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RESEARCH ARTICLE

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The Zariski topology-graph of modules over commutative rings II

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Abstract Let M be a module over a commutative ring R. In this paper, we continue our study about the Zariski topology-graph $G(\tau_T)$ which was introduced in Ansari-Toroghy et al. (Commun Algebra 42:3283–3296, 2014). For a non-empty subset T of $\mathrm{Spec}(M)$, we obtain useful characterizations for those modules M for which $G(\tau_T)$ is a bipartite graph. Also, we prove that if $G(\tau_T)$ is a tree, then $G(\tau_T)$ is a star graph. Moreover, we study coloring of Zariski topology-graphs and investigate the interplay between $\chi(G(\tau_T))$ and $\omega(G(\tau_T))$.

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1 Introduction

Throughout this paper, R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp. $N < M$) we mean that N is a submodule (resp. proper submodule) of M.

Define $(N :_R M)$ or simply $(N : M) = \{r \in R | \; rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $\mathrm{Ann}_R(M)$ or simply $\mathrm{Ann}(M)$. M is said to be faithful if $\mathrm{Ann}(M) = (0)$.

Let $N, K \leq M$. Then the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$ (see [3]).

A prime submodule of M is a submodule $P \neq M$ such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [10].

The prime spectrum of M is the set of all prime submodules of M and denoted by $\mathrm{Spec}(M)$.

If N is a submodule of M, then $V(N) = \{P \in \mathrm{Spec}(M) | \; (P : M) \supseteq (N : M)\}$ [11].

The Zariski topology on $X = \mathrm{Spec}(M)$ is the topology τ_M described by taking the set $Z(M) = \{V(N) |$ N is a submodule of $M\}$ as the set of closed sets of $\mathrm{Spec}(M)$ [11].

A topological space X is irreducible if for any decomposition $X = X_1 \cup X_2$ with closed subsets X_i of X with $i = 1, 2$, we have $X = X_1$ or $X = X_2$.

There are many papers on assigning graphs to rings or modules (see, for example, [1,5,6,9]). In [4], the present authors introduced and studied the graph $G(\tau_T)$ and $AG(M)$, called the Zariski topology-graph and the annihilating-submodule graph, respectively.

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Let T be a non-empty subset of Spec(M). The Zariski topology-graph $G(\tau_T)$ is an undirected graph with vertices $V(G(\tau_T))=\{N < M|$ there exists $K < M$ such that $V(N) \cup V(K) = T$ and $V(N), V(K) \neq T\}$ and distinct vertices N and L are adjacent if and only if $V(N) \cup V(L) = T$ (see [4, Definition 2.3]).

$AG(M)$ is an undirected graph with vertices $V(AG(M))=\{N \leq M|$ there exists $(0) \neq K < M$ with $NK = (0)\}$. In this graph, distinct vertices $N, L \in V(AG(M))$ are adjacent if and only if $NL = (0)$. Let $AG(M)^*$ be the subgraph of $AG(M)$ with vertices $V(AG(M)^*) = \{N < M$ with $(N : M) \neq \text{Ann}(M)|$ there exists a submodule $K < M$ with $(K : M) \neq \text{Ann}(M)$ and $NK = (0)\}$. By [4, Theorem 3.4], one conclude that $AG(M)^*$ is a connected subgraph.

If Spec$(M) \neq \emptyset$, the mapping ψ : Spec$(M) \to$ Spec$(R/\text{Ann}(M))$ such that $\psi(P) = (P : M)/\text{Ann}(M)$ for every $P \in$ Spec(M), is called the natural map of Spec(M) [11].

The prime radical $\sqrt{N}$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $\sqrt{N}$ is defined to be M [10].

We recall that $N < M$ is said to be a semiprime submodule of M if for every ideal I of R and every submodule K of M with $I^2K \subseteq N$ implies that $IK \subseteq N$. Further M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [17]).

The notations Nil(R), Min(M), and Min(T) will denote the set of all nilpotent elements of R and the set of all minimal prime submodules of M, and the set of minimal members of T, respectively.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $\omega(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq \omega(G)$.

In this article, we continue our studying about $G(\tau_T)$ and $AG(M)$ and we try to relate the combinatorial properties of the above mentioned graphs to the algebraic properties of M.

In Sect. 2 of this paper, we state some properties related to the Zariski topology-graph that are basic or needed in the later sections. In Sect. 3, we study the bipartite Zariski topology-graphs of modules over commutative rings (see Proposition 3.1). Also, we prove that if $G(\tau_T)$ is a tree, then $G(\tau_T)$ is a star graph (see Theorem 3.5). In Sect. 4, we study coloring of the Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau_T))$ and $\omega(G(\tau_T))$. We show that under condition over minimal submodules of $M/(\cap_{P \in T} P : M)M$, we have $\omega(G(\tau_T)) = \chi(G(\tau_T))$ (see Theorem 4.1). Moreover, we investigate some relations between the existence of cycles in the Zariski topology-graph of a cyclic module and the number of its minimal members of T (see Proposition 4.10).

Let us introduce some graphical notions and denotations that are used in what follows: A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. A path in a graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i.

A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. For some $U \subseteq V(G)$, we denote by $N(U)$, the set of all vertices of $G \backslash U$ adjacent to at least one vertex of U. For every vertex $v \in V(G)$, the size of $N(v)$ is denoted by deg(v). If all the vertices of G have the same degree k, then G is called k-regular, or simply regular. We denote by C_n a cycle of order n. Let G and G' be two graphs. A graph homomorphism from G to G' is a mapping $\phi : V(G) \longrightarrow V(G')$ such that for every edge $\{u, v\}$ of G, $\{\phi(u), \phi(v)\}$ is an edge of G'. A retract of G is a subgraph H of G such that there exists a homomorphism $\phi : G \longrightarrow H$ such that $\phi(x) = x$, for every vertex x of H. The homomorphism ϕ is called the retract (graph) homomorphism (see [13]).

Throughout the rest of this paper, we denote by T a non-empty subset of Spec(M), $F := \cap_{P \in T} P$, $Q := (F : M)M$, $\bar{M} := M/Q$, $\bar{N} := N/Q$, $\bar{m} := m + Q$, and $\bar{I} := I/(Q : M)$, where N is a submodule of M containing Q, $m \in M$, and I is an ideal of R containing $(Q : M)$.

2 Auxiliary results

In this section, we provide some properties related to the Zariski topology-graph that are basic or needed in the sequel.

Remark 2.1 Let N be a submodule of M. Set $V^*(N) := \{P \in \mathrm{Spec}(M) | \ P \supseteq N\}$. By [4, Remark 2.2], for submodules N and K of M, we have

$$V(N) \cup V(K) = V(N \cap K) = V(NK) = V^*(NK).$$

By [4, Remark 2.5], we have T is a closed subset of $\mathrm{Spec}(M)$ if and only if $T = V(F)$ and $G(\tau_T) \neq \emptyset$ if and only if $T = V(F)$ and T is not irreducible. So if N and K are adjacent in $G(\tau_T)$, then $V^*(NK) = V^*(Q)$ and hence $\sqrt{NK} = F$. Therefore, $F \subseteq \sqrt{(N : M)M}$ and $F \subseteq \sqrt{(K : M)M}$.

Lemma 2.2 (See [2, Proposition 7.6]) *Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of R. Then the following statements are equivalent:*

(a) *$R = R_1 \oplus \ldots \oplus R_n$;*
(b) *As an abelian group R is the direct sum of $R_1, \ldots, R_n$;*
(c) *There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \ldots + e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.*

Proposition 2.3 *Suppose that e is an idempotent element of R. We have the following statements.*

(a) *$R = R_1 \oplus R_2$, where $R_1 = eR$ and $R_2 = (1 - e)R$.*
(b) *$M = M_1 \oplus M_2$, where $M_1 = eM$ and $M_2 = (1 - e)M$.*
(c) *For every submodule N of M, $N = N_1 \oplus N_2$ such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and $(N :_R M) = (N_1 :_{R_1} M_1) \oplus (N_2 :_{R_2} M_2)$.*
(d) *For submodules N and K of M, $NK = N_1K_1 \oplus N_2K_2$, $N \cap K = N_1 \cap K_1 \oplus N_2 \cap K_2$ such that $N = N_1 \oplus N_2$ and $K = K_1 \oplus K_2$.*
(e) *Prime submodules of M are $P \oplus M_2$ and $M_1 \oplus Q$, where P and Q are prime submodules of M_1 and M_2, respectively.*
(f) *For submodule N of M, we have $\sqrt{N} = \sqrt{N_1 \oplus N_2} = \sqrt{N_1} \oplus \sqrt{N_2}$, where $N = N_1 \oplus N_2$.*

Proof This is clear. □

An ideal $I < R$ is said to be nil if I consist of nilpotent elements.

Lemma 2.4 (See [15, Theorem 21.28]) *Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I. Then there exists an idempotent e in uR such that $e - u \in I$.*

Lemma 2.5 (See [5, Lemma 2.4]) *Let N be a minimal submodule of M and let* Ann(M) *be a nil ideal. Then we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.*

We note that M is said to be primeful if either $M = (0)$ or $M \neq (0)$ and the natural map of $\mathrm{Spec}(M)$ is surjective (see [12]).

Proposition 2.6 *We have the following statements.*

(a) *If N, L are adjacent in $G(\tau_T)$, then $\sqrt{(N : M)M}/F$ and $\sqrt{(L : M)M}/F$ are adjacent in $AG(M/F)$.*
(b) *If M is a primeful module and N, L are adjacent in $G(\tau_T)$, then $\sqrt{N}/F$ and $\sqrt{L}/F$ are adjacent in $AG(M/F)$.*

Proof (a) First, we see easily that for any submodule N of M, $V(N) = V(\sqrt{(N : M)M})$. Suppose that N and L are adjacent in $G(\tau_T)$ so that $V(N) \cup V(L) = T$. Then we have $V^*(\sqrt{(N : M)M}\sqrt{(L : M)M}) = T$. It follows that $\sqrt{(N : M)M}\sqrt{(L : M)M} \subseteq F$ (see Remark 2.1). Also by Remark 2.1, $F \subseteq \sqrt{(N : M)M}$ and $F \subseteq \sqrt{(L : M)M}$. Therefore, $\sqrt{(N : M)M}/F$ and $\sqrt{(L : M)M}/F$ are adjacent in $AG(M/F)$.
(b) This is clear by [4, Corollary 4.5]. □

Remark 2.7 The Proposition 2.6(a) extends [4, Theorem 4.4].

Lemma 2.8 *Assume that T is a closed subset of* Spec(M)*. Then $AG(\bar{M})^*$ is isomorphic with a subgraph of $G(\tau_T)$. In particular, $AG(M/F)^*$ is isomorphic with an induced subgraph of $G(\tau_T)$.*

Proof Let $\bar{N} \in V(AG(\bar{M})^*)$. Then there exists a nonzero submodule $\bar{K}$ of $\bar{M}$ such that it is adjacent to $\bar{N}$ (if $N = K$, then $(N : M) = (Q : M)$, a contradiction). So we have $NK \subseteq Q$. Hence $V(NK) \supseteq T$. Since $Q \subseteq N$ and $Q \subseteq K$, then $V(N) \subseteq T$ and $V(K) \subseteq T$. Therefore $V(NK) = T$ (if $V(N) = T$, then $(N : M) = (Q : M)$, a contradiction). Hence N is a vertex in $G(\tau_T)$ which is adjacent to K. To see the last assertion, let N/F and K/F be two vertices of $AG(M/F)^*$. If N and K are adjacent in $G(\tau_T)$, then by Proposition 2.6, $\sqrt{(N : M)M}/F$ and $\sqrt{(K : M)M}/F$ are adjacent in $AG(M/F)$. So

$$\sqrt{(N : M)M}\sqrt{(K : M)M} \subseteq F.$$

Since

$$NK = ((N : M)M : M)((K : M)M : M)M \subseteq \sqrt{(N : M)M}\sqrt{(K : M)M},$$

we have N/F and K/F are adjacent in $AG(M/F)^*$, as desired. □

Lemma 2.9 *If $\bar{M}$ is a faithful module and T is a closed subset of* $\mathrm{Spec}(M)$*, then $G(\tau_{\mathrm{Spec}(M)})$ and $AG(M)^*$ are the same.*

Proof $\bar{M}$ is a faithful module and T is a closed subset of $\mathrm{Spec}(M)$ so that $T = \mathrm{Spec}(M)$. If $G(\tau_{\mathrm{Spec}(M)}) \neq \emptyset$, then there exist non-trivial submodules N and K of M which are adjacent in $G(\tau_{\mathrm{Spec}(M)})$. Hence $V(NK) = \mathrm{Spec}(M)$ which implies that $NK = (0)$ so that $AG(M)^* \neq \emptyset$. By Lemma 2.8, $AG(M)^*$ is isomorphic with a subgraph of $G(\tau_{\mathrm{Spec}(M)})$. One can see that the vertex map $\phi : V(G(\tau_{\mathrm{Spec}(M)})) \longrightarrow V(AG(M)^*)$, defined by $N \longrightarrow N$ is an isomorphism. □

Recall that $\Delta(G(\tau_T))$ is the maximum degree of $G(\tau_T)$ and the length of an R-module M, is denoted by $l_R(M)$.

Lemma 2.10 *Let every nontrivial submodule of M be a vertex in $G(\tau_T)$. If $\Delta(G(\tau_T)) < \infty$, then $l_R(M) \leq \Delta(G(\tau_T)) + 1$. Also, every non-trivial submodule of M has finitely many submodules.*

Proof First, we show that the descending chain of non-trivial submodules $K_1 \supsetneq K_2 \supsetneq K_3 \supsetneq \ldots$ terminates. Since $G(\tau_T)$ is connected, there exists a submodule N such that $V(N) \cup V(K_1) = T$. Hence for each i, $i \geq 1$, $V(N) \cup V(K_i) = T$ and so $\deg(N) = \infty$, a contradiction. Next, let $N_1 \subsetneq N_2 \subsetneq N_3 \subsetneq \ldots$ be an ascending chain of non-trivial submodules of M. Since $G(\tau_T)$ is connected, there exists a submodule K such that $V(K) \cup V(N_{\Delta+1}) = T$, where $\Delta = \Delta(G(\tau_T))$. Hence $V(K) \cup V(N_i) = T$ for each $1 \leq i \leq \Delta + 1$. Thus $\deg(K) \geq \Delta + 1$, a contradiction. It follows that $l_R(M) \leq \Delta + 1$. For the proof of the last assertion, let N be a non-trivial submodule of M. Since $G(\tau_T)$ is connected, there exists a submodule K such that $V(N) \cup V(K) = T$. Hence for every submodule N' of N, $V(N') \cup V(K) = T$. As $\Delta < \infty$, the number of submodules of N should be finite. □

Theorem 2.11 *Suppose that $\bar{M}$ is a multiplication module and $G(\tau_T) \neq \emptyset$. If $G(\tau_T)$ has acc (resp. dcc) on vertices, then $\bar{M}$ is a Noetherian (resp. an Artinian) module.*

Proof Suppose that $G(\tau_T)$ has acc (resp. dcc) on vertices. By [4, Remark 2.6], F is not a prime submodule of M and hence there exist $r \in R$ and $m \in M$ such that $rm \in F$ but $m \notin F$ and $r \notin (F : M)$. Now $\overline{rM} \cong \bar{M}/(\bar{0} :_{\bar{M}} r)$. Further, $\overline{rM}$ and $(\bar{0} :_{\bar{M}} r)$ are vertices in $AG(\bar{M}) = AG(\bar{M})^*$ ($\bar{M}$ is a multiplication module) because $(\bar{0} :_{\bar{M}} r)(\overline{rM}) = ((\bar{0} :_{\bar{M}} r) : \bar{M})(\overline{rM} : \bar{M})\bar{M} \subseteq \overline{rM}((\bar{0} :_{\bar{M}} r) : \bar{M}) \subseteq r(\bar{0} :_{\bar{M}} r) = \bar{0}$. Then by Lemma 2.8, $\{N \mid \bar{N} \leq \bar{M}, \bar{N} \subseteq \overline{rM}\} \cup \{N \mid \bar{N} \leq \bar{M}, \bar{N} \subseteq (\bar{0} :_{\bar{M}} r)\} \subseteq V(G(\tau_T))$. It follows that the R-modules $\overline{rM}$ and $(\bar{0} :_{\bar{M}} r)$ have acc (resp. dcc) on submodules. Since $\overline{rM} \cong \bar{M}/(\bar{0} :_{\bar{M}} r)$, $\bar{M}$ has acc on submodules and the proof is completed. □

3 Zariski topology-graph of modules

First, in this section we give the more notation to be used throughout the remainder of this article. Suppose that e $(e \neq 0, 1)$ is an idempotent element of R. Let $M_1 := eM$, $M_2 := (1-e)M$, $T_1 := \{P_1 \in \mathrm{Spec}(M_1) | P_1 \oplus M_2 \in T\}$, $T_2 := \{P_2 \in \mathrm{Spec}(M_2) | M_1 \oplus P_2 \in T\}$, $F_1 := \cap_{P_1 \in T_1} P_1$, $F_2 := \cap_{P_2 \in T_2} P_2$, $Q_1 := (F_1 : M_1)M_1$, $Q_2 := (F_2 : M_2)M_2$, $\bar{M}_1 = \overline{eM} = eM/Q_1$, and $\bar{M}_2 = \overline{(e-1)M} = (e-1)M/Q_2$. Consequently, we have, $Q = Q_1 \oplus Q_2$, where $Q = (\cap_{P \in T} P : M)M$ and $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$

We recall that a submodule N of M is a prime R-module if and only if it is a prime $R/\mathrm{Ann}(M)$-module (see [16, Result 1.2]).

Proposition 3.1 *Suppose that $\bar{M}$ does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$ and $\mathrm{Ann}(\bar{M})$ is a nil ideal. Then the following statements hold.*

(a) *If there exists a vertex of $G(\tau_T)$ which is adjacent to every other vertex, then $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module for some idempotent element $e \in R$.*
(b) *If $\bar{M}_1$ and $\bar{M}_2$ are prime modules for some idempotent element $e \in R$, then $G(\tau_T)$ is a complete bipartite graph.*

Proof • (a) Suppose that N is adjacent to every other vertex of $G(\tau_T)$. Since $V(N) = V((N : M)M)$, we have $\bar{N} = \overline{(N : M)M}$ and hence $V(\bar{N}) = V^*(N)$. Thus, $N = \sqrt{N}$ because $V(N) = V(\sqrt{N})$. We claim that $\bar{N}$ is a minimal submodule of $\bar{M}$. Let $\bar{Q} \subsetneq \bar{K} \subsetneq \bar{N}$. If $V(K) \neq T$, then K is adjacent to N and hence $V(K) = T$, a contradiction. So $\bar{N}$ is a minimal submodule of $\bar{M}$. We have $(\bar{N})^2 \neq (\bar{0})$ because $V(N) \neq T$. Then Lemma 2.5, implies that $\bar{M} \cong \overline{eM} \oplus \overline{(e-1)M}$ for some idempotent element e of R. Without loss of generality we may assume that $M_1 \oplus Q_2$ is adjacent to every other vertex. Since $V(F_1 \oplus Q_2) = V(Q_1 \oplus F_2) = T$, the assumption of theorem implies that $F = Q$. We claim that $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module. Let $Q_1 \subsetneq K < M_1$. We have $V(K \oplus Q_2) \neq T$ because $Q_1 \oplus Q_2 \subsetneq K \oplus Q_2$. Since $V(K \oplus Q_2) \cup V(Q_1 \oplus M_2) = T$, we have $K \oplus Q_2$ is a vertex and hence is adjacent to $M_1 \oplus Q_2$. Therefore $V(K \oplus Q_2) \cup V(M_1 \oplus Q_2) = V(K \oplus Q_2) = T$, a contradiction. It implies that $\bar{M}_1$ is a simple module. Now, we show that $\bar{M}_2$ is a prime module. It is enough to show that it is a prime $R/(Q_2 : M_2)$-module. Otherwise, $\bar{I}\bar{K} = (\bar{0})$, where $(Q_2 : M_2) \subsetneq I < R$ and $Q_2 \subsetneq K < M_2$. It follows that $V(M_1 \oplus K) \cup V(Q_1 \oplus IM_2) = V(Q_1 \oplus K(IM_2)) = T$ because $K(IM_2) \subseteq IK \subseteq Q_2$ (note that $(Q_2 : M_2) \subseteq (K : M_2)$ and $(Q_2 : M_2) \subseteq I$). Therefore, $M_1 \oplus K$ is a vertex and hence is adjacent to $M_1 \oplus Q_2$. So $V(M_1 \oplus K) \cup V(M_1 \oplus Q_2) = T = V(M_1 \oplus Q_2)$, a contradiction (note that $M_1 \oplus K$ is properly containing $Q_1 \oplus Q_2$).

• (b) Assume that $N_1 \oplus N_2$ is adjacent to $K_1 \oplus K_2$. One can see that $\sqrt{N_1 K_1} \oplus \sqrt{N_2 K_2} = \sqrt{Q_1} \oplus \sqrt{Q_2}$. It implies that $\overline{(\sqrt{(K_1 : M_1)M_1} : M_1)}\,\overline{\sqrt{(N_1 : M_1)M_1}} = (\bar{0})$ and $\overline{(\sqrt{(K_2 : M_2)M_2} : M_2)}\,\overline{\sqrt{(N_2 : M_2)M_2}} = (\bar{0})$. Since $\bar{M}_1$ and $\bar{M}_2$ are prime modules, $(\sqrt{(K_1 : M_1)M_1} : M_1) = (Q_1 : M_1)$ or $\sqrt{(N_1 : M_1)M_1} = Q_1$ and $(\sqrt{(K_2 : M_2)M_2} : M_2) = (Q_2 : M_2)$ or $\sqrt{(N_2 : M_2)M_2} = Q_2$. Therefore $G(\tau_T)$ is a complete bipartite graph with two parts U and V such that $N \in U$ if and only if $V(N) = V(M_1 \oplus Q_2)$ and $K \in V$ if and only if $V(K) = V(Q_1 \oplus M_2)$.

□

Corollary 3.2 *Let $\bar{M}$ be a faithful module which does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$. Then the following statements are equivalent.*

(a) *There is a vertex of $G(\tau_{\mathrm{Spec}(M)})$ which is adjacent to every other vertex of $G(\tau_{\mathrm{Spec}(M)})$.*
(b) *$G(\tau_{\mathrm{Spec}(M)})$ is a star graph.*
(c) *$M = F \oplus D$, where F is a simple module and D is a prime module.*

Proof (a) ⇒ (b) Let $\bar{M}$ be a faithful module. Then $Q = (0)$ and we have $T = \mathrm{Spec}(M)$. By Proposition 3.1, $M = M_1 \oplus M_2$, where M_1 is a simple module and M_2 is a prime module. Then every non-zero submodule of M is of the form $M_1 \oplus N_2$ and $(0) \oplus N_2$, where N_2 is a non-zero submodule of M_2. We show that non of the submodules of the form $(0) \oplus N_2$ can be adjacent to each other. Assume that $(0) \oplus N_2$ and $(0) \oplus K_2$ are adjacent in $G(\tau_{\mathrm{Spec}(M)})$, where $(0) \neq N_2 \leq M_2$ and $(0) \neq K_2 \leq M_2$. Since (0) is a prime submodule of M_2, by Remark 2.1, we have $N_2 K_2 = (0)$. Hence $V((0) \oplus N_2) = \mathrm{Spec}(M)$ or $V((0) \oplus K_2) = \mathrm{Spec}(M)$, a contradiction. Similarly, we can not have any vertex of the form $M_1 \oplus N_2$, where N_2 is a non-zero proper submodule of M_2. Now it is easy to see that $M_1 \oplus (0)$ is adjacent to every other vertex and so $G(\tau_{\mathrm{Spec}(M)})$ is a star graph.

(b) ⇒ (c) This follows by Proposition 3.1(a).

(c) ⇒ (a) Assume that $M = F \oplus D$, where F is a simple module and D is a prime module. Using the Proposition 3.1 (b), $G(\tau_{\mathrm{Spec}(M)})$ is a complete bipartite graph with two parts U and V such that $N \in U$ if and only if $V(N) = V(F \oplus (0))$ and $K \in V$ if and only if $V(K) = V((0) \oplus D)$. We claim that $|U| = 1$. Otherwise, $V(F \oplus (0)) = V(N \oplus K)$, where $N = (0)$ or $N = F$ and $(0) \neq K < D$. Therefore $V(N \oplus K) \cup V((0) \oplus D) = \mathrm{Spec}(M)$ and hence $V((0) \oplus K) = \mathrm{Spec}(M)$ that is a contradiction with our assumption. So $F \oplus (0)$ is adjacent to every other vertex of $G(\tau_{\mathrm{Spec}(M)})$ □

Lemma 3.3 *Let $e \in R$ be an idempotent element of R and suppose that $\bar{M}$ does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$. If $G(\tau_T)$ is a triangle-free graph, then both $\bar{M}_1$ and $\bar{M}_2$ are prime R-modules. Moreover, if $G(\tau_T)$ has no cycle, then $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module.*

Proof First recall that if $\bar{M}$ does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$, then $F = Q$ because $V(F_1 \oplus Q_2) = V(Q_1 \oplus F_2) = T$. Without loss of generality, we can assume that $\bar{M}_1$ is not a prime module. Then $\bar{I}\bar{K} = (\bar{0})$, where $(Q_1 : M_1) \subsetneq I < R$ and $Q_1 \subsetneq K < M_1$. It follows that $Q_1 \oplus M_2$, $K \oplus Q_2$, and $IM_1 \oplus Q_2$ form a triangle in $G(\tau_T)$, a contradiction (note that $V(K \oplus Q_2) \cup V(IM_1 \oplus Q_2) = V(K(IM_1) \oplus Q_2) = T$. Also $IM_1 \neq K$. Otherwise, $V(K \oplus Q_2) = V(K^2 \oplus Q_2) = V(K(IM_1) \oplus Q_2) = T$, a contradiction). So both $\bar{M}_1$ and $\bar{M}_2$ are prime R-modules. Now suppose that $G(\tau_T)$ has no cycle. If none of $\bar{M}_1$ and $\bar{M}_2$ is a simple module, then we choose non-trivial submodules N_i in M_i for some $i = 1, 2$. So $N_1 \oplus Q_2$, $Q_1 \oplus N_2$, $M_1 \oplus Q_2$, and $Q_1 \oplus M_2$ form a cycle, a contradiction. □

Corollary 3.4 *Assume that M is a multiplication module or a primeful module,* $\mathrm{Ann}(\bar{M})$ *is a nil ideal, and $\bar{M}$ does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$. Then $G(\tau_T)$ is a star graph if and only if $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime module for some idempotent $e \in R$.*

Proof The necessity is clear by Proposition 3.1(a). For the converse, assume that $\bar{M} = \bar{M}_1 \oplus \bar{M}_2$, where $\bar{M}_1$ is a simple module and $\bar{M}_2$ is a prime for some idempotent $e \in R$. Using the Proposition 3.1(b), $G(\tau_T)$ is a complete bipartite graph with two parts U and V such that $N \in U$ if and only if $V(N) = V(M_1 \oplus Q_2)$ and $K \in V$ if and only if $V(K) = V(Q_1 \oplus M_2)$. We claim that $|U| = 1$. Otherwise, $V(M_1 \oplus Q_2) = V(N_1 \oplus N_2)$, where $N_1 \leq M_1$ and $N_2 \leq M_2$. If $N_1 \neq M_1$, then $\sqrt{(N_1 : M_1)M_1} = M_1$, a contradiction (note that if M is a multiplication module or a primeful module, then $\sqrt{(N : M)M} \neq M$, where $N < M$). If $N_2 \neq Q_2$, then $V(Q_1 \oplus N_2) = T$, a contradiction. So $G(\tau_T)$ is a star graph. □

Theorem 3.5 *If $G(\tau_T)$ is a tree, then $G(\tau_T)$ is a star graph.*

Proof Suppose that $G(\tau_T)$ is not a star graph. Then $G(\tau_T)$ has at least four vertices. Obviously, there are two adjacent vertices L and K of $G(\tau_T)$ such that $|N(L)\backslash\{K\}| \geq 1$ and $|N(K)\backslash\{L\}| \geq 1$. Let $N(L)\backslash\{K\} = \{L_i\}_{i\in\Lambda}$ and $N(K)\backslash\{L\} = \{K_j\}_{j\in\Gamma}$. Since $G(\tau_T)$ is a tree, we have $N(L) \cap N(K) = \emptyset$. By [4, Theorem 2.10], $diam(G(\tau_T)) \leq 3$. So every edge of $G(\tau_T)$ is of the form $\{L, K\}$, $\{L, L_i\}$ or $\{K, K_j\}$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, Pick $p \in \Lambda$ and $q \in \Gamma$. Since $G(\tau_T)$ is a tree, L_pK_q is a vertex of $G(\tau_T)$. If $L_pK_q = L_u$ for some $u \in \Lambda$, then $V(KL_u) = T$, a contradiction. If $L_pK_q = K_v$, for some $v \in \Gamma$, then $V(LK_v) = T$, a contradiction. If $L_pK_q = L$ or $L_pK_q = K$, then $V(L^2) = T$ or $V(K^2) = T$, respectively, and hence $V(L) = T$ or $V(K) = T$, a contradiction. So the claim is proved. □

Theorem 3.6 *Let R be an Artinian ring, M be a multiplication or a primeful module, and suppose that $\bar{M}$ does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$. If $G(\tau_T)$ is a bipartite graph, then $|T| = 2$ and $G(\tau_T) \cong K_2$.*

Proof At first we recall that if $G(\tau_T) \neq \emptyset$, then $|E(G(\tau_T))| \geq 1$. Assume that $G(\tau_T)$ is a bipartite graph. Therefore $G(\tau_T)$ is not empty. We show that R can not be a local ring. Otherwise, m is the unique maximal ideal of R and hence is the unique prime ideal. Then [14, Corollary 2.11] implies that mM is the only prime submodule of M so that $G(\tau_T) = \emptyset$, a contradiction. Hence by [8, Theorem 8.7], $R = R_1 \oplus \ldots \oplus R_n$, where R_i is an Artinian local ring for $i = 1, \ldots, n$ and $n \geq 2$. By Lemma 2.2 and Proposition 2.3, since $G(\tau_T)$ is a bipartite graph, we have $n = 2$ and hence $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$ for some idempotent $e \in R$ (for example, if $n = 3$, then $M_1 \oplus Q_2 \oplus Q_3$, $Q_1 \oplus M_2 \oplus Q_3$, and $Q_1 \oplus Q_2 \oplus M_3$ form a triangle that is a contradiction). By Lemma 3.3, $\bar{M}_1$ and $\bar{M}_2$ are prime modules. Then it is easy to see that $\bar{M}_1$ and $\bar{M}_2$ are vector spaces over $R/\mathrm{Ann}(\bar{M}_1)$ and $R/\mathrm{Ann}(\bar{M}_2)$, respectively and so are semisimple R-modules. Since $G(\tau_T)$ is a bipartite graph, $\bar{M}_1$ and $\bar{M}_2$ are simple R-modules. A Similar argument as we did in proof of Corollary 3.4 implies that $T = \{M_1 \oplus Q_2, Q_1 \oplus M_2\}$ and $G(\tau_T) \cong K_2$. □

Proposition 3.7 *Assume that M is a multiplication module,* $\mathrm{Ann}(\bar{M})$ *is a nil ideal, and $\bar{M}$ does not have a non-zero submodule $\overline{F} \neq \bar{N}$ with $V(N) = T$.*

(a) *If $G(\tau_T)$ is a finite bipartite graph, then $|T| = 2$ and $G(\tau_T) \cong K_2$.*
(b) *If $G(\tau_T)$ is a regular graph of finite degree, then $|T| = 2$ and $G(\tau_T) \cong K_2$.*

Proof (a) By Theorem 2.11, $\bar{M}$ is an Artinian and Noetherian module so that $R/\mathrm{Ann}(\bar{M})$ is an Artinian ring. A similar arguments in Theorem 3.6 says that, $R/\mathrm{Ann}(\bar{M})$ is a non-local ring. So by [8, Theorem 8.7] and Lemma 2.2, there exist pairwise orthogonal idempotents modulo $\mathrm{Ann}(\bar{M})$. By lemma 2.4, $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$, for some idempotent e of R. Now, the proof that $G(\tau_T) \cong K_2$ is similar to the proof of Theorem 3.6.
(b) We may assume that $G(\tau_T)$ is not empty. So F is not a prime submodule by [4, Remark 2.6] and hence there exist $r \in R$ and $m \in M$ such that $rm \in F$ but $m \notin F$ and $r \notin (F : M)$. A similar manner in proof of

Theorem 2.11, shows that if the set of R-submodules of $\overline{rM}$ (resp. $(\bar{0} :_{\bar{M}} r)$ is infinite, then $(\bar{0} :_{\bar{M}} r)$ (resp. $\overline{rM}$) has infinite degree, a contradiction. Thus $\overline{rM}$ and $(\bar{0} :_{\bar{M}} r)$ have finite length so that $\bar{M}$ has a finite length. Therefore $R/\text{Ann}(\bar{M})$ is an Artinian ring. As in the proof of part (a), $\bar{M} \cong \bar{M}_1 \oplus \bar{M}_2$ for some idempotent $e \in R$. If $\bar{M}_1$ has one non-trivial submodule $\bar{N}$, then $\deg(Q_1 \oplus M_2) > \deg(N \oplus M_2)$ (we note that by [6, Proposition 2.5], $\bar{N}\bar{K} = (\bar{0})$ for some $(\bar{0}) \neq \bar{K} < \bar{M}_1$) and this contradicts the regularity of $G(\tau_T)$. Hence $\bar{M}_1$ is a simple module. Similarly, $\bar{M}_2$ is a simple module. Finally a similar argument as we have seen in Theorem 3.6 gives $G(\tau_T) \cong K_2$.

□

4 Coloring of the Zariski-topology graph of modules

The purpose of this section is to study the coloring of the Zariski topology-graph of modules and investigate the interplay between $\chi(G(\tau_T))$ and $\omega(G(\tau_T))$. We note that since $E(G(\tau_T)) \geq 1$ when $G(\tau_T) \neq \emptyset$, then $\chi(G(\tau_T))) \geq 2$.

Theorem 4.1 *Let $\bar{M}$ be an Artinian module such that for every minimal submodule $\bar{N}$ of $\bar{M}$, N is a vertex in $G(\tau_T)$. Then $\omega(G(\tau_T)) = \chi(G(\tau_T))$.*

Proof $\bar{M}$ is Artinian, so it contains a minimal submodule. Since for every minimal submodule $\bar{N}$ of $\bar{M}$, N is a vertex in $G(\tau_T)$, we have $V(N) \neq T$. Also, $N \cap L = Q$, where $\bar{N}$ and $\bar{L}$ are minimal submodules of $\bar{M}$. It follows that N and L are adjacent in $G(\tau_T)$, where $\bar{N}$ and $\bar{L}$ are minimal submodules of $\bar{M}$. First, suppose that $\bar{M}$ has infinitely many minimal submodules. Then $\omega(G(\tau_T)) = \infty$ and there is nothing to prove. Next, assume that $\bar{M}$ has k minimal submodules, where k is finite. We conclude that $\chi(G(\tau_T)) = k = \omega(G(\tau_T))$. Obviously, $\omega(G(\tau_T)) \geq k$. If possible, assume that $\omega(G(\tau_T)) > k$. Let $\Sigma = \{N_\lambda\}_{\lambda \in I}$, where $|I| = \omega(G(\tau_T))$ be a maximum clique in $G(\tau_T)$. As for every $N_\lambda \in \Sigma$, $\overline{\sqrt{(N_\lambda : M)M}}$ contains a minimal submodule, there exists a minimal submodule $\bar{K}$ and submodules N_i and N_j in Σ, such that $\bar{K} \subset \overline{\sqrt{(N_i : M)M}} \cap \overline{\sqrt{(N_j : M)M}}$, and hence $V(K) = T$, a contradiction. Hence $\omega(G(\tau_T)) = k$. Next, we claim that $G(\tau_T)$ is k-colorable. In order to prove, put $A = \{\bar{K}_1, \ldots, \bar{K}_k\}$ be the set of all minimal submodules of $\bar{M}$. Now, we define a coloring f on $G(\tau_T)$ by setting $f(N) = min\{i | \ K_i \subseteq \overline{\sqrt{(N : M)M}}\}$ for every vertex N of $G(\tau_T)$. Let N and L be adjacent in $G(\tau_T)$ and $f(N) = f(L) = j$. Thus $K_j \subseteq \overline{\sqrt{(N : M)M}} \cap \overline{\sqrt{(L : M)M}}$, a contradiction. It implies that f is a proper k coloring of $G(\tau_T)$ and hence $\chi(G(\tau_T)) \leq k = \omega(G(\tau_T))$, as desired. □

Theorem 4.2 *Assume that $\bar{M}$ is a faithful module. Then the following statements are equivalent.*

(a) $\chi(G(\tau_{\text{Spec}(M)})) = 2$.
(b) $G(\tau_{\text{Spec}(M)})$ *is a bipartite graph with two non-empty parts.*
(c) $G(\tau_{\text{Spec}(M)})$ *is a complete bipartite graph with two non-empty parts.*
(d) Either R is a reduced ring with exactly two minimal prime ideals or $G(\tau_{\text{Spec}(M)})$ is a star graph with more than one vertex.

Proof By using Lemma 2.8, $G(\tau_{\text{Spec}(M)})$ and $AG(M)^*$ are the same and so [5, Theorem 3.3] completes the proof. □

Lemma 4.3 *Assume that T is a finite set. Then $\chi(G(\tau_T)))$ is finite. In particular, $\omega(G(\tau_T)))$ is finite.*

Proof Suppose that $T = \{P_1, P_2, \ldots, P_k\}$ is a finite set of distinct prime submodules of M. Define a coloring $f(N) = min\{n \in \mathbb{N} | \ P_n \notin V(N)\}$, where N is a vertex of $G(\tau_T)$. We can see that $\chi(G(\tau_T))) \leq k$. □

Theorem 4.4 *For every module M, $\omega(G(\tau_T)) = 2$ if and only if $\chi(G(\tau_T)) = 2$. In particular, $G(\tau_T)$ is bipartite if and only if $G(\tau_T)$ is triangle-free.*

Proof Let $\omega(G(\tau_T)) = 2$. On the contrary assume that $G(\tau_T)$ is not bipartite. So $G(\tau_T)$ contains an odd cycle. Suppose that $C := N_1 - N_2 - \cdots - N_{2k+1} - N_1$ be a shortest odd cycle in $G(\tau_T)$ for some natural number k. Clearly, $k \geq 2$. Since C is a shortest odd cycle in $G(\tau_T)$, N_3N_{2k+1} is a vertex. Now consider the vertices N_1, N_2, and N_3N_{2k+1}. If $N_1 = N_3N_{2k+1}$, then $V(N_4N_1) = T$. This implies that $N_1 - N_4 - \cdots - N_{2k+1} - N_1$ is an odd cycle, a contradiction. Thus $N_1 \neq N_3N_{2k+1}$. If $N_2 = N_3N_{2k+1}$, then we have $C_3 = N_2 - N_3 - N_4 - N_2$, again a contradiction. Hence $N_2 \neq N_3N_{2k+1}$. It is easy to check N_1, N_2, and N_3N_{2k+1} form a triangle in $G(\tau_T)$, a contradiction. The converse is clear. In particular, we note that empty graphs are bipartite graphs. □

Corollary 4.5 *Assume that $e \in R$ is an idempotent element and $\bar{M}$ does not have a non-zero submodule $\bar{F} \neq \bar{N}$ with $V(N) = T$. Then $G(\tau_T)$ is a complete bipartite graph if and only if $\bar{M}_1$ and $\bar{M}_2$ are prime modules.*

Proof Assume that $G(\tau_T)$ is a complete bipartite graph. Therefore Theorem 4.4 states that $G(\tau_T)$ is a triangle-free graph. So Lemma 3.3 follows that $\bar{M}_1$ and $\bar{M}_2$ are prime modules. The conversely holds by Proposition 3.1(b). □

Remark 4.6 Assume that S is a multiplicatively closed subset of R such that $S \cap (\cup_{P \in T}(P : M)) = \emptyset$. Let T_S={$S^{-1}P| \ P \in T$}. One can see that $V(N) = T$ if and only if $V(S^{-1}N) = T_S$, where M is a finitely generated module.

Theorem 4.7 *Let S be a multiplicatively closed subset of R defined as in Remark 4.6 and M is a finitely generated module. Then $G(\tau_{T_S})$ is a retract of $G(\tau_T)$ and $\omega(G(\tau_{T_S})) = \omega(G(\tau_T))$.*

Proof Consider a vertex map $\phi : V(G(\tau_T)) \longrightarrow V(G(\tau_{T_S})), N \longrightarrow N_S$. Clearly, $N_S \neq K_S$ implies that $N \neq K$ and $V(N) \cup V(K) = T$ if and only if $V(N_S) \cup V(K_S) = T_S$. Thus ϕ is surjective and hence $\omega(G(\tau_{T_S})) \leq \omega(G(\tau_T))$. If $N \neq K$ and $V(N) \cup V(K) = T$, then we show that $N_S \neq K_S$. On the contrary suppose that $N_S = K_S$. Then $V(N_S^2) = V(N_S K_S) = V(N_S) \cup V(K_S) = T_S$ and so $V(N^2) = T$, a contradiction. This shows that the map ϕ is a graph homomorphism. Now, for any vertex N_S of $G(\tau_{T_S})$, we can choose a fixed vertex N of $G(\tau_T)$. Then ϕ is a retract (graph) homomorphism which clearly implies that $\omega(G(\tau_{T_S})) = \omega(G(\tau_T))$ under the assumption. □

Corollary 4.8 *Let S be a multiplicatively closed subset of R defined as in Remark 4.6 and let M be a finitely generated module. Then $\chi(AG(M_S)) = \chi(AG(M))$.*

Corollary 4.9 *Assume that M is a semiprime module and $AG(M)^*$ does not have an infinite clique. Then M is a faithful module and $0 = (P_1 \cap \ldots \cap P_k : M)$, where P_i is a prime submodule of M for $i = 1, \ldots, k$.*

Proof By [5, Theorem 3.8 (b)], M is a faithful module and the last assertion follows directly from the proof of [5, Theorem 3.8 (b)]. □

Recall that the girth of a graph G is the length of a shortest cycle in G and denoted by $gr(G)$.

Proposition 4.10 *Let R be an Artinian ring, $\bar{M}$ be a multiplication module, and let T be a closed subset of* Spec(M). *Then we have the following statements.*

(a) *If S is a finite subset of T, then there exists a clique of size $|S|$ in $G(\tau_T)$.*
(b) *We have $\omega(G(\tau_T)) \geq |Min(T)|$ and if $|Min(T)| \geq 3$, then $gr(G(\tau_T)) = 3$.*
(c) *If $\sqrt{(\bar{0})} = (\bar{0})$, then $\chi(G(\tau_{\text{Spec}(M)})) = \omega(G(\tau_{\text{Spec}(M)})) = |Min(T)|$.*

Proof (a) Let R be an Artinian ring and let $\bar{M}$ be a multiplication module. Then [14, Corollary 2.9] implies that $\bar{M}$ is a cyclic module. We show that $T = Min(T)$. Suppose that $P_1 \subseteq P_2$, where $P_1, P_2 \in T$. Then $(P_1 : M) = (P_2 : M)$ because every prime ideal in R is maximal. Since $\bar{M}$ is multiplication, we have $P_1 = P_2$ and finally the proof is straightforward by the facts that $AG(\bar{M}) = AG(\bar{M})^*$, [6, Theorem 3.6], and $AG(\bar{M})$ is isomorphic with a subgraph of $G(\tau_T)$ by Lemma 2.8.
(b) This is clear by item (a).
(c) If $|Min(T)| = \infty$, then by part (b), there is nothing to prove. Otherwise, [6, Theorem 3.8] implies that $AG(\bar{M})$ does not have an infinite clique. So $\bar{M}$ is a faithful module by Corollary 4.9. Next, Lemma 2.8 says that $G(\tau_{\text{Spec}(M)})$ and $AG(M)^*$ are the same. Now the result follows by [6, Theorem 3.8].

□

Lemma 4.11 *Assume that $\bar{M}$ is a semiprime module. Then the following statements are equivalent.*

(a) *$\chi(G(\tau_{\text{Spec}(M)})))$ is finite.*
(b) *$\omega(G(\tau_{\text{Spec}(M)})))$ is finite.*
(c) *$G(\tau_{\text{Spec}(M)}))$ does not have an infinite clique.*

Proof $(a) \Longrightarrow (b) \Longrightarrow (c)$ is clear.

$(c) \Longrightarrow (a)$ Suppose that $G(\tau_{\text{Spec}(M)}))$ does not have an infinite clique. By Lemma 2.8, $AG(\bar{M})^*$ does not have an infinite clique and so by Corollary 4.9, there exists a finite number of prime submodules $P_1, \ldots, P_k$ of M such that $(F : M) = (P_1 \cap \ldots \cap P_k : M)$. Define a coloring $f(N) = min\{n \in \mathbb{N}| \ P_n \notin V(N)\}$, where N is a vertex of $G(\tau_T)$. Then we have $\chi(G(\tau_{\text{Spec}(M)}))) \leq k$. □

Corollary 4.12 *Assume that $AG(M/F)^*$ does not have an infinite clique. Then $G(\tau_{\mathrm{Spec}(M)})$ and $AG(M)^*$ are the same. Also, $\chi(G(\tau_{\mathrm{Spec}(M)}))$ is finite.*

Proof Since M/F is a semiprime module, by Corollary 4.9, M/F is a faithful module and there exists a finite number of prime submodules $P_1, \ldots, P_k$ of M such that $(F : M) = (P_1 \cap \ldots \cap P_k : M)$. So the result follows by Lemma 2.8 and from the proof of (c) $\Longrightarrow$ (a) of Lemma 4.11. □

We recall that M is said to be X-injective if either $\mathrm{Spec}(M) = \emptyset$ or the natural map of $X = \mathrm{Spec}(M)$ is injective (see [7]).

Proposition 4.13 *Suppose that $\sqrt{(\bar{0})} = (\bar{0})$, for every minimal member P of $\mathrm{Spec}(M)$, $(P : M)$ is a minimal ideal of R, and $\bar{M}$ is an X-injective module. Then the following statements are equivalent.*

(a) *$\chi(G(\tau_{\mathrm{Spec}(M)}))$ is finite.*
(b) *$\omega(G(\tau_{\mathrm{Spec}(M)}))$ is finite.*
(c) *$G(\tau_{\mathrm{Spec}(M)})$ does not have an infinite clique.*
(d) *$Min(\mathrm{Spec}(M))$ is a finite set.*

Proof (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) is clear.

(c) $\Longrightarrow$ (d) Suppose $G(\tau_{\mathrm{Spec}(M)})$ does not have an infinite clique. By Lemma 2.8, $AG(\bar{M})^*$ does not have an infinite clique and hence by Corollary 4.9, there exists a finite number of prime submodules $P_1, \ldots, P_k$ of M such that $(F : M) = (P_1 \cap P_2 \cap \cdots \cap P_k : M)$. By assumptions, one can see that $Min(\mathrm{Spec}(M))$ is a finite set.

(d) $\Longrightarrow$ (a) Assume that $Min(\mathrm{Spec}(M))$ is a finite set (equivalently, $\bar{M}$ has a finite number of minimal prime submodules) so that $(F : M) = (P_1 \cap P_2 \cap \cdots \cap P_k : M)$, where $\mathrm{Min}(\mathrm{Spec}(M)) = \{P_1, \ldots, P_k\}$. Define a coloring $f(N) = \min\{n \in N \mid P_n \notin V(N)\}$, where N is a vertex of $G(\tau_{\mathrm{Spec}(M)})$. Then we have $\chi(G(\tau_{\mathrm{Spec}(M)})) \leq k$. □

Example 4.14 If M is a faithfully flat R-module (for example, free modules), then pM is a p-prime submodule of M, where p is a prime ideal of R by [10, Theorem 3]. So for every minimal prime submodule P of M, $(P : M)$ is a minimal ideal of R.

Proposition 4.15 *Assume that $\sqrt{(\bar{0})} = (\bar{0})$ and $\bar{M}$ is a faithful module. Then the following statements are equivalent.*

(a) *$\chi(G(\tau_{\mathrm{Spec}(M)}))$ is finite.*
(b) *$\omega(G(\tau_{\mathrm{Spec}(M)}))$ is finite.*
(c) *$G(\tau_{\mathrm{Spec}(M)})$ does not have an infinite clique.*
(d) *R has a finite number of minimal prime ideals.*
(e) *$\chi(G(\tau_{\mathrm{Spec}(M)})) = \omega(G(\tau_{\mathrm{Spec}(M)})) = |Min(R)| = k$, where k is finite.*

Proof This is clear by Lemma 2.8, [5, Proposition 3.10], and [5, Corollary 3.11]. □

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References

1. Anderson, D.F.; Livingston, P.S.: The zero-divisor graph of a commutative ring. J. Algebra **217**, 434–447 (1999)
2. Anderson, W.; Fuller, K.R.: Rings and Categories of Modules. Springer, New York (1974)
3. Ansari-Toroghy, H.; Farshadifar, F.: Product and dual product of submodules. Far East J. Math. Sci **25**(3), 447–455 (2008)
4. Ansari-Toroghy, H.; Habibi, S.: The Zariski topology-graph of modules over commutative rings. Commun. Algebra **42**, 3283–3296 (2014)

5. Ansari-Toroghy, H.; Habibi, S.: The annihilating-submodule graph of modules over commutative rings. Math. Rep. **20**(70), 245–262 (2018)
6. Ansari-Toroghy, H.; Habibi, S.: The annihilating-submodule graph of modules over commutative rings II. Arab. J. Math. **5**, 187–194 (2016)
7. Ansari-Toroghy, H.; Ovlyaee-Sarmazdeh, R.: On the prime spectrum of X-injective modules. Commun. Algebra **38**, 2606–2621 (2010)
8. Atiyah, M.F.; Macdonald, I.G.: Introduction to Commutative Algebra. Addison-Wesley, Boston (1969)
9. Beck, I.: Coloring of commutative rings. J. Algebra **116**, 208–226 (1988)
10. Chin-Pi, L.: Prime submodules of modules. Comment. Math. Univ. St. Pauli **33**(1), 61–69 (1984)
11. Chin-Pi, L.: The Zariski topology on the prime spectrum of a module. Houst. J. Math. **25**(3), 417–432 (1999)
12. Chin-Pi, L.: A module whose prime spectrum has the surjective natural map. Houst. J. Math **33**(1), 125–143 (2007)
13. Diestel, R.: Graph theory. In: Grad, Texts in Math. Springer (2005)
14. Elbast, Z.A.; Smith, P.F.: Multiplication modules. Commun. Algebra **16**, 755–779 (1988)
15. Lam, T.Y.: A First Course in Non-commutative Rings. Springer, New York (1991)
16. McCasland, R.L.; Moor, M.E.: Prime submodules. Commun. Algebra **20**(6), 1803–1817 (1992)
17. Tavallaee, H.A.; Varmazyar, R.: Semi-radicals of submodules in modules. IUST Int. J. Eng. Sci. **19**, 21–27 (2008)