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Existence of periodic solutions for some quasilinear parabolic problems with variable exponents

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Abstract In this paper, we prove the existence of at least one periodic solution for some nonlinear parabolic boundary value problems associated with Leray–Lions's operators with variable exponents under the hypothesis of existence of well-ordered sub- and supersolutions.

Mathematics Subject Classification 35B10 · 35K10 · 35K59

الملخص

في هذه الورقة، نقوم بدراسة وجود حل دوري واحد على الأقل لبعض المسائل غير الخطية ذات قيم حدية مكافئية والمرفقة لمؤثرات من نوع ليراي-ليونز (Leray-Lions) ذات أسس متغيرة وذلك تحت فرضية وجود حلول سفلية وعلوية مرتبة جيدا.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, and fixed $T > 0$. Our aim here is to prove the existence of periodic solutions for the following nonlinear parabolic problem

$$(P)\begin{cases} \partial_t u + \mathcal{A}u = f(x,t,u,\nabla u) & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(0) = u(T) & \text{in } \Omega, \end{cases}$$

where $\mathcal{A}u = -\text{div}(\mathbf{A}(\cdot,\cdot,u,\nabla u))$ is a Leray–Lions's type operator with variable exponents acting from some functional space V_0 into its topological dual V_0' and where f is a nonlinear Carathéodory function, whose growth with respect to $|\nabla u|$ is at most of order $p(x)$ in the sense defined below (Hypothesis A4).

The suitable functional spaces to deal with in this type of problems are generalized Lebesgue and Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$, respectively. There are many differences between Lebesgue and Sobolev spaces with constant exponents and those with variable exponents. For instance, $p(x)$ needs to satisfy the log-Höder condition (see [10,12]) in order that the Poincaré's inequality and the density of smooth functions in $W^{1,p(x)}(\Omega)$ hold. Many difficulties arise in the case of variable exponents. One typical difficulty when dealing with problems like (P) is to define adequate functional spaces for solutions. When $p(x) = p$ is a

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constant, it is well known that $L^p(0,T;W_0^{1,p}(\Omega))$ can be taken as a space of solutions. However, when $p(x)$ is nonconstant, then nor $L^{p(x)}(0,T;W_0^{1,p(x)}(\Omega))$ neither $L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))$, where $p_- = \min_\Omega p(x)$, constitute a suitable space of solutions (see [5].) Henceforth, to overcome this difficulty, we shall define below our functional space of solutions V_0 as it was done by Bendahmane in [5].

Nonlinear problems defined by (P) arise in many applications; for instance, in electrorheological fluids (see [18]), where the essential part of the energy is given by $\int_\Omega |Du(x)|^{p(x)}\mathrm{d}x$ (Du being symmetric part of ∇u). This type of fluids has the ability to change its mechanical properties (for example becoming a solid gel) when an electric field is applied. Another important application is when f depends only on (x,t) and $\mathbf{A}(x,t,s,\xi) = |\xi|^{p(x)-2}\xi$; then the problem (P) can be seen as a sort of nonlinear diffusion equation whose coefficient of diffusion takes the form $|\nabla u|^{p(x)-2}$ by analogy with Fick's diffusion model (see [2]). For other applications, we refer the reader to [7,21].

There is by now an extensive literature on the existence of solutions for problem (P). Let us recall some known results in the case where $p(x) := p$ is a real constant. In [9], by applying a penalty method to an appropriately associated auxiliary parabolic variational inequality, J. Deuel and P. Hess proved the existence of at least one periodic solution for problem (P) in the case where the natural growth of f with respect to $|\nabla u|$ is of order less than p, that means $|f(x,t,u,\nabla u)| \leq k(x,t) + c|\nabla u|^{p-\delta}$ for some $\delta > 0$ and $k(x,t) \in L^{1+\delta}(\Omega \times (0,T))$, c being a positive constant. In [14], N. Grenon extends the result of [9] to the case where the natural growth of f with respect to $|\nabla u|$ is at most of order p; but instead of a periodicity condition the author considered an initial one. The proof therein is based on some regularization techniques used in [6,17].

Let us point out that in the two previous works, the hypothesis of existence of well-ordered sub- and supersolutions is supposed. Following [9], the results in [14] were extended by El Hachimi and Lamrani in [11], where the authors obtained the existence of periodic solutions, under the same hypotheses as in [14]. For variable exponents, this kind of problems has been studied by many authors [2,5,13,20], by means of different methods such as: subdifferential operators, Galerkin scheme, semigroup theory, etc.

The main goal of this paper is to extend the results in [11] to the variable exponents case under the hypothesis of existence of well-ordered sub- and supersolutions. It is well known that this method, when it is applicable, has more advantages compared to other methods. For example, we can give some order properties of the solutions. Nevertheless, this method is quite complicated because it requires well-ordered sub- and supersolutions, which is not usually easy to get. Indeed, in many application cases, sub- and supersolutions are obtained from eigenfunction associated with the first eigenvalue of some operators (say the p-Laplacian.) But, when dealing with variable exponents, it is well known that the $p(x)$-Laplacian does not have in general a first eigenvalue (see [12]) and therefore, we have to find sub- and supersolution by means of other ideas (see our application example in Sect. 5).

Now, we explain how this paper is organized. In Sect. 2 we introduce some notations and properties of Lebesgue–Sobolev spaces with variable exponents. Then, we give in Sect. 3 the main result, Theorem 3.2. Section 4 is devoted to prove the main result. Finally, in Sect. 5 we give an application of our main result.

2 Preliminaries

In this section, we briefly recall some definitions and basic properties of the generalized Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$, when Ω is a bounded open set of $\mathbb{R}^N$ ($N \geq 1$) with a smooth boundary. For the details see [8,10,12].

Let $p : \overline{\Omega} \mapsto [1,+\infty)$ be a continuous, real-valued function. Denote by $p_- = \min_{x\in\overline{\Omega}} p(x)$ and $p_+ = \max_{x\in\overline{\Omega}} p(x)$.

We introduce the variable exponents Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \mapsto \mathbb{R};\; u \text{ is measurable with } \int_\Omega |u(x)|^{p(x)}\mathrm{d}x < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

The following inequality will be used later

$$\min\left\{\|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)}\right\} \leq \int_\Omega |u(x)|^{p(x)}\mathrm{d}x \leq \max\left\{\|u\|^{p_-}_{L^{p(x)}(\Omega)}, \|u\|^{p_+}_{L^{p(x)}(\Omega)}\right\}. \tag{2.1}$$

Lemma 2.1 [8,10,12]

- *$(L^{p(x)}(\Omega), \|\cdot\|_{L^{p(x)}(\Omega)})$ is a Banach space.*
- *If $p_- > 1$, then $L^{p(x)}(\Omega)$ is reflexive and its conjugate space can be identified with $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Moreover, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the Hölder inequality*

$$\int_\Omega |uv|\mathrm{d}x \leq \left(\frac{1}{p_-} + \frac{1}{(p')_-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)} \leq 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}.$$

- *If $p_+ < +\infty$, then $L^{p(x)}(\Omega)$ is separable.*
- *The inclusion between Lebesgue spaces also generalizes naturally; if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere in Ω, then we have the following continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.*

Now, we define also the variable Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the following norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

Definition 2.2 The variable exponents $p : \overline{\Omega} \mapsto [1, +\infty)$ is said to satisfy the log-Hölder continuous condition if

$$\forall\, x, y \in \overline{\Omega}, \quad |x - y| < 1, \quad |p(x) - p(y)| < \omega(|x - y|),$$

where $\omega : (0, \infty) \mapsto \mathbb{R}$ is a nondecreasing function with $\limsup_{\alpha\to 0} \omega(\alpha) \ln\left(\frac{1}{\alpha}\right) < \infty$.

Lemma 2.3 [8,10,12]

- *If $1 < p_- \leq p_+ < \infty$, then the space $(W^{1,p(x)}(\Omega), \|\cdot\|_{W^{1,p(x)}(\Omega)})$ is a separable and reflexive Banach space.*
- *If $p(x)$ satisfies the log-Hölder continuous condition, then $C^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$. Moreover, we can define the Sobolev space with zero boundary values, $W_0^{1,p(x)}(\Omega)$ as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{1,p(x)}(\Omega)}$.*
- *For all $u \in W_0^{1,p(x)}(\Omega)$, the Poincaré inequality*

$$\|u\|_{L^{p(x)}(\Omega)} \leq C\|\nabla u\|_{L^{p(x)}(\Omega)},$$

holds. Moreover, $\|u\|_{W_0^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)}$ is a norm in $W_0^{1,p(x)}(\Omega)$.

Throughout this paper, we shall assume that the variable exponents $p(x)$ satisfy the log-Hölder condition and that $1 < p_- \leq p_+ < \infty$.

3 Hypotheses and main result

We suppose that Ω is a bounded open set of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $Q = \Omega \times (0, T)$ where $T > 0$ is fixed and $\Sigma = \partial\Omega \times (0, T)$.

Let

$$V_0 = \{f \in L^{p_-}(0, T; W_0^{1,p(x)}(\Omega));\ |\nabla f| \in L^{p(x)}(Q)\},$$

endowed with the norm

$$\|f\|_{V_0} = \|\nabla f\|_{L^{p(x)}(Q)},$$

or, the equivalent norm

$$\|f\|_{V_0} = \|f\|_{L^{p_-}(0,T;W_0^{1,p(x)}(\Omega))} + \|\nabla f\|_{L^{p(x)}(Q)}.$$

The equivalence of the two norms comes from Poincaré's inequality and the continuous embedding $L^{p(x)}(Q) \hookrightarrow L^{p_-}(0, T; L^{p(x)}(\Omega))$.

We set

$$V = \{f \in L^{p_-}(0, T; W^{1,p(x)}(\Omega));\ |\nabla f| \in L^{p(x)}(Q)\}.$$

We state some further properties of V_0 in the following lemma.

Lemma 3.1 [5] *We denote by V_0' the dual space of V_0. Then*

- *We have the following continuous dense embeddings:*

$$L^{p_+}(0, T; W_0^{1,p(x)}(\Omega)) \overset{d}{\hookrightarrow} V_0 \overset{d}{\hookrightarrow} L^{p_-}(0, T; W_0^{1,p(x)}(\Omega)).$$

In particular, since $\mathscr{D}(Q)$ is dense in $L^{p_+}(0, T; W_0^{1,p(x)}(\Omega))$, it is also dense in V_0 and for the corresponding dual spaces we have

$$L^{(p_-)'}(0, T; (W_0^{1,p(x)}(\Omega))') \hookrightarrow V_0' \hookrightarrow L^{(p_+)'}(0, T; (W_0^{1,p(x)}(\Omega))').$$

- *One can represents the elements of V_0' as follows: let $G \in V_0'$, then there exists $F = (f_1, f_2, \cdots, f_N) \in (L^{p'(x)}(Q))^N$ such that $G = -\mathrm{div}(F)$ and*

$$\langle G, u\rangle_{V',V_0} = \int_0^T \int_\Omega F \cdot \nabla u \, dx dt,$$

for any $u \in V_0$.

Now, let us give the hypotheses which concern $\mathbf{A}$ and f.

(A1) $\mathbf{A}$ is a Carathéodory function defined on $Q \times \mathbb{R} \times \mathbb{R}^N$, with values in $\mathbb{R}^N$ such that there exist $\lambda > 0$, and $l \in L^{p'(x)}(Q)$, $l \geq 0$, so that for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$: (say growth condition of $\mathbf{A}$)

$$|\mathbf{A}(x, t, s, \xi)| \leq \lambda(l(x, t) + |s|^{p(x)-1} + |\xi|^{p(x)-1}), \quad \text{a.e. in } Q.$$

(A2) For all $s \in \mathbb{R}$ and for all $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$: (say monotonicity condition of $\mathbf{A}$)

$$(\mathbf{A}(x, t, s, \xi) - \mathbf{A}(x, t, s, \xi')) \cdot (\xi - \xi') > 0, \quad \text{a.e. in } Q.$$

(A3) There exists $\alpha > 0$, so that for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$: (say coercivity condition of $\mathbf{A}$)

$$\mathbf{A}(x, t, s, \xi) \cdot \xi \geq \alpha|\xi|^{p(x)}, \quad \text{a.e. in } Q.$$

(A4) f is a Carathéodory function on $Q \times \mathbb{R} \times \mathbb{R}^N$, and there exist a function $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ increasing, and $h \in L^1(Q)$, $h \geq 0$, such that: (say natural growth condition on f respect to $|\xi|$ of order $p(x)$)

$$|f(x, t, s, \xi)| \leq b(|s|)(h(x, t) + |\xi|^{p(x)}), \quad \text{for } (x, t, s, \xi) \in Q \times \mathbb{R} \times \mathbb{R}^N.$$

Remark 3.2 If $u \in V_0 \cap L^\infty(Q)$, then under the assumptions $(A1)$, $(A2)$, and $(A3)$ we have $\mathcal{A}u \in V_0'$. Moreover, under the assumption $(A4)$ we have $f(x, t, u, \nabla u) \in L^1(Q)$.

Definition 3.3 A periodic solution for problem (P) is a measurable function $u : Q \mapsto \mathbb{R}$ satisfying the following conditions

$$u \in V_0 \cap L^{\infty}(Q), \quad \partial_t u \in V_0' + L^1(Q), \tag{3.1}$$

$$\langle \partial_t u, \phi \rangle_{V_0'+L^1(Q), V_0 \cap L^{\infty}(Q)} + \int_Q \mathbf{A}(x,t,u,\nabla u)\cdot \nabla \phi = \int_Q f(x,t,u,\nabla u)\phi \quad \text{for all } \phi \in V_0 \cap L^{\infty}(Q), \tag{3.2}$$

$$u(x,0) = u(x,T) \quad \text{for all } x \in \Omega. \tag{3.3}$$

Thanks to the previous remark and (3.2), we have $\partial_t u \in V_0' + L^1(Q)$. Moreover, the periodicity condition (3.3) makes sense according to the following lemma.

Lemma 3.4 [5] *We set* $\mathcal{W} := \{u \in V_0;\ \partial_t u \in V_0' + L^1(Q)\}$. *Then, we have the following embedding*

$$\mathcal{W} \cap L^{\infty}(Q) \hookrightarrow C([0,T]; L^2(\Omega)).$$

Now, we can ensure that all terms of (P) have a meaning.

Definition 3.5 A subsolution (in the distributional sense) of problem (P) is a function $\varphi \in V \cap L^{\infty}(Q)$ such that $\partial_t \varphi \in V_0' + L^1(Q)$ and

$$\begin{cases} \partial_t \varphi + \mathcal{A}\varphi \le f(x,t,\varphi,\nabla\varphi) & \text{in } Q, \\ \varphi \le 0 & \text{on } \Sigma, \\ \varphi(0) \le \varphi(T) & \text{in } \Omega, \end{cases}$$

A supersolution of problem (P) is obtained by reversing the inequalities.

We can now state the main result of this paper.

Theorem 3.6 *Suppose that* $\mathbf{A}$ *verifies the hypotheses* $A1), A2), A3)$, *and that* f *satisfies* $A4)$. *Moreover, assume the existence of a subsolution* φ, *and a supersolution* ψ, *such that* $\varphi \le \psi$ *a.e. in* Q. *Then, there exists at least one periodic solution* u *of problem* (P), *such that* $\varphi \le u \le \psi$ *a.e. in* Q.

Before we start the proof, we give some technical lemmas which will be used later.

Lemma 3.7 [15] *Let* $\pi : \mathbb{R} \to \mathbb{R}$ *be a* C^1 *piecewise function with* $\pi(0) = 0$ *and* $\pi' = 0$ *outside a compact set. Let* $\Pi(s) = \int_0^s \pi(\sigma)\mathrm{d}\sigma$. *If* $u \in V_0 \cap L^{\infty}(Q)$ *with* $\partial_t u \in V_0' + L^1(Q)$, *then*

$$\int_0^T \langle \partial_t u, \pi(u)\rangle = \langle \partial_t u, \pi(u)\rangle_{V_0'+L^1(Q), V_0\cap L^{\infty}(Q)} = \int_{\Omega} \Pi(u(T))\mathrm{d}x - \int_{\Omega} \Pi(u(0))\mathrm{d}x.$$

Lemma 3.8 [1] *Assume that* $(A1)$, $(A2)$, *and* $(A3)$ *are satisfied and let* (u_n) *be a sequence in* V_0 *which converges weakly to* u *in* V_0, *and*

$$\limsup_{n\to\infty} \int_Q (\mathbf{A}(x,t,u_n,\nabla u_n) - \mathbf{A}(x,t,u_n,\nabla u))\cdot(\nabla u_n - \nabla u) \le 0.$$

Then,

$$u_n \to u \quad \text{strongly in } V_0.$$

4 Proof of Theorem 3.6

4.1 Truncation of problem (P)

Let φ be a subsolution and ψ a supersolution of problem (P), such that $\varphi \le \psi$ a.e. in Q. Let us define for $u \in V$ the truncation function $T(u)$ by

$$T(u) = u - (u - \psi)^+ + (\varphi - u)^+.$$

We shall denote by

$$\mathbf{A}(u, \nabla u)(x,t) = \mathbf{A}(x,t,u(x,t),\nabla u(x,t)) \quad \text{and} \quad F(u,\nabla u)(x,t) = f(x,t,u(x,t),\nabla u(x,t)),$$

the Nemyskii operators associated, respectively, with the functions A and f.

For almost everywhere (x, t) in Q, we define

$$\mathbf{A}^\star(u, \nabla u)(x,t) = \mathbf{A}(Tu, \nabla u)(x,t), \quad \text{and} \quad F^\star(u,\nabla u)(x,t) = F(Tu, \nabla Tu)(x,t).$$

Note that the $F^\star$ is not a Carathéodory function since it is not continuous with respect to ∇u. This constraint will be overcome thanks to the following lemma.

Lemma 4.1 *The operator $F^\star : u \to F^\star(u, \nabla u)$ is defined and continuous from V into $L^1(Q)$. Moreover, there exists a constant $C > 0$ such that*

$$|F^\star(u,\nabla u)(x,t)| \le C(h^\star(x,t) + |\nabla u|^{p(x)}), \quad \text{a.e. in } Q, \tag{4.1}$$

where $h^\star$ is a nonnegative function in $L^1(Q)$.

Proof The proof of this lemma is similar to the case when $p(x)$ is a constant (see [14].) □

Denote by $\mathcal{A}^\star u = -\text{div}(\mathbf{A}^\star(u, \nabla u))$; then $\mathcal{A}^\star$ is a Leray–Lions's type operator from V_0 into its dual V_0', that means $\mathbf{A}^\star$ satisfies the assumptions $A1$), $A2$) and $A3$).

4.2 Penalization and regularization of problem (P)

Let $k > 0$ be a constant such that

$$-k \le \varphi - 1 \le \psi + 1 \le k \ \text{ a.e. in } Q.$$

We set

$$K = \{v \in V_0, -k \le v \le k \ \text{ a.e. in } Q\}.$$

Then, K is a closed convex set of V_0 . For $u \in V$, we define

$$\beta(u) = [(u-k)^+]^{(p_-)-1} - [(u+k)^-]^{(p_-)-1}.$$

Let $\eta > 0$, the penalization operator related to K is defined by $\frac{1}{\eta}\beta(u)$. Moreover, it is clear that

$$\beta(u)u \ge 0 \quad \text{a.e. in } Q, \ \text{ and } K \equiv \{v \in V_0, \beta(v) = 0 \quad \text{a.e. in } Q\}.$$

Let $\epsilon > 0$, for $u \in V$, and for almost everywhere (x, t) in Q we set

$$F^\star_\epsilon(u,\nabla u)(x,t) = \frac{|F^\star(u,\nabla u)(x,t)|}{1 + \epsilon|F^\star(u,\nabla u)(x,t)|}.$$

It is clear that $F^\star_\epsilon(u, \nabla u) \in L^\infty(Q)$ for all $u \in V$. Moreover, the mapping $u \to F^\star_\epsilon(u, \nabla u)$ is continuous from V into $L^1(Q)$, and from (4.1) we can easily verify that

$$|F^\star_\epsilon(u,\nabla u)(x,t)| \le C(h^\star(x,t) + |\nabla u|^{p(x)}), \quad \text{a.e. in } Q. \tag{4.2}$$

where C is a constant which is independent of ϵ.

We will now consider the following penalized–regularized problem

$$(P^\star_{\eta,\epsilon}) \begin{cases} u_{\eta,\epsilon} \in V_0, \ \partial_t u_{\eta,\epsilon} \in V_0', \\ \partial_t u_{\eta,\epsilon} + \mathcal{A}^\star u_{\eta,\epsilon} - F^\star_\epsilon(u_{\eta,\epsilon}, \nabla u_{\eta,\epsilon}) + \frac{1}{\eta}\beta(u_{\eta,\epsilon}) = 0 & \text{in } Q, \\ u_{\eta,\epsilon} = 0 & \text{in } \Sigma, \\ u_{\eta,\epsilon}(0) = u_{\eta,\epsilon}(T) & \text{in } \Omega. \end{cases}$$

By application of Theorem 1.2, p.319 in [16] we can ensure the existence of a solution of problem $(P^\star_{\eta,\epsilon})$. Indeed, we have:

Proposition 4.2 *Let* $D = \{u \in V_0, \text{suchthat} \partial_t u \in V_0' \text{and} u(0) = u(T)\}$. *Then, the operator* $\mathcal{N}u = \mathcal{A}^\star u - F_\epsilon^\star(u, \nabla u) + \frac{1}{\eta}\beta(u)$ *defined from* D *into* V_0' *is bounded, coercive and pseudo-monotone. Moreover, there exists at least one solution* $(u_{\eta,\epsilon})$ *of problem* $(P_{\eta,\epsilon}^\star)$.

Proof Boundedness of $\mathcal{N}$. From the assumption $A1)$ and the definition of $\mathcal{N}$, we have

$$|\langle \mathcal{N}u, v\rangle| \le \lambda \left(\int_Q l(x,t)|\nabla v| + \int_Q |Tu|^{p(x)-1}|\nabla v| + \int_Q |\nabla u|^{p(x)-1}|\nabla v|\right) + \int_Q |F_\epsilon^\star(u, \nabla u)||v| + \frac{1}{\eta}\int_Q |\beta(u)||v|. \tag{4.3}$$

We treat each integral in the right-side member of (4.3).

By Remark 3.2 (recall that φ, ψ are in the $L^\infty(Q)$), we have

$$\int_Q l(x,t)|\nabla v| + \int_Q |Tu|^{p(x)-1}|\nabla v| \le C\|v\|_{V_0}.$$

By using Hölder's inequality, we get

$$\int_Q |\nabla u|^{p(x)-1}|\nabla v| \le c_2 \||\nabla u|^{p(x)-1}\|_{L^{p'(x)}(Q)}\|v\|_{V_0}.$$

Then, if $\||\nabla u|^{p(x)-1}\|_{L^{p'(x)}(Q)} \le 1$, it's over. Otherwise, from inequality (2.1), we have

$$\||\nabla u|^{p(x)-1}\|^{(p')_-}_{L^{p'(x)}(Q)} = \||\nabla u|^{p(x)-1}\|^{p'_+}_{L^{p'(x)}(Q)} \le \int_Q \left(|\nabla u|^{p(x)-1}\right)^{p'(x)},$$

and

$$\int_Q \left(|\nabla u|^{p(x)-1}\right)^{p'(x)} \le \max\left\{\|\nabla u\|^{p_-}_{L^{p(x)}(Q)}, \|\nabla u\|^{p_+}_{L^{p(x)}(Q)}\right\} = \max\left\{\|u\|^{p_-}_{V_0}, \|u\|^{p_+}_{V_0}\right\}.$$

Hence,

$$\int_Q |\nabla u|^{p(x)-1}|\nabla v| \le c\max\left\{\|u\|_{V_0}^{\frac{p_-}{p'_+}}, \|u\|_{V_0}^{(p_+)-1}\right\}\|v\|_{V_0}.$$

Since $|F_\epsilon^\star| \le \frac{1}{\epsilon}$ and $V_0 \hookrightarrow L^1(Q)$, then we get

$$\int_Q |F_\epsilon^\star(u, \nabla u)||v| \le (c_1/\epsilon)\|v\|_{V_0}.$$

Moreover, we have

$$\frac{1}{\eta}\int_Q |\beta(u)||v| \le \frac{1}{\eta}\left(\int_Q |(u-k)^+|^{(p_-)-1}|v| + \int_Q |(u+k)^-|^{(p_-)-1}|v|\right). \tag{4.4}$$

Now, since $u \in V_0 \hookrightarrow L^{p_-}(Q)$, we get $((u-k)^+)^{(p_-)-1} \in L^{(p_-)'}(Q)$. Then, by using Hölder's inequality in (4.4), we obtain

$$\int_Q |(u-k)^+|^{(p_-)-1}|v| \le c_3\|(u-k)^+\|^{(p_-)-1}_{V_0}\|v\|_{V_0} \le c_3\|u\|^{(p_-)-1}_{V_0}\|v\|_{V_0}.$$

Similarly, we obtain

$$\int_Q |(u+k)^-|^{(p_-)-1}|v| \le c_4\|u\|^{(p_-)-1}_{V_0}\|v\|_{V_0}.$$

Whence,

$$\|\mathcal{N}u\|_{V_0'} \le \gamma\left(1 + \|u\|^{(p_-)-1}_{V_0} + \max\left\{\|u\|_{V_0}^{\frac{p_-}{p'_+}}, \|u\|_{V_0}^{(p_+)-1}\right\}\right).$$

where γ is a positive constant.

Coercivity of $\mathcal{N}$. From the definition of $\mathcal{N}$, we have

$$\langle \mathcal{N}u, u\rangle = \langle \mathcal{A}^\star u, u\rangle - \langle F_\epsilon^\star(u, \nabla u), u\rangle + \left\langle \frac{1}{\eta}\beta(u), u\right\rangle.$$

Furthermore, we have

$$\langle \mathcal{A}^\star u, u\rangle = \int_Q \mathbf{A}^\star(u, \nabla u)\cdot\nabla u \geq \alpha\int_Q |\nabla u|^{p(x)} \geq \alpha \min\left\{\|u\|_{V_0}^{p_-}, \|u\|_{V_0}^{p_+}\right\},$$

$$\langle F_\epsilon^\star(u, \nabla u), u\rangle = \int_Q \frac{F^\star(u, \nabla u)}{1+\epsilon|F^\star(u, \nabla u)|}u \leq \int_Q \frac{|F^\star(u, \nabla u)|}{1+\epsilon|F^\star(u, \nabla u)|}|u| \leq \frac{1}{\epsilon}\|u\|_{L^1(Q)} \leq \frac{c}{\epsilon}\|u\|_{V_0},$$

and

$$\left\langle \frac{1}{\eta}\beta(u), u\right\rangle = \frac{1}{\eta}\int_Q \beta(u)u \geq 0.$$

Hence,

$$\frac{\langle \mathcal{N}u, u\rangle}{\|u\|_{V_0}} \geq \alpha\min\left\{\|u\|_{V_0}^{(p_-)-1}, \|u\|_{V_0}^{(p_+)-1}\right\} - \frac{c}{\epsilon}.$$

Whence,

$$\frac{\langle \mathcal{N}u, u\rangle}{\|u\|_{V_0}} \to +\infty, \quad \text{when } \|u\|_{V_0} \to +\infty.$$

Pseudo-monotonicity of $\mathcal{N}$.

Let $(u_n) \in D$ and $u \in D$, such that u_n converges weakly to u in V_0 (then u_n converges weakly to u in $L^{p_-}(0, T; W_0^{1,p(x)}(\Omega))$, see Lemma 3.1), and $\partial_t u_n$ converges weakly to $\partial_t u$ in V_0' (then $\partial_t u_n$ converges weakly to $\partial_t u$ in $L^{(p_+)'}(0, T; W^{-1,p'(x)}(\Omega))$ see Lemma 3.1 again).

Moreover, we suppose that

$$\lim_{n\to\infty}\sup\langle \mathcal{N}u_n, u_n - u\rangle \leq 0. \tag{4.5}$$

We shall prove that

$$\lim_{n\to\infty}\inf\langle \mathcal{N}u_n, u_n - v\rangle \geq \langle \mathcal{N}u, u - v\rangle \quad \text{for all } v \in V_0. \tag{4.6}$$

Choose $s > \frac{N}{2} + 1$ such that $W^{-1,p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$; then $\partial_t u_n$ converges weakly to $\partial_t u$ in $L^{(p_+)'}(0, T; H^{-s}(\Omega))$.

We set $B_0 = W_0^{1,p(x)}(\Omega)$, $B = L^{p(x)}(\Omega)$ and $B_1 = H^{-s}(\Omega)$. We have the following embeddings

$$B_0 \overset{c}{\hookrightarrow} B \hookrightarrow B_1, \tag{4.7}$$

where $B_0 \overset{c}{\hookrightarrow} B$ means that B_0 is compactly embedded in B.

By a theorem of Aubin–Lions's, pp. 57–58 in [16], we deduce that u_n converges strongly to u in $L^{p_-}(0, T; L^{p(x)}(\Omega))$, which embedded into $L^{p_-}(Q)$.

Furthermore, we have

$$\frac{1}{\eta}\int_Q |\beta(u_n)||u_n - u| \leq \frac{1}{\eta}\left[\int_Q |(u_n - k)^+|^{(p_-)-1}|u_n - u| + \int_Q |(u_n + k)^-|^{(p_-)-1}|u_n - u|\right]. \tag{4.8}$$

By using Hölder's inequality and the embedding of V_0 into $L^{p_-}(Q)$ in (4.8), we get

$$\frac{1}{\eta}\int_Q |\beta(u_n)||u_n - u| \leq \frac{c}{\eta}\left(\|u_n\|_{V_0}^{(p_-)-1}\|u_n - u\|_{L^{p_-}(Q)}\right).$$

Since (u_n) is bounded in V_0 and u_n converges strongly to u in $L^{p_-}(Q)$, we obtain

$$\frac{1}{\eta}\int_Q |\beta(u_n)||u_n-u| \to 0 \quad \text{when } n\to+\infty. \tag{4.9}$$

On the other hand, since $|F^\star_\epsilon| \le \frac{1}{\epsilon}$ and $L^{p_-}(Q)$ is embedded in $L^1(Q)$, we get

$$\int_Q |F^\star_\epsilon(u_n,\nabla u_n)||u_n-u| \le \frac{c}{\epsilon}\|u_n-u\|_{L^{p_-}(Q)} \to 0 \quad \text{when } n\to\infty. \tag{4.10}$$

We develop each term of $\langle \mathcal{N}u_n, u_n-u\rangle$ and use (4.5), (4.9) and (4.10), to obtain

$$\lim_{n\to\infty}\sup\langle \mathcal{N}u_n, u_n-u\rangle = \lim_{n\to\infty}\sup\int_Q \mathbf{A}(Tu_n,\nabla u_n)\cdot\nabla(u_n-u) \le 0. \tag{4.11}$$

Applying Vitali's theorem and weak convergence of $\mathbf{A}(Tu_n,\nabla u)$ in $(L^{p'(x)}(Q))^N$, we obtain

$$\lim_{n\to\infty}\int_Q \mathbf{A}(Tu_n,\nabla u)\cdot\nabla(u_n-u) = 0. \tag{4.12}$$

By using (4.11) and (4.12), then we obtain

$$\lim_{n\to\infty}\int_Q [\mathbf{A}(Tu_n,\nabla u_n)-\mathbf{A}(Tu_n,\nabla u)]\cdot(\nabla u_n-\nabla u) \le 0.$$

Now, thanks to Lemma 3.8, we get

$$u_n \to u \quad \text{strongly in } V_0 \quad \text{that means } \nabla u_n \to \nabla u \quad \text{strongly in } (L^{p(x)}(Q))^N.$$

Hence, the inequality (4.6) desired.

Finally, by Theorem 1.2, p. 319 in [16], we deduce the existence of at least one solution $(u_{\eta,\epsilon})$ of problem $(P^\star_{\eta,\epsilon})$. □

4.3 A-priori-estimates

In this section, we are going to obtain some estimations on the sequence solutions $(u_{\eta,\epsilon})$ of problem $(P^\star_{\eta,\epsilon})$ independently of η and ϵ.

4.3.1 Estimates on $(u_{\eta,\epsilon})_\eta$ in V_0 and $L^\infty(Q)$

Let us fix ϵ, and denote by $(u_\eta) \equiv: (u_{\eta,\epsilon})$.

Lemma 4.3 *The sequences $\left(\frac{1}{\eta}\beta(u_\eta)\right)_\eta$ and $(u_\eta)_\eta$ are bounded respectively in $L^{(p_-)'}(Q)$ and V_0 .*

Proof From the definition of $\frac{1}{\eta}\beta(u_\eta)$, we deduce that

$$\left\|\frac{1}{\eta}\beta(u_\eta)\right\|_{L^{(p_-)'}(Q)} \le \left\|\frac{(u_\eta-k)^+}{\eta^{\frac{1}{(p_-)-1}}}\right\|^{(p_-)-1}_{L^{p_-}(Q)} + \left\|\frac{(u_\eta+k)^-}{\eta^{\frac{1}{(p_-)-1}}}\right\|^{(p_-)-1}_{L^{p_-}(Q)}.$$

Then, we only need to show that:

$$\left(\frac{(u_\eta-k)^+}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta \quad \text{and} \quad \left(\frac{(u_\eta+k)^-}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta \quad \text{are bounded in } L^{p_-}(Q).$$

Since $(u_\eta-k)^+ \in V_0$, then by multiplying $(P^\star_{\eta,\epsilon})$ by $(u_\eta-k)^+$, we get

$$\langle \partial_t u_\eta, (u_\eta-k)^+\rangle + \int_Q \mathbf{A}^\star(u_\eta,\nabla u_\eta)\cdot\nabla(u_\eta-k)^+ - \int_Q F^\star_\epsilon(u_\eta,\nabla u_\eta)(u_\eta-k)^+ + \frac{1}{\eta}\int_Q ((u_\eta-k)^+)^{p_-} = 0.$$

Since $u_\eta(0) = u_\eta(T)$, then from Lemma 3.7, we deduce that $\langle \partial_t u_\eta, (u_\eta - k)^+\rangle = 0$. Hence

$$\frac{1}{\eta}\int_Q ((u_\eta - k)^+)^{p_-} = \int_Q F^\star_\epsilon(u_\eta, \nabla u_\eta)(u_\eta - k)^+ - \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta)\cdot \nabla (u_\eta - k)^+. \tag{4.13}$$

Under the assumption $A3$), the second integral in the right-hand side of equality (4.13) is nonnegative. On the other hand, we have $|F^\star_\epsilon| \le 1/\epsilon$. Let us divide both sides of equality (4.13) by $\eta^{1/(p_{-1})-1}$. By Hölder's inequality, we obtain

$$\int_Q \frac{((u_\eta - k)^+)^{p_-}}{\eta^{(p_-)'}} \le C\left[\int_Q \frac{((u_\eta - k)^+)^{p_-}}{\eta^{(p_-)'}}\right]^{1/p_-}, \quad \text{where } C \text{ is indepedent on } \eta,$$

whence $\left(\frac{(u_\eta - k)^+}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta$ is bounded in $L^{p_-}(Q)$.

Using $(-(u_\eta + k)^-)$ as a test function, we prove in the same way that $\left(\frac{(u_\eta + k)^-}{\eta^{\frac{1}{(p_-)-1}}}\right)_\eta$ is bounded in $L^{p_-}(Q)$.

Now we prove that $(u_\eta)_\eta$ is bounded in V_0. Multiplying $(P^\star_{\eta,\epsilon})$ by u_η and using the assumption $A3$), Lemma 3.7, and the fact that $\beta(u_\eta)u_\eta \ge 0$, we obtain

$$\alpha\int_Q |\nabla u_\eta|^{p(x)} \le \int_Q \mathbf{A}^\star(u_\eta, \nabla u_\eta)\cdot \nabla u_\eta \le \int_Q F^\star_\epsilon(u_\eta, \nabla u_\eta)u_\eta \le \frac{c}{\epsilon}\|u_\eta\|_{V_0}.$$

By using the inequality (2.1), hence, we get

$$\min\left\{\|u_\eta\|_{V_0}^{(p_-)-1}, \|u_\eta\|_{V_0}^{(p_+)-1}\right\} \le c(\epsilon, \alpha).$$

Since $(p_-) - 1$ and $(p_+) - 1$ are strictly greater than 0, we deduce that $(u_\eta)_\eta$ is bounded in V_0. □

Lemma 4.4 *The sequence* $(\partial_t u_\eta)_\eta$ *is bounded in* V_0'.

Proof Let $v \in V_0$, from the first equation of problem $(P^\star_{\eta,\epsilon})$, we get

$$\langle \partial_t u_\eta, v\rangle = -\langle \mathcal{A}^\star u_\eta, v\rangle + \langle F^\star_\epsilon(u_\eta, \nabla u_\eta), v\rangle - \left\langle \frac{1}{\eta}\beta(u_\eta), v\right\rangle.$$

Thus,

$$|\langle \partial_t u_\eta, v\rangle| \le \int_Q |\mathbf{A}^\star(u_\eta, \nabla u_\eta)||\nabla v| + \int_Q |F^\star_\epsilon(u_\eta, \nabla u_\eta)||v| + \int_Q \frac{1}{\eta}|\beta(u_\eta)||v|. \tag{4.14}$$

We treat each integral in the right-hand side of (4.14). We claim first that $\mathbf{A}^\star(u_\eta, \nabla u_\eta)$ is bounded in $(L^{p'(x)}(Q))^N$. Indeed, if $\|\mathbf{A}^\star(u_\eta, \nabla u_\eta)\|_{L^{p'(x)}(Q)} \le 1$, the claim is obvious. Otherwise, we have

$$\|\mathbf{A}^\star(u_\eta, \nabla u_\eta)\|_{L^{p'(x)}(Q)}^{(p')_-} \le \int_Q |\mathbf{A}^\star(u_\eta, \nabla u_\eta)|^{p'(x)},$$

since $p'(x) \le (p')_+$, and $(a+b)^p \le 2^{p-1}(a^p + b^p)$ for all $a, b \ge 0$ and $p > 1$, then according to the assumption $A1$), we get

$$\int_Q |\mathbf{A}^\star(u_\eta, \nabla u_\eta)|^{p'(x)} \le c(\lambda, (p')_+)\left[\int_Q l(x,t)^{p'(x)} + \int_Q |Tu_\eta|^{p(x)} + \int_Q |\nabla u_\eta|^{p(x)}\right]. \tag{4.15}$$

By the inequality (2.1) for the third integral in the right-hand side of (4.15), and the fact that (u_η) is bounded in V_0 (by Lemma 4.3), we can deduce the boundedness of $\mathbf{A}^\star(u_\eta, \nabla u_\eta)$ in $(L^{p'(x)}(Q))^N$.

Since $|F^\star_\epsilon| \le \frac{1}{\epsilon}$, $\frac{1}{\eta}\beta(u_\eta)$ is bounded in $L^{(p_-)'}(Q)$ (by Lemma 4.3) and $v \in V_0 \hookrightarrow L^{p_-}(Q) \hookrightarrow L^1(Q)$, then we use Hölder's inequality in (4.14), to obtain the desired result. □

As in (4.7), by Aubin–Lions's theorem, we can extract a subsequence, still denoted by (u_η) which is relatively compact in $L^{p_-}(0,T;L^{p(x)}(\Omega)) \hookrightarrow L^{p_-}(Q)$. Furthermore, there exists $u_\epsilon \in V_0$ such that: for all $\epsilon > 0$ fixed, we have as $\eta \to 0$

$$u_\eta \to u_\epsilon \quad \text{strongly in } L^{p_-}(Q) \quad \text{and a.e. in } Q, \tag{4.16}$$

$$u_\eta \rightharpoonup u_\epsilon \quad \text{weakly in } V_0, \tag{4.17}$$

$$\partial_t u_\eta \rightharpoonup \partial_t u_\epsilon \quad \text{weakly in } V_0'. \tag{4.18}$$

Now, as $F^\star_\epsilon(u_\eta,\nabla u_\eta)$ and $\frac{1}{\eta}\beta(u_\eta)$ are bounded in $L^{(p_-)'}(Q)$, independently of η, then there exist β_ϵ and F_ϵ in $L^{(p_-)'}(Q)$, such that

$$\frac{1}{\eta}\beta(u_\eta) \rightharpoonup \beta_\epsilon \quad \text{in } L^{(p_-)'}(Q), \tag{4.19}$$

and

$$F^\star_\epsilon(u_\eta,\nabla u_\eta) \rightharpoonup F_\epsilon \quad \text{in } L^{(p_-)'}(Q). \tag{4.20}$$

In addition, as $\mathbf{A}^\star(u_\eta,\nabla u_\eta)$ is bounded in $(L^{p'(x)}(Q))^N$, then there exists χ_ϵ in $(L^{p'(x)}(Q))^N$ such that

$$\mathbf{A}^\star(u_\eta,\nabla u_\eta) \rightharpoonup \chi_\epsilon \quad \text{in } (L^{p'(x)}(Q))^N \hookrightarrow (L^{(p_-)'}(Q))^N. \tag{4.21}$$

The estimations in V_0 and $L^\infty(Q)$ obtained above do not allow us to pass directly to the limit in the problem $(P^\star_{\eta,\epsilon})$, because we can not pass to the limit in the term $F^\star_\epsilon(u_\eta,\nabla u_\eta)$ (which is bounded only in $L^1(Q)$, see (4.2).) To overcome this difficulty we need the strong convergence in V_0 of the sequence solutions (u_η). To this end, we shall prove the following lemma.

Lemma 4.5 *(u_η) converges strongly to (u_ϵ) in V_0, when η tends to zero.*

Proof The proof is almost the same as in the case when the exponents $p(x) = p$ is a constant(see [14]). Thus, we give here only a sketch. The general idea is to use Lemma 3.8, since $\mathbf{A}^\star$ satisfies the hypothesis $A1), A2), A3)$, and the weak convergence of u_η to u_ϵ in V_0. Hence, it suffices to show that

$$\limsup_{\eta\to 0}\int_Q \big(\mathbf{A}^\star(u_\eta,\nabla u_\eta) - \mathbf{A}^\star(u_\eta,\nabla u_\epsilon)\big)\cdot\big(\nabla u_\eta - \nabla u_\epsilon\big) = 0. \tag{4.22}$$

We consider $\mu > 0$ and we subtract $(P^\star_{\eta,\epsilon})$ from $(P^\star_{\mu,\epsilon})$, we get

$$\partial_t u_\eta - \partial_t u_\mu + \mathcal{A}^\star u_\eta - \mathcal{A}^\star u_\mu - F^\star_\epsilon(u_\eta,\nabla u_\eta) + F^\star_\epsilon(u_\mu,\nabla u_\mu) + \frac{1}{\eta}\beta(u_\eta) - \frac{1}{\mu}\beta(u_\mu) = 0.$$

We multiply this equation by $u_\eta - u_\mu$, and use Lemma 3.7, to obtain

$$\int_Q \big(\mathbf{A}^\star(u_\eta,\nabla u_\eta) - \mathbf{A}^\star(u_\mu,\nabla u_\mu)\big)\cdot(\nabla u_\eta - \nabla u_\mu) - \int_Q \big(F^\star_\epsilon(u_\eta,\nabla u_\eta) - F^\star_\epsilon(u_\mu,\nabla u_\mu)\big)(u_\eta - u_\mu)$$
$$+ \int_Q \left(\frac{1}{\eta}\beta(u_\eta) - \frac{1}{\mu}\beta(u_\eta)\right)(u_\eta - u_\mu) = 0. \tag{4.23}$$

Firstly, we take the lim sup when η tends to 0 and secondly the lim sup when μ tends to 0 in (4.23). By using (4.16), (4.17), (4.18), (4.19) and (4.20), we obtain

$$\limsup_{\eta\to 0}\int_Q \mathbf{A}^\star(u_\eta,\nabla u_\eta)\cdot\nabla u_\eta - \int_Q \chi_\epsilon\nabla u_\epsilon - \int_Q \chi_\epsilon\nabla u_\epsilon + \limsup_{\mu\to 0}\int_Q \mathbf{A}^\star(u_\mu,\nabla u_\mu)\cdot\nabla u_\mu = 0.$$

So, for $\mu = \eta$, we get

$$\limsup_{\eta\to 0}\int_Q \mathbf{A}^\star(u_\eta,\nabla u_\eta)\cdot\nabla u_\eta = \int_Q \chi_\epsilon\nabla u_\epsilon. \tag{4.24}$$

On the other hand, since u_η converges to u_ϵ a.e. in Q, by assumption $A1$), we get

$$\mathbf{A}^\star(u_\eta, \nabla u_\epsilon) \to \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon), \quad \text{strongly in } (L^{p'(x)}(Q))^N.$$

Moreover, from (4.16) and (4.17), we obtain

$$\int_Q \mathbf{A}^\star(u_\eta, \nabla u_\epsilon)\cdot\nabla(u_\eta - u_\epsilon) \to 0, \quad \text{when } \eta \to 0. \tag{4.25}$$

Finally, we use (4.24) and (4.25) to obtain (4.22). □

Now, since the mapping $u \to F^\star_\epsilon(u, \nabla u)$ is continuous from V into $L^1(Q)$, the previous lemma permits to pass to the limit in the term $F^\star_\epsilon(u_\eta, \nabla u_\eta)$ which converges to $F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon)$ in $L^1(Q)$. Moreover, we can also deduce the strong convergence of $\mathcal{A}^\star u_\eta$ to $\mathcal{A}^\star u_\epsilon$ in V_0'.

Furthermore, since $\frac{1}{\eta}\beta(u_\eta)$ is bounded in $L^{(p_-)'}(Q)$, and u_η converges strongly to u_ϵ in V_0, then $\beta(u_\epsilon) = 0$ a.e. in Q, which implies that u_ϵ is in K. Thus, u_ϵ is in $L^\infty(Q)$, this is a fundamental difference with u_η (the role of the penalty operator $\frac{1}{\eta}\beta(u_\eta)$.)

Finally, we pass to the limit in $(P^\star_{\eta,\epsilon})$, when η tends to zero, to obtain the following problem

$$(P^\star_\epsilon)\begin{cases} u_\epsilon \in V_0 \cap L^\infty(Q), \;\; \partial_t u_\epsilon \in V_0' + L^1(Q), \\ \partial_t u_\epsilon + \mathcal{A}^\star u_\epsilon - F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon) + \beta_\epsilon = 0 & \text{in} Q, \\ u_\epsilon(0) = u_\epsilon(T) & \text{in}\Omega, \end{cases}$$

and one can easily deduce that

$$\forall\, v \in K, \;\; \langle \partial_t u_\epsilon, v - u_\epsilon\rangle + \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)\cdot\nabla(v - u_\epsilon) - \int_Q F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon)(v - u_\epsilon) \ge 0. \tag{4.26}$$

4.3.2 Estimates on $(u_\epsilon)_\epsilon$ in V_0

At this stage, we got a nonlinear problem $(P^\star_\epsilon)$ which only depends on the parameter ϵ. So, to pass to the limit when ϵ tends to zero, we need some a priori estimates in V_0.

Lemma 4.6 *The sequence (u_ϵ) is bounded in V_0.*

Proof We prove this result by using the test function $z_s(u_\epsilon) = \exp(s u_\epsilon^2) u_\epsilon$, where s is such that

$$\alpha z_s'(u_\epsilon) - C|z_s(u_\epsilon)| \ge \frac{\alpha}{2}, \tag{4.27}$$

where α is defined in $A3$) and C in (4.2). As u_ϵ is in $V_0 \cap L^\infty(Q)$, then $z_s(u_\epsilon)$ is in $V_0 \cap L^\infty(Q)$.

By multiplying $(P^\star_\epsilon)$ by $z_s(u_\epsilon)$, we obtain

$$\langle \partial_t u_\epsilon, z_s(u_\epsilon)\rangle + \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)\cdot\nabla z_s(u_\epsilon) + \int_Q \beta_\epsilon z_s(u_\epsilon) = \int_Q F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon) z_s(u_\epsilon). \tag{4.28}$$

From the periodicity condition of u_ϵ, the first term in the left-hand side of (4.28) equals zero. We use (4.16), (4.19) and the sign condition of β, we get $\int_Q \beta_\epsilon z_s(u_\epsilon) \ge 0$. Moreover, by (4.2), the coercivity assumption $A3$), and the fact that u_ϵ is in K, we obtain

$$\alpha \int_Q z_s'(u_\epsilon)|\nabla u_\epsilon|^{p(x)} \le C\left(1 + \int_Q |z_s(u_\epsilon)||\nabla u_\epsilon|^{p(x)}\right).$$

Now, by the (4.27) and the inequality (2.1), we get

$$\min\left\{\|u_\epsilon\|_{V_0}^{p_-}, \|u_\epsilon\|_{V_0}^{p_+}\right\} \le \frac{2C}{\alpha},$$

where C is independent of ϵ. Hence, (u_ϵ) is bounded in V_0. □

Lemma 4.7 *The sequence* $(\partial_t u_\epsilon)$ *is bounded in* $V_0' + L^1(Q)$.

To prove this lemma, it suffices to show from the problem $(P_\epsilon^\star)$ that β_ϵ is bounded in $L^1(Q)$. In other words, we need the following estimate, whose proof is similar to that in [14, p. 296]

$$\|\frac{1}{\eta}\beta(u_{\eta,\epsilon})\|_{L^1(Q)} \leq C_1 + \int_Q C(h^\star(x,t) + |\nabla u_{\eta,\epsilon}|^{p(x)}), \tag{4.29}$$

where C_1 is independent of η and ϵ, and where C is defined in (4.2).

Proof Let $v \in V_0 \cap L^\infty(Q)$, then from the equation of problem $(P_\epsilon^\star)$, we have

$$|\langle \partial_t u_\epsilon, v\rangle| \leq \int_Q |\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)||\nabla v| + \int_Q |F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)||v| + \int_Q |\beta_\epsilon||v|. \tag{4.30}$$

In a similar way as in the proof of Lemma 4.4, and since (u_ϵ) is bounded in V_0, we obtain

$$\int_Q |\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)||\nabla v| \leq C\|v\|_{V_0}.$$

We use (4.2), inequality (2.1) and the boundedness of (u_ϵ) in V_0, to obtain

$$\int_Q |F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)||v| \leq C'\|v\|_{V_0}.$$

Now, by using (4.29) and since $v \in L^\infty(Q)$, we obtain

$$\int_Q |\beta_\epsilon||v| \leq C''\|v\|_{L^\infty(Q)}.$$

Finally, we have

$$\|\partial_t u_\epsilon\|_{V_0'+L^1(Q)} \leq C, \quad \text{where } C \text{ is independent of } \epsilon.$$

□

Passage to the limit in ϵ.

We fix $s > \frac{N}{2} + 1$, so that $H_0^s(\Omega) \hookrightarrow L^\infty(\Omega)$, and then $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$. We have also, $H_0^s(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$, and consequently, $W^{-1,p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$. From Lemma 3.1 we have $V_0' \hookrightarrow L^{(p_+)'}(0,T;(W_0^{1,p(x)}(\Omega))')$. Thus, from the previous lemma $(\partial_t u_\epsilon)$ is bounded in $L^1(0,T;H^{-s}(\Omega))$. Moreover, from the compactness theorem of [19] (p. 85, Corollary 4) and (4.7), the sequence (u_ϵ) is relatively compact in $L^{p_-}(Q)$. So, we can extract a subsequence still denoted by (u_ϵ), such that, when ϵ tends to zero we have

$$u_\epsilon \to u \ \text{ strongly in } L^{p_-}(Q), \ \text{ and a.e. in } Q, \tag{4.31}$$
$$u_\epsilon \to u \ \text{ weak}^*\text{in } L^\infty(Q), \tag{4.32}$$
$$\partial_t u_\epsilon \rightharpoonup \partial_t u \ \text{ weakly in } V_0' + L^1(Q). \tag{4.33}$$

Now, as $\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)$ is bounded in $(L^{p'(x)}(Q))^N$, there exists χ in $(L^{p'(x)}(Q))^N$ such that

$$\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \rightharpoonup \chi \ \text{ in } (L^{p'(x)}(Q))^N \hookrightarrow (L^{(p_-)'}(Q))^N. \tag{4.34}$$

In addition, by using (4.31), it is clear that u is in K.

Lemma 4.8 *The sequence* (u_ϵ) *converges strongly to some* u *in* V_0.

Proof The idea of proof is to apply the Lemma 3.8, since u_ϵ converges weakly to u in V_0 and $\mathbf{A}^\star$ satisfies $A1$), $A2$) and $A3$).

We consider $\epsilon' > 0$ and we subtract $(P^\star_\epsilon)$ from $(P^\star_{\epsilon'})$, we obtain

$$\partial_t(u_\epsilon - u_{\epsilon'}) + \mathcal{A}^\star u_\epsilon - \mathcal{A}^\star u_{\epsilon'} - F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon) + F^\star_{\epsilon'}(u_{\epsilon'}, \nabla u_{\epsilon'}) + \beta_\epsilon - \beta_{\epsilon'} = 0. \tag{4.35}$$

Now, we multiply (4.35) by the same type of test function $z_s(u_\epsilon - u_{\epsilon'})$ used in the proof of Lemma 4.6, we get

$$\begin{aligned}&\langle \partial_t(u_\epsilon - u_{\epsilon'}), z_s(u_\epsilon - u_{\epsilon'})\rangle + \int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) z_s'(u_\epsilon - u_{\epsilon'})\\ &\quad + \int_Q (F^\star_{\epsilon'}(u_{\epsilon'}, \nabla u_{\epsilon'}) - F^\star_\epsilon(u_\epsilon, \nabla u_\epsilon)) z_s(u_\epsilon - u_{\epsilon'}) + \int_Q (\beta_\epsilon - \beta_{\epsilon'}) z_s(u_\epsilon - u_{\epsilon'}) = 0. \end{aligned}\tag{4.36}$$

Thanks to the periodicity condition of u_ϵ, the first term of (4.36) equals zero. By (4.16), (4.19) and the sign condition of β, the last term of (4.36) is nonnegative. By (4.2), the Eq. (4.36), then implies that

$$\begin{aligned}&\int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) z_s'(u_\epsilon - u_{\epsilon'}) \le C \int_Q (h^\star(x,t) + |\nabla u_\epsilon|^{p(x)}) |z_s(u_\epsilon - u_{\epsilon'})|\\ &\quad + C \int_Q (h^\star(x,t) + |\nabla u_{\epsilon'}|^{p(x)}) |z_s(u_\epsilon - u_{\epsilon'})|. \end{aligned}\tag{4.37}$$

Using the coercivity condition $A3$), we get

$$\begin{aligned}&\int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) z_s'(u_\epsilon - u_{\epsilon'}) \le 2C \int_Q h^\star(x,t) |z_s(u_\epsilon - u_{\epsilon'})|\\ &\quad + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_\epsilon |z_s(u_\epsilon - u_{\epsilon'})| + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla u_{\epsilon'} |z_s(u_\epsilon - u_{\epsilon'})|\\ &\le 2C \int_Q h^\star(x,t) |z_s(u_\epsilon - u_{\epsilon'})| + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla(u_\epsilon - u_{\epsilon'}) |z_s(u_\epsilon - u_{\epsilon'})|\\ &\quad + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_{\epsilon'} |z_s(u_\epsilon - u_{\epsilon'})| - \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla(u_\epsilon - u_{\epsilon'}) |z_s(u_\epsilon - u_{\epsilon'})|\\ &\quad + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla u_\epsilon |z_s(u_\epsilon - u_{\epsilon'})|. \end{aligned}\tag{4.38}$$

By condition (4.27), we deduce that

$$\begin{aligned}&\frac{1}{2} \int_Q (\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'})) \cdot \nabla(u_\epsilon - u_{\epsilon'}) \le 2C \int_Q h^\star |z_s(u_\epsilon - u_{\epsilon'})|\\ &\quad + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla u_{\epsilon'} |z_s(u_\epsilon - u_{\epsilon'})| + \frac{C}{\alpha} \int_Q \mathbf{A}^\star(u_{\epsilon'}, \nabla u_{\epsilon'}) \cdot \nabla u_\epsilon |z_s(u_\epsilon - u_{\epsilon'})|. \end{aligned}\tag{4.39}$$

Following the same steps of Lemma 4.5, we obtain the desired result, namely

$$\limsup_{\epsilon \to 0} \int_Q \big(\mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) - \mathbf{A}^\star(u_\epsilon, \nabla u)\big) \cdot (\nabla u_\epsilon - \nabla u) \le 0.$$

□

Now, we prove that u is between φ and ψ almost everywhere in Q, where φ and ψ are, respectively, sub- and supersolution of problem (P) with $\varphi \le \psi$ a.e. in Q.

Proposition 4.9 *We have* $\varphi \le u \le \psi$ *a.e. in* Q.

Proof We shall prove that $\varphi \leq u$ a.e. in Q. One can verify easily that: $v = u_\epsilon + (\varphi - u_\epsilon)^+$ is in K. Then, we can take it as a function test in (4.26). Hence, we obtain

$$\langle \partial_t u_\epsilon, (\varphi - u_\epsilon)^+ \rangle + \int_Q \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon) \cdot \nabla(\varphi - u_\epsilon)^+ - \int_Q F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(\varphi - u_\epsilon)^+ \geq 0. \tag{4.40}$$

Since φ is a subsolution, and $(\varphi - u_\epsilon)^+$ in $V_0 \cap L^\infty(Q)$, we obtain

$$\langle \partial_t \varphi, (\varphi - u_\epsilon)^+ \rangle + \int_Q \mathbf{A}^\star(\varphi, \nabla \varphi) \cdot \nabla(\varphi - u_\epsilon)^+ - \int_Q F(\varphi, \nabla \varphi)(\varphi - u_\epsilon)^+ \leq 0. \tag{4.41}$$

By subtracting (4.40) from (4.41), and by Lemma 3.7, we get

$$\int_Q (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}^\star(u_\epsilon, \nabla u_\epsilon)) \cdot \nabla(\varphi - u_\epsilon)^+ + \int_Q (F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon) - F(\varphi, \nabla \varphi))(\varphi - u_\epsilon)^+ \leq 0. \tag{4.42}$$

Thanks to Lemma 4.8, we pass to the limit when ϵ tends to zero in (4.42) and get

$$\int_Q (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}^\star(u, \nabla u)) \cdot \nabla(\varphi - u)^+ + \int_Q (F^\star(u, \nabla u) - F(\varphi, \nabla \varphi))(\varphi - u)^+ \leq 0. \tag{4.43}$$

Furthermore, from the definition of $\mathbf{A}^\star$ and $F^\star$, we have

$$(F^\star(u, \nabla u) - F(\varphi, \nabla \varphi))(\varphi - u)^+ = 0 \quad \text{a.e.in } Q \quad \text{and} \quad \mathbf{A}^\star(u, \nabla u) \cdot \nabla(\varphi - u)^+ = \mathbf{A}(\varphi, \nabla u) \cdot \nabla(\varphi - u)^+.$$

Therefore, we obtain

$$\int_Q (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}(\varphi, \nabla u)) \cdot \nabla(\varphi - u)^+ \leq 0,$$

that is

$$\int_{\{\varphi \geq u\}} (\mathbf{A}(\varphi, \nabla \varphi) - \mathbf{A}(\varphi, \nabla u)) \cdot \nabla(\varphi - u) \leq 0.$$

According to $A2$), this implies that $\nabla(\varphi - u) = 0$ a.e. in $\{(x, t) \in Q, \varphi \geq u\}$. Then, $\varphi - u = 0$ a.e. in $\{(x, t) \in Q, \varphi \geq u\}$ which means that $\varphi \leq u$ a.e. in Q. By a similar proof, we can obtain $u \leq \psi$ a.e. in Q. □

To complete the proof of Theorem 3.6 we need the following lemma.

Lemma 4.10 *β_ϵ tends to zero in $L^1(Q)$.*

Proof By taking $(u_\epsilon - k + 1)^+ \in V_0 \cap L^\infty(Q)$ as a test function in $(P_\epsilon^\star)$, and using the periodicity condition of u_ϵ and assumption $A3$), we obtain

$$\int_Q \beta_\epsilon (u_\epsilon - k + 1)^+ \leq F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(u_\epsilon - k + 1)^+. \tag{4.44}$$

On the other hand, we have

$$\int_Q |\beta_\epsilon| = \int_{\{|u_\epsilon| < k\}} |\beta_\epsilon| + \int_{\{u_\epsilon = k\}} |\beta_\epsilon| + \int_{\{u_\epsilon = -k\}} |\beta_\epsilon|. \tag{4.45}$$

The definition of β gives $\beta(u_{\epsilon,\eta}) = 0$ if $|u_{\epsilon,\eta}| \leq k$, hence, $\frac{1}{\eta}\beta(u_{\epsilon,\eta})$ tends to 0, when η tends to 0 that is

$$\frac{1}{\eta}\beta(u_{\epsilon,\eta})\chi_{\{|u_\epsilon| < k\}} \to 0 \quad \text{a.e. in } Q. \tag{4.46}$$

From (4.46), the boundedness of $\frac{1}{\eta}\beta(u_{\epsilon,\eta})\chi_{\{|u_\epsilon| < k\}}$ in $L^{(p_-)'}(Q)$ (see Lemma 4.3) and by using Lemma 4.2 of [3], we get

$$\frac{1}{\eta}\beta(u_{\epsilon,\eta})\chi_{\{|u_\epsilon| < k\}} \to 0 \quad \text{when } \eta \to 0 \quad \text{in } L^{(p_-)'}(Q) \quad \text{weakly}. \tag{4.47}$$

By (4.19), we deduce that

$$\beta_\epsilon = 0 \quad \text{a.e. in } \{|u_\epsilon| < k\}. \tag{4.48}$$

Since $|u_\epsilon| < k$ a.e. in Q, then from (4.48), we obtain

$$\int_Q \beta_\epsilon (u_\epsilon - k + 1)^+ = \int_{\{u_\epsilon = k\}} |\beta_\epsilon|.$$

So, (4.44) becomes

$$\int_{\{u_\epsilon = k\}} |\beta_\epsilon| \le F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)(u_\epsilon - k + 1)^+. \tag{4.49}$$

Since $-k+1 \le \varphi \le u \le \psi \le k-1$, $(u_\epsilon - k + 1)^+$ tends to 0 almost everywhere in Q and in $L^\infty(Q)$ weak*, and $F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon)$ converges strongly to $F(u, \nabla u)$ in $L^1(Q)$. Then, we can deduce from (4.49) that

$$\lim_{\epsilon \to 0} \int_{\{u_\epsilon = k\}} |\beta_\epsilon| = 0.$$

In the same way, we show that $\lim_{\epsilon \to 0} \int_{\{u_\epsilon = -k\}} |\beta_\epsilon| = 0$. Whence, the desired result. □

4.3.3 Conclusion

Now, we can pass to the limit in each term of problem $(P_\epsilon^\star)$. In other words, we have

$$\begin{aligned}
&\mathcal{A}^\star(u_\epsilon, \nabla u_\epsilon) \to \mathcal{A}^\star(u, \nabla u) \ \text{ in } V_0' \ \text{ strongly},\\
&F_\epsilon^\star(u_\epsilon, \nabla u_\epsilon) \to F^\star(u, \nabla u) \ \text{ in } L^1(Q) \ \text{ strongly},\\
&\beta_\epsilon \to 0 \ \text{ in } L^1(Q) \ \text{ strongly},\\
&\partial_t u_\epsilon \to \partial_t u \ \text{ in } V_0' + L^1(Q) \ \text{ strongly}.
\end{aligned}$$

Therefore, u satisfies

$$\partial_t u + \mathcal{A}^\star(u, \nabla u) - F^\star(u, \nabla u) = 0.$$

From Proposition 4.9 we have $\varphi \le u \le \psi$. Then, we get $\mathcal{A}^\star(u, \nabla u) = \mathcal{A}(u, \nabla u)$ and $F^\star(u, \nabla u) = F(u, \nabla u)$.

Concerning the periodicity condition, since (u_ϵ) is bounded in $V_0 \cap L^\infty(Q)$ and $(\partial_t u_\epsilon)$ is bounded in $V_0' + L^1(Q)$, then $(\partial_t u_\epsilon)$ is bounded in $L^1(0, T; H^{-s}(\Omega))$. So, (u_ϵ) is relatively compact in $L^{p-}(Q)$. Hence, $u_\epsilon(0) \to u(0)$ in $L^{p-}(Q)$ and $u_\epsilon(T) \to u(T)$ in $L^{p-}(Q)$. As $u_\epsilon(0) = u_\epsilon(T)$, then we deduce that $u(0) = u(T)$.

Finally, u is a periodic solution of problem (P).

5 Applications

In this section, we construct a subsolution and a supersolution for the following nonlinear parabolic problem associated with $p(x)$-Laplacian (concerning their physical interpretation see our introduction or [2] for more details):

$$(P) \begin{cases} \partial_t u - \Delta_{p(x)} u = f(x, t) & \text{in } Q,\\ u = 0 & \text{on } \Sigma,\\ u(0) = u(T) & \text{in } \Omega, \end{cases}$$

where $\Omega \equiv B(0, R) = \{x \in \mathbb{R}^N |\ |x| < R\}$ is the unit ball, with $R > 0$ large enough. Moreover, assume that $p(x) \in C^1(\mathbb{R}^N)$ is radial, that means $p(x) = p(|x|) = p(r)$, with $|x| = r < R$, and satisfies the assumptions of Sect. 2.

Let $M = \|f\|_{L^\infty(Q)} < \infty$. We set

$$\psi(r) = \int_r^R \left[\frac{M}{N}t\right]^{\frac{1}{p(t)-1}} \mathrm{d}t, \quad \text{and} \quad \varphi(r) = -\psi(r).$$

It is clear that $\varphi(r) \le 0 \le \psi(r)$. Moreover, ψ and φ are supersolution and subsolution, respectively of problem (P). Indeed, we have

$$-\Delta_{p(r)}\psi(r) = -\frac{1}{r^{N-1}}(r^{N-1}|\psi'(r)|^{p(r)-2}\psi'(r))'.$$

Since

$$\psi'(r) = -\left(\frac{M}{N}r\right)^{\frac{1}{p(r)-1}},$$

then

$$|\psi'(r)|^{p(r)-2}\psi'(r) = -\frac{M}{N}r.$$

Now, since $\psi(r)$ is independent of t, then we obtain

$$\partial_t\psi(r) - \Delta_{p(r)}\psi(r) = -\Delta_{p(r)}\psi(r) = M = \|f\|_{L^\infty(Q)} \ge f(x,t).$$

Moreover, if $r \in \partial\Omega$ (ie. $r = R$), then $\psi(r) = 0$. Hence, ψ is a supersolution of problem (P) in the sense of Definition 3.5.

We repeat the same previous calculations, to obtain

$$\partial_t\varphi(r) - \Delta_{p(r)}\varphi(r) = -\Delta_{p(r)}\varphi(r) = -M = -\|f\|_{L^\infty(Q)} \le f(x,t),$$

as far as $\varphi(r) = 0$ if $r \in \partial\Omega$. Hence, φ is a subsolution of problem (P) in the sense of Definition 3.5.

Hence, applying our main result, Theorem 3.6, we deduce the existence of at least one periodic solution $u(x,t)$ of problem (P) such that $\varphi \le u \le \psi$ a.e. in Q.

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