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On approximation of functions by rational functions in weighted generalized grand Smirnov classes

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Abstract Let G be a doubly connected domain in the complex plane $\mathbb{C}$, bounded by Ahlfors 1-regular curves. In this study the approximation of the functions by Faber–Laurent rational functions in the ω-weighted generalized grand Smirnov classes $\mathcal{E}^{p),\theta}(G,\omega)$ in the term of the rth, $r = 1, 2 \ldots$, mean modulus of smoothness are investigated.

Mathematics Subject Classification 30E10 · 41A10 · 41A25 · 46E30

1 Introduction

Let $\Gamma \subset \mathbb{C}$ be a Jordan rectifiable curve and let $\omega : \Gamma \to [0, \infty]$ be a weight function, that is a positive almost everywhere (a.e.) and integrable function on Γ. For $1 < p < \infty$ we define a class $L^p(\Gamma, \omega)$ of Lebesgue measurable functions f on Γ satisfying the condition

$$\left(\frac{1}{|\Gamma|} \int\limits_{\Gamma} |f(z)|^p \, \omega(z) \, |dz| < \infty \right)^{\frac{1}{p}} < \infty,$$

where $|\Gamma|$ is the length of Γ. We denote by $L^{p),\theta}(\Gamma, \omega)$, $\theta \geq 0$, the Lebesgue space of all measurable functions f on Γ, that is, the space of all such functions for which

$$\|f\|_{L^{p),\theta}(\Gamma,\omega)} := \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon^{\theta}}{|\Gamma|} \int\limits_{\Gamma} |f(z)|^{p-\varepsilon} \, \omega(z) \, |dz| \right)^{\frac{1}{p-\varepsilon}} < \infty.$$

The space $L^{p),\theta}(\Gamma, \omega)$ is called the generalized grand Lebesgue space. $L^{p),\theta}(\Gamma, \omega)$ is Banach function space, nonreflexive and nonseparable. The grand and generalized grand Lebesgue space were introduceed in the works [13,26], respectively. If $\theta_1 < \theta_2$ then for $0 < \varepsilon < p - 1$ the embeddings:

$$L^{p}(\Gamma, \omega) \subset L^{p),\theta_1}(\Gamma, \omega) \subset L^{p),\theta_2}(\Gamma, \omega) \subset L^{p-\varepsilon}(\Gamma, \omega), \quad 1 < p < \infty$$

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hold. Note that the information about properties and applications of the grand Lebesgue spaces can be found in [11,13,26,33,35,36].

A Jordan curve Γ is called *Ahlfors 1-regular* [37], if there exists a number $c > 0$ such that for every $r > 0$, $\sup\{|\Gamma \cap D(z,r)| : z \in \Gamma\} \le cr$, , where $D(z,r)$ is an open disk with radius r and centered at z and $|\Gamma \cap D(z,r)|$ is the length of the set $\Gamma \cap D(z,r)$.

Let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p-condition on Γ if

$$\sup_{z\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int_{\Gamma\cap D(z,r)}\omega(\zeta)\,|d\zeta|\right)\left(\frac{1}{r}\int_{\Gamma\cap D(z,r)}[\omega(\zeta)]^{-\frac{1}{p-1}}\,|d\zeta|\right)^{p-1}<\infty$$

Let us further assume that B is a simply connected domain with a rectifiable Jordan boundary Γ and $B^- := \text{ext}\Gamma$. Without loss of generality we assume that $0 \in B$. Let

$$\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}, \quad D := \text{int}\mathbb{T}, \quad D^- := \text{ext}\mathbb{T}.$$

Also, ϕ^* stand for the conformal mapping of B^- onto D^- normalized by

$$\phi^*(\infty) = \infty$$

and

$$\lim_{z\to\infty}\frac{\phi^*(z)}{z} > 0,$$

and let ψ^* be the inverse of ϕ^*. Let ϕ_1^* be the conformal mapping of B onto D^-, normalized by

$$\phi_1^*(0) = \infty$$

and

$$\lim_{z\to 0} z\phi_1^*(z) > 0.$$

The inverse mapping of ϕ_1^* will be denoted by ψ_1^*.

Note that the mappings ψ^* and ψ_1^* have in some deleted neighborhood of ∞ representations:

$$\psi^*(w) = \alpha w + \alpha_0 + \frac{\alpha_1}{w} + \frac{\alpha_2}{w^2} + \cdots + \frac{\alpha_k}{w^k} + \cdots, \quad \alpha > 0$$

and

$$\psi_1^*(w) = \frac{\beta_1}{w} + \frac{\beta_2}{w^2} + \cdots + \frac{\beta_k}{w^k} + \cdots, \quad \beta_1 > 0.$$

For $1 < p < \infty$ and $0 < \varepsilon < p-1$ the functions:

$$\frac{\left(\frac{d\psi^*(w)}{dw}\right)^{1-\frac{1}{p-\varepsilon}}}{\psi^*(w) - z}, \quad z \in B$$

and

$$\frac{w^{-\frac{2}{p-\varepsilon}}\left(\frac{d\psi_1^*(w)}{dw}\right)^{1-\frac{1}{p-\varepsilon}}}{\psi_1^*(w) - z}, \quad z \in B^-.$$

are analytic in the domain D^-. The following expansions hold:

$$\frac{\left(\frac{d\psi^*(w)}{dw}\right)^{1-\frac{1}{p-\varepsilon}}}{\psi^*(w) - z} = \sum_{k=0}^{\infty}\frac{\Phi_{k,p-\varepsilon}(z)}{w^{k+1}}, \quad z \in B, \quad w \in D^-$$

and

$$\frac{w^{-\frac{2}{p-\epsilon}}\left(\frac{d\psi_1^*(w)}{dw}\right)^{1-\frac{1}{p-\varepsilon}}}{\psi_1^*(w) - z} = \sum_{k=1}^{\infty}-\frac{F_{k,p-\varepsilon}(\frac{1}{z})}{w^{k+1}}, \quad z \in B^-, \quad w \in D^-,$$

where $\Phi_{k,p-\varepsilon}(z)$ and $F_{k,p-\varepsilon}(\frac{1}{z})$ are the $p-\varepsilon$ Faber polynomials of degree k with respect to z and $\frac{1}{z}$ for the continuums $\overline{B}$ and $\overline{B}\backslash B$, respectively (see also [5,20,23] and ([34], pp. 255–257).

Let $E^1(B)$ be a classical Smirnov class of analytic functions in B. The set $E^{p),\theta}(B,\omega) := \left\{f \in E^1(B) : f \in L^{p),\theta}(\Gamma,\omega)\right\}$ is called the ω-weighted generalized grand Smirnov class in B.

Let $\omega \in A_p(\mathbb{T})$. For $f \in L^{p),\theta}(\Gamma,\omega)$ we define the operator

$$\left(\nu_h^r f\right)(w) := \frac{1}{h}\int_{-0}^{h} \left|\Delta_t^r f(w)\right| \mathrm{d}t, \quad h > 0,$$

where

$$\Delta_t^r f(w) := \sum_{k=0}^{r} (-1)^{r+k+1} \binom{r}{k} f\left(w\mathrm{e}^{ikt}\right), \quad r \in \mathbb{N} = \{1, 2, \ldots\}, \quad w \in \mathbb{T}, \quad t > 0.$$

If $\omega \in A_p(\mathbb{T})$ and $f \in L^p(\mathbb{T}, \omega)$, then the operator ν_h is a bounded on $L^{p),\theta}(\mathbb{T},\omega)$ [24]:

$$\sup_{|h|\le\delta} \left\|\nu_h^r(f)\right\|_{L^{p),\theta}(\mathbb{T},\omega)} \le c_1 \|f\|_{L^{p),\theta}(\mathbb{T},\omega)}.$$

Let $1 < p < \infty$, $\omega \in A_p(\mathbb{T})$ and $f \in L^{p),\theta}(\mathbb{T},\omega)$, $\theta > 0$. The function

$$\Omega^r_{p),\theta,\omega}(f,\delta) := \sup_{|h|\le\delta} \left\|\nu_h^r f(w)\right\|_{L^{p),\theta}(\mathbb{T},\omega)}, \quad \delta > 0$$

is called *the r-th mean modulus* of $f \in L^{p),\theta}(\mathbb{T},\omega)$.

It can be easily shown that $\Omega^r_{p),\theta,\omega}(f,\cdot)$ is a continuous, non-negative and nondecreasing function satisfying the conditions:

$$\lim_{\delta\to 0} \Omega^r_{p),\theta,\omega,}(f,\delta) = 0, \ \Omega^r_{p),\theta,\omega}(f+g,\delta) \le \Omega^r_{p),\theta,\omega}(f,\delta) + \Omega^r_{p),\theta,\omega}(g,\delta), \quad \delta > 0$$

for $f, g \in L^{p),\theta}(\mathbb{T},\omega)$.

Let G be a doubly connected domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curves Γ_1 and Γ_2 (the closed curve Γ_2 is in the closed curve Γ_1). Without loss of generality we assume $0 \in \mathrm{int}\Gamma_2$. Let $G_1^0 := \mathrm{int}\Gamma_1$, $G_1^\infty := \mathrm{ext}\Gamma_1$, $G_2^0 := \mathrm{int}\Gamma_2$, $G_2^\infty := \mathrm{ext}\Gamma_2$.

We denote by $w = \phi(z)$ the conformal mapping of G_1^∞ onto domain D^- normalized by the conditions:

$$\phi(\infty) = \infty, \quad \lim_{z\to\infty} \frac{\phi(z)}{z} > 0$$

and let ψ be the inverse mapping of ϕ.

We denote by $w = \phi_1(z)$ the conformal mapping of G_2^0 onto domain D^- normalized by the conditions:

$$\phi_1(0) = \infty, \quad \lim_{z\to 0}(z.\phi_1(z)) > 0$$

and let ψ_1 be the inverse mapping of ϕ_1.

Let us take

$$C_{\rho_0} := \{z : |\phi(z)| = \rho_0 > 1\}, \quad \Gamma_{r_0} := \{z : |\phi_1(z)| = r_0 > 1\}.$$

For $\Phi_{k,p-\varepsilon}(z)$ and $F_{k,p-\varepsilon}\left(\frac{1}{z}\right)$ the following integral representations hold [5,20,23,34], pp. 255–257:

(1) If $z \in intC_{\rho_0}$, then

$$\Phi_{k,p-\varepsilon}(z) = \frac{1}{2\pi i}\int_{C_{\rho_0}} \frac{[\phi(\zeta)]^k \left(\phi'(\zeta)\right)^{\frac{1}{p-\varepsilon}}}{\zeta - z}\mathrm{d}\zeta. \tag{1.1}$$

(2) If $z \in \text{ext}C_{\rho_0}$, then

$$\Phi_{k,p-\varepsilon}(z) = [\phi(z)]^k \left(\phi'(z)\right)^{\frac{1}{p-\varepsilon}} + \frac{1}{2\pi i}\int\limits_{C_{\rho_0}} \frac{[\phi(\zeta)]^k \left(\phi'(\zeta)\right)^{\frac{1}{p-\varepsilon}}}{\zeta - z}\mathrm{d}\zeta. \tag{1.2}$$

(3) If $z \in intC_{r_0}$, then

$$F_{k,p-\varepsilon}\left(\frac{1}{z}\right) = [\phi_1(z)]^{k-\frac{2}{p-\varepsilon}} \left(\phi'(z)\right)^{\frac{1}{p-\varepsilon}} - \frac{1}{2\pi i}\int\limits_{C_{r_0}} \frac{[\phi_1(\zeta)]^{k-\frac{2}{p-\varepsilon}} \left(\phi_1'(\zeta)\right)^{\frac{1}{p-\varepsilon}}}{\zeta - z}\mathrm{d}\zeta. \tag{1.3}$$

(4) If $z \in \text{ext}C_{r_0}$, then

$$F_{k,p-\varepsilon}\left(\frac{1}{z}\right) = -\frac{1}{2\pi i}\int\limits_{C_{r_0}} \frac{[\phi_1(\zeta)]^{k-\frac{2}{p-\varepsilon}} \left(\phi_1'(\zeta)\right)^{\frac{1}{p-\varepsilon}}}{\zeta - z}\mathrm{d}\zeta. \tag{1.4}$$

If a function $f(z)$ is analytic in the doubly connected domain bounded by the curves C_{ρ_0} and Γ_{r_0}, then the following series expansion holds:

$$f(z) = \sum_{k=0}^{\infty} a_k \Phi_{k,p-\varepsilon}(z) + \sum_{k=1}^{\infty} b_k F_{k,p-\varepsilon}\left(\frac{1}{z}\right), \tag{1.5}$$

where

$$a_k = \frac{1}{2\pi i}\int\limits_{|w|=\rho_1} \frac{f[\psi(w)]\left(\psi'(w)\right)^{\frac{1}{p-\varepsilon}}}{w^{k+1}} dw, \quad (1 < \rho_1 < \rho_0), \; k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{1}{2\pi i}\int\limits_{|w|=r_1} \frac{f[\psi_1(w)]\left(\psi_1'(w)\right)^{\frac{1}{p-\varepsilon}} w^{\frac{2}{p-\varepsilon}}}{w^{k+1}} \mathrm{d}w, \quad (1 < r_1 < r_0), \quad k = 1, 2, \ldots$$

The series (1.5) is called the $p-\varepsilon$ Faber–Laurent series of f, and the coefficients a_k and b_k are said to be the $p-\varepsilon$ Faber–Laurent coefficients of f. For $z \in G$ by Cauchy's integral formulae we have

$$f(z) = \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta - \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z}\mathrm{d}\xi.$$

If $z \in \text{int}\Gamma_2$ and $z \in \text{ext}\Gamma_1$, then

$$\frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta - \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z}\mathrm{d}\xi = 0. \tag{1.6}$$

Let us consider

$$I_1(z) := \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta, \quad I_2(z) := \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z}\mathrm{d}\xi.$$

The function $I_1(z)$ determines the functions $I_1^+(z)$ and $I_1^-(z)$, while the function $I_2(z)$ determines the functions $I_2^+(z)$ and $I_2^-(z)$. The functions $I_1^+(z)$ and $I_1^-(z)$ are analytic in $\text{int}\Gamma_1$ and $\text{ext}\Gamma_1$, respectively. The functions $I_2^+(z)$ and $I_2^-(z)$ are analytic in $\text{int}\Gamma_2$ and $\text{ext}\Gamma_2$, respectively.

Let B be a finite domain in the complex plane bounded by a rectifiable Jordan curve Γ and $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B$$

and

$$f^-(z) = \frac{1}{2\pi i} \int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B^-$$

are analytic in B and B^-respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon \to \infty} \frac{1}{2\pi i} \int\limits_{\Gamma \cap \{\zeta : |\zeta - z| > \varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the Cauchy singular integral of f at $z \in \Gamma$.

According to the Privalov theorem ([12], p. 431), if one of the functions f^+ or f^- has the non-tangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2} f(z), \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2} f(z)$$

and hence

$$f = f^+ - f^- \tag{1.7}$$

holds a.e. on Γ. From the results given in [33], it follows that if Γ is an Ahlfors 1- regular curve, then S_Γ is bounded on $L^{p),\theta}(\Gamma, \omega)$.

We will say that the doubly connected domain G is bounded by the Ahlfors 1-regular curve if the domains G_1^0 and G_2^0 are bounded by the closed Ahlfors 1-regular curves.

Let Γ_i $(i = 1, 2)$ be a regular curve and let $f_0 := f[\psi(w)]\, \psi'(w)^{\frac{1}{p-\varepsilon}}$ for $f \in L^{p),\theta}(\Gamma_1, \omega)$ and let $f_1(w) := f[\psi_1(w)]\, (\psi_1'(w))^{\frac{1}{p-\varepsilon}} w^{\frac{2}{p-\varepsilon}}$ for $f \in L^{p),\theta}(\Gamma_2, \omega)$. We also set $\omega_0(w) := \omega[\psi(w)]$, $\omega_1(w) := \omega[\psi_1(w)]$. Then , if $f \in L^{p),\theta}(\Gamma_1, \omega)$ and $f \in L^{p),\theta}(\Gamma_2, \omega)$ we obtain $f_0 \in L^{p),\theta}(\mathbb{T}, \omega_0)$ and $f_1 \in L^{p),\theta}(\mathbb{T}, \omega_1)$.

Moreover, $f_0^-(\infty) = f_1^-(\infty) = 0$ and by (1.7)

$$\left.\begin{aligned} f_0(w) &= f_0^+(w) - f_0^-(w) \\ f_1(w) &= f_1^+(w) - f_1^-(w) \end{aligned}\right\} \tag{1.8}$$

a.e. on $\mathbb{T}$.

Now, in the doubly connected domain we define the ω-weighted generalized grand Smirnov class . Let $E^1(G)$ be a classical Smirnov class of analytic functions in G. The set $E^{p),\theta}(G, \omega) := \left\{ f \in E^1(G) : f \in L^{p),\theta}(\Gamma, \omega) \right\}$ is called the ω-weighted generalized grand Smirnov class in G. We denote by $\mathcal{E}^{p),\theta}(G, \omega)$ the closure of Smirnov class $E^p(G, \omega)$ in the space $E^{p),\theta}(G, \omega)$.

Lemma 1.1 [23,24]. *Let $g \in \mathcal{E}^{p),\theta}(D, \omega)$, $\omega \in A_p(\mathbb{T})$, $1 < p < \infty$ and $\theta > 0$. If $\sum_{k=0}^{n} d_k(g) w^k$ is the nth partial sum of the Taylor series of g at the origin, then there exists a constant $c_2 > 0$ such that*

$$\left\| g(w) - \sum_{k=0}^{n} d_k(w) w^k \right\|_{L^{p),\theta}(\mathbb{T},\omega)} \le c_2 \Omega^r_{p),\theta,\omega}\left(g, \frac{1}{n}\right), \quad r \in \mathbb{N}$$

for every natural number n.

We set

$$R_n(f,z) := \sum_{k=0}^{n} a_k \Phi_{k,p-\varepsilon}(z) + \sum_{k=1}^{n} b_k F_{k,p-\varepsilon}\left(\frac{1}{z}\right).$$

The rational function $R_n(f,z)$ is called the $p-\varepsilon$ Faber–Laurent rational function of degree n of f.

Since series of Faber polynomials are a generalization of Taylor series to the case of a simply connected domain, it is natural to consider the construction of a similar generalization of Laurent series to the case of a doubly-connected domain.

The problems of approximation of the functions in the non-weighted and weighted grand Lebesgue spaces were investigated in [6–10,23,24]. In this study the approximation problems of the functions by Faber–Laurent rational functions in the weighted generalized grand Smirnov classes $\mathcal{E}^{p),\theta}(G,\omega)$, $\theta > 0$, defined in the doubly connected domains with the regular boundaries are studied. Similar problems in the different spaces were investigated by several authors (see for example, [1–5,14–23,25,27–32,38,39]).

Our main result can be formulated as following.

Theorem 1.2 *Let G be a finite doubly connected domain with the Ahlfors 1-regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. If $\omega \in A_p(\Gamma), \omega_0, \omega_1 \in A_p(\mathbb{T})$, $1 < p < \infty$ and $f \in \mathcal{E}^{p),\theta}(G,\omega)$, $\theta > 0$, then there is a constant $c_3 > 0$ such that for any $n = 1, 2, 3, \ldots$*

$$\|f - R_n(\cdot, f)\|_{L^{p),\theta}(\Gamma,\omega)} \le c_3 \left\{ \Omega^r_{p),\theta,\omega_0}(f_0, 1/n) + \Omega^r_{p),\theta,\omega_1}(f_1, 1/n) \right\},$$

where $R_n(., f)$ is the $p-\varepsilon$ Faber–Laurent rational function of degree n of f.

2 Proof of main result

Proof of Theorem 1.1 We take the curves Γ_1, Γ_2 and $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$ as the curves of integration in the formulas (1.2)–(1.5) and (1.6), respectively. (This is possible due to the conditions of Theorem 1.2). Let $f \in \mathcal{E}^{p),\theta}(G,\omega)$. Then $f_0 \in L^{p),\theta}(\mathbb{T},\omega_0)$, $f_1 \in L^{p),\theta}(\mathbb{T},\omega_1)$. According to (1.8)

$$\left. \begin{aligned} f(\zeta) &= [f_0^+(\phi(\zeta)) - f_0^-(\phi(\zeta))](\phi(\zeta))^{\frac{1}{p-\varepsilon}} \\ f(\xi) &= [f_1^+(\phi_1(\xi)) - f_1^-(\phi_1(\xi))](\phi_1(\xi))^{-\frac{2}{p-\varepsilon}}(\phi_1'(\xi))^{\frac{1}{p-\varepsilon}}. \end{aligned} \right\} \tag{2.1}$$

Let $z \in \text{ext}\Gamma_1$. Using (1.2) and (2.1) we have

$$\begin{aligned} &\sum_{k=0}^{n} a_k \Phi_{k,p}(z) \\ &= \sum_{k=0}^{n} a_k [\phi(z)]^k (\phi'(z))^{\frac{1}{p-\varepsilon}} + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p-\varepsilon}} \sum_{k=0}^{n} a_k [\phi(\zeta)]^k}{\zeta - z} d\zeta \\ &= \sum_{k=0}^{n} a_k [\phi(z)]^k (\phi'(z))^{\frac{1}{p-\varepsilon}} \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p-\varepsilon}} \sum_{k=0}^{n} a_k [\phi(\zeta)]^k - f_0^+[\phi(\zeta)]}{\zeta - z} d\zeta \\ &\quad + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - f_0^-[\phi(z)](\phi'(z))^{\frac{1}{p}}. \end{aligned} \tag{2.2}$$

For $z \in \text{ext}\Gamma_2$, the relations(1.4) and (2.1) imply that

$$\sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)$$

$$= -\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{(\phi_1'(\xi))^{\frac{1}{p-\varepsilon}}\phi_1(\xi)^{-\frac{2}{p-\varepsilon}}\sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k}{\xi - z}\mathrm{d}\xi$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\sum_{k=0}^{n} b_k\left[\phi_1(\xi)\right]^k}{\xi - z}\mathrm{d}\xi$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{((\phi_1(\xi))^{-\frac{2}{p-\varepsilon}}(\phi_1'(\xi))^{\frac{1}{p-\varepsilon}}\left[f_1^{+}(\phi_1(\xi)) - \sum_{k=0}^{n} b_k\left[\phi_1(\xi)\right]^k\right]}{\xi - z}\mathrm{d}\xi$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z}\mathrm{d}\xi. \tag{2.3}$$

For $z \in \mathrm{ext}\Gamma_1$, by virtue (2.2), (2.3) we obtain

$$\sum_{k=0}^{n} a_k\left[\Phi_k(z)\right]^k + \sum_{k=1}^{n} a_k F_k\left(\frac{1}{z}\right)$$

$$= \sum_{k=0}^{n} a_k\left[\phi(z)\right]^k(\phi'(z))^{\frac{1}{p-\varepsilon}} + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p-\varepsilon}}\sum_{k=0}^{n} a_k\left[\phi(\zeta)\right]^k - f_0^{+}\left[\phi(\zeta)\right]}{\zeta - z}\mathrm{d}\zeta$$

$$-f_0^{-}\left[\phi(z)\right] + \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{((\phi_1(\xi))^{-\frac{2}{p-\varepsilon}}(\phi_1'(\xi))^{\frac{1}{p-\varepsilon}}\left[f_1^{+}(\phi_1(\xi)) - \sum_{k=0}^{n} b_k\left[\phi_1(\xi)\right]^k\right]}{\xi - z}\mathrm{d}\xi.$$

Taking limit as $z \to z^* \in \Gamma_1$ along all non-tangential paths outside Γ_1, it appears that

$$f\left(z^*\right) - \sum_{k=0}^{n} a_k\Phi_k\left(z^*\right) - \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z^*}\right)$$

$$= f_0^{+}\left[\phi\left(z^*\right)\right] - \sum_{k=0}^{n} a_k\left[\phi\left(z^*\right)\right]^k(\phi'(z^*))^{\frac{1}{p-\varepsilon}}$$

$$+\frac{1}{2}(\phi'(z^*))^{\frac{1}{p-\varepsilon}}\left(f_0^{+}\left[\phi\left(z^*\right)\right] - \sum_{k=0}^{n} a_k\left[\phi\left(z^*\right)\right]^k\right)$$

$$+S_{\Gamma_1}\left[(\phi')^{\frac{1}{p-\varepsilon}}(f_0^{+}\circ\phi - \sum_{k=0}^{n} a_k\phi^k)\right](z^*)$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f_1^{+}\left[\phi_1(\xi)\right] - \sum_{k=1}^{n} b_k\left[\phi_1(\xi)\right]^k}{\xi - z^*}\mathrm{d}\xi \tag{2.4}$$

a.e. on Γ_1.

Now using (2.4), Minkowski's inequality and the boundedness of S_{Γ_1} in $L^{p),\theta}(\Gamma_1,\omega)$ [33] we get

$$\|f - R_n(f,z)\|_{L^{p),\theta}(\Gamma_1,\,\omega)}$$

$$\leq c_4\left\|f_0^{+}(\omega) - \sum_{k=0}^{n} a_k\omega^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_0)} + c_5\left\|f_1^{+}(w) - \sum_{k=0}^{n} b_k\omega^k\right\|_{L^{p),\theta}(\mathbb{T},\omega_1)}. \tag{2.5}$$

That is, the Faber–Laurent coefficients a_k and b_k of the function f are the Taylor coefficients of the functions f_0^{+} and f_1^{+}, respectively. Then by (2.5), Lemma 1 and [23] we obtain

$$\|f - R_n(.,f)\|_{L^{p),\theta}}(\Gamma_1,\ \omega) \leq c_6(p)\left\{\Omega^r_{p),\,\theta,\omega_0}(f_0,1/n) + \Omega^r_{p),\,\theta,\omega_1}(f_1,1/n)\right\}.$$

Let $z \in \mathrm{int}\Gamma_2$. Then from (1.3) and (2.1) we have

$$\begin{aligned}
&\sum_{k=1}^{n} b_k F_{k,p}\left(\frac{1}{z}\right)\\
&\quad=(\phi_1'(z))^{\frac{1}{p-\varepsilon}}(\phi_1(z))^{-\frac{2}{p-\varepsilon}}\sum\nolimits_{k=1}^{n} b_k[\phi_1(z)]^k\\
&\qquad-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{(\phi_1'(\zeta))^{\frac{1}{p-\varepsilon}}(\phi_1(\zeta))^{-\frac{2}{p-\varepsilon}}\sum_{k=1}^{n} b_k[\phi_1(\xi)]^k}{\xi-z}\mathrm{d}\xi\\
&\quad=(\phi_1'(z))^{\frac{1}{p-\varepsilon}}(\phi_1(z))^{-\frac{2}{p-\varepsilon}}\sum\nolimits_{k=1}^{n} b_k[\phi_1(z)]^k\\
&\qquad-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{(\phi_1'(\zeta))^{\frac{1}{p-\varepsilon}}(\phi_1(\zeta))^{-\frac{2}{p-\varepsilon}}\left(\sum_{k=1}^{n} b_k[\phi_1(\xi)]^k-f_1^{+}[\phi_1(\xi)]\right)}{\xi-z}\mathrm{d}\xi\\
&\qquad-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi-z}\mathrm{d}\xi-f_1^{-}[\phi_1(z)](\phi_1'(z))^{\frac{1}{p-\varepsilon}}(\phi_1(z))^{-\frac{2}{p-\varepsilon}}
\end{aligned}\tag{2.6}$$

For $z \in \mathrm{int}\Gamma_1$, using (1.1) and (2.1) we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} a_k\Phi_k(z)\\
&\quad=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{(\phi'(\zeta))^{\frac{1}{p}}\sum_{k=1}^{n} a_k[\phi(\zeta)]^k}{\zeta-z}\mathrm{d}\zeta.\\
&\quad=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{(\phi'(\zeta))^{\frac{1}{p}}\left(\sum_{k=1}^{n} a_k[\phi(\zeta)]^k-f_0^{+}[\phi(\zeta)]\right)}{\zeta-z}\mathrm{d}\zeta\\
&\qquad+\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta.
\end{aligned}\tag{2.7}$$

Now, by virtue of (2.6) and (2.7) for $z \in \mathrm{int}\Gamma_2$, we conclude that

$$\begin{aligned}
&\sum_{k=0}^{n} a_k\Phi_k(z)\\
&\quad+\sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right)\\
&=\frac{1}{2\pi i}\int\limits_{\Gamma_1}\frac{(\phi'(\zeta))^{\frac{1}{p-\varepsilon}}\left(\sum_{k=0}^{n} a_k[\phi(\zeta)]^k-f_0^{+}[\phi(\zeta)]\right)}{\zeta-z}\mathrm{d}\zeta\\
&\quad+(\phi_1'(z))^{\frac{1}{p-\varepsilon}}(\phi_1(z))^{-\frac{2}{p-\varepsilon}}\sum\nolimits_{k=1}^{n} b_k[\phi_1(z)]^k\\
&\quad-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{(\phi_1'(\zeta))^{\frac{1}{p-\varepsilon}}(\phi_1(\zeta))^{-\frac{2}{p-\varepsilon}}\left(\sum_{k=1}^{n} b_k[\phi_1(\xi)]^k-f_1^{+}[\phi_1(\xi)]\right)}{\xi-z}\mathrm{d}\xi\\
&\quad-f_1^{-}[\phi_1(z)](\phi_1'(z))^{\frac{1}{p-\varepsilon}}(\phi_1(z))^{-\frac{2}{p-\varepsilon}}
\end{aligned}$$

Taking the limit as $z \to z^* \in \Gamma_2$ along all non-tangential paths inside Γ_2, we reach

$$\begin{aligned} & f\left(z^*\right) - \sum_{k=0}^{n} a_k \Phi_{k,p}\left(z^*\right) - \sum_{k=1}^{n} b_k F_{k,p}\left(\frac{1}{z^*}\right) \\ &= f_1^+\left[\phi_1\left(z^*\right)\right] - \frac{1}{2}(\phi_1'(z^*))^{\frac{1}{p-\varepsilon}} (\phi_1(z^*))^{-\frac{2}{p-\varepsilon}} \left[\sum_{k=1}^{n} b_k \left[\phi_1\left(z^*\right)\right]^k - f_1^+\left[\phi_1\left(z^*\right)\right]\right] \\ & - S_{\Gamma_2}\left[(\phi_1')^{\frac{1}{p}} (\phi_1)^{-\frac{2}{p-\varepsilon}} \left(\sum\nolimits_{k=1}^{n} b_k \phi_1^k - \left(f_1^+ \circ \phi_1\right)\right)\right](z^*) \\ & - \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p-\varepsilon}} \left(\sum\nolimits_{k=0}^{n} a_k \left[\phi\left(\zeta\right)\right]^k - f_0^+\left[\phi\left(\zeta\right)\right]\right)}{\zeta - z^*} \mathrm{d}\zeta \end{aligned} \tag{2.8}$$

a.e. on Γ_2.

Using (2.8), Minkowski's inequality and the boundedness of S_{Γ_2} in $L^{p),\theta}(\Gamma_2, \omega)$ [33] we get

$$\begin{aligned} & \|f - R_n(f,z)\|_{L^{p),\theta}(\Gamma_2,\omega)} \\ & \leq_7 \left\| f_1^+(w) - \sum_{k=1}^{n} b_k w^k \right\|_{L^{p),\theta}(\mathbb{T},\omega_1)} + c_8 \left\| f_0^+(w) - \sum_{k=0}^{n} a_k w^k \right\|_{L^{p),\theta}(\mathbb{T},\omega_0)}. \end{aligned} \tag{2.9}$$

Use of (2.9), Lemma 1.1 and [23] leads to

$$\| f - R_n\left(., f\right) \|_{L^{p),\theta}(\Gamma,\omega)} \leq c_9 \left\{ \Omega^r_{p),\theta\omega_1}\left(f_1, 1/n\right) + \Omega^r_{p),\theta,\omega_0}\left(f_0, 1/n\right) \right\}.$$

The proof is complete. □

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References

1. Andrievskii, V.V.; Israfilov, D.M.: Approximations of functions on quasiconformal curves by rational functions. Izv. Akad. Nauk Azerb. SSR ser. Fiz.-Tekhn. Math. **36**(4), 416–428 (1980) **(in Russian)**
2. Andrievskii, V.V.: Jackon's approximation theorem for biharmonic functions in a multiply connected domain. East J. Approx. **6**(2), 229–239 (2000)
3. Akgün, R.; Israfilov, D.M.: Approximation by interpolating polynomials in Smirnov–Orlicz classs. J. Korean Math. Soc. **43**(2), 413–424 (2006)
4. Akgün, R.; Israfilov, D.M.: Approximation and moduli of fractional orders in Smirnov–Orlicz classes. Glasnik Matematič ki **43**(63), 121–136 (2008)
5. Cavus, A.; Israfilov, D.M.: Approximation by Faber–Laurent rational functions in the mean of functions of the class $L_p(\Gamma)$ with $1 < p < \infty$. Approx. Theory Appl. **11**(1), 105–118 (1995)
6. Danella, N.; Kokilashvili, V.: On the approximation of periodic functions with in the frame of grand Lebesgue spaces. Bull. Georgian Natl. Acad. Sci **6**(2), 11–16 (2012)
7. Danella, N.; Kokilashvili, V.: Approximation by trigonometric polynomials in subspace of weighted grand Lebesgue space. Bull. Georgian Natl. Acad. Sci. **7**(1), 11–15 (2013)
8. Danelia, N.; Kokilashvili, V.; Tsanava, T.S.: Some approximation results in subspace of weighted grand Lebesgue spaces. Proc. A. Razmadze Math. Inst. **164**, 104–108 (2014)
9. Danelia, N.; Kokilashvili, V.: Approximation of periodic functions in grand variable exponent Lebesgue spaces. Proc. A. Razmadze Math. Inst. **164**, 100–103 (2014)
10. Danelia, N.; Kokilashvili, V.: Approximation by trigonometric polynomials in the framework of variable exponent grand Lebesgue spaces. Georgian Math. J. **23**(1), 43–53 (2016)
11. D'onofrio, L.; Sbordone, C.; Schiattarella, R.: Grand Sobolev spaces and their application in geometric function theory and PDEs. J. Fixed Point Theory Appl. **13**, 309–340 (2013)
12. Goluzin, G.M.: Geometric Theory of Functions of a Complex Variable, Translation of Mathematical Monographs, 26. AMS, Providence (1968)

13. Greco, L.; Iwaniec, T.; Sbordone, C.: Inverting the p harmonic operator. Manuscr. Math. **92**, 249–258 (1997)
14. Guven, A.; Israfilov, D.M.: Improved inverse theorems in weighted Lebesgue and Smirnov spaces. Bul. Belg. Math. Soc. Simon Stevin **14**, 681–692 (2007)
15. Guven, A.; Israfilov, D.M.: Approximation in rearrangement invariant spaces on Carleson curves. East J. Approx. **12**(4), 381–395 (2006)
16. Ibragimov, I.I.; Mamedkanov, J.I.: A constructive characterization of a certain class of functions. Sov. Math. Dokl. **4**, 820–823 (1976)
17. Israfilov, D.M.: Approximate properties of the generalized Faber series in an integral metric. Izv. Akad. Nauk Az. SSR Fiz.-Tekh. Math. Nauk **2**, 10–14 (1987) **(in Russian)**
18. Israfilov, D.M.: Approximation by p-Faber polynomials in the weighted Smirnov class $E^p(G, w)$ and the Bieberbach polynomials. Constr. Approx. **17**, 335–351 (2001)
19. Israfilov, D.M.; Akgün, R.: Approximation by polynomials and rational functions in weighted rearrangement invariant spaces. J. Math. Anal. Appl. **346**, 489–500 (2008)
20. Israfilov, D.M.: Approximation by $p-$ Faber–Laurent rational functions in weighted Lebesgue spaces. Chechoslovak Math. J. **54**(129), 751–765 (2004)
21. Israfilov, D.M.; Guven, A.: Approximation in weighted Smirnov classes. East J. Approx. **11**(1), 91–102 (2005)
22. Israfilov, D.M.; Testici, A.: Approximation by Faber-Laurent rational functions in Lebesgue spaces with variable exponent. Indagationes Math. **27**, 914–922 (2016)
23. Israfilov, D.M.; Testici, A.: Approximation in weighted generalized grand Smirnov classes. Stud. Sci. Math. Hung. **54**(4), 471–488 (2017)
24. Israfilov, D.M.; Testici, A.: Approximation in weighted generalized grand Lebesgue spaces. Colloq. Math. **143**(1), 113–126 (2016)
25. Israfilov, D.M.; Oktay, B.; Akgun, R.: Approximation in Smirnov–Orlicz classes. Glasnik Matematicki **40**(6), 87–102 (2005)
26. Iwaniec, T.; Sbordone, C.: On integrability of the Jacobian under minimal hypotheses. Arch Ration. Mech. Anal. **119**, 129–143 (1992)
27. Jafarov, S.Z.: Approximation of functions by rational functions on closed curves of the complex plane. Arab. J. Sci. Eng. **36**, 1529–1534 (2011)
28. Jafarov, S.Z.: On approximation of functions by $p-$ Faber–Laurent rational functions. Complex Var. Elliptic Equ. **60**(3), 416–428 (2015)
29. Jafarov, S.Z.: Approximation of harmonic functions classes with singularities on quasiconformal curves. Taiwan. J. Math. **12**(3), 829–840 (2008)
30. Kocharyan, G.S.: On a generalization of the Laurent and Fourier series. Izv. Akad. Nauk Arm. SSr Ser Fiz.-Mat. Nauk **11**(1), 3014 (1958)
31. Kokilashvili, V.M.: On approximation of analytic functions from E_p classes. Trudy Tbiliss. Mat. Inst. im Razmadze Akad. Nauk Gruzin SSR **34**, 82–102 (1968) **(in Russian)**
32. Kokilashvili, V.M.: A direct theorem on mean approximation of analytic functions by polynomials. Sov. Math. Dokl. **10**, 411–414 (1969)
33. Kokilashvili, V.M.: Boundedness criteria for singular integrals in weighted grand Lebesgue spaces. J. Math. Sci. **170**(3), 20–33 (2010)
34. Suetin, P.K.: Series of Faber Polynomials. Gordon and Breach Science Publishers, London (1998)
35. Sbordine, C.: Grand Sobolev spaces and their applications to variational problems. Le Mathematiche LI **2**, 335–347 (1996)
36. Sbordine, C.: Nonlinear elliptic equations with right hand side in nonstandard spaces. Rend. Sem. Math. Fis. Modena **Supplemento al XLVI**, 361–368 (1998)
37. Schul, R.: Ahlfors-regular curves in metric spaces. Ann. Acad. Sci. Fenn. Math. **32**(2), 437–460 (2007)
38. Yurt, H.; Guven, A.: On rational approximation of functions in rearrangement invariant spaces. J. Class. Anal. **3**(1), 69–83 (2013)
39. Yurt, H.; Guven, A.: Approximation by Faber–Laurent rational functions on doubly connected domains. N. Z. J. Math. **44**, 113–124 (2014)