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Some algebraic structures on the generalization general products of monoids and semigroups

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Abstract For arbitrary monoids A and B, in Cevik et al. (Hacet J Math Stat 2019:1–11, 2019), it has been recently defined an extended version of the general product under the name of *a higher version of Zappa products for monoids* (or *generalized general product*) $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ and has been introduced an implicit presentation as well as some theories in terms of finite and infinite cases for this product. The goals of this paper are to present some algebraic structures such as regularity, inverse property, Green's relations over this new generalization, and to investigate some other properties and the product obtained by a left restriction semigroup and a semilattice.

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1 Introduction and preliminaries

The notion of Zappa–Szép products generalizes those of direct and semidirect products; the key property is that every element of the Zappa–Szép product can be written uniquely as a product of two elements, one from each factor, in any given order. In the literature, there are some key stone studies on the general product which is also referred as *bilateral semidirect products* (see [11]), *Zappa products* (see [7,12,16,18]) or *knit products* (see [1,14]). As a next step of general product, in [4], the same authors of this paper have recently introduced the generalization of the general product under the name of *a higher version of Zappa products for monoids* as in the following:

For arbitrary monoids A and B, it is known that the $A^{\times B}$ denotes the Cartesian product of the number of B copies of the monoid A while the set $A^{\oplus B}$ denotes the corresponding direct product. Then a generalization of the general products (both restricted and unrestricted) of the monoid $A^{\oplus B}$ by the monoid $B^{\oplus A}$ is defined on $A^{\times B} \times B^{\times A}$ and $A^{\oplus B} \times B^{\oplus A}$, respectively, with the multiplication $(f, h)\left(f', h'\right) = \left(f\ {}^{h}f',\ h^{f'}\ h'\right)$, where $f, f' \in A^{\oplus B}$, $h, h' \in B^{\oplus A}$, $\delta : B^{\oplus A} \longrightarrow \tau\left(A^{\oplus B}\right)$, $\left(f'\right)\delta_h = {}^{h}f'$ and $\psi : A^{\oplus B} \longrightarrow \tau\left(B^{\oplus A}\right)$, $(h)\,\psi_{f'} = h^{f'}$

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are defined by, for $a \in A$ and $b \in B$, ${}^{h}f' = {}^{(h^a)} f'$ and $h^{f'} = h^{({}^b f')}$. Also, for $x \in A$ and $y \in B$, we define $(x)\, h^a = (ax)\, h$ and $(y)^b\, f' = (yb)\, f'$ such that, for all $c \in A$, $d \in B$,

$$(d)^{(h^a)}\, f' = \left(dh^a\right) f' \text{ and } (c)\, h^{({}^b f')} = \left({}^b f' c\right) h$$

are held. Moreover, for all $f, f' \in A^{\oplus B}$ and $h, h' \in B^{\oplus A}$, the following properties are satisfied:

$$\begin{cases} p1^{\bullet} : {}^{(hh')}f = {}^{h}\left({}^{h'}f\right), & p2^{\bullet} : {}^{h}(ff') = ({}^{h}f)({}^{h^f}f'), \\ p3^{\bullet} : (h^f)^{f'} = h^{(ff')}, & p4^{\bullet} : (hh')^f = h^{({}^{h'}f)}(h')^f, \\ p5^{\bullet} : {}^{h}\overline{1} = \overline{1}, & p6^{\bullet} : h^{\overline{1}} = h, \\ p7^{\bullet} : {}^{\widetilde{1}}f = f, & p8^{\bullet} : \widetilde{1}^f = \widetilde{1}. \end{cases} \tag{1}$$

It is easy to show that (both restricted and unrestricted) the generalized general product $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi}\ B^{\oplus A}$ is a monoid with the identity $\left(\overline{1}, \widetilde{1}\right)$, where $\overline{1} : B \longrightarrow A$, $(b)\,\overline{1} = 1_A$ and $\widetilde{1} : A \longrightarrow B$, $(a)\,\widetilde{1} = 1_B$, for all $a \in A$ and $b \in B$. We note that throughout this paper all generalized general products will be assumed to be restricted. We also note that this above definition of the generalized general product should be considered as the external generalized general product as similar .

In the remaining parts of this paper, we will first investigate the isomorphism between the internal and external generalized general products as a generalization of the ordinary general products (in Sect. 2) and then using the result in this section, we will state and prove some results on regularity as well as inverse property (in Sect. 3). After that, in Sect. 4, we will study on Green's relations over this new generalization. Additionally, in Sect. 5, we will investigate some other properties while the generalized general product obtained by a left restriction semigroup and a semilattice.

2 Correspondence between internal and external cases

A monoid M is named as the internal Zappa–Szép product of two submonoids if every element of M admits a unique factorization as the product of one element of each of the submonoids in a given order. This definition yields actions of the submonoids on each other that must be structure preserving (see details, for instance, in [6]). In [17], the author made a detailed investigation between the internal and external Zappa–Szép products (or equivalently, general products) of any two monoids in terms of general products, and then presented some results dealing with the isomorphism of internal and external cases. Thus, it is natural to transfer these decompositions into the generalized general products. In fact, by taking into account the main result of this section (see Theorem 2.2 below) which is about the isomorphism between internal and external cases, we will state and prove the regularity over generalized general products of monoids in Sect. 3.

A simple calculation shows that the product $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi}\ B^{\oplus A}$ cannot be a group in general except the cases A is a group and B is the trivial group which is not useful for studying the group properties on it since it only becomes the group A up to isomorphism. However, by keeping our mind A and B are any monoids, we expect to obtain an equivalence between internal and external generalized general product as in the ordinary general product of monoids. To do that, we will present the following lemma and theorem which are the generalization of [11, Proposition 2.1].

Lemma 2.1 *Suppose that the monoid M is the internal generalized general product $M = A^{\oplus B} B^{\oplus A}$ of $A^{\oplus B}$ and $B^{\oplus A}$. Then there is an action of $B^{\oplus A}$ on the left of $A^{\oplus B}$ and an action of $A^{\oplus B}$ on the right of $B^{\oplus A}$ such that $p1^{\bullet}$ - $p4^{\bullet}$ hold in (1) and $M \cong A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi}\ B^{\oplus A}$.*

Proof Since $M = A^{\oplus B} B^{\oplus A}$, each element $m \in M$ is uniquely expressible as $m = fg$ with $f \in A^{\oplus B}$ and $g \in B^{\oplus A}$. We must have unique elements $g' \in B^{\oplus A}$ and $f' \in A^{\oplus B}$ such that $gf = f'g'$. Writing $f' = {}^{g}f$ and $g' = g^f$, we have mutual actions defined by the multiplication

$$\begin{array}{ccc} B^{\oplus A} \times A^{\oplus B} \longrightarrow A^{\oplus B} & \text{and} & B^{\oplus A} \times A^{\oplus B} \longrightarrow B^{\oplus A} \\ (g, f) \longmapsto {}^{g}f & & (g, f) \longmapsto g^f \end{array}$$

(see [3] for similar actions). Thus, these actions unique subject to the relation $gf = ({}^{g}f)\left(g^f\right)$ which clearly gives $(fg)\left(f'g'\right) = f\left({}^{g}f'\right) g^{f'} g'$ for all $f, f' \in A^{\oplus B}$ and $g, g' \in B^{\oplus A}$. Now, according to the associativity of the monoid M and the uniqueness property of the decomposition, we certainly obtain the properties $p1^{\bullet}$–$p4^{\bullet}$ in (1) for these actions. In detail, by the associativity, we have $g\left(ff'\right) = (gf)\,f'$ that implies

$$g\left(ff'\right) = {}^{g}\left(ff'\right)\left(g^{ff'}\right) \text{ and } (gf)\,f' = \left({}^{g}f\right)\left({}^{g^{f}}f'\right)\left(g^{f}\right)^{f'}.$$

On the other hand, by uniqueness, we have ${}^{g}\left(ff'\right) = \left({}^{g}f\right)\left({}^{g^{f}}f'\right)$ and $g^{(ff')} = \left(g^{f}\right)^{f'}$.

Now, we can form the external generalized general product $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ of $A^{\oplus B}$ and $B^{\oplus A}$. Let us define a map $\alpha : M \longrightarrow A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ by $(fg)\,\alpha = (f, g)$. Clearly α is well defined, one-to-one and onto. Since

$$\begin{aligned}\left((fg)\left(f'g'\right)\right)\alpha &= \left(f\left({}^{g}f'\right)g^{f'}g'\right)\alpha = \left(f\left({}^{g}f'\right), g^{f'}g'\right)\\ &= (f, g)\left(f', g'\right) = (fg)\,\alpha\left(f'g'\right)\alpha\,,\end{aligned}$$

it is also a homomorphism. Hence, $M \cong A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$, as required. □

The following theorem is an extended version of Lemma 2.1.

Theorem 2.2 *Let M be a monoid and $A^{\oplus B}$, $B^{\oplus A}$ be submonoids of M. Suppose that $M = A^{\oplus B}B^{\oplus A}$ is the internal generalized general product of $A^{\oplus B}$ and $B^{\oplus A}$. Then there is an action of $B^{\oplus A}$ on the left of $A^{\oplus B}$ and an action of $A^{\oplus B}$ on the right of $B^{\oplus A}$ such that $p1^{\bullet}$-$p8^{\bullet}$ in (1) hold and also M is isomorphic to the external generalized general product $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$.*

Proof Suppose that $M = A^{\oplus B}B^{\oplus A}$ is the internal generalized general product of $A^{\oplus B}$ and $B^{\oplus A}$. Then by Lemma 2.1, there is an action of $B^{\oplus A}$ on the left of $A^{\oplus B}$ and an action of $A^{\oplus B}$ on the right of $B^{\oplus A}$ such that the properties $p1^{\bullet}$-$p4^{\bullet}$ are satisfied. Also, since $M = A^{\oplus B}B^{\oplus A}$ is a monoid and $1_M \in A^{\oplus B} \cap B^{\oplus A}$, we certainly have

$$(1_M)g = g = g(1_M) = \left({}^{g}1_M\right)g^{1_M} \text{ and } f(1_M) = f = (1_M)f = \left({}^{1_M}f\right)(1_M)^{f}.$$

Therefore, by uniqueness, the properties $p5^{\bullet}$-$p8^{\bullet}$ are held as well. Thus we obtain the monoid $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ as the external generalized general product. With the same approach as in Lemma 2.1, by defining a map $\beta : M \longrightarrow A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ with the rule $(fg)\,\beta = (f, g)$, it is easy to see that $M \cong A_{\delta}^{\oplus B} \bowtie_{\psi} B^{\oplus A}$.

Conversely, let us consider the external generalized general product $M = A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ of the monoids $A^{\oplus B}$ and $B^{\oplus A}$. By denoting two submonoids $\overline{A^{\oplus B}} = \left\{(f, \widetilde{1}) : f \in A^{\oplus B}\right\}$ and $\overline{B^{\oplus A}} = \left\{(\overline{1}, g) : g \in B^{\oplus A}\right\}$ of M and taking into account the maps $f \longmapsto (f, \widetilde{1})$ and $g \longmapsto (\overline{1}, g)$, we can easily see that the submonoids $\overline{A^{\oplus B}}$ and $\overline{B^{\oplus A}}$ are isomorphic to $A^{\oplus B}$ and $B^{\oplus A}$, respectively. Additionally, since each element $(f, g) \in M$ can be written as a unique decomposition $(f, g) = (f, \widetilde{1})(\overline{1}, g)$, we finally obtain the internal generalized general product $M = \overline{A^{\oplus B}B^{\oplus A}}$ of $\overline{A^{\oplus B}}$ and $\overline{B^{\oplus A}}$.

Hence, the result. □

We note that, for semigroups, there is no such a correspondence between the internal and external generalized general products (as proved in Lemma 2.1 and Theorem 2.2) and indeed not even for the general product as remarked by Brin ([3]). In fact, in Sect. 3, we will use this important correspondence to discuss the regularity for only monoids by considering the internal generalized general product of monoids which will also be true for the external generalized general product of monoids.

3 Regularity and inverse properties

In this section, we determine the all criterion when the generalized general product $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ to be a regular and to be an inverse monoid.

First, we will recall some basic definitions as in the following: a semigroup S is called regular if for each $x \in S$, there exists an element $y \in S$ such that $xyx = x$ and $yxy = y$ ([10]) in which the element y is called the inverse of x. The set of regular elements of S is denoted by $Reg\,(S)$ while the set of inverses of the element x is denoted by $V\,(x)$. (Remind that the inverse element is not unique in semigroups unless the semigroup is not an inverse semigroup). A simple fact says that, in a semigroup S, if $xyx = x$, then $y = yxy \in V(x)$ and so to show the regularity of S, we need only to find an element y such that $xyx = x$. In addition, an idempotent in S is an element $e \in S$ such that $e^2 = e$ and the set of all idempotent elements of S is denoted by $E(S)$. Clearly, if $y \in V(x)$ then xy, yx are idempotents. By Hall's theorem ([9, Theorem 3.3.3]), S is regular if and only if the product of any two idempotent element is regular.

Proposition 3.1 *If A is a regular monoid and B is a group, then $A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is a regular monoid.*

Proof Since A is regular, $A^{\oplus B}$ is a regular monoid ([15]) and since B is a group, $B^{\oplus A}$ is a group ([13]). For an element $(f, h) \in A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$, where $f \in A^{\oplus B}$, $h \in B^{\oplus A}$, our aim is to find a suitable element $(g, k) \in A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$, where $g \in A^{\oplus B}$ and $k \in B^{\oplus A}$, such that the equality $(f, h)(g, k)(f, h) = (f, h)$ holds.

Set $(g, k) = \left({}^{h^{-1}}f', \left(h^{{}^{h^{-1}}f'}\right)^{-1}\right)$, where $f' \in V(f)$. Then

$$\begin{aligned}
(f, h)(g, k)(f, h) &= (f, h)\left({}^{h^{-1}}f', \left(h^{{}^{h^{-1}}f'}\right)^{-1}\right)(f, h)\\
&= \left(f\ {}^{h}\left({}^{h^{-1}}f'\right), h^{{}^{h^{-1}}f'}\left(h^{{}^{h^{-1}}f'}\right)^{-1}\right)(f, h)\\
&\overset{p1^{\bullet}}{=} \left(f\ {}^{hh^{-1}}f', 1_B\right)(f, h) \overset{p7^{\bullet}}{=} \left(f\ {}^{1_B}f', 1_B\right)(f, h)\\
&= \left(ff', 1_B\right)(f, h) \overset{p8^{\bullet}}{=} \left(ff'\ {}^{1_B}f, (1_B)^{f}\, h\right)\\
&= \left(ff'f, h\right) = (f, h)
\end{aligned}$$

since $f' \in V(f)$. Thus, $(f, h)\left({}^{h^{-1}}f', \left(h^{{}^{h^{-1}}f'}\right)^{-1}\right)(f, h) = (f, h)$, and so $A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is regular. □

The proof of the following result is quite similar as the proof of Zappa–Szép product (i.e ordinary general product) version which has been done by Wazzan in [17].

Proposition 3.2 *Let $A^{\oplus B}$ be a left zero semigroup and $B^{\oplus A}$ be a regular semigroup. For all $g \in B^{\oplus A}$, suppose there exists some $f \in A^{\oplus B}$ such that $g^f = g$ and, for all $t \in A^{\oplus B}$, suppose there exists some $g' \in V(g)$ such that $\left(g'\right)^t = g'$. Therefore, $A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is regular.*

In the following theorem, we will present necessary and sufficient conditions on regularity of the monoid $A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$ using the method *sandwich set* which was defined by Howie in [10, Proposition 2.5.1]. We should note that, by Theorem 2.2, since there exists an isomorphism between the internal and external generalized general products, we will use the internal forms in the proofs of some results at the remaining part of this section.

Theorem 3.3 *For regular monoids A and B, the generalized general product $A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is regular if and only if $fh \in Reg\left(A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}\right)$ for all $f \in E\left(A^{\oplus B}\right)$ and $h \in E\left(B^{\oplus A}\right)$.*

Proof By the proof of Proposition 3.1, we know that $A^{\oplus B}$ and $B^{\oplus A}$ are regular since A and B are regular.

Let us prove the sufficiency part. Now let $(f, h) \in A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$, where $f \in A^{\oplus B}$ and $h \in B^{\oplus A}$. Since both $A^{\oplus B}$ and $B^{\oplus A}$ are regular, there must exist $f' \in V(f)$ and $h' \in V(h)$ having $f'f \in E\left(A^{\oplus B}\right)$ and $hh' \in E\left(B^{\oplus A}\right)$. Then by keeping our minds the assumption $\left(f'f\right)\left(hh'\right) \in Reg\left(A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}\right)$, the sandwich set ([10, Proposition 2.5.1]) of the elements $f'f$ and hh' is defined by

$$\begin{aligned}
S\left(f'f,\, hh'\right) = \Big\{ g \in \Big[V\left(\left(f'f\right)\left(hh'\right)\right) \cap E\left(A^{\oplus B}{}_{\delta}\bowtie_{\psi} B^{\oplus A}\right)\Big]\\
: \ g\left(f'f\right) = \left(hh'\right) g = g \Big\}\ .
\end{aligned}$$

We aim now to show that this set really exists, and then by [10, Proposition 2.5.3], we will say that the generalized general product $A^{\oplus B}\,{}_{\delta}\bowtie_{\psi} B^{\oplus A}$ is regular.

By the assumption $\left(f'f\right)\left(hh'\right) \in Reg\left(A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}\right)$, we definitely have an element $k \in V\left(\left(f'f\right)\left(hh'\right)\right)$ such that $g = \left(hh'\right)k\left(f'f\right)$. Then

$$\begin{aligned}
\left(f'f\right)\left(hh'\right)g\left(f'f\right)\left(hh'\right) &= \left(f'f\right)\left(hh'\right)\left(hh'\right)k\left(f'f\right)\left(f'f\right)\left(hh'\right)\\
&= \left(f'f\right)\left(hh'\right)^2 k\left(f'f\right)^2\left(hh'\right)\\
&\quad \text{since } f'f \in E\left(A^{\oplus B}\right) \text{ and } hh' \in E\left(B^{\oplus A}\right)\\
&= \left(f'f\right)\left(hh'\right)k\left(f'f\right)\left(hh'\right) \text{ since } k \in V\left(\left(f'f\right)\left(hh'\right)\right)\\
&= \left(f'f\right)\left(hh'\right)
\end{aligned}$$

and

$$\begin{aligned}
g\left(\left(f'f\right)\left(hh'\right)\right)g &= \left(hh'\right)k\left(f'f\right)\left(\left(f'f\right)\left(hh'\right)\right)\left(hh'\right)k\left(f'f\right)\\
&= \left(hh'\right)k\left(f'f\right)^2\left(hh'\right)^2 k\left(f'f\right)\\
&= \left(hh'\right)\left(k\left(f'f\right)\left(hh'\right)k\right)\left(f'f\right)\\
&\quad \text{since } f'f \in E\left(A^{\oplus B}\right) \text{ and } hh' \in E\left(B^{\oplus A}\right)\\
&= \left(hh'\right)k\left(f'f\right) \text{ since } k \in V\left(\left(f'f\right)\left(hh'\right)\right)\\
&= g
\end{aligned}$$

which yields $g \in V\left(\left(f'f\right)\left(hh'\right)\right)$. Moreover,

$$\begin{aligned}
g^2 &= \left(hh'\right)k\left(f'f\right)\left(hh'\right)k\left(f'f\right) = \left(hh'\right)\left(k\left(f'f\right)\left(hh'\right)k\right)\left(f'f\right)\\
&= \left(hh'\right)k\left(f'f\right) = g
\end{aligned}$$

and so $g \in E\left(A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}\right)$. Also we obtain $g \in V\left(\left(f'f\right)\left(hh'\right)\right)$, since

$$g\left(f'f\right) = \left(hh'\right)k\left(f'f\right)\left(f'f\right) = g \text{ and } \left(hh'\right)g = \left(hh'\right)\left(hh'\right)k\left(f'f\right) = g\,.$$

Furthermore, we can write

$$\begin{aligned}
(fh)\left(h'gf'\right)(fh) &= f\left(hh'\right)g\left(f'f\right)h\\
&= fgh \qquad \text{since } g \in V\left(\left(f'f\right)\left(hh'\right)\right)\\
&= ff'fghh'h \text{ since } f' \in V(f),\ h' \in V(h)\\
&= f\left(f'fghh'\right)h\,.
\end{aligned} \tag{2}$$

Now, in (2), we have $f'fghh' = f'f\left(hh'gf'f\right)hh' = \left(f'fhh'\right)g\left(f'fhh'\right) = f'fhh'$ since $g \in V\left(\left(f'f\right)\left(hh'\right)\right)$. Then

$$\begin{aligned}
&(fh)\left(h'gf'\right)(fh) = f\left(f'fhh'\right)h = fh \quad \text{and}\\
&\left(h'gf'\right)fh\left(h'gf'\right) = h'g^2f' = h'gf',
\end{aligned}$$

and so $h'gf' \in V(fh)$. Thus, fh is a regular element which implies that $A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}$ is regular.

The necessity part of the proof is clear. □

Corollary 3.4 *If A and B are regular and $E\left(A^{\oplus B}\right)$ and $E\left(B^{\oplus A}\right)$ act trivially, then $A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}$ is regular.*

Proof Let us consider an element $(f,h) \in A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}$ with $f \in E\left(A^{\oplus B}\right)$ and $h \in E\left(B^{\oplus A}\right)$. Since $E\left(A^{\oplus B}\right)$ and $E\left(B^{\oplus A}\right)$ act trivially, we get

$$(f,h)(f,h) = \left(f\left({}^{h}f\right), \left(h^{f}\right)h\right) = (ff, hh) = (f,h)$$

which implies (f,h) is an idempotent in $A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}$. Therefore, $(f,h) \in Reg\left(A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}\right)$, and hence $A^{\oplus B}{}_{\delta} \bowtie_{\psi} B^{\oplus A}$ is regular. □

In the next theorem, we give necessary conditions for $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ to be an inverse monoid.

Theorem 3.5 *$A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ is an inverse monoid if*

(i) *$A^{\oplus B}$ and $B^{\oplus A}$ are inverse monoids,*
(ii) *$E\left(B^{\oplus A}\right)$ and $E\left(B^{\oplus A}\right)$ act trivially,*
(iii) *for each $(f,h)\in A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$, where $f\in A^{\oplus B}$ and $h\in B^{\oplus A}$, the elements f and h act trivially on each other.*

Proof By Corollary 3.4, $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ is regular. Since a regular semigroup is an inverse semigroup if and only if its idempotents commute, it actually suffices to show that the idempotents of $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ commutes.

Assume that (f,h), (g,k) are idempotents of $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$. On the other hand, $(f,h)(f,h)=(f,h)=\left(f\left({}^{h}f\right),h^{f}h\right)$ and $(g,k)(g,k)=(g,k)=\left(g\left({}^{k}g\right),k^{g}k\right)$ which yield $f=f\left({}^{h}f\right)$, $h=h^{f}h$ and $g=g\left({}^{k}g\right)$, $k=k^{g}k$. By (iii), since f and h as well as g and k act trivially on each others, we get $f=f^2$, $g=g^2$, $h=h^2$ and $k=k^2$. But, by (i), since $A^{\oplus B}$ and $B^{\oplus A}$ are inverse monoids, the idempotents commutes that is $fg=gf\in A^{\oplus B}$ and $hk=kh\in B^{\oplus A}$. Therefore,

$$\begin{aligned}(f,h)(g,k)&=\left(f\left({}^{h}g\right),h^{g}k\right)\\&=(fg,hk)\ \text{ since } h \text{ and } g \text{ are idempotents,}\\&\quad\ \text{they act trivially by } (ii)\\&=(gf,kh)=\left(g\left({}^{k}f\right),k^{f}h\right)\\&=(g,k)(f,h)\,.\end{aligned}$$

Thus, $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ is an inverse monoid, as required. □

Remark 3.6 There also exists a particular class of regular semigroup, namely coregular semigroups. An element α of a semigroup S is called coregular if there is a $\beta\in S$ such that $\alpha=\alpha\beta\alpha=\beta\alpha\beta$ as well as the semigroup S is called coregular if each element of it is coregular ([2,5]). In fact, we leave the coregularity and its properties over generalized general products as an open problem for the future studies.

4 Some Green's relations on generalized general product

Green's relations $\mathcal{R}$ and $\mathcal{L}$ on Zappa–Szép products (general products) of semigroups have been first investigated in the paper [11]. Nevertheless, Wazzan ([17]) studied some related results on the same topic as well.

As a next step of the studies in [11,17,20], in this section, we will study on some Green's relations for generalized general product $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$.

The following proposition is the generalized version of a result in [11] over semigroups.

Proposition 4.1 *Let $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ be the generalized general product of semigroups $A^{\oplus B}$ and $B^{\oplus A}$. Then*

(i) $(f_1,g_1)\,\mathcal{L}\,(f_2,g_2)\Rightarrow g_1\mathcal{L}g_2$ *in* $B^{\oplus A}$;
(ii) $(f_1,g_1)\,\mathcal{R}\,(f_2,g_2)\Rightarrow f_1\mathcal{R}f_2$ *in* $A^{\oplus B}$.

Proof Suppose $(f_1,g_1)\,\mathcal{L}\,(f_2,g_2)$ in $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$. Then there exists any two elements (h_1,l_1), $(h_2,l_2)\in A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ such that $(h_1,l_1)(f_1,g_1)=(f_2,g_2)$ and $(h_2,l_2)(f_2,g_2)=(f_1,g_1)$. In other words, we must have

$$\left(h_1\left({}^{l_1}f_1\right),l_1^{f_1}g_1\right)=(f_2,g_2)\quad\text{ and }\quad\left(h_2\left({}^{l_2}f_2\right),l_2^{f_2}g_2\right)=(f_1,g_1)\,,$$

which imply $h_1\left({}^{l_1}f_1\right)=f_2$, $l_1^{f_1}g_1=g_2$, $h_2\left({}^{l_2}f_2\right)=f_1$ and $l_2^{f_2}g_2=g_1$. It follows that $g_1\mathcal{L}g_2$ in $B^{\oplus A}$. Similar argument can be discussed for the proof of (ii). □

Theorem 4.2 *Let $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ be the generalized general product of a monoid $A^{\oplus B}$ and a group $B^{\oplus A}$. Then*

$$(f_1,g_1)\,\mathcal{R}\,(f_2,g_2)\Longleftrightarrow f_1\mathcal{R}f_2\ \text{ in } A^{\oplus B}\,.$$

Proof The necessity part is clear by Proposition 4.1-(ii).

To prove the sufficiency part, let us suppose that $f_1 \mathcal{R} f_2$ in $A^{\oplus B}$. So there must exist t_1 and t_2 in $A^{\oplus B}$ such that $f_1 t_1 = f_2$ and $f_2 t_2 = f_1$. To show the existence of $(f_1, g_1)\, \mathcal{R}\, (f_2, g_2)$, we have to find (h_1, l_1) and (h_2, l_2) in $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi}\ B^{\oplus A}$ such that $(f_1, g_1)\,(h_1, l_1) = (f_2, g_2)$ and $(f_2, g_2)\,(h_2, l_2) = (f_1, g_1)$, or equivalently, $f_1\,({}^{g_1}h_1) = f_2$, $g_1^{h_1} l_1 = g_2$, $f_2\,({}^{g_2}h_2) = f_1$ and $g_2^{h_2} l_2 = g_1$. Notice that the second and forth equalities can also be written as

$$l_1 = (g_1^{h_1})^{-1} g_2 \quad \text{and} \quad l_2 = (g_2^{h_2})^{-1} g_1.$$

In fact, by setting $h_1 = {}^{g_1^{-1}}t_1$ and $h_2 = {}^{g_2^{-1}}t_2$, we obtain

$$\begin{aligned}
(f_1, g_1)\,(h_1, l_1) &= (f_1, g_1)\left({}^{g_1^{-1}}t_1, (g_1^{{}^{g_1^{-1}}t_1})^{-1} g_2\right) \\
&= \left(f_1\left({}^{g_1}\left({}^{g_1^{-1}}t_1\right)\right), g_1^{{}^{g_1^{-1}}t_1}(g_1^{{}^{g_1^{-1}}t_1})^{-1} g_2\right) \\
&= \left(f_1\left({}^{\left(g_1 g_1^{-1}\right)}t_1\right), g_2\right) = (f_1 t_1, g_2) = (f_2, g_2)\,.
\end{aligned}$$

With a similar approximation, we also obtain the equality $(f_2, g_2)\,(h_2, l_2) = (f_1, g_1)$. Therefore, $(f_1, g_1)\, \mathcal{R}\, (f_2, g_2)$, as required. □

Theorem 4.3 *If* $\left({}^{g_1^{-1}}f_1\right) \mathcal{L} \left({}^{g_2^{-1}}f_2\right)$ *such that* $(g_1^{-1})^{f_1} = g_1^{-1}$ *and* $(g_2^{-1})^{f_2} = g_2^{-1}$ *in* $B^{\oplus A}$*, then* $(f_1, g_1)\, \mathcal{L}\, (f_2, g_2)$ *in the product* $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi}\ B^{\oplus A}$*, where* $A^{\oplus B}$ *is a monoid and* $B^{\oplus A}$ *is a group.*

Proof Suppose $\left({}^{g_1^{-1}}f_1\right) \mathcal{L} \left({}^{g_2^{-1}}f_2\right)$ holds with its conditions. Then there exist t_1 and t_2 in $A^{\oplus B}$ such that $t_1\left({}^{g^{-1}}f_1\right) = {}^{g_2^{-1}}f_2$ and $t_2\left({}^{g_2^{-1}}f_2\right) = {}^{g_1^{-1}}f_1$, respectively. In here, clearly $f_2 = ({}^{g_2}t_1)\left({}^{g_2^{t_1} g_1^{-1}}f_1\right)$.

We set $(h_1, l_1) = \left({}^{g_2}t_1, g_2^{t_1} g_1^{-1}\right)$ and $(h_2, l_2) = \left({}^{g_1}t_2, g_1^{t_2} g_2^{-1}\right)$. Then we obtain

$$\begin{aligned}
(h_1, l_1)\,(f_1, g_1) &= \left(h_1\left({}^{l_1}f_1\right), l_1^{f_1} g_1\right) = \left(({}^{g_2}t_1)\left({}^{g_2^{t_1} g_1^{-1}}f_1\right), (g_2^{t_1} g_1^{-1})^{f_1} g_1\right) \\
&= \left(f_2, g_2^{t_1\left({}^{g_1^{-1}}f_1\right)}(g_1^{-1})^{f_1} g_1\right) = \left(f_2, g_2^{{}^{g_2^{-1}}f_2} g_1^{-1} g_1\right) \\
&= \left(f_2, ((g_2^{-1})^{f_2})^{-1}\right) = (f_2, g_2)\,.
\end{aligned}$$

Similarly, one can obtain $(h_2, l_2)\,(f_2, g_2) = (f_1, g_1)$. Hence, $(f_1, g_1)\, \mathcal{L}\, (f_2, g_2)$ in $A^{\oplus B}\ {}_{\delta}\!\bowtie_{\psi}\ B^{\oplus A}$. □

Remark 4.4 It is known that there also exist some other types of Green's relations. One may study those relations with their properties over generalized general products for a future project.

5 Generalized general product of a left restriction semigroup by a semilattice

In this section, by considering the generalized general product of a left restriction semigroup with a semilattice of projections, we will determine some algebraic properties of it. Recall that left restriction semigroups are a class of semigroups which generalize inverse semigroups. A semigroup S is called a *semilattice* if all its elements are idempotents and commute. For inverse semigroups A and B, by [13, Proposition 3], if A and B are semilattices then $A^{\oplus B}$ and $B^{\oplus A}$ are semilattices, respectively. On the other hand, an inverse semigroup S is an *unary semigroup* $\left(S, \cdot, ^{-1}\right)$, where $^{-1}$ represents the inverse unary operation on S.

Definition 5.1 ([19]) A left restriction semigroup S is a unary semigroup $\left(S,\cdot,^{+}\right)$, where $(S,\cdot)$ is a semigroup and $^{+}$ is an unary operation such that the following identities hold:

$$a^{+}a=a,\quad a^{+}b^{+}=b^{+}a^{+},\quad \left(a^{+}b\right)^{+}=a^{+}b^{+}\quad \text{and}\quad ab^{+}=(ab)^{+}a.$$

Putting $E=\{a^{+}:a\in S\}$, it is easy to see that E becomes a semilattice. These idempotents are called projections of S and we call E is the semilattice of projections of S. If S is a left restriction semigroup with semilattice of projections E, then a natural partial order on S is defined by the rule

$$a\leq b\Longleftrightarrow a=eb\quad \text{or, equivalently,}\quad a\leq b\Longleftrightarrow a=a^{+}b$$

for some $e\in E$ and all $a,b\in S$. We refer the reader to [19] for detailed study on left (right, two sided) restriction semigroups.

If S is a semigroup and E is a non-empty subset of $E\,(S)$ which is called the distinguished set of idempotents, then the relations $\leq_{\widetilde{\mathcal{R}}_E}$ and $\leq_{\widetilde{\mathcal{L}}_E}$ are defined by the rules

$$a\leq_{\widetilde{\mathcal{R}}_E} b\Longleftrightarrow \{e\in E: eb=b\}\subseteq\{e\in E: ea=a\}\quad \text{and}$$
$$a\leq_{\widetilde{\mathcal{L}}_E} b\Longleftrightarrow \{e\in E: be=b\}\subseteq\{e\in E: ae=a\}\ ,$$

respectively, for all $a,b\in S$. It is clear that $\leq_{\widetilde{\mathcal{R}}_E}$ and $\leq_{\widetilde{\mathcal{L}}_E}$ are pre-order on S. The associated equivalence relations are denoted by $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$. Thus, for any $a,b\in S$, we have $a\widetilde{\mathcal{R}}_E b$ if and only if a and b have the same set of left identities and $a\widetilde{\mathcal{L}}_E b$ if and only if a and b have the same set of right identities in E.

In fact, this section can be thought as a generalization of the results in [8, Lemmas 4.1.1, 4.1.2 and Proposition 4.1.4]. We will consider a left restriction semigroup $B^{\oplus A}$ with semilattice of projections $A^{\oplus B}$. By defining a left action of $B^{\oplus A}$ on $A^{\oplus B}$ and a right action of $A^{\oplus B}$ on $B^{\oplus A}$, we will see that $A^{\oplus B}\bowtie B^{\oplus A}$ becomes a generalized general product. We will also determine the set of idempotents of $A^{\oplus B}\,{}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$. We will actually see that $A^{\oplus B}\,{}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ is not itself left restriction but it contains a subsemigroup which is left restriction.

Lemma 5.2 *Let $B^{\oplus A}$ be a left restriction semigroup with semilattice of projections $A^{\oplus B}$. Define an action of $B^{\oplus A}$ on $A^{\oplus B}$ by ${}^{f}g=(fg)^{+}$ and an action of $A^{\oplus B}$ on $B^{\oplus A}$ by $f^{g}=fg$. Then $A^{\oplus B}\,{}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ is the generalized general product of $B^{\oplus A}$ and $A^{\oplus B}$.*

Proof To proof this lemma, we need to check these two actions whether they actually satisfy the properties defined in (1).

For $f_1,f_2\in B^{\oplus A}$ and $g\in A^{\oplus B}$, since we have

$${}^{f_1}\left({}^{f_2}g\right)={}^{f_1}(f_2g)^{+}=\left(f_1(f_2g)^{+}\right)^{+}=(f_1(f_2g))^{+}=((f_1f_2)g)^{+}={}^{f_1f_2}g,$$

condition $p1^{\bullet}$ holds.

Let $f\in B^{\oplus A}$ and $g_1,g_2\in A^{\oplus B}$. Then

$$\begin{aligned}\left({}^{f}g_1\right)\left({}^{f^{g_1}}g_2\right)&=(fg_1)^{+}\left({}^{fg_1}g_2\right)=(fg_1)^{+}((fg_1)g_2)^{+}\\&=((fg_1)g_2)^{+}\ \text{ using }\ (ab)^{+}\leq a^{+}\text{ for any }a,b\in S\\&=(f(g_1g_2))^{+}={}^{f}(g_1g_2)\,.\end{aligned}$$

Thus, $p2^{\bullet}$ holds.

For $f\in B^{\oplus A}$ and $g_1,g_2\in A^{\oplus B}$, we have $(f^{g_1})^{g_2}=(fg_1)^{g_2}=(fg_1)\,g_2=f\,(g_1g_2)=f^{g_1g_2}$. So $p3^{\bullet}$ holds.

For $f_1,f_2\in B^{\oplus A}$ and $g\in A^{\oplus B}$, we get

$$f_1^{\,{}^{f_2}g}f_2^{g}=f_1^{(f_2g)^{+}}(f_2g)=f_1(f_2g)^{+}(f_2g)=f_1(f_2g)=(f_1f_2)\,g=(f_1f_2)^{g}\,.$$

Thus, $p4^{\bullet}$ holds.

Therefore, $A^{\oplus B}\,{}_{\delta}\!\bowtie_{\psi} B^{\oplus A}$ is the generalized general product under the binary operation $(f_1,g_1)\,(f_2,g_2)=\left(f\,(g_1f_2)^{+},g_1f_2g_2\right)$, as required. □

We now compute the set of idempotents of $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$, where $B^{\oplus A}$ is a left restriction semigroup with semilattice of projections $A^{\oplus B}$.

Lemma 5.3 *Let $B^{\oplus A}$ is a left restriction semigroup with semilattice of projections $A^{\oplus B}$. Then*

$$E\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)=\left\{(f,g): f\leq g^{+},\ f=fgf\right\}.$$

Moreover, $\overline{A^{\oplus B}}=\left\{(f,f): f\in A^{\oplus B}\right\}$ is a semilattice isomorphic to $A^{\oplus B}$, and if $E\left(B^{\oplus A}\right)=B^{\oplus A}$ then $\overline{A^{\oplus B}}=E\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)$.

Proof Let $(f,g)\in A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$. Then

$$\begin{aligned}(f,g)\in E\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right) &\Leftrightarrow (f,g)^{2}=(f,g)\ \Leftrightarrow\ (f,g)(f,g)=(f,g)\\ &\Leftrightarrow \left(f(gf)^{+},gfg\right)=(f,g)\\ &\Leftrightarrow f=f(gf)^{+}\ \text{and}\ g=gfg.\end{aligned}$$

Now $g=gfg\Longrightarrow g\mathcal{R}gf\widetilde{\mathcal{R}}_{A^{\oplus B}}(gf)^{+}$, so that $g^{+}=(gf)^{+}$. Hence,

$$(f,g)\Longleftrightarrow\left(fg^{+},gfg\right)=(f,g)\Longleftrightarrow f\leq g^{+}\ \text{ and }\ gfg=g.$$

Clearly $\overline{A^{\oplus B}}\subseteq E\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)$, and easy to check that $\overline{A^{\oplus B}}$ is a semilattice isomorphic to $A^{\oplus B}$.

Now, if $E\left(B^{\oplus A}\right)=B^{\oplus A}$ then $\overline{A^{\oplus B}}=E\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)$. Also, if (f,g) is an element of $E\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)$ then we obtain $gf=gfgf=(gf)^{+}=g^{+}$ by the equality $g=gfg$. Thus, we have $g=gfg=(gf)fg=g^{+}fg=fg$ which gives $g^{+}\leq f$. So, since $f\leq g^{+}$, it follows that $g^{+}=f$. As a result of that $g=gfg=g^{2}=g^{+}=f$. □

As a main result of this section, we now record some properties of the generalized general product of a left restriction semigroup $B^{\oplus A}$ with semilattice of projections $A^{\oplus B}$.

Theorem 5.4 *Let us consider the product $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$, where $B^{\oplus A}$ is a left restriction semigroup with semilattice of projections $A^{\oplus B}$, and let $(f,g)\in A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$. Then the followings hold:*

(a) *$\overline{A^{\oplus B}}=\left\{(f,f): f\in A^{\oplus B}\right\}$ is a semilattice isomorphic to $E\left(B^{\oplus A}\right)$;*
(b) *there is a morphism $\alpha:\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)\longrightarrow B^{\oplus A}$ separating the idempotents of $\overline{A^{\oplus B}}$;*
(c) *$(h,h)(f,g)=(f,g)$ if and only if $hf=f$ and $fg=g$;*
(d) *(f,g) has a left identity in $\overline{A^{\oplus B}}$ if and only if $fg=g$;*
$\left\lfloor \text{in this case } (f,g)\widetilde{\mathcal{R}}_{\overline{A^{\oplus B}}}(f,f) \text{ if and only if } fg=g\right\rfloor$;
(e) *$(f,g)(l,l)=(f,g)$ if and only if $f\leq g^{+}$, $g=gl$;*
(f) *for $(f,g)\in A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$, $(f,g)\widetilde{\mathcal{L}}_{\overline{A^{\oplus B}}}(l,l)$, where $(l,l)\in\overline{A^{\oplus B}}$ if and only if $f\leq g^{+}$ and $g\widetilde{\mathcal{L}}_{A^{\oplus B}}l$;*
(g) *for some $g,l\in A^{\oplus B}$, the relations $(h,h)\widetilde{\mathcal{R}}_{\overline{A^{\oplus B}}}(f,g)\widetilde{\mathcal{L}}_{\overline{A^{\oplus B}}}(l,l)$ implies $(f,g)=\left(g^{+},g\right)$. Moreover, there is a canonical imbedding of $B^{\oplus A}$ into $A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$ under $g\mapsto\left(g^{+},g\right)$.*

Proof (a) From Lemma 5.3, we know that $\overline{A^{\oplus B}}$ is a semilattice which is isomorphic to $E\left(B^{\oplus A}\right)$.
(b) Define $\alpha:\left(A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}\right)\to B^{\oplus A}$ by $(f,g)\alpha=fg$. Clearly α is surjective. Also, for any elements $(f,g),(h,l)\in A^{\oplus B}\ {}_{\delta}\bowtie_{\psi}\ B^{\oplus A}$, we write

$$\begin{aligned}((f,g)(h,l))\alpha &=\left(f(gh)^{+},ghl\right)\alpha=f(gh)^{+}ghl\\ &=f(gh)l=fghl=(f,g)\alpha(h,l)\alpha\end{aligned}$$

so α is a homomorphism. Further, for any $(f,f),(h,h)\in\overline{A^{\oplus B}}$, since

$$(f,f)\alpha=(h,h)\alpha\Longleftrightarrow f=h,$$

the homomorphism α separates idempotents of $\overline{A^{\oplus B}}$.

(c) Let $(f, g) \in A^{\oplus B}\ {}_{\delta} \bowtie_{\psi} B^{\oplus A}$ and $(h, h) \in \overline{A^{\oplus B}}$. Then

$$\begin{aligned}(h, h)(f, g) = (f, g) &\Leftrightarrow (h(hf), hfg) = (f, g)\\ &\Leftrightarrow hf = f \text{ and } hfg = g\\ &\Leftrightarrow hf = f \text{ and } fg = g.\end{aligned}$$

(d) Suppose now $(f, g)\, \widetilde{\mathcal{R}}_{\overline{A\oplus B}}\, (f, f)$. By (c), we have $fg = g$.
Conversely, if $fg = g$ then (f, f) is a left identity of (f, g) since $(f, f)(f, g) = (f, fg) = (f, g)$. Now suppose that $(h, h) \in \overline{A^{\oplus B}}$ exists with $(h, h)(f, g) = (f, g)$. Then $hf = f$ by (c), so that we have $(h, h)(f, f) = (f, f)$ since $A^{\oplus B} \cong \overline{A^{\oplus B}}$. Hence, $(f, f)\, \widetilde{\mathcal{R}}_{\overline{A\oplus B}}\, (f, g)$.

(e) For $(f, g) \in A^{\oplus B}\ {}_{\delta} \bowtie_{\psi} B^{\oplus A}$ and $(h, h) \in \overline{A^{\oplus B}}$, we get

$$\begin{aligned}(f, g)(h, h) = (f, g) &\Leftrightarrow \left(f(gh)^{+}, ghh\right) = (f, g)\\ &\Leftrightarrow f(gh)^{+} = f \text{ and } gh = g\\ &\Leftrightarrow f \leq g^{+} \text{ and } gh = g\,.\end{aligned}$$

(f) Let $(f, g)\, \widetilde{\mathcal{L}}_{\overline{A\oplus B}}\, (l, l)$. Then $(f, g)(l, l) = (f, g)$ gives $f \leq g^{+}$ and $gl = g$. Now suppose that $gh = g$ for some $h \in A^{\oplus B}$. Thus $(f, g)(h, h) = \left(f(gh)^{+}, gh\right) = (f, g)$. Moreover, since $(f, g)\, \widetilde{\mathcal{L}}_{\overline{A\oplus B}}\, (l, l)$ we actually obtain $(l, l)(h, h) = (l, l)$. On the other hand, by the isomorphism $\overline{A^{\oplus B}} \cong A^{\oplus B}$, we have $lh = l$. So that $g \widetilde{\mathcal{L}}_{A\oplus B} l$.
Conversely, if $f \leq g^{+}$ and $g \widetilde{\mathcal{L}}_{A\oplus B} l$, then $gl = g \Longrightarrow (f, g)(l, l) = (f, g)$, and if $(f, g)(h, h) = (f, g)$ then $gh = g$ and so $lh = l$ which gives $(l, l)(h, h) = (l, l)$. Therefore, $(f, g)\, \widetilde{\mathcal{L}}_{\overline{A\oplus B}}\, (l, l)$.

(g) For some $g, l \in A^{\oplus B}$, it is a direct proof to show that

$$(h, h)\, \widetilde{\mathcal{R}}_{\overline{A\oplus B}}\, (f, g)\, \widetilde{\mathcal{L}}_{\overline{A\oplus B}}\, (l, l) \ \text{ implies } \ (f, g) = \left(g^{+}, g\right)$$

using (c) and (e). Now suppose $S = \left\{\left(g^{+}, g\right) : g \in B^{\oplus A}\right\}$. To prove that S is a subsemigroup of $A^{\oplus B}\ {}_{\delta} \bowtie_{\psi} B^{\oplus A}$, let $\left(g^{+}, g\right), \left(f^{+}, f\right) \in S$. Then

$$\left(g^{+}, g\right), \left(f^{+}, f\right) = \left(g^{+}(gf)^{+}, gf^{+}f\right) = \left(\left(g^{+}gf\right)^{+}, gf\right) = \left((gf)^{+}, gf\right) \in S.$$

Obviously, $B^{\oplus A} \cong S$ under $g \mapsto \left(g^{+}, g\right)$. Therefore, S is a left restriction subsemigroup of $A^{\oplus B}\ {}_{\delta} \bowtie_{\psi} B^{\oplus A}$, where $\left(g^{+}, g\right)^{+} = \left(\left(g^{+}\right)^{+}, g^{+}\right) \in S$.
Hence, the result. □

6 Conclusions

In this paper, we investigated some specific theories such as internal, external, regularity, inverse, and Green's relations over generalized general products $A^{\oplus B}\ {}_{\delta} \bowtie_{\psi} B^{\oplus A}$. Of course, there are still so many different properties that can be checked on this important product. On the other hand, in Remarks 3.6 and 4.4, we indicated some problems for the future studies.

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