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The $M^X/M/c$ Bernoulli feedback queue with variant multiple working vacations and impatient customers: performance and economic analysis

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Abstract The present paper deals with an $M^X/M/c$ Bernoulli feedback queueing system with variant multiple working vacations and impatience timers which depend on the states of the servers. Whenever a customer arrives at the system, he activates an random impatience timer. If his service has not been completed before his impatience timer expires, the customer may abandon the system. Using certain customer retention mechanism, the impatient customer can be retained in the system. After getting incomplete or unsatisfactory service, with some probability, each customer may comeback to the system as a Bernoulli feedback. Using the probability generating functions (PGFs), we derive the steady-state solution of the model. Then, we obtain useful performance measures. Moreover, we carry out an economic analysis. Finally, numerical study is performed to explore the effects of the model parameters on the behavior of the system.

Mathematics Subject Classification 60K25 · 68M20 · 90B22

1 Introduction

Vacation queueing models with impatient customers are very helpful in providing basic framework for efficient design and study of diverse practical situations including telephone switchboard, inventory problems with perishable goods, computer and communication network, data/voice transmission, manufacturing system, etc.

In recent past, vacation queueing models have been widely studied. Doshi [3], Takagi [17], and Tian and Zhang [18] are excellent survey works on the subject. An extensive amount of the literature is available on queueing models with server vacation and batch arrival and can be found in Madan and AI-Rawwash [13], Wang et al. [20], Haridass and Arrumuganathan [5], etc.

Working vacation queues with customer impatience have attracted the interest of many researchers. Altman and Yechiali [1] treated the infinite-server queueing model with system's additional tasks and impatient customers. Perel and Yechiali [14] considered a two-phase Markovian random environment with impatient customers. Working vacation queueing model with customer impatience has been analyzed by Yue et al. [22]. Then, Zhang et al. [25] presented an equilibrium balking strategies in Markovian queues with working vacations. Sun et al. [16] gave the equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations. Goswami [4] analyzed a queueing model with impatient customers with Bernoulli

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schedule working vacations and vacation interruption. Laxmi and Jyothsna [10] dealt with balking and reneging multiple working vacations queue with heterogeneous servers. Later, Tian et al. [19] presented equilibrium and optimal strategies in $M/M/1$ queueing model with working vacations and vacation interruptions. Recently, in Bouchentouf and Yahiaoui [2], a study on queueing system with Bernoulli feedback, reneging and retention of reneged customers, multiple working vacations, and Bernoulli schedule vacation interruption has been done. For more literature on customer impatience in working vacation queueing models, the authors may be referred to Selvaraju and Goswami [15] and Laxmi and Jyothsna [8,9].

Variant of multiple vacation policy is relatively a recent one where it is permitted to the server to take a certain fixed number of consecutive vacations, if the system remains empty at the end of a vacation. This sort of vacation schedule was carried out by Zhang and Tian [24]. In their paper, a Geo/G/1 queueing model with multiple adaptive vacations has been analyzed. Literature related to variant multiple working vacations can be found in Ke [6], Ke et al. [7], Wang et al. [21], and Yue et al. [23]. Recently, Laxmi and Rajesh [11] studied a variant working vacations queue with customer impatience. Furthermore, the performance measures of batch arrival queue with variant working vacations and reneging have been presented in Laxmi and Rajesh [12].

In the present investigation, we carry out the analysis of an $M^X/M/c$ Bernoulli feedback queueing model with variant of multiple working vacations, reneging which depend on the states of the servers and retention of reneged customers. The queueing model presented in this paper has many practical applications. Moreover, as the impatience has strongly bad effect on the economy of any firm, a great idea of retention of impatient customers is incorporated in this work. Besides, to the best of our knowledge, modeling of multi-server queueing system with Bernoulli feedback, variant of working vacations, impatience timers which depend on the states of the servers, and retention of reneged customers has not been attempted in the literature. This paper makes a contribution in this sense.

The paper is arranged as follows. We describe the model in Sect. 2. The theoretical analysis of the system is presented in Sect. 3. Useful measures of effectiveness and the cost of our model are given in Sect. 4. To validate the analytical results and to facilitate the sensitivity analysis, we present some numerical results for system performance measures and cost model in Sect. 5. Some concluding remarks and notable features of investigation done are highlighted in Sect. 6.

2 The model formulation

We consider an $M^X/M/c$ queueing system with variant of working vacations, Bernoulli feedback, impatient customers which depend on the states of the servers, and retention of reneged customers. For the mathematical formulation of the queueing model, the following notations and assumptions are given:

Customers arrive in batches according to a Poisson process with rate λ. The arrival batch size X is a random variable with probability mass function $P(X = l) = b_l;\ \ l = 1, 2, \ldots$ The service times during normal busy period is assumed to be exponentially distributed with mean $1/\mu$. During the vacation time, the service is provided according to an exponential distribution with parameter η, such that $\eta \leq \mu$. The queueing system consists of c servers, all the servers go on working vacation synchronously once the system becomes empty, and they also return to the system as one at the same time. If the servers return from working vacation period to find an empty queue, they immediately leave all together for another working vacation. Otherwise, they return to serve the queue. Working vacation periods are assumed to be exponentially distributed with mean $1/\phi$.

At a working vacation completion instant, if there are customers in the system, the servers come back to regular busy period. Otherwise, they take all together working vacations sequentially until the maximum number of working vacations, denoted by K is taken. When the K consecutive working vacations are complete, all servers switch to normal busy period and stay idle or busy depending on the availability of the arriving batches of customers. Therefore, in variant multiple vacation policy, if the system remains empty at the end of a working vacation, the servers are permitted to take a finite number, say K, of consecutive working vacations.

Whenever a batch of customers arrives to the system and finds the servers on working vacation (respectively. busy), an independent impatience timer T_1 (respectively. T_2) is activated, which is exponentially distributed with parameter ξ_1 (respectively. ξ_2). If the customer's service has not been completed before the customer's timer expires, the customer may leave the system. Each impatient customer may abandon the system with probability α, and can be retained in the queue with complementary probability $\alpha' = (1 - \alpha)$. If the service is incomplete or unsatisfactory, the customer can either abandon the system with probability β or rejoin the tail of the queue of the system for another service with probability β', where $\beta + \beta' = 1$. Note that, both customers, the newly arrived and those that are fed back are served in order in which they join the tail of the primary queue.

The inter-arrival times, batch sizes, working vacation times, service times, and impatience times are independent of each other.

3 Theoretical analysis of the model

Let $N(t)$ denote the number of customers in the system at time t, and let $\kappa(t)$ be the status of the servers at time t. For the mathematical representation of the proposed model at an instant t, we consider the following states of the system based on the status of the servers:

$$\kappa(t) = \begin{cases} j, & \text{the servers are taking the } (j+1)\text{th vacation at time } t,\ j = \overline{0, 1, K-1}, \\ K, & \text{the servers are idle or busy at time } t. \end{cases}$$

Figure 1 depicts the state-transition diagram. The bi-variate process $\{(N(t), \kappa(t)),\ \ t \geq 0\}$ represents two-dimensional infinite state Markov chain in continuous time with state space:

$$\Omega = \left\{(n, j) :\ \ n \geq 0;\ \ j = \overline{0, K}\right\}.$$

Let $P_{n,j} = \lim_{t\to\infty} P\left(N(t) = n; \kappa(t) = j\right),\ \ n \geq 0;\ \ j = \overline{0, K}$ be the steady-state probabilities of the process $\{(N(t); \kappa(t)); t \geq 0\}$.

3.1 PGFs and balance equations

Define the probability generating functions as follows:

$$G_j(z) = \sum_{n=0}^{\infty} P_{n,j} z^n,\ \ |z| \leq 1,\ \ j = 0, \ldots, K,$$

$$G'_j(z) = \frac{\mathrm{d}}{\mathrm{d}z} G_j(z) = \sum_{n=1}^{\infty} n P_{n,j} z^{n-1},\ \ j = 0, \ldots, K,$$

and

$$B(z) = \sum_{n=1}^{\infty} b_n z^n, \quad \text{with} \quad B(1) = \sum_{n=1}^{\infty} b_n = 1.$$

To develop the model, the steady-state Chapman–Kolmogorov equations for the system states are constructed as follows:

$$(\lambda + \phi) P_{0,0} = (\beta\nu + \alpha\xi_1) P_{1,0} + (\beta\mu + \alpha\xi_2) P_{1,K}, \tag{1}$$

$$(\lambda + \phi + \beta\nu + \alpha\xi_1) P_{1,0} = \lambda b_1 P_{0,0} + 2(\beta\nu + \alpha\xi_1) P_{2,0}, \tag{2}$$

$$(\lambda + \phi + n(\beta\nu + \alpha\xi_1)) P_{n,0} = \lambda \sum_{m=1}^{n} b_m P_{n-m,0} + (n+1)(\beta\nu + \alpha\xi_1) P_{n+1,0}, \quad 2 \leq n \leq c-1, \tag{3}$$

$$(\lambda + \phi + c\beta\nu + n\alpha\xi_1) P_{n,0} = \lambda \sum_{m=1}^{n} b_m P_{n-m,0} + (c\beta\nu + (n+1)\alpha\xi_1) P_{n+1,0}, \quad n \geq c, \tag{4}$$

$$(\lambda + \phi) P_{0,j} = (\beta\nu + \alpha\xi_1) P_{1,j} + \phi P_{0,j-1},\ \ 1 \leq j \leq K, \tag{5}$$

$$(\lambda + \phi + \beta\nu + \alpha\xi_1) P_{1,j} = \lambda b_1 P_{0,j} + 2(\beta\nu + \alpha\xi_1) P_{2,j},\ \ 1 \leq j \leq K-1, \tag{6}$$

$$(\lambda + \phi + n(\beta\nu + \alpha\xi_1)) P_{n,j} = \lambda \sum_{m=1}^{n} b_m P_{n-m,j} + (n+1)(\beta\nu + \alpha\xi_1) P_{n+1,j}, \quad 2 \leq n \leq c-1,\ \ 1 \leq j \leq K-1, \tag{7}$$

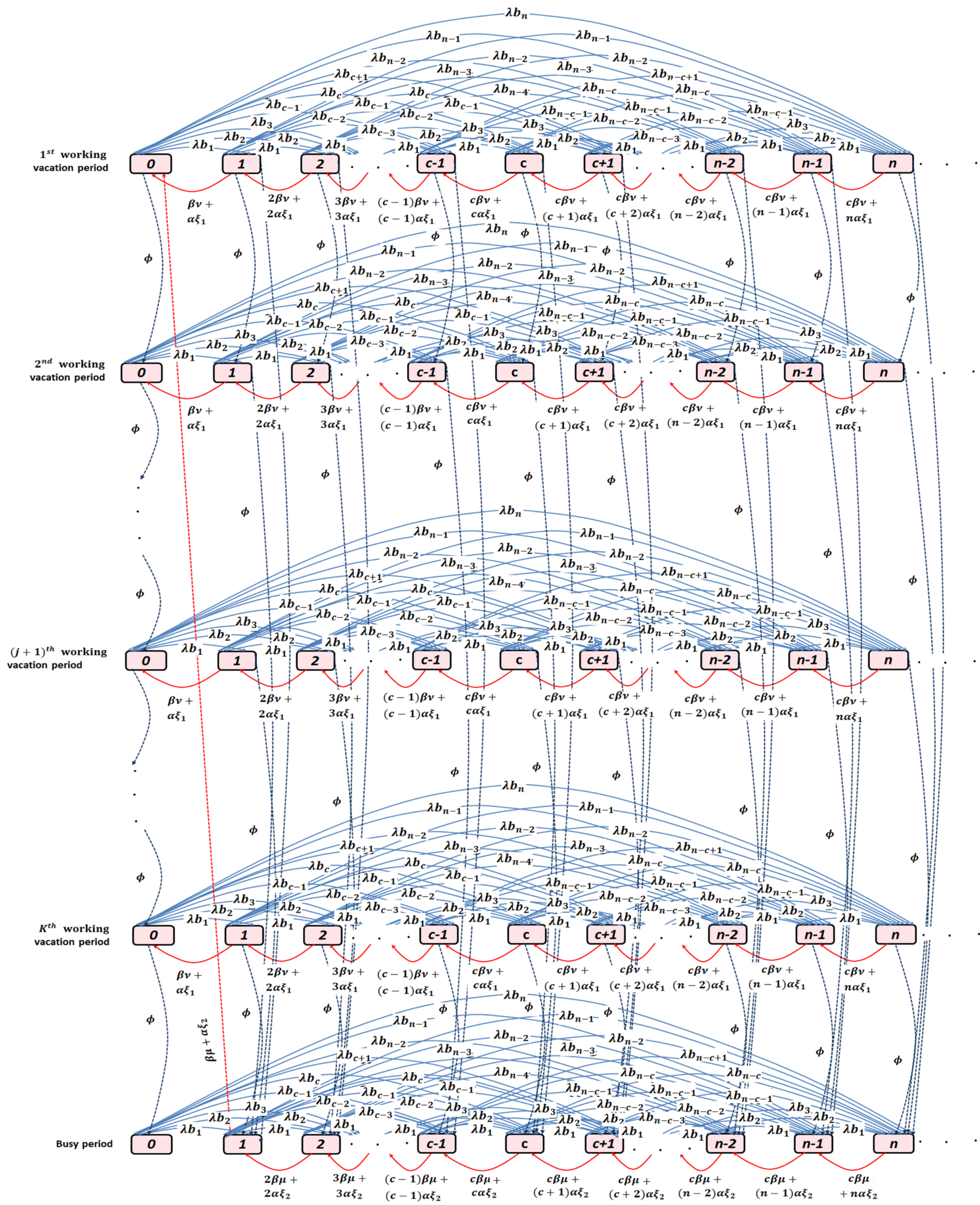

Fig. 1 State-transition-rate diagram

$$(\lambda + \phi + c\beta\nu + n\alpha\xi_1)P_{n,j} = \lambda \sum_{m=1}^{n} b_m P_{n-m,j} + (c\beta\nu + (n+1)\alpha\xi_1)P_{n+1,j},$$

$$n \geq c,\ 1 \leq j \leq K-1, \tag{8}$$

$$\lambda P_{0,K} = \phi P_{0,K-1}, \tag{9}$$

$$(\lambda+\beta\mu+\alpha\xi_2)P_{1,K}=\lambda b_1P_{0,K}+2(\beta\mu+\alpha\xi_2)P_{2,K}+\phi\sum_{j=0}^{K-1}P_{1,j}, \tag{10}$$

$$\begin{aligned}(\lambda+n\,(\beta\mu+\alpha\xi_2))\,P_{n,K}&=\lambda\sum_{m=1}^{n}b_mP_{n-m,K}+(n+1)(\beta\mu+\alpha\xi_2)P_{n+1,K}\\&\quad+\phi\sum_{j=0}^{K-1}P_{n,j},\ 2\le n\le c-1,\end{aligned} \tag{11}$$

$$\begin{aligned}(\lambda+c\beta\mu+n\alpha\xi_2)\,P_{n,K}&=\lambda\sum_{m=1}^{n}b_mP_{n-m,K}+(c\beta\mu+(n+1)\alpha\xi_2)P_{n+1,K}\\&\quad+\phi\sum_{j=0}^{K-1}P_{n,j},\ n\ge c.\end{aligned} \tag{12}$$

The normalizing condition is given as follows:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{K}P_{n,j}=1. \tag{13}$$

Multiplying Eqs. (1)–(4) by z^n and summing all possible values of n, we get the following:

$$\begin{aligned}&(1-z)z\alpha\xi_1G_0'(z)+[\lambda z(B(z)-1)-z(\phi+c\beta\nu)+c\beta\nu]\,G_0(z)\\&=\beta\nu(1-z)\sum_{n=0}^{c-1}(c-n)P_{n,0}z^n-(\alpha\xi_2+\beta\mu)zP_{1,K}.\end{aligned} \tag{14}$$

In a similar manner, we get from Eqs. (5)–(8):

$$\begin{aligned}&(1-z)z\alpha\xi_1G_j'(z)+[\lambda z(B(z)-1)-z(\phi+c\beta\nu)+c\beta\nu]\,G_j(z)\\&=\beta\nu(1-z)\sum_{n=0}^{c-1}(c-n)P_{n,j}z^n-\phi zP_{0,j-1},\ \ 1\le j\le K-1.\end{aligned} \tag{15}$$

In the same way, from Eqs. (9)–(12), we find the following:

$$\begin{aligned}&(1-z)z\alpha\xi_2G_K'(z)+[\lambda z(B(z)-1)+c\beta\mu(1-z)]G_K(z)=-z\phi\sum_{j=0}^{K-1}G_j(z)\\&\beta\mu(1-z)\sum_{n=0}^{c-1}(c-n)P_{n,K}z^n+z(\beta\mu+\alpha\xi_2)P_{1,K}+z\phi\sum_{j=0}^{K-2}P_{0,j}.\end{aligned} \tag{16}$$

Next, using the recursive method, we get the following:

$$\begin{cases}P_{n,0}=\gamma_nP_{0,0}+\varphi_nP_{1,K},\\ P_{n,j}=\gamma_nP_{0,j}+\omega_nP_{0,j-1},\end{cases}$$

where

$$\gamma_n=\begin{cases}1, & \text{if } n=0;\\ \dfrac{\lambda+\phi}{\beta\nu+\alpha\xi_1}, & \text{if } n=1.\\ \psi_{n-1}\gamma_{n-1}-\dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1}b_i\gamma_{n-1-i} & \text{if } 2\le n\le c-1,\end{cases}$$

$$\varphi_n = \begin{cases} 0, & \text{if } n = 0; \\ -\dfrac{\beta\mu + \alpha\xi_2}{\beta\nu + \alpha\xi_1}, & \text{if } n = 1. \\ \psi_{n-1}\varphi_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i \varphi_{n-1-i} & \text{if } 2 \le n \le c-1, \end{cases}$$

$$\omega_n = \begin{cases} 0, & \text{if } n = 0; \\ -\dfrac{\phi}{\beta\nu + \alpha\xi_1}, & \text{if } n = 1. \\ \psi_{n-1}\omega_{n-1} - \dfrac{A}{n}\displaystyle\sum_{i=1}^{n-1} b_i \omega_{n-1-i} & \text{if } 2 \le n \le c-1, \end{cases}$$

with

$$A = \frac{\lambda}{\beta\nu + \alpha\xi_1} \text{ and } \psi_n = \frac{\lambda + \phi + n(\beta\nu + \alpha\xi_1)}{(n+1)(\beta\nu + \alpha\xi_1)}.$$

3.2 Solutions of the differential equations

For $z \neq 1$ and $z \neq 0$, Eqs. (14) and (15) can be written, respectively, as follows:

$$G_0'(z) + \left(\frac{\lambda}{\alpha\xi_1}H'(z) - \frac{(\phi + c\beta\nu)}{(1-z)\alpha\xi_1} + \frac{c\beta\nu}{(1-z)z\alpha\xi_1}\right)G_0(z)$$
$$= \frac{\beta\nu}{z\alpha\xi_1}Q_0(z)P_{0,0} + \left(\frac{\beta\nu}{\alpha\xi_1 z}Q_1(z) - \frac{\alpha\xi_2 + \beta\mu}{(1-z)\alpha\xi_1}\right)P_{1,K}, \tag{17}$$

for $j = \overline{1, K-1}$.

$$G_j'(z) + \left(\frac{\lambda}{\alpha\xi_1}H'(z) - \frac{(\phi + c\beta\nu)}{(1-z)\alpha\xi_1} + \frac{c\beta\nu}{(1-z)z\alpha\xi_1}\right)G_j(z)$$
$$= \frac{\beta\nu}{z\alpha\xi_1}Q_0(z)P_{0,j} + \left(\frac{\beta\nu}{\alpha\xi_1 z}Q_2(z) - \frac{\phi}{(1-z)\alpha\xi_1}\right)P_{0,j-1}, \tag{18}$$

where

$$Q_0(z) = \sum_{n=0}^{c-1}(c-n)\gamma_n z^n,\ Q_1(z) = \sum_{n=0}^{c-1}(c-n)\varphi_n z^n,\ Q_2(z) = \sum_{n=0}^{c-1}(c-n)\omega_n z^n,$$

with

$$H(z) = \int_0^z \frac{B(x) - 1}{1 - x}\mathrm{d}x \text{ and } H'(z) = \frac{B(z) - 1}{1 - z}.$$

Now, by taking $z = 1$ in Eqs. (14) and (15), we, respectively, have the following:

$$\phi G_0(1) = (\alpha\xi_2 + \beta\mu)P_{1,K}, \tag{19}$$

and

$$G_j(1) = P_{0,j-1}. \tag{20}$$

Next, to solve the linear differential Eqs. (17) and (18), we multiply both sides of the above equations by $e^{\frac{\lambda}{\alpha\xi_1}H(z)}(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}$. Then, integrating form 0 to z, we obtain the following:

$$G_0(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}}z^{\frac{c\beta\nu}{\alpha\xi_1}}}\left\{\frac{\beta\nu}{\alpha\xi_1}K_0(z)P_{0,0} + \left(\frac{\beta\nu}{\alpha\xi_1}K_1(z) - \frac{\alpha\xi_2 + \beta\mu}{\alpha\xi_1}K_2(z)\right)P_{1,K}\right\}, \tag{21}$$

for $j = \overline{1, K-1}$.

$$G_j(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}} z^{\frac{c\beta\nu}{\alpha\xi_1}}} \left\{ \frac{\beta\nu}{\alpha\xi_1} K_0(z) P_{0,j} + \left(\frac{\beta\nu}{\alpha\xi_1} K_3(z) - \frac{\phi}{\alpha\xi_1} K_2(z) \right) P_{0,j-1} \right\}, \tag{22}$$

where

$$K_0(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}} x^{\frac{c\beta\nu}{\alpha\xi_1}-1} Q_0(x)\mathrm{d}x,$$
$$K_1(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}} x^{\frac{c\beta\nu}{\alpha\xi_1}-1} Q_1(x)\mathrm{d}x,$$
$$K_2(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}-1} x^{\frac{c\beta\nu}{\alpha\xi_1}} \mathrm{d}x,$$

and

$$K_3(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_1}H(x)} (1-x)^{\frac{\phi}{\alpha\xi_1}} x^{\frac{c\beta\nu}{\alpha\xi_1}-1} Q_2(x)\mathrm{d}x.$$

Next, $z = 0$ and $z = 1$ are the roots of the numerator of the right-hand sides of (21) and (22). Thus, taking $z = 1$ in (21) and (22), respectively, we get the following:

$$P_{1,K} = \theta_1 P_{0,0}, \quad \text{where} \quad \theta_1 = \frac{\beta\nu K_0(1)}{(\beta\mu + \alpha\xi_2)K_2(1) - \beta\nu K_1(1)}, \tag{23}$$

and

$$P_{0,j} = C^j P_{0,0}, \ \ 1 \le j \le K-1, \quad \text{where} \quad C = \frac{\phi K_2(1) - \beta\nu K_3(1)}{\beta\nu K_0(1)}. \tag{24}$$

Via Eqs. (9) and (24), we obtain the following:

$$P_{0,K} = \theta_0 P_{0,0}, \tag{25}$$

where

$$\theta_0 = \frac{\phi}{\lambda} C^{K-1}.$$

Substituting Eqs. (23) and (24) in Eqs. (21) and (22), respectively, we get the following:

$$G_0(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}} z^{\frac{c\beta\nu}{\alpha\xi_1}}} \left\{ \frac{\beta\nu K_0(z) + (\beta\nu K_1(z) - (\beta\mu + \alpha\xi_2)K_2(z))\theta_1}{\alpha\xi_1} \right\} P_{0,0}, \tag{26}$$

and for $j = \overline{1, K-1}$

$$G_j(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}} z^{\frac{c\beta\nu}{\alpha\xi_1}}} \left\{ \beta\nu K_0(z) + \frac{\beta\nu K_3(z) - \phi K_2(z)}{C} \right\} \frac{C^j}{\alpha\xi_1} P_{0,0}. \tag{27}$$

Thus

$$\sum_{j=0}^{K-1} G_j(z) = \Psi(z) P_{0,0}, \quad j = \overline{0, K-1}, \tag{28}$$

with

$$\Psi(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_1}H(z)}}{(1-z)^{\frac{\phi}{\alpha\xi_1}} z^{\frac{c\beta\nu}{\alpha\xi_1}}} \left\{ \frac{\beta\nu K_0(z) + (\beta\nu K_1(z) - (\beta\mu + \alpha\xi_2)K_2(z))\theta_1}{\alpha\xi_1} \right.$$
$$\left. + \frac{C}{\alpha\xi_1} \left(\beta\nu K_0(z) + \frac{\beta\nu K_3(z) - \phi K_2(z)}{C} \right) \left(\frac{1 - C^{K-1}}{1 - C} \right) \right\}.$$

By taking $z = 1$ in Eq. (16), we find the following:

$$\phi \sum_{j=0}^{K-1} G_j(1) = (\beta\mu + \alpha\xi_2) P_{1,K} + \phi \sum_{j=0}^{K-2} P_{0,j}. \tag{29}$$

Consequently, we have the following:

$$\sum_{j=0}^{K-1} G_j(1) = \left\{ \frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1 - C^{K-1}}{1 - C} \right\} P_{0,0}. \tag{30}$$

Next, we have to solve the differential Eq. (16). Therefore, we must express recursively the quantity $P_{n,K}$ in terms of $P_{0,0}$. In the same manner as previously, it yields the following:

$$P_{n,K} = \theta_n P_{0,0},$$

with

$$\theta_n = \begin{cases} \theta_0, & \text{if } n = 0; \\ \theta_1, & \text{if } n = 1. \\ \sigma_{n-1}\theta_{n-1} - \dfrac{B}{n}\displaystyle\sum_{i=1}^{n-1} b_i \theta_{n-1-i} - \dfrac{E}{n}(\gamma_{n-1} H(K) + \omega_{n-1} h(K)) & \text{if } 2 \le n \le c-1, \end{cases}$$

where

$$\sigma_n = \frac{\lambda + n(\beta\mu + \alpha\xi_2)}{(n+1)(\beta\mu + \alpha\xi_2)}, \quad B = \frac{\lambda}{\beta\mu + \alpha\xi_2}, \quad E = \frac{\phi}{\beta\mu + \alpha\xi_2},$$
$$H(K) = \sum_{j=0}^{K-1} C^j = \frac{1 - C^K}{1 - C}, \quad \text{and } h(K) = \sum_{j=0}^{K-1} C^{j-1} = \frac{1 - C^K}{C(1 - C)}.$$

By substituting Eqs. (29) in (16), we have the following:

$$G'_K(z) + \left(\frac{\lambda}{\alpha\xi_2} H(z)' + \frac{c\beta\mu}{z\alpha\xi_2} \right) G_K(z) = \frac{\beta\mu}{z\alpha\xi_2} Q_3(z) P_{0,0} - \frac{\phi \sum_{j=0}^{K-1} \left[G_j(z) - G_j(1) \right]}{(1 - z)\alpha\xi_2}, \tag{31}$$

where

$$Q_3(z) = \sum_{n=0}^{c-1} (c - n)\theta_n z^n.$$

Multiplying Eq. (31) by $e^{\frac{\lambda}{\alpha\xi_2} H(z)} z^{\frac{c\beta\mu}{\alpha\xi_2}}$ and integrating from 0 to z, then using Eqs. (28) and (30), we get the following:

$$G_K(z) = \frac{e^{-\frac{\lambda}{\alpha\xi_2} H(z)}}{z^{\frac{c\beta\mu}{\alpha\xi_2}}} \left\{ -\frac{\phi}{\alpha\xi_2} \left(K_4(z) - \left(\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1 - C^{K-1}}{1 - C} \right) K_5(z) \right) + \frac{\beta\mu}{\alpha\xi_2} K_6(z) \right\} P_{0,0}, \tag{32}$$

where

$$K_4(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_2} H(x)} x^{\frac{c\beta\mu}{\alpha\xi_2}} (1 - x)^{-1} \Psi(x) \mathrm{d}x,$$
$$K_5(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_2} H(x)} x^{\frac{c\beta\mu}{\alpha\xi_2}} (1 - x)^{-1} \mathrm{d}x,$$

and

$$K_6(z) = \int_0^z e^{\frac{\lambda}{\alpha\xi_2} H(x)} x^{\frac{c\beta\mu}{\alpha\xi_2} - 1} Q_3(x) \mathrm{d}x.$$

Now, taking $z = 1$ in Eq. (32), and using the normalization condition:

$$\sum_{j=0}^{K-1} G_j(1) + G_K(1) = 1,$$

we obtain the following:

$$P_{0,0} = \left\{ e^{-\frac{\lambda}{\alpha\xi_2}H(1)} \left\{ -\frac{\phi}{\alpha\xi_2}\left(K_4(1) - \left(\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C}\right)K_5(1)\right) \right.\right.$$
$$\left.\left. + \frac{\beta\mu}{\alpha\xi_2}K_6(1)\right\} + \left(\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1-C^{K-1}}{1-C}\right)\right\}^{-1}. \tag{33}$$

4 Performance measures and cost model

4.1 Measures of effectiveness

Performance measures are significant features of queueing systems as they reflect the effectiveness of the considered queueing system. The queueing model developed may be of great importance using some useful characteristics which can be in the future employed for the prediction, development, and improvement of the concerned real-world queueing system. In this section, we formulate some important system performance measures in terms of steady-state probabilities.

– The average number of customers in the system ($E(L)$).

$$E(L) = E(L_{\mathrm{WV}}) + E(L_{\mathrm{K}}),$$

where $E(L_{\mathrm{WV}})$ is the mean system size when the servers are on working vacation and $E(L_{\mathrm{K}})$ represents the mean system size when the servers are in busy period. Differentiating Eq. (14), taking $z = 1$, and using Eq. (19), we get the following:

$$(\alpha\xi_1 + \phi)\, G_0'(1) = \left(\lambda B'(1) - c\beta\nu\right) G_0(1) + \beta\nu\left(Q_0(1)P_{0,0} + Q_1(1)P_{1,K}\right). \tag{34}$$

In the same manner, for $j = \overline{1, K-1}$, differentiating Eq. (15), taking $z = 1$ and using Eq. (20), we get the following:

$$(\alpha\xi_1 + \phi)\, G_j'(1) = \left(\lambda B'(1) - c\beta\nu\right) G_j(1) + \beta\nu\left(Q_0(1)P_{0,j} - Q_2(1)P_{0,j-1}\right), \tag{35}$$

where

$$Q_0(1) = \sum_{n=0}^{c-1}(c-n)\gamma_n,\ Q_1(1) = \sum_{n=0}^{c-1}(c-n)\varphi_n,\ Q_2(1) = \sum_{n=0}^{c-1}(c-n)\omega_n.$$

From Eq. (34), we obtain the following:

$$G_0'(1) = \left\{\frac{\lambda B'(1) - c\beta\nu}{\alpha\xi_1 + \phi}\left(\frac{\beta\mu + \alpha\xi_2}{\phi}\right)\theta_1 + \frac{\beta\nu(Q_0(1) + \theta_1 Q_1(1))}{\alpha\xi_1 + \phi}\right\}P_{0,0}. \tag{36}$$

From Eq. (35), summing over all the possible values of j, $j = \overline{1, K-1}$, we obtain the following:

$$\sum_{j=1}^{K-1} G_j'(1) = \left\{\left(\frac{1-C^{K-1}}{C(1-C)}\right)\frac{\lambda B'(1) + \beta\nu(Q_0(1)C - Q_2(1) - c)}{\alpha\xi_1 + \phi}\right\}P_{0,0}. \tag{37}$$

Furthermore, $E(L_{\mathrm{WV}})$ is obtained as follows:

$$E(L_{\mathrm{WV}}) = G_0'(1) + \sum_{j=1}^{K-1} G_j'(1). \tag{38}$$

Substituting Eqs. (36) and (37) in (38), we get the following:

$$\begin{aligned} E(L_{\mathrm{WV}}) = \Bigg\{ & \frac{\lambda B'(1) - c\beta\nu}{\alpha\xi_1 + \phi}\left(\frac{\beta\mu + \alpha\xi_2}{\phi}\right)\theta_1 + \frac{\beta\nu(Q_0(1) + \theta_1 Q_1(1))}{\alpha\xi_1 + \phi} \\ & + \left(\left(\frac{1 - C^{K-1}}{C(1-C)}\right)\frac{\lambda B'(1) + \beta\nu(Q_0(1)C - Q_2(1) - c)}{\alpha\xi_1 + \phi}\right)\Bigg\} P_{0,0}. \end{aligned} \tag{39}$$

Next, from Eq. (16), using L'Hospital rule, we find the following:

$$\begin{aligned} E(L_{\mathrm{K}}) = \lim_{z\to 1} G_K'(z) = \lim_{z\to 1} \Bigg\{ & \frac{-(\lambda z(B(z) - 1) + c\beta\mu(1-z))}{(1-z)\alpha\xi_2} G_K(z) \\ & - \frac{\phi \sum_{j=0}^{K-1}[G_j(z) - G_j(1)]}{(1-z)\alpha\xi_2} + \frac{\beta\mu}{z\alpha\xi_2} Q_3(z) P_{0,0}\Bigg\}. \end{aligned}$$

This implies that

$$E(L_{\mathrm{K}}) = \frac{\lambda B'(1) - \beta\mu}{\alpha\xi_2} G_K(1) + \frac{\phi}{\alpha\xi} \sum_{j=1}^{K-1} G_j'(1) + \frac{\beta\mu}{\alpha\xi_2} Q_3(1) P_{0,0}, \tag{40}$$

where

$$Q_3(1) = \sum_{n=0}^{c-1}(c-n)\theta_n.$$

– The mean number of customers in the queue $(E(L_{\mathrm{q}}))$:

$$\begin{aligned} E(L_{\mathrm{q}}) &= \sum_{j=0}^{K} \sum_{n=c+1}^{\infty} (n-c) P_{n,j} \\ &= E(L) - c + \left\{\left(Q_0(1) + \frac{Q_2(1)}{C}\right)\left(\frac{1 - C^K}{1 - C}\right) + Q_3(1)\right\} P_{0,0}. \end{aligned}$$

– The probability that the servers are in working vacation period (P_{WV}):

$$P_{\mathrm{WV}} = \sum_{j=0}^{K-1} G_j(1) = \left\{\frac{\beta\mu + \alpha\xi_2}{\phi}\theta_1 + \frac{1 - C^{K-1}}{1 - C}\right\} P_{0,0}.$$

– The probability that the servers are idle during working vacation period (P_{idle}):

$$P_{\mathrm{idle}} = \sum_{j=0}^{K-1} P_{0,j} = \frac{1 - C^K}{1 - C} P_{0,0}.$$

– The probability that the servers are busy (P_{busy}):

$$P_{\mathrm{busy}} = 1 - P_{0,K} - P_{\mathrm{WV}}.$$

Table 1 Total costs vs. λ

K	1			5			9		
λ	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
2.9	120.4041	531.6658	411.2618	195.2688	568.5601	373.2913	195.3084	568.5796	373.2712
3.0	122.1671	534.7137	412.5467	192.7104	569.8081	377.0976	192.7494	569.8274	377.0781
3.1	123.9192	537.6298	413.7105	190.3974	571.0168	380.6194	190.4356	571.0360	380.6003
3.2	125.6620	540.4197	414.7576	188.3150	572.1867	383.8717	188.3524	572.2056	383.8533
3.3	126.7224	545.4875	418.7651	182.1972	573.0985	390.8506	182.2479	573.0985	390.5806

– The mean number of customers served per unit time (N_s):

$$N_s = \beta\mu\sum_{n=1}^{c-1} nP_{n,K} + c\beta\mu\sum_{n=c}^{\infty} P_{n,K} + \beta\nu\sum_{j=0}^{K}\sum_{n=1}^{c-1} nP_{n,j} + c\beta\nu\sum_{j=0}^{K}\sum_{n=c}^{\infty} P_{n,j}$$

$$= c\beta\left(\mu(P_{\text{busy}} + P_{0,K}) + \nu P_{\text{WV}}\right) + \beta\left(\mu Q_3(1) + \nu(Q_0(1)H(K) + Q_2(1)h(K))\right) P_{0,0}.$$

∗ The average rate of abandonment of a customer due to impatience (R_a):

$$R_a = \sum_{j=0}^{K-1}\sum_{n=0}^{\infty} n\alpha\xi_1 P_{n,j} + \sum_{n=0}^{\infty} n\alpha\xi_2 P_{n,K}$$

$$= \alpha\xi_1 E(L_{\text{WV}}) + \alpha\xi_2 E(L_{\text{K}}).$$

∗ The average rate of retention of impatient customers (R_e):

$$R_e = \sum_{j=0}^{K-1}\sum_{n=0}^{\infty} n(1-\alpha)\xi_1 P_{n,j} + \sum_{n=0}^{\infty} n(1-\alpha)\xi_2 P_{n,K}$$

$$= (1-\alpha)\xi_1 E(L_{\text{WV}}) + (1-\alpha)\xi_2 E(L_{\text{K}}).$$

4.2 Economic model

To construct the cost model, we consider the following cost (in unit) elements associated with different events:

- C_1 Cost per unit time when the servers are busy.
- C_2 Cost per unit time when the servers are idle during busy period.
- C_3 Cost per unit time when the servers are idle during working vacation period.
- C_4 Cost per unit time when the servers are on working vacation period.
- C_5 Cost per unit time when a customer joins the queue and waits for service.
- C_6 Cost per unit time when a customer reneges.
- C_7 Cost per unit time when a customer is retained.
- C_8 Cost per service per unit time when the servers are in busy period.
- C_9 Cost per service per unit time when the servers are in working vacation period.
- C_{10} Cost per unit time when a customer returns to the system as a feedback customer.
- C_{11} Fixed server purchase cost per unit.

Let R be the revenue earned by providing service to a customer.
Γ be the total expected cost per unit time of the system:

$$\Gamma = C_1 P_{\text{busy}} + C_2 P_{0,K} + C_3 P_{\text{idle}} + C_4 P_{\text{WV}} + C_5 E(L_q) + C_6 R_a + C_7 R_e$$

$$+ c\mu C_8 + c\nu C_9 + c\beta'(\mu+\nu)C_{10} + cC_{11}.$$

Table 2 Total costs vs. ϕ

q	0.5			0.7			0.9		
ϕ	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
0.07	160.2763	595.4973	435.2210	172.8560	572.8560	400.0426	216.3829	545.9906	329.6077
0.09	160.7400	596.2907	435.5507	170.2858	578.0479	407.7620	219.4602	554.4544	334.9942
0.11	160.8474	596.8863	436.0389	166.8535	581.9042	415.0506	213.1461	565.2968	352.1507
0.13	160.7960	597.3476	436.5516	161.7721	584.5145	422.7425	205.0141	571.0750	366.0609
0.15	160.6748	597.7125	437.0377	158.9608	586.6848	427.7240	197.1398	575.5170	378.3772

Table 3 Total costs vs. μ

ξ_2	0.80			0.94			1.04		
μ	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
4.60	183.9987	644.5488	460.5501	191.9106	635.7698	443.8591	197.0989	630.0409	432.9421
5.00	187.7598	691.1525	503.3926	196.2569	679.9235	483.6665	198.8866	676.0608	477.1742
5.40	191.4837	737.2152	545.7281	200.5359	723.5063	522.9703	203.3135	718.9175	515.6040
5.80	195.1493	782.8152	587.6659	204.7636	766.5681	561.8045	207.7011	761.2520	553.5509
6.20	198.7750	827.9931	629.2181	208.9442	809.1618	600.2176	212.0479	803.1188	591.0709

Table 4 Total costs vs. ν

α	0.5			0.7			0.9		
ν	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
0.35	98.7895	548.2676	449.4781	95.5000	498.4809	402.9810	92.9471	458.5866	365.6395
0.55	100.1176	554.3015	454.1939	96.4025	508.2268	411.8243	93.7156	472.5236	378.8080
0.75	101.9522	558.9834	457.0311	97.8672	516.6151	418.7479	95.1487	484.0831	388.9344
0.95	104.2273	562.3944	458.1671	99.8645	523.7082	423.8435	97.1571	494.2884	397.1314
1.15	106.8859	565.1780	458.2921	102.3833	529.8369	427.4536	99.7588	502.9528	403.1940

Table 5 Total costs vs. ξ_2

β	0.5			0.7			0.9		
ξ_2	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
0.79	195.2539	580.2338	384.9799	190.4259	572.1482	381.7223	199.3447	564.7339	365.3892
0.81	195.4810	579.6329	384.1518	191.5068	571.3736	379.8668	201.4988	563.8195	362.3207
0.83	195.7182	579.0377	383.3195	190.1446	572.0249	381.8803	203.6107	562.9214	359.3108
0.85	195.9633	578.4492	382.4859	191.1776	571.2889	380.1113	205.6878	562.0369	356.3491
0.87	196.2156	577.8671	381.6515	192.1779	570.5641	378.3661	207.7247	561.1681	353.4434

Table 6 Total costs vs. ξ_1

c	1			2			3		
ξ_1	Γ	Δ	Θ	Γ	Δ	Θ	Γ	Δ	Θ
4.50	58.1448	198.6960	140.5512	97.5554	397.3824	299.8269	138.0011	596.6616	458.6605
5.00	58.1369	198.6854	140.5485	97.5312	397.2814	299.7502	137.9293	596.3670	458.4377
5.50	58.1297	198.6729	140.5431	97.5028	397.1844	299.6816	137.8431	596.0944	458.2513
6.00	58.1230	198.6588	140.5358	97.4712	397.0918	299.6207	137.7470	595.8445	458.0976
6.50	57.2817	193.4057	136.1240	91.7429	385.8117	294.0688	126.7755	578.6476	451.8720

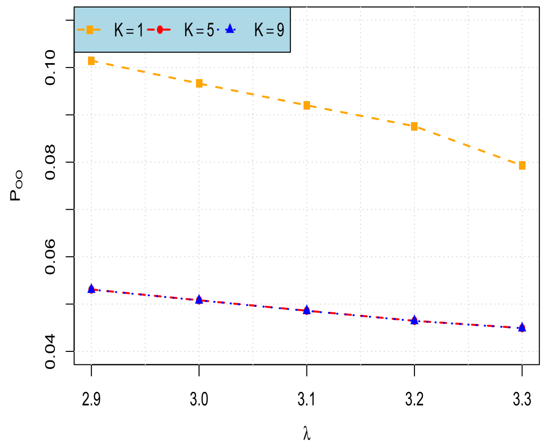

Fig. 2 Impact of λ on $P_{0,0}$

Δ be the total expected revenue per unit time of the system.

$$\Delta = R \times N_s.$$

Θ be the total expected profit per unit time of the system.

$$\Theta = \Delta - \Gamma.$$

5 Numerical analysis

In this section, we present numerical examples to analyze the parameter impact on the system performance as well as on total expected cost, total expected revenue, and total expected profit. The characteristics and different costs of the queueing model are obtained using R program coded by the authors. We assume that the arrival batch size X follows a geometric distribution with parameter q, that is $P(X = l) = (1-q)^{l-1}q$, with $0 < q < 1$, and $l = 1, 2, \ldots$ Consequently, $B(z) = \dfrac{qz}{1-(1-q)z}$.

To illustrate the system numerically, the values for default parameters are considered as follows: First, we consider the following cases:

- Table 1 : $\lambda = 2.9 : 0.1 : 3.3$, $K = (1, 5, 9)$, $c = 3$, $q = 0.8$, $\mu = 4$, $\nu = 3.8$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 0.5$, and $\xi_2 = 0.8$.
- Table 2 : $\lambda = 3$, $K = 3$, $c = 3$, $q = (0.5, 0.7, 0.9)$, $\mu = 4$, $\nu = 3.8$, $\phi = 0.07 : 0.02 : 0.15$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 0.5$, and $\xi_2 = 0.8$.
- Table 3 : $\lambda = 3$, $K = 3$, $c = 3$, $q = 0.8$, $\mu = 4.6 : 0.4 : 6.2$, $\nu = 3.8$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 0.5$, and $\xi_2 = (0.8, 0.94, 1.04)$.
- Table 4 : $\lambda = 3.4$, $K = 3$, $c = 3$, $q = 0.8$, $\mu = 4.0$, $\nu = 0.35 : 0.2 : 1.15$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = (0.5, 0.7, 0.9)$, $\xi_1 = 0.5$, and $\xi_2 = 0.8$.
- Table 5 : $\lambda = 2.9$, $K = 3$, $c = 3$, $q = 0.8$, $\mu = 4.0$, $\nu = 3.8$, $\phi = 0.1$, $\beta = (0.5, 0.7, 0.9)$, $\alpha = 0.8$, $\xi_2 = 0.79 : 0.02 : 0.87$, and $\xi_1 = 0.5$.
- Table 6: $\lambda = 3.4$, $K = 3$, $c = (1, 2, 3)$, $q = 0.8$, $\mu = 4.0$, $\nu = 3.8$, $\phi = 0.1$, $\beta = 0.8$, $\alpha = 0.8$, $\xi_1 = 4.5 : 0.5 : 6.5$, and $\xi_2 = 0.8$.

Second, for economic cost results, we consider the following situations: $C_1 = 5$, $C_2 = 3$, $C_3 = 4$, $C_4 = 5$, $C_5 = 5$, $C_6 = 5$, $C_7 = 5$, $C_8 = 4$, $C_9 = 4$, $C_{10} = 5$, $C_{11} = 4$, and $R = 50$. Numerical results are presented in the following tables and figures.

5.1 Discussion on the results

- From Table 1 and Figs. 2, 3, we see that for different values of K, along the increase of the arrival rate λ, the probability that the system becomes empty $P_{0,0}$ decreases. Thus, the mean number of customers

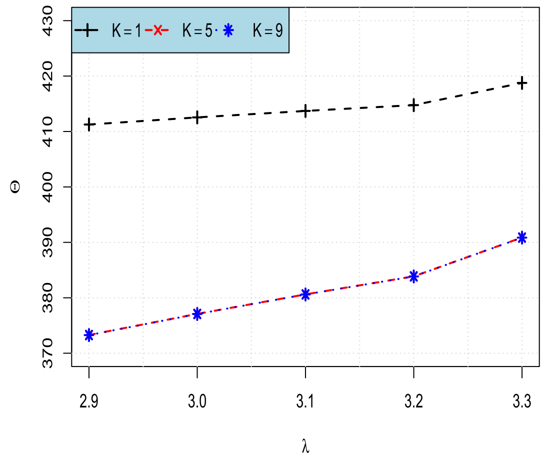

Fig. 3 Impact of λ on Θ

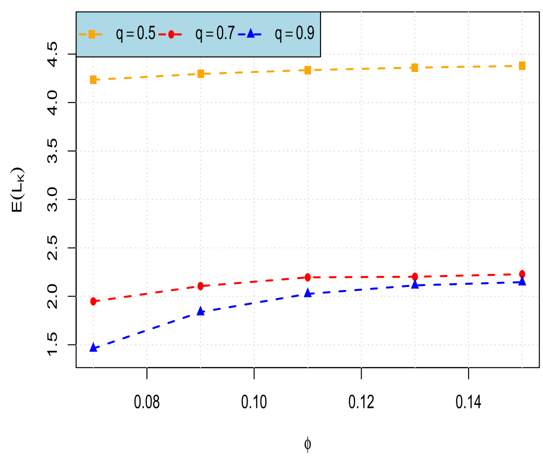

Fig. 4 Impact of ϕ on $E(L_{\mathrm{K}})$

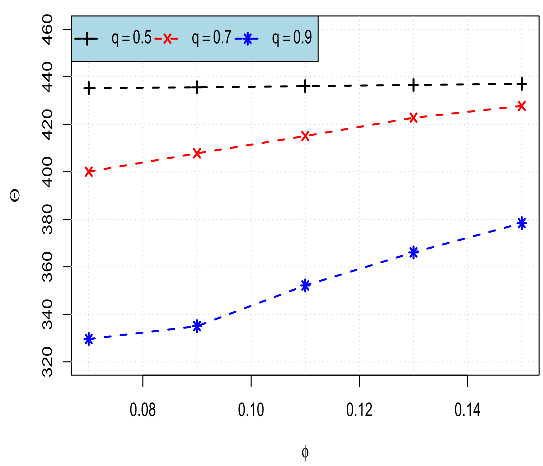

Fig. 5 Impact of ϕ on Θ

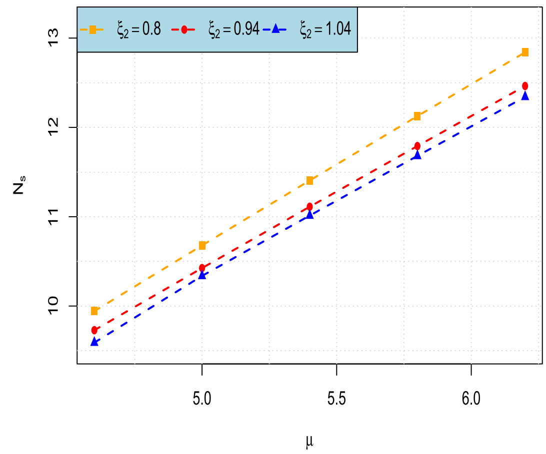

Fig. 6 Impact of μ on N_s

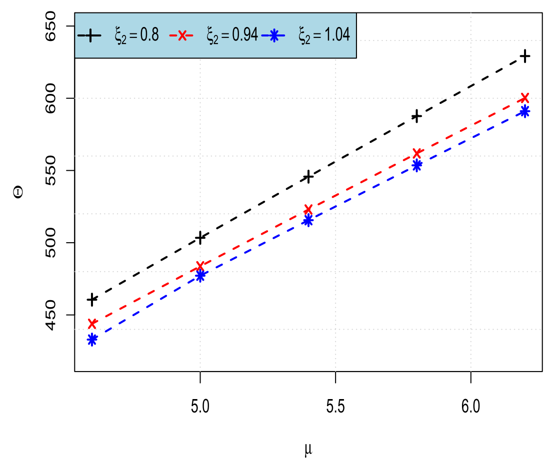

Fig. 7 Impact of μ on Θ

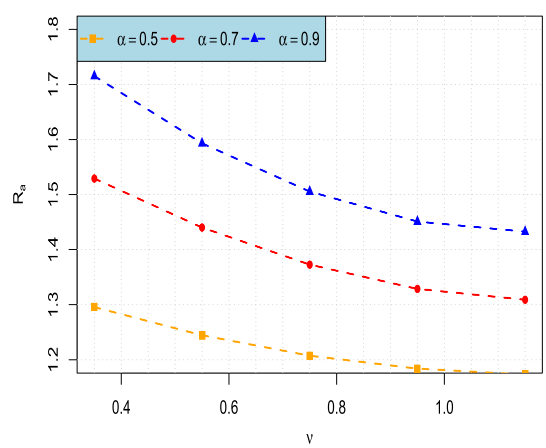

Fig. 8 Impact of ν on R_a

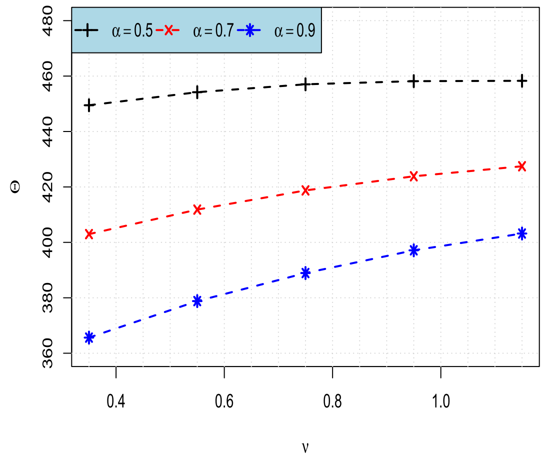

Fig. 9 Impact of ν on Θ

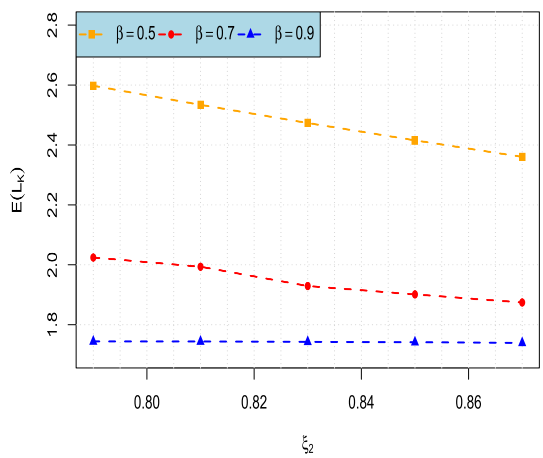

Fig. 10 Impact of ξ_2 on $E(L_{\mathrm{K}})$

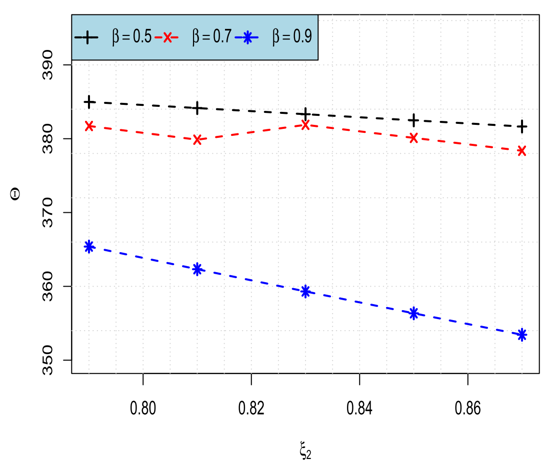

Fig. 11 Impact of ξ_2 on Θ

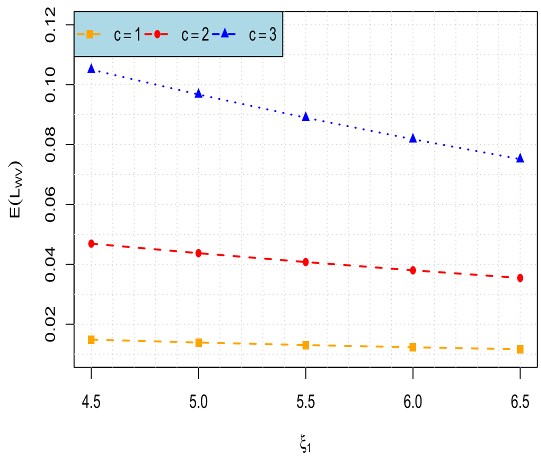

Fig. 12 Impact of ξ_1 on $E(L_{\mathrm{WV}})$

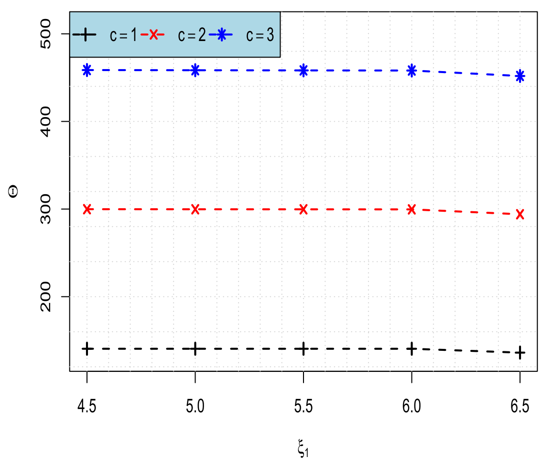

Fig. 13 Impact of ξ_1 on Θ

served increases. This implies an increase in the total expected profit Θ. Furthermore, it is well observed that the increase of the number of variant vacation has a bad effect on the system.

- The impact of vacation rate ϕ is depicted in Table 2 and Figs. 4, 5 for different mean batch sizes $1/q$. It can be observed that, for fixed q, as ϕ increases, the mean size of the system when the servers are in normal busy period $E(L_{\mathrm{K}})$ increases, as intuitively expected. On the other hand, for fixed ϕ, $E(L_{\mathrm{K}})$ increases with $1/q$, as it should be. Thus, it is clearly obvious that the total expected profit Θ increases with the increasing of ϕ, while the augmentation of q implies a lost in Θ.
- In Table 3 and Figs. 6, 7, we illustrate the effect of service rate during busy period μ, for various impatience rate during busy period ξ_2. It is quite clear that with the increase in the service rate μ, the mean number of customers served augments. Thus, the total expected profit Θ increases. Obviously, the number of customers served decreases when ξ_2 increases. Thus, we have a significant total expected profit Θ for large values of μ and small values of ξ_2.
- According to the results presented in Table 4 and Figs. 8, 9, we see that the average rate of abandonment R_{a} decreases with the increases in the service rate during vacation period ν. This is because the mean number of customers served augments with ν. Consequently, the average rate of abandonment is reduced. Furthermore, the increase in the probability of non-retention α implies an increase in R_{a}. Finally, it is well observed that the increases in the service rate during vacation period ν and in the retention probability α' have a nice impact on the total expected profit Θ.
- The impact of the impatience rate during busy period ξ_2 for different values of non-feedback probabilities β is illustrated in Table 5 and Figs. 10, 11. It is clearly shown that, with the increase in impatience rate during normal busy period ξ_2, the mean size on the system when the servers are in normal busy period $E(L_{\mathrm{K}})$

decreases, this implies a diminution in the mean number of customers served. Consequently, the total expected profit Θ decreases. Furthermore, from the above presentations, it is well seen that the feedback probability β' has a nice effect on the economy of the system.

– Figures 12 and 13 plot the impatience rate during working vacation period ξ_1 for different values of number of servers c. It is well observed that when the impatience rate ξ_1 is large, the mean size of the system when the servers are on working vacation period decreases. Therefore, the mean number of customers served is reduced. This leads to a decrease in Θ. On the other hand, from Table 6, we observe that when the number of servers becomes large, the total expected profit is significant. This is due to the fact that the mean number of customers served increases with c, while the average rate of abandonment decreases with the increasing of the number of the servers.

6 Conclusions and future scope

In the present study, we explored reneging behavior in multi-server Bernoulli feedback queueing system with batch arrival, variant of multiple working vacations and retention of the reneged customers. For the analysis purpose, we investigated various system characteristics in terms of steady-state probabilities using the probability generating functions (PGFs). Reneging and retention probabilities incorporated in our model may play an important role in the economy of the concerned system. Numerical experiments performed can be useful and benefic to explore the impacts of system parameters on the performance measures in different contexts. The model developed may provide lucrative perspicacity to the production managers, system engineers, etc. To make the system modeling more closer to the real-world problems, an extension of our results for a non-Markovian models is a pointer to future research. Moreover, we can extend this study by incorporating the bulk failure.

References

1. Altman, E.; Yechiali, U.: Infinite-server queues with system's additional tasks and impatient customers. Probab. Eng. Inf. Sci. **22**(4), 477–493 (2008)
2. Bouchentouf, A.A.; Yahiaoui, L.: On feedback queueing system with reneging and retention of reneged customers, multiple working vacations and Bernoulli schedule vacation interruption. Arab. J. Math. **6**(1), 1–11 (2017)
3. Doshi, B.T.: Single server queues with vacation: a survey. Q. Syst. **1**(1), 29–66 (1986)
4. Goswami, V.: Analysis of impatient customers in queues with Bernoulli schedule working vacations and vacation interruption. J. Stoch. **2014**, 10 (2014). Article ID 207285
5. Haridass, M.; Arrumuganathan, R.: Analysis of a bulk queue with unreliable server and single vacation. Int. J. Open Probl. Comput. Math. **1**(2), 130–148 (2008)
6. Ke, J.C.: Operating characteristic analysis on the $M^{[X]}/G/1$ system with a variant vacation policy and balking. Appl. Math. Model. **31**(7), 1321–1337 (2007)
7. Ke, J.C.; Huang, K.B.; Pearn, W.L.: Randomized policy of a Poisson input queue with J vacations. J. Syst. Sci. Syst. Eng. **19**(1), 50–71 (2010)
8. Laxmi, V.P.; Goswami, V.; Jyothsna, K.: Analysis of finite buffer Markovian queue with balking, reneging and working vacations. Int. J. Strategic Decis. Sci. **4**(1), 1–24 (2013)
9. Laxmi, V.P.; Jyothsna, K.: Performance analysis of variant working vacation queue with balking and reneging. Int. J. Math. Oper. Res. **6**(4), 548–565 (2014)
10. Laxmi, V.P.; Jyothsna, K.: Balking and reneging multiple working vacations queue with heterogeneous servers. J. Math. Model. Algorithm **14**(3), 267–285 (2015)
11. Laxmi, V.P.; Rajesh, P.: Analysis of variant working vacations queue with customer impatience. Int. J. Manag. Sci. Eng. Manag. **12**(3), 186–195 (2016)
12. Laxmi, V.P.; Rajesh, P.: Performance measures of variant working vacation on batch arrival queue with reneging. Int. J. Math. Arch. **8**(8), 85–96 (2017)
13. Madan, K.C.; AI-Rawwash, M.: On the $M^X/G/1$ queue with feedback and optional server vacations based on a single vacation policy. Appl. Math. Comput. **160**(3), 909–919 (2005)
14. Perel, N.; Yechiali, U.: Queues with slow servers and impatient customers. Eur. J. Oper. Res. **201**(1), 247–258 (2010)
15. Selvaraju, N.; Goswami, C.: Impatient customers in an $M/M/1$ queue with single and multiple working vacations. Comput. Ind. Eng. **65**(2), 207–215 (2013)

16. Sun, W.; Li, S.: Equilibrium and optimal behavior of customers in Markovian queues with multiple working vacations. TOP **22**(2), 694–715 (2014)
17. Takagi, H.: Queueing Analysis, A Foundation of Performance Evaluation, Volume 1: Vacation and Priority Systems, vol. 1. Elsevier, Amsterdam (1991)
18. Tian, N.; Zhang, Z.G.: Vacation Queueing Models: Theory and Applications. International Series in Operations Research and Management Science. Springer, New York (2006)
19. Tian, R.; Hu, L.; Wu, X.: Equilibrium and optimal strategies in $M/M/1$ queues with working vacations and vacation interruptions. Math. Probl. Eng. **2016**, 10 (2016). (Article ID 9746962)
20. Wang, K.H.; Chan, M.C.; Ke, J.C.: Maximum entropy analysis of the $M^x/M/1$ queueing system with multiple vacations and server breakdowns. Comput. Ind. Eng. **52**(2), 192–202 (2007)
21. Wang, T.Y.; Ke, J.C.; Chang, F.M.: On the discrete-time $Geo/G/1$ queue with randomized vacations and at most J vacations. Appl. Math. Model. **35**(5), 2297–2308 (2011)
22. Yue, D.; Yue, W.; Xu, G.: Analysis of customers impatience in an $M/M/1$ queue with working vacations. J. Ind. Manag. Optim. **8**(4), 895–908 (2012)
23. Yue, D.; Yue, W.; Saffer, Z.; Chen, X.: Analysis of an $M/M/1$ queueing system with impatient customers and a variant of multiple vacation policy. J. Ind. Manag. Optim. **10**(1), 89–112 (2014)
24. Zhang, Z.G.; Tian, N.: Discrete time $Geo/G/1$ queue with multiple adaptive vacations. Q Syst. **38**(4), 419–429 (2001)
25. Zhang, F.; Wang, J.; Liu, B.: Equilibrium balking strategies in Markovian queues with working vacations. Appl. Math. Model. **37**(16–17), 8264–8282 (2013)