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Lie symmetry analysis of some conformable fractional partial differential equations

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Abstract In this article, Lie symmetry analysis is used to investigate invariance properties of some nonlinear fractional partial differential equations with conformable fractional time and space derivatives. The analysis is applied to Korteweg–de Vries, modified Korteweg–de Vries, Burgers, and modified Burgers equations with conformable fractional time and space derivatives. For each equation, all the vector fields and the Lie symmetries are obtained. Moreover, exact solutions are given to these equations in terms of solutions of ordinary differential equations. In particular, it is shown that the fractional Korteweg–de Vries can be reduced to the first Painlevé equation and to the fractional second Painlevé equation. In addition, a solution of the fractional modified Korteweg–de Vries is given in terms of solutions of the fractional second Painlevé equation.

Mathematics Subject Classification 26A33 · 35R11

1 Introduction

In recent years, the interest in fractional calculus has increased due to its applications in many fields, such as Mathematics, Physics, Chemistry, Engineering, Finance, and Social sciences. As a result, several definitions for fractional derivatives appear in the literature to present more accurate models for real life phenomena. Some of known fractional derivatives are Riemann–Liouville, modified Riemann–Liouville, Caputo, Hadmard, Erdélyi–Kober, Riesz, Grünwald–Letnikov, Marchaud and others (see [11,12,26,27,29,31,33,34,37,40]). All known fractional derivatives satisfy one of the well-known properties of classical derivative, namely the linear property. However, the other properties of classical derivative, such as the derivative of a constant is zero; the product rule, quotient rule, and the chain rule, either do not hold or are too complicated for many fractional derivatives.

Recently, a new definition of fractional derivative that extends the familiar limit definition of the derivative of a function has been introduced by Khalil et al. [28]. The new definition is called the conformable fractional derivative. Unlike other definitions, this new definition is prominently compatible with the classical derivative and seems to satisfy all the requirements of the usual derivative. The importance of the conformable fractional derivative lies in satisfying the product and the quotient formulas. In [1], many properties of the conformable fractional derivative were proved, such as chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, and Laplace transforms. In [24], more results on the conformable fractional integral and derivative were given such as the extended mean value theorem and the Racetrack type principle. A conformable time-scale fractional calculus was introduced in [6] and a generalization of the conformable fractional derivative was given in [50]. Many studies related to this new fractional derivative were published [2–4,15,19,32,41,43,51].

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The Lie symmetry theory plays a significant role in the analysis of differential equations. The Norwegian mathematician Sophus Lie devoted the first work exclusively to the subject of Lie symmetry in the 19th century. It is regarded as the most important approach for constructing analytical solutions of nonlinear differential equations. After that, many papers and excellent textbooks have been devoted to the theory of Lie symmetry groups and their applications to differential equations; for examples, see [7–9,14,23,35]. Lie group analysis of fractional differential equations was investigated recently in [5,10,13,16–18,20–22,25,30,36,38,39,44–49]. The Lie symmetry analysis of time-fractional Burgers and Korteweg–de Vries (KdV) equations with Riemann–Liouville time derivative was studied in [39]. The Lie symmetry analysis of the KdV equations with modified Riemann–Liouville time-fractional derivative was investigated in [45]. It was shown that each of these equations can be reduced to a nonlinear ordinary differential equation of fractional order with a new independent variable. The fractional derivative in the reduced equations turned out to be the Erdelyi–Kober fractional derivative. In [42], the Lie symmetry analysis of Korteweg–de Vries, modified Korteweg–de Vries, Burgers, and modified Burgers equations with conformable fractional time derivative and classical space derivative has been investigated.

In this article, we derive the prolongation formulas for conformable fractional derivatives and apply the method of Lie group to conformable fractional partial differential equations (CFPDEs). We study the Lie symmetry analysis of Korteweg-de Vries, modified Korteweg–de Vries, Burgers, and modified Burgers equations with conformable fractional time and space partial derivatives. For each equation, all the vector fields and the Lie symmetries are obtained. We show that the equations under consideration can be reduced to ordinary differential equations with classical or fractional derivatives. In particular, we derive solutions of the conformable fractional Korteweg–de Vries and modified Korteweg–de Vries equations in terms of conformable fractional Painlevé equations.

2 Conformable fractional calculus

Definition 2.1 [28] Given a function $f : [0,\infty) \to \mathbb{R}$, the conformable fractional derivative of order α of f is defined by

$$D^{\alpha}[f(t)] = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{1}$$

for all $t > 0,\ \alpha \in (0,1]$. If $D^{\alpha}[f(t)]$ exists for t in some interval $(0,a),\ a > 0$, and $\lim_{t\to 0^+} D^{\alpha}[f(t)]$ exists, then $D^{\alpha}[f(0)] = \lim_{t\to 0^+} D^{\alpha}[f(t)]$.

If $D^{\alpha}[f(t)]$ exists for $t \in [0,\infty)$, then f is said to be α-differentiable at t. One should notice that a function could be α-differential at a point but not differentiable at the same point.

As an example $f(t) = \sqrt{t}$, $\quad D^{\frac{1}{2}}[f(t)] = \frac{1}{2}$. Consequently, $D^{\frac{1}{2}}[f(0)] = \frac{1}{2}$, but the first derivative is given by $D[f(0)]$, does not exist.

Theorem 2.2 [28] *Let $\alpha \in (0,1]$ and $f,\ g$ be α-differentiable at a point $t > 0$. Then,*

1. $D^{\alpha}[af(t) + bg(t)] = a[D^{\alpha}f(t)] + b[D^{\alpha}g(t)]$, *for all* $a, b \in \mathbb{R}$.
2. *If* $f(t) = t^p$, *then* $D^{\alpha}[f(t)] = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$.
3. *If f is the constant function defined by* $f(t) = c$, *then* $D^{\alpha}[f(t)] = 0$.
4. $D^{\alpha}[f(t)g(t)] = f(t)D^{\alpha}[g(t)] + g(t)D^{\alpha}[f(t)]$.
5. $D^{\alpha}\left[\frac{f(t)}{g(t)}\right] = \frac{g(t)D^{\alpha}[f(t)] - f(t)D^{\alpha}[g(t)]}{[g(t)]^2}$.
6. *If, in addition, f is differentiable, then* $D^{\alpha}f(t) = t^{1-\alpha}\frac{\mathrm{d}f(t)}{\mathrm{d}t}$.

Definition 2.3 [28] $I^{\alpha}[f(t)] = I[t^{\alpha-1}f(t)] = \int_0^t \frac{f(\tau)}{\tau^{1-\alpha}}d\tau$, where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 2.4 [28] $D^{\alpha}I^{\alpha}[f(t)] = f(t)$, *for* $t \geq 0$, *where f is any continuous function in the domain of* I^{α}.

Lemma 2.5 [1] *Let $f : [0,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have* $I^{\alpha}D^{\alpha}[f(t)] = f(t) - f(0)$.

Lemma 2.6 [1] *Let $0 < \alpha \leq 1$, f be α-differentiable at $g(t) > 0$, and g be α-differentiable at $t > 0$. Then,* $D^{\alpha}[(fog)(t)] = D^{\alpha}[f(g(t))]D^{\alpha}[g(t)][g(t)]^{\alpha-1}$.

Lemma 2.7 [32] *Let* $0 < \alpha \leq 1$, f *be differentiable at* $g(t)$, *and* g *be* α-*differentiable at* $t > 0$. *Then,* $D^{\alpha}[(fog)(t)] = [f'(g(t))]D^{\alpha}[g(t)]$.

3 Lie symmetry analysis of CFPDEs

Consider a conformable fractional partial differential equation in the form

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = F\left(t, x, u, \frac{\partial^{\alpha} u}{\partial x^{\alpha}}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}, \cdots\right), \quad 0 < \beta,\ \alpha \leq 1, \tag{2}$$

where $u = u(x, t)$, $F(t, x, u, \frac{\partial^{\alpha} u}{\partial x^{\alpha}}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}, \cdots)$ is a nonlinear function, $\frac{\partial^{\beta} u}{\partial t^{\beta}}$ and $\frac{\partial^{\alpha} u}{\partial x^{\alpha}}$ are the conformable fractional derivatives of order β and α, respectively. Here, $\frac{\partial^{n\alpha} u}{\partial x^{n\alpha}}$ are the sequential fractional derivatives given by

$$\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{\alpha} u}{\partial x^{\alpha}}, \qquad \frac{\partial^{n\alpha} u}{\partial x^{n\alpha}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}} \frac{\partial^{(n-1)\alpha}}{\partial x^{(n-1)\alpha}}, \quad n = 3, 4, \cdots .$$

Our aim is to study the symmetry transformations of Eq. (2).

The invertible point transformations

$$\hat{x} = X(t, x, u, \varepsilon), \quad \hat{t} = T(t, x, u, \varepsilon), \quad \hat{u} = U(t, x, u, \varepsilon), \tag{3}$$

depending on a continuous parameter ε, are said to be symmetry transformations of Eq. (2), if Eq. (2) has the same form in the new variables $\hat{x}$, $\hat{t}$, $\hat{u}$. The set G of all such transformations forms a continuous group. The symmetry group G is also known as the group admitted by Eq. (2).

The key step in obtaining a Lie group of symmetry transformations is to find the infinitesimal generator of the group. To provide a basis of group generators, one has to create and solve the so-called determining system of equations.

The infinitesimal transformations of (3) read

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi(t, x, u) + o(\varepsilon^2), \\ \hat{t} &= t + \varepsilon\tau(t, x, u) + o(\varepsilon^2), \\ \hat{u} &= u + \varepsilon\eta(t, x, u) + o(\varepsilon^2). \end{aligned} \tag{4}$$

It is convenient to introduce the operator

$$V = \xi(t, x, u)\frac{\partial}{\partial x} + \tau(t, x, u)\frac{\partial}{\partial t} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{5}$$

which is known as the infinitesimal operator or the generator of the group G. The group transformations (3) corresponding to operator (5) can be obtained by solving the Lie equations

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{t}, \hat{x}, \hat{u}), \qquad \frac{d\hat{t}}{d\varepsilon} = \tau(\hat{t}, \hat{x}, \hat{u}), \qquad \frac{d\hat{u}}{d\varepsilon} = \eta(\hat{t}, \hat{x}, \hat{u}), \tag{6}$$

subject to the initial conditions

$$\hat{x}|_{\varepsilon=0} = x, \qquad \hat{t}|_{\varepsilon=0} = t, \qquad \hat{u}|_{\varepsilon=0} = u. \tag{7}$$

A surface $u = u(t, x)$ is mapped to itself by the group of transformations generated by V if

$$V(u - u(t, x)) = 0 \qquad \text{when} \qquad u = u(t, x). \tag{8}$$

By definition, the transformations (3) form a symmetry group G of Eq. (2) if the function $\hat{u}(\hat{t}, \hat{x})$ satisfies the equation

$$\frac{\partial^{\beta} \hat{u}}{\partial \hat{t}^{\beta}} = F\left(\hat{t}, \hat{x}, \hat{u}, \frac{\partial^{\alpha} \hat{u}}{\partial \hat{x}^{\alpha}}, \frac{\partial^{2\alpha} \hat{u}}{\partial \hat{x}^{2\alpha}}, \frac{\partial^{3\alpha} \hat{u}}{\partial \hat{x}^{3\alpha}}, \cdots\right), \quad 0 < \beta,\ \alpha \leq 1, \tag{9}$$

whenever the function $u = u(t, x)$ satisfies Eq. (2). Extending transformation (4) to the operator of fractional differentiation $\frac{\partial^\beta u}{\partial t^\beta}$ and to the operator of fractional differentiation of various orders $\frac{\partial^\alpha u}{\partial x^\alpha}$, $\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}$, $\frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}, \ldots$, one can obtain

$$\begin{aligned}
\frac{\partial^\beta \hat{u}}{\partial \hat{t}^\beta} &= \frac{\partial^\beta u}{\partial t^\beta} + \varepsilon \eta_\beta^t(t, x, u) + o(\varepsilon^2),\\
\frac{\partial^\alpha \hat{u}}{\partial \hat{x}^\alpha} &= \frac{\partial^\alpha u}{\partial x^\alpha} + \varepsilon \eta_\alpha^x(t, x, u) + o(\varepsilon^2),\\
\frac{\partial^{2\alpha} \hat{u}}{\partial \hat{x}^{2\alpha}} &= \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} + \varepsilon \eta_\alpha^{xx}(t, x, u) + o(\varepsilon^2),\\
\frac{\partial^{3\alpha} \hat{u}}{\partial \hat{x}^{3\alpha}} &= \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} + \varepsilon \eta_\alpha^{xxx}(t, x, u) + o(\varepsilon^2),\\
&\vdots
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\eta_\beta^t &= t^{1-\beta}\eta^t + (1-\beta)\tau t^{-\beta} u_t,\\
\eta_\alpha^x &= x^{1-\alpha}\eta^x + (1-\alpha)\xi x^{-\alpha} u_x,\\
\eta_\alpha^{xx} &= x^{2-2\alpha}\eta^{xx} + (1-\alpha)x^{1-2\alpha}\eta^x + (2-2\alpha)x^{1-2\alpha}\xi u_{xx}\\
&\quad + (1-\alpha)(1-2\alpha)x^{-2\alpha}\xi u_x,\\
\eta_\alpha^{xxx} &= x^{3-3\alpha}\eta^{xxx} + (3-3\alpha)x^{2-3\alpha}\eta^{xx} + (1-\alpha)(1-2\alpha)x^{1-3\alpha}\eta^x\\
&\quad + (3-3\alpha)\xi x^{2-3\alpha} u_{xxx} + (3-3\alpha)(2-3\alpha)\xi x^{1-3\alpha} u_{xx}\\
&\quad + (1-\alpha)(1-2\alpha)(1-3\alpha)\xi x^{-3\alpha} u_x,\\
&\vdots
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
\eta^t &= D_t(\eta) - u_x D_t(\xi) - u_t D_t(\tau),\\
\eta^x &= D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),\\
\eta^{xx} &= D_x(\eta^x) - u_{xx} D_x(\xi) - u_{xt} D_x(\tau),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxx} D_x(\xi) - u_{xxt} D_x(\tau),\\
&\vdots
\end{aligned} \tag{12}$$

Here, D_t and D_x denote the total derivative operators and are defined as:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots\\
D_x &= \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots
\end{aligned}$$

If the vector field (5) generates a symmetry of (2), then V must satisfy Lie symmetry condition

$$pr^{(n\alpha,\beta)} V(\Delta_1)\Big|_{\Delta_1=0} = 0, \tag{13}$$

where $\Delta_1 = \frac{\partial^\beta u}{\partial t^\beta} - F\left(t, x, u, \frac{\partial^\alpha u}{\partial x^\alpha}, \frac{\partial^{2\alpha} u}{\partial x^{2\alpha}}, \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}, \cdots\right)$.

4 The fractional Korteweg–de Vries equation

In this section, we consider the following fractional Korteweg–de Vries (KdV) equation of the form

$$\frac{\partial^\beta u}{\partial t^\beta}+6u\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}=0, \tag{14}$$

where $0<\beta,\ \alpha\leq 1,\ \beta$ and α are parameters describing the order of the conformable fractional time and space derivatives, respectively. According to the Lie theory, applying the $(3\alpha,\ \beta)-$ prolongation $pr^{(3\alpha,\ \beta)}V$ to (14), we find the infinitesimal criterion (13) to be

$$6\eta\frac{\partial^\alpha u}{\partial x^\alpha}+\eta^t_\beta+6u\eta^x_\alpha+\eta^{xxx}_\alpha=0, \tag{15}$$

which must be satisfied whenever $\frac{\partial^\beta u}{\partial t^\beta}+6u\frac{\partial^\alpha u}{\partial x^\alpha}+\frac{\partial^{3\alpha} u}{\partial x^{3\alpha}}=0$. It is worth to note that using Theorem (2.2), we find that (14) is equivalent to the following equation:

$$\begin{aligned}&t^{1-\beta}u_t+6x^{1-\alpha}uu_x+(1-\alpha)(1-2\alpha)x^{1-3\alpha}u_x\\&+3(1-\alpha)x^{2-3\alpha}u_{xx}+x^{3-3\alpha}u_{xxx}=0.\end{aligned} \tag{16}$$

Substituting the general formulae for $\eta^x_\alpha,\ \eta^{xxx}_\alpha$ and η^t_β from (11) and (12) into (15), using (16) to replace u_{xxx} whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, we can get the full determining equations for the symmetry group of (14). Solving these equations, we obtain

$$\tau=\frac{-3c_1}{2\beta}t+c_2t^{1-\beta},\quad \xi=\frac{-c_1}{2\alpha}x+\frac{6c_3}{\beta}t^\beta x^{1-\alpha}+c_4x^{1-\alpha},\quad \eta=c_1u+c_3, \tag{17}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Therefore, the symmetry group of (14) is spanned by the four vector fields

$$\begin{aligned}V_1&=t^{1-\beta}\tfrac{\partial}{\partial t}, & V_2&=x^{1-\alpha}\tfrac{\partial}{\partial x},\\ V_3&=\tfrac{6t^\beta x^{1-\alpha}}{\beta}\tfrac{\partial}{\partial x}+\tfrac{\partial}{\partial u}, & V_4&=\tfrac{-3t}{2\beta}\tfrac{\partial}{\partial t}-\tfrac{x}{2\alpha}\tfrac{\partial}{\partial x}+u\tfrac{\partial}{\partial u}.\end{aligned} \tag{18}$$

The commutation relations between these vector fields are given by

$$[V_1,V_2]=0,\quad [V_1,V_3]=6V_2,\quad [V_1,V_4]=\frac{-3}{2}V_1, \tag{19}$$

$$[V_2,V_3]=0,\quad [V_2,V_4]=\frac{-1}{2}V_2,\quad [V_3,V_4]=V_3, \tag{20}$$

where the Lie bracket of two vector fields is defined by $[\rho,\sigma]=\rho\sigma-\sigma\rho$. Thus, we see that the set of these vector fields is closed under the Lie bracket.

The similarity variables for the infinitesimal generator V_4 can be found by solving the corresponding characteristic equation

$$\frac{\alpha \mathrm{d}x}{x}=\frac{\beta \mathrm{d}t}{3t}=\frac{-\mathrm{d}u}{2u}, \tag{21}$$

and the corresponding invariants are

$$\zeta=xt^{\frac{-\beta}{3\alpha}},\qquad u=t^{\frac{-2\beta}{3}}\Psi(\zeta). \tag{22}$$

Substituting transformation (22) into (14), we find that (14) can be reduced to

$$\frac{d^{3\alpha}\Psi(\zeta)}{d\zeta^{3\alpha}}+6\Psi(\zeta)\frac{d^\alpha\Psi(\zeta)}{d\zeta^\alpha}-\frac{\beta}{3}\frac{\zeta^\alpha}{\alpha}\frac{d^\alpha\Psi(\zeta)}{d\zeta^\alpha}-\frac{2\beta}{3}\Psi(\zeta)=0. \tag{23}$$

Equation (23) is a nonlinear fractional ordinary differential equation with conformable derivative. The scale $\omega = \left(\frac{\beta}{3}\right)^{\frac{1}{3\alpha}} \zeta$, $\Psi(\zeta) = \left(\frac{\beta}{3}\right)^{\frac{2}{3}} W(\omega)$ transforms (23) to an equivalent form

$$K_1(W) = \frac{d^{3\alpha} W(\omega)}{d\omega^{3\alpha}} + 6W(\omega)\frac{d^{\alpha} W(\omega)}{d\omega^{\alpha}} - \frac{\omega^{\alpha}}{\alpha}\frac{d^{\alpha} W(\omega)}{d\omega^{\alpha}} - 2W(\omega) = 0. \tag{24}$$

Equation (24) can be integrated once using the following identity:

$$\frac{d^{\alpha}}{d\omega^{\alpha}}\left[\left(2W - \frac{\omega^{\alpha}}{\alpha}\right) K_2(W)\right] = \left(2W - \frac{\omega^{\alpha}}{\alpha}\right) K_1(W), \tag{25}$$

where

$$K_2(W) = \frac{d^{2\alpha} W}{d\omega^{2\alpha}} + 2W^2 - \frac{\omega^{\alpha}}{\alpha} W + \frac{\gamma(\gamma+1) + \frac{d^{\alpha} W}{d\omega^{\alpha}} - (\frac{d^{\alpha} W}{d\omega^{\alpha}})^2}{2W - \frac{\omega^{\alpha}}{\alpha}} = 0. \tag{26}$$

Equation (26), under the transformation $\Theta = (W - \frac{\omega^{\alpha}}{2\alpha})/(4\gamma+1)$, is reduced to the fractional thirty fourth Painlevé equation (FP_{34})

$$\frac{d^{2\alpha}\Theta}{d\omega^{2\alpha}} - \frac{1}{2\Theta}\left(\frac{d^{\alpha}\Theta}{d\omega^{\alpha}}\right)^2 - 4\sigma\Theta^2 + \frac{\omega^{\alpha}}{\alpha}\Theta + \frac{1}{2\Theta} = 0 \tag{27}$$

with $\sigma = \frac{-1}{6}$. The solutions of (26) are also expressible in terms of solutions of second fractional Painlevé equation (FP_{II}). There exists the following one-to-one correspondence between solutions of (26) and those of FP_{II}, given by

$$W = -\frac{d^{\alpha}\Phi}{d\omega^{\alpha}} - \Phi^2, \qquad \Phi = \frac{\frac{d\alpha W}{d\omega^{\alpha}} + \gamma}{2W - \frac{\omega^{\alpha}}{\alpha}}, \tag{28}$$

where Φ satisfies the FP_{II} equation

$$\frac{d^{2\alpha}\Phi(\omega)}{d\omega^{2\alpha}} = 2\Phi^3(\omega) + \frac{\omega^{\alpha}}{\alpha}\Phi(\omega) + \gamma. \tag{29}$$

As a second example, we consider the linear combination $V_3 + aV_1$, where a is a constant, to obtain another similarity reduction by solving the corresponding characteristic equation

$$\frac{\beta dx}{6t^{\beta}x^{1-\alpha}} = \frac{dt}{at^{1-\beta}} = \frac{du}{1}. \tag{30}$$

The corresponding invariants are

$$\zeta = \frac{x^{\alpha}}{\alpha} - \frac{3}{a\beta^2}t^{2\beta}, \qquad u = \frac{1}{a\beta}t^{\beta} + \Psi(\zeta). \tag{31}$$

Substituting transformation (31) into Eq. (14), one can find that (14) can be reduced to the following nonlinear ordinary differential equation with classical derivative:

$$\frac{d^3\Psi(\zeta)}{d\zeta^3} + 6\Psi(\zeta)\frac{d\Psi(\zeta)}{d\zeta} + \frac{1}{a} = 0. \tag{32}$$

After a first integration, we get

$$\frac{d^2\Psi(\zeta)}{d\zeta^2} + 3\Psi^2(\zeta) + \frac{1}{a}\zeta = \gamma, \tag{33}$$

where γ is a constant of integration. Equation (33) is a second-order nonlinear differential equation with classical derivative and it can be reduced to the first Painlevé equation (P_I)

$$\frac{d^2\Phi(z)}{dz^2} = 6\Phi^2(z) + z, \tag{34}$$

via the change of variables $z = \left(\frac{1}{2a}\right)^{\frac{1}{5}}(\zeta - \gamma a)$ and $\Psi(\zeta) = -2\left(\frac{1}{2a}\right)^{\frac{2}{5}}\Phi(z)$.

5 The fractional modified Korteweg–de Vries equation

This section investigates the Lie symmetry analysis of the fractional modified Korteweg–de Vries (mKdV) equation

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} - 6u^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} = 0, \tag{35}$$

where $0 < \beta,\ \alpha \leq 1$, and $\beta,\ \alpha$ are parameters describing the order of the conformable fractional time and space derivatives, respectively. According to the Lie theory, applying the $(3\alpha, \beta)-$ prolongation $pr^{(3\alpha,\ \beta)}V$ to (35), we find the infinitesimal criterion (13) to be

$$-12\eta u \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + \eta^{t}_{\beta} - 6\eta^{x}_{\alpha} u^2 + \eta^{xxx}_{\alpha} = 0, \tag{36}$$

which must be satisfied whenever $\frac{\partial^{\beta} u}{\partial t^{\beta}} - 6u^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + \frac{\partial^{3\alpha} u}{\partial x^{3\alpha}} = 0$. Now, we write equation (35) in the equivalent form

$$\begin{aligned} &t^{1-\beta} u_t - 6x^{1-\alpha} u^2 u_x + (1-\alpha)(1-2\alpha) x^{1-3\alpha} u_x \\ &+3(1-\alpha) x^{2-3\alpha} u_{xx} + x^{3-3\alpha} u_{xxx} = 0. \end{aligned} \tag{37}$$

Direct substitution of $\eta^{x}_{\alpha},\ \eta^{xxx}_{\alpha}$ and η^{t}_{β} from (11) and (12) into (36), using (37) to replace u_{xxx} whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, we get the full determining equations for the symmetry group of (35). Solving these equations, we obtain

$$\tau = \frac{-3c_1}{\beta} t + c_2 t^{1-\beta}, \qquad \xi = -\frac{c_1}{\alpha} x + c_3 x^{1-\alpha}, \qquad \eta = c_1 u, \tag{38}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants. Therefore, the symmetry group of (35) is spanned by the three vector fields

$$\begin{aligned} &V_1 = t^{1-\beta} \frac{\partial}{\partial t}, \quad V_2 = x^{1-\alpha} \frac{\partial}{\partial x}, \\ &V_3 = \frac{3t}{\beta} \frac{\partial}{\partial t} + \frac{x}{\alpha} \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}. \end{aligned} \tag{39}$$

These vector fields satisfy Lie bracket relations

$$[V_1, V_2] = 0, \qquad [V_1, V_3] = 3V_1, \qquad [V_2, V_3] = V_2. \tag{40}$$

Note that when $\beta = \alpha = 1$, the vector fields of the fractional mKdV equation reduce to the vector fields of the classical mKdV equation [7].

The similarity variables for the infinitesimal generator V_3 can be found by solving the corresponding characteristic equations

$$\frac{\alpha \mathrm{d}x}{x} = \frac{\beta \mathrm{d}t}{3t} = \frac{\mathrm{d}u}{-u}. \tag{41}$$

The corresponding invariants are

$$\zeta = x t^{\frac{-\beta}{3\alpha}}, \qquad u = t^{\frac{-\beta}{3}} \Psi(\zeta). \tag{42}$$

Using the transformation (42), Eq. (35) can be reduced to the nonlinear $FODE$

$$\frac{d^{3\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{3\alpha}} - 6\Psi^2(\zeta) \frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} - \frac{\beta}{3} \frac{\zeta^{\alpha}}{\alpha} \frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} - \frac{\beta}{3} \Psi(\zeta) = 0. \tag{43}$$

As a result, we have

$$\frac{d^{2\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{2\alpha}} - 2\Psi^3(\zeta) - \frac{\beta}{3} \frac{\zeta^{\alpha}}{\alpha} \Psi(\zeta) = \gamma, \tag{44}$$

where γ is a constant of integration. Equation (44) can be converted by the scale $\omega = \left(\frac{\beta}{3}\right)^{\frac{1}{3\alpha}} \zeta,\ \Psi(\zeta) = \left(\frac{\beta}{3}\right)^{\frac{1}{3}} \Phi(\omega)$ to the fractional Painlevé equation FP_{II}

$$\frac{d^{2\alpha}\Phi(\omega)}{\mathrm{d}\omega^{2\alpha}} - 2\Phi^3(\omega) - \frac{\omega^\alpha}{\alpha}\Phi(\omega) = \mu, \tag{45}$$

where $\mu = 3\gamma$.

6 The fractional Burgers equation

This section is devoted to the Lie symmetry analysis of the following fractional Burgers equation

$$\frac{\partial^\beta u}{\partial t^\beta} + au\frac{\partial^\alpha u}{\partial x^\alpha} + b\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0, \tag{46}$$

where $0 < \beta,\ \alpha \leq 1$, β and α are parameters describing the order of the conformable fractional time and space derivatives. According to the Lie theory, applying the $(2\alpha,\ \beta)-$prolongation $pr^{(2\alpha,\ \beta)}V$ to (46), the infinitesimal criterion (13) reads

$$a\eta\frac{\partial^\alpha u}{\partial x^\alpha} + \eta^t_\beta + au\eta^x_\alpha + b\eta^{xx}_\alpha = 0. \tag{47}$$

The condition (47) must be satisfied whenever $\frac{\partial^\beta u}{\partial t^\beta} + au\frac{\partial^\alpha u}{\partial x^\alpha} + b\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0$. Equation (46) has the equivalent form

$$t^{1-\beta}u_t + ax^{1-\alpha}uu_x + b(1-\alpha)x^{1-2\alpha}u_x + bx^{2-2\alpha}u_{xx} = 0. \tag{48}$$

Next, we use (11) and (12) to substitute $\eta^x_\alpha,\ \eta^{xx}_\alpha$ and η^t_β into (47), and (48) to replace u_{xx} whenever it occurs. After equating the coefficients of the various monomials in partial derivatives of u, we get the full determining equations for the symmetry group of (46). Solving these equations, we obtain

$$\begin{aligned}\tau &= \frac{c_1 t^{1+\beta}}{\beta^2} - \frac{2c_2 t}{\beta} + c_4 t^{1-\beta},\\ \xi &= \frac{c_1 x t^\beta}{\alpha\beta} - \frac{c_2 x}{\alpha} + \frac{ac_3}{\beta}t^\beta x^{1-\alpha} + c_5 x^{1-\alpha},\\ \eta &= \left(\frac{-c_1 t^\beta}{\beta} + c_2\right)u + \left(\frac{c_1 x^\alpha}{a\alpha} + c_3\right),\end{aligned} \tag{49}$$

where $c_1,\ c_2,\ c_3,\ c_4$ and c_5 are arbitrary constants. Therefore, the symmetry group of (46) is spanned by the five vector fields

$$\begin{aligned}&V_1 = t^{1-\beta}\frac{\partial}{\partial t}, \quad V_2 = x^{1-\alpha}\frac{\partial}{\partial x},\\ &V_3 = \frac{at^\beta x^{1-\alpha}}{\beta}\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad V_4 = \frac{-2t}{\beta}\frac{\partial}{\partial t} - \frac{x}{\alpha}\frac{\partial}{\partial x} + u\frac{\partial}{\partial u},\\ &V_5 = \frac{t^{1+\beta}}{\beta^2}\frac{\partial}{\partial t} + \frac{xt^\beta}{\alpha\beta}\frac{\partial}{\partial x} + \left(\frac{-t^\beta u}{\beta} + \frac{x^\alpha}{a\alpha}\right)\frac{\partial}{\partial u}.\end{aligned} \tag{50}$$

It is easily checked that these five vector fields satisfy

$$\begin{aligned}&[V_1, V_2] = [V_2, V_3] = [V_3, V_5] = 0, \quad [V_2, V_4] = -V_2,\\ &[V_2, V_5] = \frac{1}{a}V_3, [V_1, V_3] = aV_2, \quad [V_1, V_4] = -2V_1,\\ &[V_1, V_5] = -V_4, \quad [V_3, V_4] = V_3, \quad [V_4, V_5] = -2V_5.\end{aligned} \tag{51}$$

Thus, the Lie algebra of infinitesimal symmetries of equation (46) is spanned by these five vector fields. The number of the vector fields coincides with that of the classical Burgers equation and when $\beta = \alpha = 1$ these vector fields reduce to that of the classical Burgers equation [14].

The similarity variables for the infinitesimal generator V_4 can be found by solving the corresponding characteristic equation

$$\frac{-\alpha \mathrm{d}x}{x} = \frac{\beta \mathrm{d}t}{-2t} = \frac{\mathrm{d}u}{u}, \tag{52}$$

and the corresponding invariants are

$$\zeta = x t^{\frac{-\beta}{2\alpha}}, \qquad u = t^{\frac{-\beta}{2}} \Psi(\zeta). \tag{53}$$

The transformation (53) reduces Eq. (46) to the following nonlinear $FODE$

$$b\frac{d^{2\alpha}\Psi(\zeta)}{d\zeta^{2\alpha}} + a\Psi(\zeta)\frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} - \frac{\beta}{2}\frac{\zeta^{\alpha}}{\alpha}\frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} - \frac{\beta}{2}\Psi(\zeta) = 0. \tag{54}$$

Consequently, we have

$$b\frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} + \frac{a}{2}\Psi^2(\zeta) - \frac{\beta}{2}\frac{\zeta^{\alpha}}{\alpha}\Psi(\zeta) = \gamma, \tag{55}$$

where γ is a constant of integration.

The fractional Riccati equation (55) can be transformed by the transform $\Psi(\zeta) = \frac{2b}{a}\Phi^{-1}(\zeta)\frac{d^{\alpha}\Phi(\zeta)}{d\zeta^{\alpha}}$ to the linear equation

$$\frac{d^{2\alpha}\Phi(\zeta)}{\mathrm{d}\zeta^{2\alpha}} - \frac{\beta}{2b}\frac{\zeta^{\alpha}}{\alpha}\frac{d^{\alpha}\Phi(\zeta)}{\mathrm{d}\zeta^{\alpha}} + \frac{\gamma}{b}\Phi(\zeta) = 0. \tag{56}$$

From the linear combination $V_3 + \mu V_1$, where μ is a constant, another similarity reduction can be found by solving the corresponding characteristic equation

$$\frac{\beta \mathrm{d}x}{a t^{\beta} x^{1-\alpha}} = \frac{\mathrm{d}t}{\mu t^{1-\beta}} = \frac{\mathrm{d}u}{1}, \tag{57}$$

and the corresponding invariants are

$$\zeta = \frac{x^{\alpha}}{\alpha} - \frac{a}{2\mu\beta^2}t^{2\beta}, \qquad u = \frac{1}{\mu\beta}t^{\beta} + \Psi(\zeta). \tag{58}$$

Substituting transformation (58) into Eq. (46), we find that (46) can be reduced to a nonlinear ODE with the classical derivative

$$b\Psi''(\zeta) + a\Psi(\zeta)\Psi'(\zeta) + \frac{1}{\mu} = 0, \tag{59}$$

where $\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$. From which we obtain the Riccati equation

$$b\Psi'(\zeta) + \frac{a}{2}\Psi^2(\zeta) + \frac{1}{\mu}\zeta = \gamma, \tag{60}$$

where γ is a constant of integration.

7 The fractional modified Burgers equation

In this section, we will consider the Lie symmetry analysis of the following nonlinear fractional modified Burgers equation

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} + au^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + b\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0, \tag{61}$$

where $0 < \beta,\ \alpha \leq 1,\ \beta$ and α are parameters describing the order of the conformable fractional time and space derivatives. According to the Lie theory, applying the $(2\alpha,\ \beta)$−prolongation $pr^{(2\alpha,\ \beta)}V$ to (61), one can find the infinitesimal criterion (13) to be

$$2a\eta x^{1-\alpha} u u_x + \eta^t_{\beta} + a\eta^x_{\alpha} u^2 + b\eta^{xx}_{\alpha} = 0, \tag{62}$$

which must be satisfied whenever $\frac{\partial^{\beta} u}{\partial t^{\beta}} + au^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + b\frac{\partial^{2\alpha} u}{\partial x^{2\alpha}} = 0$. Equation (61) has the equivalent form

$$t^{1-\beta} u_t + ax^{1-\alpha} u^2 u_x + b(1-\alpha)x^{1-2\alpha} u_x + bx^{2-2\alpha} u_{xx} = 0. \tag{63}$$

Using (11) and (12) to substitute $\eta^x_{\alpha},\ \eta^{xx}_{\alpha}$ and η^t_{β} into (62), replacing u_{xx} by $\frac{-1}{b} t^{1-\beta} x^{2\alpha-2} \frac{\partial u}{\partial t} - \frac{a}{b} u^2 x^{\alpha-1} \frac{\partial u}{\partial x} - (1-\alpha)x^{-1}\frac{\partial u}{\partial x}$ whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, we get the full determining equations for the symmetry group of (61). Solving these equations, we obtain

$$\tau = \frac{-4c_1 t}{\beta} + c_2 t^{1-\beta}, \qquad \xi = \frac{-2c_1}{\alpha} x + c_3 x^{1-\alpha}, \qquad \eta = c_1 u, \tag{64}$$

where $c_1,\ c_2$ and c_3 are arbitrary constants. Therefore, the symmetry group of (61) is spanned by the three vector fields

$$V_1 = t^{1-\beta}\frac{\partial}{\partial t}, \qquad V_2 = x^{1-\alpha}\frac{\partial}{\partial x}, \qquad V_3 = \frac{4t}{\beta}\frac{\partial}{\partial t} + \frac{2x}{\alpha}\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{65}$$

The commutation relations between these vector fields are given by

$$[V_1, V_2] = 0, \qquad [V_1, V_3] = 4V_1, \qquad [V_2, V_3] = 2V_2. \tag{66}$$

Once again the vector fields of the fractional modified Burgers equation reduce to those of the classical equations as: $\beta = \alpha = 1$ [44].

The one-parameter group generated by V_3 can be found by solving the corresponding characteristic equations

$$\frac{\alpha \mathrm{d}x}{2x} = \frac{\beta \mathrm{d}t}{4t} = \frac{-\mathrm{d}u}{u}, \tag{67}$$

and the corresponding invariants are

$$\zeta = xt^{\frac{-\beta}{2\alpha}}, \qquad u = t^{\frac{-\beta}{4}}\Psi(\zeta). \tag{68}$$

Direct substitution of transformation (68) into equation (61) reduces (61) to a nonlinear FODE with a new independent variable. As a result, we get

$$b\frac{d^{2\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{2\alpha}} + a\Psi^2(\zeta)\frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} - \frac{\beta}{2}\frac{\zeta^{\alpha}}{\alpha}\frac{d^{\alpha}\Psi(\zeta)}{\mathrm{d}\zeta^{\alpha}} - \frac{\beta}{4}\Psi(\zeta) = 0. \tag{69}$$

Equation (69) can be converted by the scale $\omega = \left(\frac{\beta}{2}\right)^{\frac{1}{2\alpha}}\zeta,\ \Psi(\zeta) = \left(\frac{\beta}{2}\right)^{\frac{1}{4}}\Phi(\omega)$ to the fractional equation

$$b\frac{d^{2\alpha}\Phi(\omega)}{\mathrm{d}w^{2\alpha}} + \left(a\Phi^2(\omega) - \frac{\omega^{\alpha}}{\alpha}\right)\frac{d^{\alpha}\Phi(\omega)}{\mathrm{d}\omega^{\alpha}} - \frac{1}{2}\Phi(\omega) = 0. \tag{70}$$

8 Conclusion

We have applied the Lie group analysis to the time–space fractional Korteweg–de Vries, modified Korteweg–de Vries, Burgers, and modified Burgers equations, where the time and space derivatives are the conformable fractional derivatives. All the generating vector fields for each equation have been calculated. Thus, it is evident that the Lie group analysis can be used successfully to study conformal fractional partial differential equations. It is worth to note that the number of the generating vector fields for each of the four time–space fractional equations is the same as that of the classical equation and the generating vector fields of each of these equations reduce to that of the corresponding classical equation when $\alpha = 1$ and $\beta = 1$.

Using the obtained Lie symmetries, we have shown that the equations under consideration can be transformed to fractional ordinary differential equations with conformable derivative or to ordinary differential equations with classical derivative. More precisely, we have shown that the time–space fractional KdV equation can be transformed into the conformable fractional second Painlevé equation and to classical first Painlevé equation. For the time–space fractional modified KdV equation, we obtained a solution in terms of the conformable fractional second Painlevé equation. In the case of Burgers equation, we derived solutions in terms of conformable fractional Riccati and classical Riccati equations.

It should be noted that the similarity reduction method converts the time–space partial differential equation with conformable fractional derivatives to ordinary differential equations with conformable fractional derivative or with classical derivative. However, time fractional partial differential equation with conformable fractional derivative is transformed to an ordinary differential equation with classical derivative, also time fractional partial differential equation with Riemann–Liouville fractional derivative is transformed to an ordinary fractional differential equation with an Erdélyi–Kober derivative depending on a parameter α.

It is interesting to apply the Lie group analysis to other partial differential equations with time and time–space fractional derivatives with more than two independent variables.

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