme is presented along with their convergence analysis and numerical examples. In Sect. 3, we present extension of sixth-order scheme and their numerical examples. Section 4 and Section 5 are devoted to the efficiency of methods and concluding remarks.

Saima Yaseen and Fiza Zafar have contributed equally to this work.

S. Yaseen · F. Zafar (✉)
Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan 60800, Pakistan
E-mail: fizazafar@bzu.edu.pk

S. Yaseen
E-mail: saimayasin08@gmail.com

2 Development of a sixth-order scheme for nonlinear scalar equation

We introduce a new sixth-order method for solving nonlinear equations.

2.1 Derivation of the scheme

For the development of our scheme, we use weight function approach. Our method is defined by the following three steps:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - P(u_n)\frac{f(x_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - Q(v_n)\frac{f(z_n)}{f'(x_n)}, \end{aligned} \tag{1}$$

where $P : \mathbb{C} \to \mathbb{C}$ and $Q : \mathbb{C} \to \mathbb{C}$ are weight functions that are analytic in the neighborhood of 1 and $u_n = \frac{f'(x_n)}{f'(y_n)}$, $v_n = \frac{f'(y_n)}{f'(x_n)}$. Theorem 2.1 demonstrates that the order of convergence reaches at six using particular conditions on these weight functions.

Theorem 2.1 *Suppose $f : D \subset \mathbb{C} \to \mathbb{C}$ be a sufficiently differentiable function in D containing a simple root γ of the equation $f(x) = 0$. Moreover, we suppose that an initial guess x_0 is sufficiently close to γ. Then, the family of iterative methods (1) attains order of convergence six using the following conditions on weight functions:*

$$\begin{aligned} &P(1) = 1,\ P'(1) = -\frac{1}{4},\ P''(1) = \frac{5}{4},\ |\ P'''\ |\ (1)\ < \infty, \\ &Q(1) = 1,\ Q'(1) = -\frac{3}{2},\ |\ Q''\ |\ (1)\ < \infty. \end{aligned}$$

The error equation is given as

$$\begin{aligned} e_{n+1} = \frac{1}{729}&(-54c_2^2 + 9c_3 + 8Q''(1)c_2^2)(32P'''(1)c_2^3 + 3c_2^3 + 81c_2c_3 - 9c_4)e_n^6 \\ &+O(e_n^7). \end{aligned}$$

Proof Let us consider that $e_n = x_n - \gamma$ be the error in the nth iteration. The Taylor's series expansion of the function $f(x_n)$ and its first-order derivative $f'(x_n)$ about $x = \gamma$ with the assumption $f'(\gamma) \neq 0$ lead us to

$$f(x_n) = f'(\gamma)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)), \tag{2}$$

where

$$c_i = \frac{f^i(\gamma)}{i!\, f'(\gamma)},$$

for $i = 2, 3, \ldots$ and

$$f'(x_n) = f'(\gamma)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)). \tag{3}$$

Now

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \tag{4}$$

Using (2) and (3) in (4), we get

$$y_n = \gamma + \frac{1}{3}e_n + \frac{2}{3}c_2e_n^2 + (\frac{4}{3}c_3 - \frac{4}{3}c_2^2)e_n^3 + (2c_4 - \frac{14}{3}c_2c_3 + \frac{8}{3}c_2^3)e_n^4$$

$$+(\frac{8}{3}c_5 - \frac{20}{3}c_2c_4 - 4c_3^2 + \frac{40}{3}c_3c_2^2 - \frac{16}{3}c_2^4)e_n^5$$
$$+(-\frac{34}{3}c_3c_4 + 22c_2c_3^2 - \frac{104}{3}c_3c_2^3 + \frac{56}{3}c_4c_2^2 - \frac{26}{3}c_2c_5 + \frac{10}{3}c_6 + \frac{32}{3}c_2^5)e_n^6 + O(e_n^7).$$

In view of the fact that $f'(y_n) = f'(x_n)\mid_{e_n \longrightarrow y_n-\gamma}$, we obtain

$$f'(y_n) = f'(\gamma)(1 + \frac{2}{3}c_2e_n + \frac{1}{3}(4c_2^2 + c_3)e_n^2 + (\frac{8}{3}c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}c_4)e_n^3 + \sum_{i=3}^{5} a_ie_n^i + O(e_n^6)), \tag{5}$$

where

$$a_i = a_i(c_2, c_3, ...c_6), 4 \leqslant i \leqslant 5.$$

With the help of (3) and (5), we obtain $u_n = \frac{f'(x_n)}{f'(y_n)}$ as

$$u_n = \frac{f'(\gamma)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6))}{f'(\gamma)(1 + \frac{2}{3}c_2e_n + \frac{1}{3}(4c_2^2 + c_3)e_n^2 + (\frac{8}{3}c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}c_4)e_n^3 + \sum_{i=3}^{5} a_ie_n^i + O(e_n^6))} \tag{6}$$

$$= 1 + \frac{4}{3}c_2e_n + (\frac{8}{3}c_3 - \frac{20}{9}c_2^2)e_n^2 + (\frac{104}{27}c_4 - \frac{56}{9}c_2c_3 + \frac{64}{27}c_2^3)e_n^3 + \sum_{i=4}^{5} b_ie_n^i + O(e_n^6),$$

where

$$b_i = b_i(c_2, c_3, ...c_6), 4 \leqslant i \leqslant 5.$$

Next, we use Taylor's series expansion of $P(u_n)$ about $u_n = 1$ up to fifth-order terms, as follows:

$$P(u_n) = P(1) + P'(1)(u_n - 1) + P''(1)(u_n - 1)^2 + P'''(1)(u_n - 1)^3 + P^{iv}(1)(u_n - 1)^4 + P^{v}(1)(u_n - 1)^5 + O(e_n^6).$$

Therefore, we have

$$P(u_n) = P(1) + \frac{4}{3}P'(1)c_2e_n + (\frac{8}{3}P'(1)c_3 - \frac{20}{9}P'(1)c_2^2 + \frac{8}{9}P''(1)c_2^2)e_n^2 + \sum_{i=3}^{5} d_ie_n^i + O(e_n^6),$$

where

$$d_i = d_i(c_2, c_3, ...c_6, P(1), P'(1), P''(1), P'''(1), P^{iv}(1)), 3 \leqslant i \leqslant 5.$$

Moreover

$$\frac{f(x_n)}{f'(y_n)} = \frac{f'(\gamma)(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7))}{f'(\gamma)(1 + \frac{2}{3}c_2e_n + \frac{1}{3}(4c_2^2 + c_3)e_n^2 + (\frac{8}{3}c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}c_4)e_n^3 + \sum_{i=3}^{5} a_ie_n^i + O(e_n^6))}$$
$$= e_n + \frac{1}{3}c_2e_n^2 + (\frac{2}{3}c_3 - \frac{14}{9}c_2^2)e_n^3 + \sum_{i=3}^{5} t_ie_n^i + O(e_n^6),$$

where

$$t_i = t_i(c_2, c_3, ...c_6), 3 \leqslant i \leqslant 5.$$

Consequently, the second substep becomes

$$z_n = x_n - P(u_n).\frac{f(x_n)}{f'(y_n)} = \gamma + (1 - P(1))e_n - \frac{1}{3}c_2(P(1) + 4P'(1))e_n^2 + \sum_{i=3}^{6} g_ie_n^i + O(e_n^7), \tag{7}$$

such that

$$g_i = g_i(c_2, c_3, ...c_6, P(1), P'(1), P''(1), P'''(1), P^{iv}(1)), 3 \leqslant i \leqslant 6.$$

Using the conditions on P and its derivatives as

$$P(1) = 1, P'(1) = -\frac{1}{4}, P''(1) = \frac{5}{4}, \mid P'''(1) \mid < \infty,$$

(7) becomes

$$z_n = (-c_2c_3 + \frac{1}{9}c_4 - \frac{1}{27}c_2^3 - \frac{32}{81}P'''(1)c_2^3)e_n^4 + \sum_{i=5}^{6} h_i e_n^i + O(e_n^7)),$$

where

$$h_i = h_i(c_2, c_3, ...c_6, P'''(1), P^{iv}(1), P^v(1)), 5 \leqslant i \leqslant 6.$$

As $f(z_n) = f(x_n)\mid_{e_n \longrightarrow z_n - \gamma}$, we obtain

$$f(z_n) = f'(\gamma)((-c_2c_3 + \frac{1}{9}c_4 - \frac{1}{27}c_2^3 - \frac{32}{81}P'''(1)c_2^3)e_n^4 + \sum_{i=5}^{6} j_i e_n^i + O(e_n^7)),$$

where

$$j_i = j_i(c_2, c_3, ...c_6, P'''(1), P^{iv}(1), P^v(1)), 5 \leqslant i \leqslant 6.$$

With the help of (3) and (5), $v_n = \frac{f'(y_n)}{f'(x_n)}$ is given by

$$\begin{aligned} v_n &= \frac{f'(\gamma)(1 + \frac{2}{3}c_2e_n + \frac{1}{3}(4c_2^2 + c_3)e_n^2 + (\frac{8}{3}c_2c_3 - \frac{8}{3}c_2^3 + \frac{4}{27}c_4)e_n^3 + \sum_{i=3}^{5} a_i e_n^i + O(e_n^6))}{f'(\gamma)(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6))} \\ &= 1 - \frac{4}{3}c_2e_n + (-\frac{8}{3}c_3 + 4c_2^2)e_n^2 + (-\frac{104}{27}c_4 + \frac{40}{3}c_2c_3 - \frac{32}{3}c_2^3)e_n^3 + \sum_{i=4}^{5} k_i e_n^i + O(e_n^6), \end{aligned}$$

where

$$k_i = k_i(c_2, c_3, ...c_6), 4 \leqslant i \leqslant 5.$$

Let us consider Taylor's expansion for the weight function Q about $v_n = 1$ up to fifth-order terms as

$$\begin{aligned} Q(v_n) = Q(1) + Q'(1)(v_n - 1) + Q''(1)(v_n - 1)^2 + Q'''(1)(v_n - 1)^3 + Q^{iv}(1)(v_n - 1)^4 + \\ Q^v(1)(v_n - 1)^5 + O(e_n^6). \end{aligned}$$

Thus

$$Q(v_n) = Q(1) - \frac{4}{3}Q'(1)c_2e_n + (-\frac{8}{3}Q'(1)c_3 + 4Q'(1)c_2^2 + \frac{8}{9}Q''(1)c_2^2)e_n^2 + \sum_{i=3}^{5} l_i e_n^i + O(e_n^6),$$

where

$$l_i = l_i(c_2, c_3, ...c_6, Q(1), Q'(1), Q''(1), Q'''(1), Q^{iv}(1)), 3 \leqslant i \leqslant 5.$$

Therefore, the final step takes the form

$$\begin{aligned} x_{n+1} = (-c_2c_3 + \frac{1}{9}c_4 - \frac{1}{27}c_2^3 - \frac{32}{81}P'''(1)c_2^3 + Q(1)c_2c_3 - \frac{1}{9}Q(1)c_4 \\ + \frac{1}{27}c_2^3Q(1) + \frac{32}{81}c_2^3P'''(1)Q(1))e_n^4 + \sum_{i=5}^{6} m_i e_n^i + O(e_n^7), \end{aligned} \quad (8)$$

where

$$m_i = m_i(c_2, c_3, ...c_6, P'''(1), P^{iv}(1), P^v(1), Q(1), Q'(1)), 5 \leqslant i \leqslant 6.$$

From (8), it is clear that for the following conditions on Q and on its derivatives:

$$Q(1) = 1,\ Q'(1) = -\frac{3}{2},\ |\ Q''(1)\ | < \infty, \tag{9}$$

our proposed scheme has the following error equation:

$$\begin{aligned} e_{n+1} = {} & \frac{1}{729}\left(-54c_2^2 + 9c_3 + 8Q''(1)c_2^2\right)\left(32P'''(1)c_2^3 + 3c_2^3 + 81c_2c_3 - 9c_4\right)e_n^6 \\ & + O(e_n^7). \end{aligned} \tag{10}$$

The error equation shows that the proposed scheme (1) approaches the sixth-order of convergence. □

2.2 Particular cases of weight functions

Here are some particular cases of weight functions written as Case 1, Case 2, and Case 3.

Case 2.2 If we take the weight functions $P(u)$ and $Q(v)$ of the following form:

$$P(u) = \frac{a_0}{1 + a_1 u + a_2 u^2},$$

and

$$Q(v) = b_0 + b_1 v + b_2 v^2,$$

with

$$a_0 = \frac{16}{3},\ a_1 = \frac{22}{3},\ a_2 = -3,$$

$$b_0 = \frac{5}{2} + b_2,\ b_1 = -\frac{3}{2} - 2b_2,\ b_2 = b_2.$$

For $b_2 = 4$

$$b_0 = \frac{13}{2},\ b_1 = -\frac{19}{2}.$$

Then, for $u_n = \frac{f'(x_n)}{f'(y_n)}$ and $v_n = \frac{f'(y_n)}{f'(x_n)}$, we get a new sixth-order scheme, called as $FS1$

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(\frac{16}{3 + 22u_n - 9u_n^2}\right)\frac{f(x_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - \left(\frac{13}{2} - \frac{19}{2}v_n + 4v_n^2\right)\frac{f(z_n)}{f'(x_n)}. \end{aligned} \tag{11}$$

Case 2.3 When the weight functions $P(u)$ and $Q(v)$ are the rational function of the following form:

$$P(u) = \frac{a_0}{1 + a_1 u + a_2 u^2},$$

$$Q(v) = \frac{b_0}{1 + b_1 v + b_2 v^2},$$

with

$$a_0 = \frac{16}{3},\ a_1 = \frac{22}{3},\ a_2 = -3,$$

$$b_0 = -2 + 2b_2,\ b_1 = -3 + b_2,\ b_2 = b_2.$$

For $b_2 = 2$

$$b_0 = 2,\ b_1 = -1.$$

Then, a new sixth-order scheme is given namely as $FS2$

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(\frac{16}{3 + 22u_n - 9u_n^2}\right)\frac{f(x_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - \left(\frac{2}{1 - v_n + 2v_n^2}\right)\frac{f(z_n)}{f'(x_n)}. \end{aligned} \tag{12}$$

Case 2.4 Next, we consider weight functions $P(u)$ and $Q(v)$ of the following form:

$$P(u) = a_0 + \frac{a_1}{u} + a_2 u,$$

and

$$Q(v) = b_0 + \frac{b_1}{v} + b_2 v,$$

with

$$a_0 = 0, a_1 = \frac{5}{8}, a_2 = \frac{3}{8},$$

$$b_0 = -\frac{1}{2} - 2b_2, b_1 = \frac{3}{2} + b_2, b_2 = b_2.$$

For $b_2 = 3$

$$b_0 = -\frac{13}{2}, b_1 = \frac{9}{2}.$$

Then, another sixth-order new scheme, namely $FS3$, is obtained as

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - \left(\frac{5}{8u_n} + \frac{3}{8}u_n\right)\frac{f(x_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - \left(-\frac{13}{2} + \frac{9}{2v_n} + 3v_n\right)\frac{f(z_n)}{f'(x_n)}. \end{aligned} \tag{13}$$

2.3 Numerical results

Now, we want to verify the numerical results of our new schemes that are presented in the previous section. To demonstrate the suitability of our suggested schemes, we have considered some examples and compared the results of our schemes, namely, $FS1$, $FS2$, and $FS3$, with respect to the number of iterations n, absolute residual error of the corresponding function $|f(x_n)|$, error in two consecutive iterations $|x_n - x_{n-1}|$, and computational order of convergence $COC = \frac{\log[f(x_{n+1})/f(x_n)]}{\log[f(x_n)/f(x_{n-1})]}$. The previous methods for comparisons are considered as the sixth-order methods given by Behl et al. (2019) in [1] and Lee and Kim (2020) in [8] denoted by BS and LK. The numerical results are given in Tables 1 and 2.

Example 2.5 We choose a function from [4], which is

$$f_1(x) = (x-1)^6 - 1.$$

The function has two real and four complex roots. We take the real root $\gamma = 2$ and an initial guess $x_0 = 2.5$.

Example 2.6 Consider the function

$$f_2(x) = x^3 - \cos(x) + 2,$$

from [11]. The desired root for the function is $\gamma = 2.759 + 6.585i$. We take an initial guess $x_0 = 3 + 7.4i$.

Table 1 Comparison of sixth-order methods for univariate function $f_1(x)$

$Cases$	n	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC
$FS1$	1	2.042	$4.578e(-1)$	$2.811e(-1)$	
	2	2.000	$4.215e(-2)$	$4.671e(-8)$	4.324
	3	1.999	$7.785e(-9)$	$3.242e(-47)$	5.776
$FS2$	1	2.047	$4.528e(-1)$	$3.183e(-1)$	
	2	2.000	$4.714e(-2)$	$2.846e(-6)$	3.335
	3	2.000	$4.744e(-7)$	$3.277e(-36)$	5.930
$FS3$	1	2.053	$4.466e(-1)$	$3.6605e(-1)$	
	2	1.999	$5.336e(-2)$	$7.233e(-6)$	3.237
	3	1.999	$1.205e(-6)$	$4.233e(-33)$	5.789
LK	1	2.107	$3.926e(-1)$	$8.436e(-1)$	
	2	2.000	$1.067e(-1)$	$3.714e(-3)$	2.160
	3	2.000	$6.180e(-4)$	$7.918e(-16)$	5.377
BS	1	2.078	$4.219e(-1)$	$5.697e(-1)$	
	2	2.000	$7.800e(-2)$	$2.753e(-4)$	2.629
	3	2.000	$4.589e(-5)$	$2.778e(-23)$	5.729

Table 2 Comparison of sixth-order methods for univariate function $f_2(x)$

$Cases$	n	x_n	$\lvert x_n - x_{n-1}\rvert$	$\lvert f(x_n)\rvert$	COC
$FS1$	1	$2.764 + 6.590i$	$8.428e(-1)$	1.792	
	2	$2.759 + 6.585i$	$7.801e(-3)$	$3.108e(-12)$	5.113
	3	$2.759 + 6.585i$	$1.359e(-14)$	$9.452e(-83)$	5.995
$FS2$	1	$2.765 + 6.593i$	$8.401e(-1)$	2.426	
	2	$2.759 + 6.585i$	$1.054e(-2)$	$4.154e(-11)$	4.965
	3	$2.759 + 6.585i$	$1.817e(-13)$	$1.103e(-75)$	5.997
$FS3$	1	$2.767 + 6.592i$	$8.405e(-1)$	2.525	
	2	$2.759 + 6.585i$	$1.097e(-2)$	$1.870e(-10)$	4.709
	3	$2.759 + 6.585i$	$8.180e(-13)$	$3.421e(-71)$	5.995
LK	1	$2.791 + 6.626i$	$8.007e(-1)$	12.480	
	2	$2.759 + 6.585i$	$5.276e(-2)$	$1.937e(-5)$	3.986
	3	$2.759 + 6.585i$	$8.474e(-8)$	$4.351e(-40)$	5.964
BS	1	$2.777 + 6.606i$	$8.240e(-1)$	6.579	
	2	$2.759 + 6.585i$	$2.827e(-2)$	$1.170e(-7)$	4.466
	3	$2.759 + 6.585i$	$5.122e(-10)$	$4.556e(-54)$	5.988

3 Extension of sixth-order method to the system of nonlinear equations

Now, we give an extension of our method to the system of nonlinear equations by preserving the order of convergence as in the case of scalar equations.

3.1 Derivation of the scheme

We use the weight function approach in the development of our scheme. Our method consists of three steps, which are given below. For the multidimensional case, the scheme (1) named as FS can be rewritten as

$$\begin{aligned} Y^{(n)} &= X^{(n)} - \frac{2}{3}\left(F'\left(X^{(n)}\right)\right)^{-1} F\left(X^{(n)}\right), \\ Z^{(n)} &= X^{(n)} - P\left(U^{(n)}\right)\left(F'\left(Y^{(n)}\right)\right)^{-1} F\left(X^{(n)}\right), \\ X^{(n+1)} &= Z^{(n)} - Q\left(V^{(n)}\right)\left(F'\left(X^{(n)}\right)\right)^{-1} F\left(Z^{(n)}\right). \end{aligned} \tag{14}$$

for the multivariate vector-valued function $F : \mathbb{D} \subseteq \mathbb{C}^n \to \mathbb{C}^n$ with $n \in \mathbb{N}$

$$U^{(n)} = (F'\left(Y^{(n)}\right))^{-1} F'(X^{(n)}),$$

and

$$V^{(n)} = (F'\left(X^{(n)}\right))^{-1}F'\left(Y^{(n)}\right).$$

Theorem 3.1 *Let us suppose that $F : \mathbb{D} \subseteq \mathbb{C}^n \to \mathbb{C}^n$ with $n \in N$ be a sufficiently Frechet differentiable function in $\mathbb{D}$ containing simple root Υ. In addition, that convergence is guaranteed if we consider that initial guess $X^{(0)}$ is close to the root Υ. Then, the numerical scheme (14) has sixth-order convergence for the following conditions on weight functions:*

$$P(I) = I, P'(I) = -\frac{1}{4}I, P''(I) = \frac{5}{4}I, \mid P'''(I) \mid < \infty,$$
$$Q(I) = I, Q'(I) = -\frac{3}{2}I, \mid Q''(I) \mid < \infty,$$

where $P, Q : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ are matrix functions, sufficiently Frechet differentiable in a neighborhood of I (I is $n \times n$ identity matrix).

Proof Let us consider that $E_n = X^{(n)} - \Upsilon$ be the error in the n^{th} iteration. The Taylor's series expansion of the function $F(X^{(n)})$ and $F'(X^{(n)})$ with the assumption $\mid F'(\Upsilon) \mid \neq 0$ leads us to

$$F(X^{(n)}) = F'(\Upsilon)(E_n + C_2E_n^2 + C_3E_n^3 + C_4E_n^4 + C_5E_n^5 + C_6E_n^6 + O(E_n^7)), \tag{15}$$

where

$$C_i = \frac{1}{i!}\left[F'(\Upsilon)\right]^{-1} F^i(\Upsilon),$$

for $i = 2, 3, ...$ and

$$F'(X^{(n)}) = F'(\Upsilon)(I + 2C_2E_n + 3C_3E_n^2 + 4C_4E_n^3 + 5C_5E_n^4 + 6C_6E_n^5 + O(E_n^6)). \tag{16}$$

Now, for the first substep

$$Y^{(n)} = X^{(n)} - \frac{2}{3}(F'(X^{(n)}))^{-1}F\left(X^{(n)}\right). \tag{17}$$

Applying Taylor's series to (17), we get

$$Y^{(n)} = \Upsilon + \frac{1}{3}E_n + \frac{2}{3}C_2E_n^2 + (\frac{4}{3}C_3 - \frac{4}{3}C_2^2)E_n^3 + (2C_4 - \frac{14}{3}C_2C_3 + \frac{8}{3}C_2^3)E_n^4 + \sum_{i=5}^{6} A_iE_n^i + O(E_n^7),$$

where

$$A_i = A_i(C_2, C_3, ...C_6), 5 \leqslant i \leqslant 6.$$

Also, $F'(Y^{(n)})$ is given by

$$F'(Y^{(n)}) = F'(\Upsilon)(I + \frac{2}{3}C_2E_n + \frac{1}{3}(4C_2^2 + C_3)E_n^2 + (\frac{8}{3}C_2C_3 - \frac{8}{3}C_2^3 + \frac{4}{27}C_4)E_n^3 + \sum_{i=4}^{5} B_iE_n^i + O(E_n^6)), \tag{18}$$

where

$$B_i = B_i(C_2, C_3, ...C_6), 4 \leqslant i \leqslant 5.$$

Next, for the Taylor's series expansion of the function $U^{(n)} = (F'(Y^{(n)}))^{-1}F'(X^{(n)})$

$$U^{(n)} = I + \frac{4}{3}C_2E_n + (\frac{8}{3}C_3 - \frac{20}{9}C_2^2)E_n^2 + \sum_{i=3}^{5} D_iE_n^i + O(E_n^6),$$

where

$$D_i = D_i(C_2, C_3, ...C_6), 3 \leqslant i \leqslant 5.$$

Moreover, $P(U^{(n)})$ is given by

$$P(U^{(n)}) = P(I) + \frac{4}{3}P'(I)C_2E_n + (\frac{8}{3}P'(I)C_3 - \frac{20}{9}P'(I)C_2^2 + \frac{8}{9}P''(I)C_2^2)E_n^2 + \sum_{i=3}^{5} G_i e_n^i + O(e_n^6),$$

where

$$G_i = G_i(C_2, C_3, ...C_6, P(I), P'(I), P''(1), P'''(1), P^{iv}(I)), 3 \leqslant i \leqslant 5.$$

Let us now consider the second substep

$$Z^{(n)} = X^{(n)} - P(U^{(n)})F'(Y^{(n)})^{-1}F(X^{(n)}), \tag{19}$$

as

$$Z^{(n)} = \Upsilon + (I - P(I))E_n - \frac{1}{3}C_2(P(I) + 4P'(I))E_n^2 + \sum_{i=3}^{6} G_i E_n^i + O(E_n^7), \tag{20}$$

where

$$G_i = G_i(C_2, C_3, ...C_6, P(I), P'(I), P''(I), P'''(I), P^{iv}(I)), 3 \leqslant i \leqslant 6.$$

Taking the conditions

$$P(I) = I, P'(I) = -\frac{1}{4}I, P''(I) = \frac{5}{4}I, | P'''(1) | < \infty,$$

(19) becomes

$$Z^{(n)} = (-C_2C_3 + \frac{1}{9}C_4 - \frac{1}{27}C_2^3 - \frac{32}{81}P'''(I)C_2^3)E_n^4 + \sum_{i=5}^{6} H_i e_n^i + O(e_n^7)),$$

where

$$H_i = H_i(C_2, C_3, ...C_6, P'''(I), P^{iv}(I), P^v(I)), 5 \leqslant i \leqslant 6.$$

Similarly, $F(Z^{(n)})$ is given as

$$F(Z^{(n)}) = F'(\Upsilon)((-C_2C_3 + \frac{1}{9}C_4 - \frac{1}{27}C_2^3 - \frac{32}{81}P'''(I)C_2^3)E_n^4 + \sum_{i=5}^{6} J_i E_n^i + O(E_n^7)),$$

where

$$J_i = J_i(C_2, C_3, ...C_6, P'''(I), P^{iv}(I), P^v(I)), 5 \leqslant i \leqslant 6.$$

Also, applying Taylor's series to $V^{(n)} = (F'(X^{(n)}))^{-1}F'(Y^{(n)})$, we get

$$V^{(n)} = I - \frac{4}{3}C_2E_n + (-\frac{8}{3}C_3 + 4C_2^2)E_n^2 + \sum_{i=3}^{5} K_i E_n^i + O(E_n^6),$$

where

$$K_i = K_i(C_2, C_3, ...C_6), 3 \leqslant i \leqslant 5.$$

Similarly, $Q(V^{(n)})$ is given as

$$Q(V^{(n)}) = Q(I) - \frac{4}{3}Q'(I)C_2E_n + (-\frac{8}{3}Q'(I)C_3 + 4Q'(I)C_2^2 + \frac{8}{9}Q''(I)C_2^2)E_n^2 + \sum_{i=3}^{5} L_i E_n^i + O(E_n^6),$$

where

$$L_i = L_i(C_2, C_3, ...C_6, Q(I), Q'(I), Q''(I), Q'''(I), Q^{iv}(I)), 3 \leqslant i \leqslant 5.$$

Finally, Taylor's expansion of the last step gives

$$X^{(n+1)} = (-C_2C_3 + \frac{1}{9}C_4 - \frac{1}{27}C_2^3 - \frac{32}{81}P'''(I)C_2^3 + Q(I)C_2C_3 - \frac{1}{9}Q(I)C_4 + \frac{1}{27}C_2^3Q(I)$$

$$+\frac{32}{81}C_2^3P'''(I)Q(I))E_n^4+\sum_{i=5}^{6}M_iE_n^i+O(E_n^7),$$

where

$$M_i=M_i(C_2,C_3,...C_6,P'''(I),P^{iv}(I),P^{v}(I),Q(I),Q'(I)),5\leqslant i\leqslant 6.$$

It is apparent that taking the following conditions on the weight function Q:

$$Q(I)=I,\,Q'(I)=-\frac{3}{2},\mid Q''(I)\mid<\infty, \tag{21}$$

we obtain the following error equation from (21):

$$E^{n+1}=\frac{1}{729}(-54C_2^2+9C_3+8Q''(I)C_2^2)(81C_2C_3+3C_2^3+32P'''(I)C_2^3-9C_4)E_n^6+O(E_n^7).$$

This asymptotic error constant reveals that the proposed scheme (14) reaches at sixth-order convergence. It completes the proof. □

Next, we take some special cases of our proposed scheme (14), which are as follows:

Case 1 When the weight functions $P(U)$ and $Q(V)$ are polynomial functions of the following form:

$$P(U)=a_0I(I+a_1U+a_2U^2)^{-1},$$
$$Q(V)=b_0I+b_1V+b_2V^2,$$

with

$$a_0=\frac{16}{3},a_1=\frac{22}{3},a_2=-3,$$
$$b_0=\frac{5}{2}+b_2,b_1=-\frac{3}{2}-2b_2,b_2=b_2.$$

for $b_2=4$

$$b_0=\frac{13}{2},b_1=-\frac{19}{2}.$$

Then, we get a sixth-order scheme, named as $FS4$ which is given below

$$Y^{(n)}=X^{(n)}-\frac{2}{3}(F'\left(X^{(n)}\right))^{-1}F\left(X^{(n)}\right),$$
$$Z^{(n)}=X^{(n)}-16I(3I+22U^{(n)}-9(U^{(n)})^2)^{-1}(F'\left(Y^{(n)}\right))^{-1}F\left(X^{(n)}\right),$$
$$X^{(n+1)}=Z^{(n)}-(\frac{13}{2}I-\frac{19}{2}V^{(n)}+4(V^{(n)})^2)(F'\left(X^{(n)}\right))^{-1}F\left(Z^{(n)}\right). \tag{22}$$

Case 2 If we take the weight functions $P(U)$ and $Q(V)$ of the following form:

$$P(U)=a_0I(I+a_1U+a_2U^2)^{-1},$$
$$Q(V)=b_0I(I+b_1V+b_2V^2)^{-1},$$

with

$$a_0=\frac{16}{3},a_1=\frac{22}{3},a_2=-3,$$
$$b_0=-2+2b_2,b_1=-3+b_2,b_2=b_2.$$

for $b_2=2$

$$b_0=2,b_1=-1.$$

Then, we obtain the following sixth-order scheme called as $FS5$:

$$Y^{(n)}=X^{(n)}-\frac{2}{3}(F'\left(X^{(n)}\right))^{-1}F\left(X^{(n)}\right),$$

$$Z^{(n)} = X^{(n)} - 16I(3I + 22U^{(n)} - 9(U^{(n)})^2)^{-1}(F'\left(Y^{(n)}\right))^{-1}F\left(X^{(n)}\right),$$
$$X^{(n+1)} = Z^{(n)} - 2I(I - V^{(n)} + 2(V^{(n)})^2)^{-1}(F'\left(X^{(n)}\right))^{-1}F\left(Z^{(n)}\right). \quad (23)$$

Case 3 If we take weight functions $P(U)$ and $Q(V)$ of the following form:

$$P(U) = a_0 + a_1U^{-1} + a_2U,$$

and

$$Q(V) = b_0 + b_1V^{-1} + b_2V,$$

with

$$a_0 = 0, a_1 = \frac{5}{8}, a_2 = \frac{3}{8},$$

$$b_0 = -\frac{1}{2} - 2b_2, b_1 = \frac{3}{2} + b_2, b_2 = b_2.$$

for $b_2 = 3$

$$b_0 = -\frac{13}{2}, b_1 = \frac{9}{2}.$$

Then, we obtain the sixth-order scheme called as $FS6$

$$Y^{(n)} = X^{(n)} - \frac{2}{3}(F'\left(X^{(n)}\right))^{-1}F\left(X^{(n)}\right),$$
$$Z^{(n)} = X^{(n)} - (\frac{5}{8}(U^{(n)})^{-1} + \frac{3}{8}U^{(n)})(F'\left(Y^{(n)}\right))^{-1}F\left(X^{(n)}\right),$$
$$X^{(n+1)} = Z^{(n)} - (-\frac{13}{2}I + \frac{9}{2}(V^{(n)})^{-1} + 3V^{(n)})(F'\left(X^{(n)}\right))^{-1}F\left(Z^{(n)}\right). \quad (24)$$

Table 3 Comparison of sixth-order methods for $F_1(X)$

Cases	n	$X^{(n)}$	$\Vert X^{(n)}-X^{(n-1)}\Vert_\infty$	$\Vert F\left(X^{(n)}\right)\Vert_\infty$
$FS4$	1	$5.184e(-1)$	$3.210e(-1)$	$4.815e(-1)$
	2	$5.044e(-1)$	$5.102e(-4)$	$9.296e(-3)$
	3	$5.044e(-1)$	$2.156e(-12)$	$3.467e(-11)$
$FS5$	1	$5.215e(-1)$	$3.211e(-1)$	$4.784e(-1)$
	2	$5.044e(-1)$	$6.076e(-4)$	$1.229e(-2)$
	3	$5.044e(-1)$	$1.729e(-11)$	$2.597e(-10)$
$FS6$	1	$5.205e(-1)$	$3.209e(-1)$	$4.794e(-1)$
	2	$5.044e(-1)$	$4.922e(-4)$	$8.993e(-3)$
	3	$5.044e(-1)$	$2.759e(-11)$	$4.217e(-10)$
BA	1	$5.279e(-1)$	$3.212e(-1)$	$4.720(-1)$
	2	$5.044e(-1)$	$7.311e(-4)$	$1.602e(-2)$
	3	$5.044e(-1)$	$2.553e(-10)$	$3.886e(-9)$
KC	1	$4.804e(-1)$	$3.192e(-1)$	$5.195e(-1)$
	2	$5.044e(-1)$	$1.258e(-3)$	$2.148e(-2)$
	3	$5.044e(-1)$	$1.063e(-9)$	$1.658e(-8)$
LK	1	$5.332e(-1)$	$3.216e(-1)$	$4.667e(-1)$
	2	$5.044e(-1)$	$1.103e(-3)$	$2.773e(-2)$
	3	$5.044e(-1)$	$1.777e(-9)$	$2.731e(-8)$
BS	1	$5.332e(-1)$	$3.216e(-1)$	$4.667e(-1)$
	2	$5.044e(-1)$	$1.103e(-3)$	$2.773e(-2)$
	3	$5.044e(-1)$	$1.777e(-9)$	$2.731e(-8)$

Table 4 Comparison of sixth-order methods for $F_2(X)$

$Cases$	n	$X^{(n)}$	$\parallel X^{(n)}-X^{(n-1)}\parallel_\infty$	$\parallel F\left(X^{(n)}\right)\parallel_\infty$
$FS4$	1	$8.956e(-1)$	$9.767e(-2)$	$1.029e(-1)$
	2	$8.956e(-1)$	$2.008e(-13)$	$3.886e(-13)$
	3	$8.956e(-1)$	$3.991e(-83)$	$8.918e(-83)$
$FS5$	1	$8.956e(-1)$	$9.767e(-2)$	$2.293e(-1)$
	2	$8.956e(-1)$	$7.087e(-13)$	$1.443e(-12)$
	3	$8.956e(-1)$	$1.090e(-78)$	$1.870e(-78)$
$FS6$	1	$8.956e(-1)$	$9.767e(-2)$	$1.029e(-1)$
	2	$8.956e(-1)$	$7.942e(-13)$	$1.870e(-12)$
	3	$8.956e(-1)$	$1.047e(-78)$	$2.077e(-78)$
BA	1	$8.956e(-1)$	$9.767e(-2)$	$2.851(-1)$
	2	$8.956e(-1)$	$1.257e(-12)$	$2.481e(-12)$
	3	$8.956e(-1)$	$7.393e(-77)$	$1.242e(-76)$
KC	1	$8.956(-1)$	$9.767(-2)$	$2.293(-1)$
	2	$8.956e(-1)$	$1.343e(-12)$	$2.851e(-12)$
	3	$8.956e(-1)$	$8.087e(-77)$	$1.430e(-76)$
LK	1	$8.956e(-1)$	$9.767e(-2)$	$2.293e(-1)$
	2	$8.956e(-1)$	$2.378e(-12)$	$4.983e(-12)$
	3	$8.956(-1)$	$4.585(-75)$	$8.028(-75)$
BS	1	$8.956e(-1)$	$9.767e(-2)$	$2.293e(-1)$
	2	$8.956e(-1)$	$2.378e(-12)$	$4.983e(-12)$
	3	$8.956e(-1)$	$4.585e(-75)$	$8.028e(-75)$

Table 5 Comparison of sixth-order methods for $F_3(X)$

$Cases$	n	$X^{(n)}$	$\parallel X^{(n)}-X^{(n-1)}\parallel_\infty$	$\parallel F\left(X^{(n)}\right)\parallel_\infty$
$FS4$	1	$-5.569e(-1)$	$4.430e(-1)$	$3.630e(-1)$
	2	$-4.795e(-1)$	$6.084e(-3)$	$8.498e(-2)$
	3	$-4.795e(-1)$	$1.592e(-9)$	$1.337e(-9)$
$FS5$	1	$-5.644e(-1)$	$4.355e(-1)$	$3.555e(-1)$
	2	$-4.795e(-1)$	$1.738e(-3)$	$2.394e(-3)$
	3	$-4.795e(-1)$	$2.177e(-10)$	$5.727e(-10)$
$FS6$	1	$-5.660e(-1)$	$4.339e(-1)$	$3.539e(-1)$
	2	$-4.795e(-1)$	$3.854e(-3)$	$1.336e(-2)$
	3	$-4.795e(-1)$	$4.747e(-8)$	$7.227e(-8)$
BA	1	$-5.809e(-1)$	$4.190e(-1)$	$3.390(-1)$
	2	$-4.795e(-1)$	$3.877e(-2)$	$5.088e(-2)$
	3	$-4.795e(-1)$	$3.341e(-8)$	$3.256e(-7)$
KC	1	$-4.342e(-1)$	$5.657e(-1)$	$4.857e(-1)$
	2	$-4.795e(-1)$	$4.532e(-2)$	$4.1470e(-2)$
	3	$-4.795e(-1)$	$8.592e(-8)$	$3.952e(-7)$
LK	1	$-5.920e(-1)$	$4.079e(-1)$	$3.279e(-1)$
	2	$-4.795e(-1)$	$2.562e(-2)$	$1.055e(-1)$
	3	$-4.795e(-1)$	$5.123e(-8)$	$2.492e(-7)$
BS	1	$-5.920e(-1)$	$4.079e(-1)$	$3.279e(-1)$
	2	$-4.795e(-1)$	$2.562e(-2)$	$1.055e(-1)$
	3	$-4.795e(-1)$	$5.123e(-8)$	$2.492e(-7)$

3.2 Numerical results

Now, we want to verify the numerical results of our iterative method. For this purpose, we consider some examples and compare the results of our scheme, namely, $FS4$, $FS5$, and $FS6$, with respect to number of iterations n, absolute residual error of the corresponding function in $\left\|F\left(X^{(n)}\right)\right\|$, and absolute error in two consecutive iterations $\left\|X^{(n)}-X^{(n-1)}\right\|$ that are given in Tables 3, 4, 5. For the sake of comparison, we consider the sixth-order methods given by Behl and Argyros (2020) [2], Kansal et al. (2021) [6], Lee and Kim (2020) [8], and Behl et al. (2019) [1], namely, BA, KC, LK, and BS, respectively.

Table 6 Comparisons of EI and CE

Method	EI	CE
KC	$6^{\frac{1}{2n^2+2n}}$	$6^{\frac{1}{\frac{2}{3}n^3+6n^2+\frac{4}{3}n}}$
BA	$6^{\frac{1}{2n^2+2n}}$	$6^{\frac{1}{\frac{2}{3}n^3+5n^2+\frac{4}{3}n}}$
FS	$6^{\frac{1}{2n^2+2n}}$	$6^{\frac{1}{\frac{2}{3}n^3+7n^2+\frac{4}{3}n}}$
LK	$6^{\frac{1}{2n^2+2n}}$	$6^{\frac{1}{\frac{1}{3}n^3+5n^2+\frac{5}{3}n}}$
BS	$6^{\frac{1}{2n^2+2n}}$	$6^{\frac{1}{n^3+6n^2+n}}$

Example 3.2 We take a 3×3 system $F_1(X)$ of nonlinear equations from [6], such that

$$F_1(X) = \begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix},$$

where

$$\begin{aligned} f_1(x) &= x_1^2 + x_2^2 + x_3^2 - 1 = 0, \\ f_2(x) &= 2x_1^2 - x_2^2 - 4x_3 = 0, \\ f_3(x) &= 3x_1^2 - 4x_2^2 + x_3^2 = 0. \end{aligned} \tag{25}$$

The exact solution for the system is $\Upsilon = (0.6982886, 0.6285243, 0.3425642)$. We choose an initial guess as $X^{(0)} = (1, 1, 1)$

Example 3.3 Let us take a substance that is under observation in a bounded domain $\Omega \in \mathbb{R}^2$ with continuous boundary $\partial\Omega$. The two-dimensional nonlinear diffusion–reaction equation for the concentration $w(x, t)$ of the substance in a bounded domain is represented by an initial-boundary value problem [8]

$$w_t - d\Delta w = w(a - w) \text{ in } \Omega \times (0, \infty), \tag{26}$$

where $w = g$ on the boundary.

Here, Δ is Laplacian operator, a is positive constant, and $d > 0$ is diffusion coefficient. Let us observe the concentration of substance in a unit square region, such that $\Omega = [0, 1] \times [0, 1]$ and let we take $d = 1, a = 1$. To get the steady state solution, Eq. (26) is converted in the following form:

$$w_{xx} + w_{yy} = w(w - 1) \text{ in } \Omega \tag{27}$$

with Dirichlet boundary conditions

$$w(x, 0) = w(x, 1) = \frac{x(x-1)}{2} + 1, \; w(0, y) = w(1, y) = \frac{y(y-1)}{2} + 1.$$

We use central-divided difference formula by taking step-length $h = 1/4$ between the space components of the unit square region, and then, we discretize Eq. (27) into a system of nonlinear equations. This system consists of 25 nodes. Among these 25 nodes, 16 are boundary nodes and 9 nodes represent the interior nodal variables. We solve the system for the interior nodal variables say $x_1, x_2, ...x_9$. The desired solution to the problem is

$$\begin{aligned} \Upsilon = (&0.90232..., \; 0.89564..., \; 0.90232..., \; 0.89564..., \; 0.89708..., \\ &0.89564..., \; 0.90232..., \; 0.89564..., \; 0.90232...). \end{aligned}$$

We take $X^{(0)} = (1, 1, 1, 1, 1, 1, 1, 1, 1)^T$ as an initial guess (Table 6).

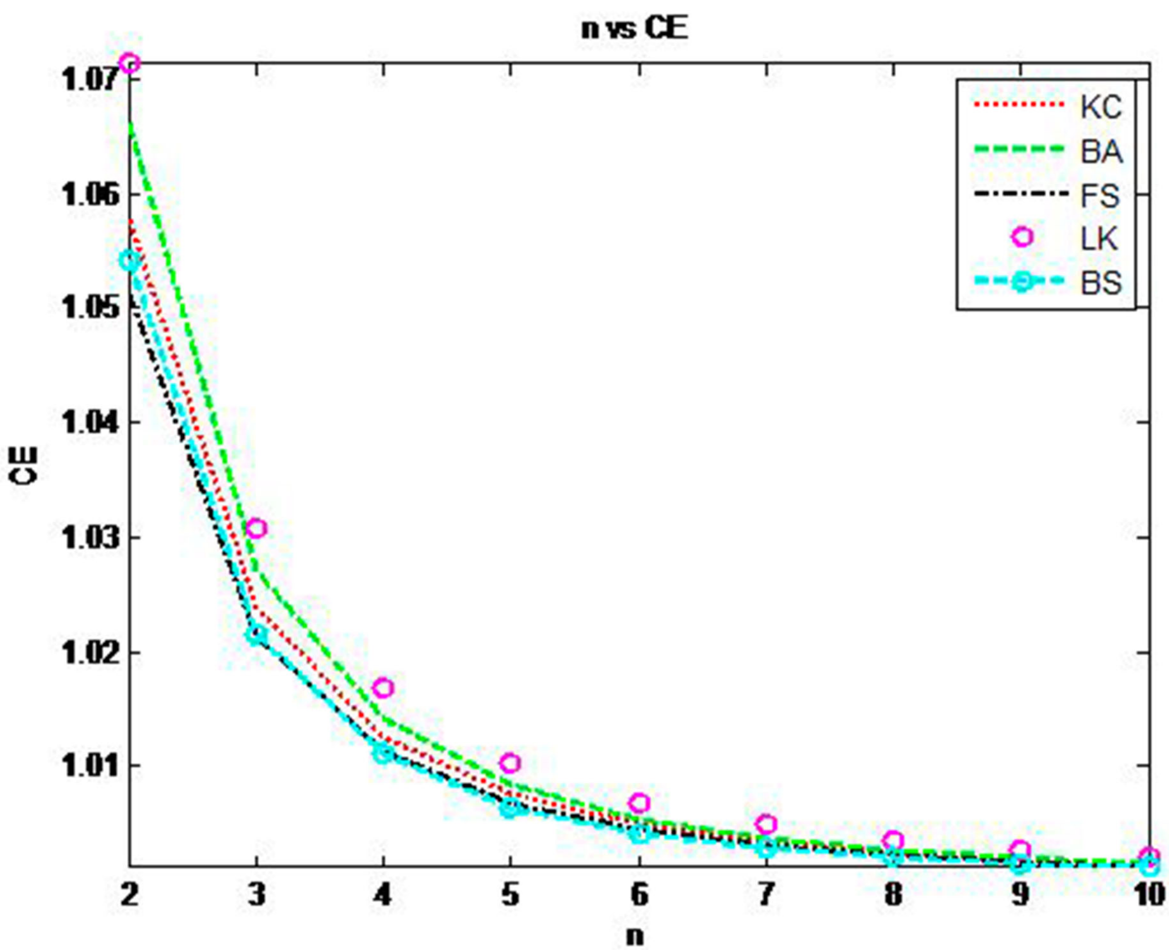

Fig. 1 Comparison of CE

Example 3.4 Consider the Van der Waals equation of state from [3]

$$s'' - \beta\left(s^2 - 1\right)s' + s = 0, \tag{28}$$

in an interval [0, 2]. Let the boundary conditions are

$$s\,(0) = 0 \text{ and } s\,(2) = 1.$$

We consider the following partition for the interval [0, 2]:

$$w_0 = 0 < w_1 < w_2 < \ldots < w_n \text{ here } w_j = w_0 + jh\,,\, h = \frac{2}{m},$$

and assume that

$$s_j = s\left(w_j\right)\,,\, j = 0, 1, 2, \ldots, m.$$

The central finite-difference formula for the first and second-order derivative is given as

$$s'_k = \frac{s_{k+1} - s_{k-1}}{2h}\,,\, s''_k = \frac{s_{k+1} - 2s_k + s_{k-1}}{h^2}, k = 1, 2, \ldots, m-1.$$

By substituting central-difference formula in Eq. (28), we obtain $(m-1) \times (m-1)$ system of nonlinear equations of the following form:

$$2\left(s_{k+1} - 2s_k + s_{k-1}\right) - \beta h\left(s_k^2 - 1\right)\left(s_{k+1} - s_{k-1}\right) + h^2 s_k = 0.$$

We take $\beta = \frac{1}{2}$ and $X^{(0)} = (-1, -2, -3, -4, -5, -6, -7, -8, -9)^T$, $k = 1, 2, \ldots, m-1$. We consider $m = 10$ and solve the system of nine nonlinear equations.

4 Efficiency of the methods

Consider the efficiency index [12] (EI), $EI = p^{\frac{1}{d}}$, where p represents the order and d represents the total number of functional evaluations. Moreover, the computational efficiency index (CE) [4] is characterized as $CE = p^{\frac{1}{(d+op)}}$ where op is the operations cost per cycle. We have made comparisons of our scheme FS for EI and CE with the sixth-order methods given in Sect. 3, namely, BA, KC, LK, and BS.

5 Conclusion

We developed a new sixth-order scheme for the univariate as well as for the multidimensional case. The numerical results of our scheme compared with those of existing families of Jarratt-type methods show that our scheme performs better than the existing ones.

Funding No funding is availed for this research work.

Declarations

Conflict of interest There is no conflict of interest in publishing this article.

Consent for publication All authors agree for its submission to AJM.

Authors' contributions All authors have contributed equally to this work.

References

1. Behl, R.; Sarría, I.; González, R.; Magreñán, A.A.: Highly efficient family of iterative methods for solving nonlinear models. J. Comput. Appl. Math. **346**, 110–132 (2019)
2. Behl, R.; Argyros, I.K.: A new higher order iterative scheme for the solutions of nonlinear systems. Mathematics (2020). https://doi.org/10.3390/math8020271
3. Burden, R.L.; Faires, J.D.: Numerical Analysis. PWS Publishing Company, New York (2001)
4. Cordero, A.; Hueso, J.L.; Martínez, E.; Torregrosa, J.R.: A modified Newton–Jarratt's composition. Numer. Algorithm **55**, 87–99 (2010)
5. Grosan, C.; Abraham, A.: A new approach for solving nonlinear equations systems. IEEE Trans. Syst. Man. Cybern. Part A: Syst. Hum. **38**, 698–714 (2008)
6. Kansal, M.; Cordero, A.; Bhalla, S.; Torregrosa, J.R.: New fourth and sixth-order classes of iterative methods for solving systems of nonlinear equations and their stability analysis. Numer. Algorithm **87**, 1017–1060 (2021)
7. Khan, Y.; Fardi, M.; Sayevand, K.: A new general eighth-order family of iterative methods for solving nonlinear equations. Appl. Math. Lett. **25**, 2262–2266 (2012)
8. Lee, M.; Kim, Y.I.: Development of a family of Jarratt-like sixth-order iterative methods for solving nonlinear systems with their basins of attraction. Algorithms (2020). https://doi.org/10.3390/a13110303
9. Lin, Y.; Bao, L.; Jia, X.: Convergence analysis of a variant of the newton method for solving nonlinear equations. Comput. Math. Appl. **59**, 2121–2127 (2010)
10. Narang, M.; Bhatia, S.; Kanwar, V.: New two-parameter Chebyshev-Halley-like family of fourth and sixth-order methods for systems of nonlinear equations. Appl. Math. Comput. **275**, 394–403 (2016)
11. Singh, A.; Jaiswal, J.P.: Several new third-order and fourth-order iterative method for solving nonlinear equations. Int. J. Eng. Math. **2014**, 1–11 (2014)
12. Ostrowski, A.M.: Solutions of Equations and System of Equations, 1st edn Academic Press, New York, NY, USA (1960)
13. Wang, X.; Kou, J.; Li, Y.: Modified Jarratt method with sixth-order convergence. Appl. Math. Lett. **22**, 1798–1802 (2009)