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Conformal metrics of prescribed scalar curvature on 4-manifolds: the degree zero case

To the memory of Prof. Abbas Bahri

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Abstract In this paper, we consider the problem of existence and multiplicity of conformal metrics on a Riemannian compact 4-dimensional manifold (M^4, g_0) with positive scalar curvature. We prove a new existence criterium which provides existence results for a dense subset of positive functions and generalizes Bahri–Coron Euler–Poincaré type criterium. Our argument gives estimates of the Morse index of the founded solutions and has the advantage to extend known existence results. Moreover, it provides, for generic K *Morse Inequalities at Infinity*, which give a lower bound on the number of metrics with prescribed scalar curvature in terms of the topological contribution of its *critical points at Infinity* to the difference of topology between the level sets of the associated Euler–Lagrange functional.

Mathematics Subject Classification 58E05 · 35J65 · 53C21 · 35B40

الملخص

في هذه الورقة، ندرس مسألة وجود وتعدد متريات محافظة على متنوعة ريمانية متراصة رباعية البعد (M^4, g_0) ذات انحناء سلّمي موجب. نثبت معيار وجود جديد يعطي نتائج وجود لمجوعة جزئية كثيفة مكونة من دوال موجبة، ويعمم المعايير من نوع بحري-كورون أويلر-بوينكاري. يعطي عرضنا تقديرا لمؤشر مورس للحل المؤسس وله إيجابية تمديد بعض النتائج المعروفة. بالإضافة إلى ذلك، يعطي لمتباينات مورس النوعية K عند اللانهاية، حدا أدنى لعدد المتريات لانحناء سلمي معطى مسبقا بدلالة المساهمة التوبولوجية للنقاط الحرجة عند اللانهاية للفرق في التوبولوجي بين مجموعات المستوى لدالية أويلر – لاجرانج المرتبطة بها.

1 Introduction and main results

Let (M^4, g_0) be a compact 4-dimensional Riemannian manifold with positive scalar curvature R_{g_0}. Given a C^2 function K defined on the manifold, the prescribed scalar curvature problem consists of finding a metric g, conformally related to g_0, such that the scalar curvature of (M, g) is given by the function K. Writing $g = u^2\, g_0$, this amounts to solve the following nonlinear partial differential equation:

$$(\mathbf{P_K}) \quad L_{g_0} u = K\, u^3, \quad u > 0 \text{ in } M^4,$$

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where L_{g_0} denotes the conformal Laplacian operator, defined as:

$$L_{g_0}u := -\Delta_{g_0}u + \frac{1}{6}u.$$

More generally the same question can be asked on every Riemannian manifold (M^n, g) of dimension $n \geq 3$. In this case the related PDE takes the form:

$$\textbf{(SC)} \quad L_{g_0}u = K\, u^{\frac{n+2}{n-2}}, \quad u > 0 \text{ in } M^n,$$

This problem has been the subject of intensive studies in the last three decades (see [2–10,12,14–16,18–22,24,26–30,33–36,38] and the references therein).

Regarding the existence results of the problem (SC), we recall that on 3-spheres, an Euler–Poincaré type criterium for the function K has been obtained by Bahri and Coron [12], see also Chang–Gursky–Yang [18]. Such a criterium has been generalized for the 4-spheres by Ben Ayed et al. [14] and on higher dimensional spheres, under a closeness to a constant assumption [19] or a "flatness condition" on the critical points of the function K [28].

For higher dimensional spheres ($n \geq 7$), Bahri [10] discovered a new topological invariant and proved new type of existence results. Some of these results have been generalized by Ben Ayed et al. [15].

The main difficulty of this problem comes from the presence of the critical Sobolev exponent, which generates blow up and lack of compactness. Indeed the problem enjoys a variational structure, however, the associated Euler Lagrange functional does not satisfy the *Palais–Smale condition*. From the variational viewpoint, it is the occurrence of *critical points at Infinity*, that are noncompact orbits of the gradient flow, along which the functional remains bounded and its gradient goes to zero, which prevents the use of variational methods.

Among approaches developed to deal with this problem, we single out the blow-up analysis of some subcritical approximation combined with the use of the Leray–Schauder topological degree, approach developed by Schoen [34], Li [28,29], Lin and Chen [20–22], among others. The second one is based on a careful study of the critical point at Infinity, through a Morse type reduction and the identification of their contribution to the difference of topology between the level sets of the associated Euler–Lagrange functional, has been initiated by Bahri and Coron [11] and developed through the works of Bahri [10], Ben Ayed et al., see [14,15], Ben Ayed and Ould Ahmedou [16], among others. Other approaches include perturbations methods of Chang–Yang [19] and Ambrosetti [2] and the flow approach of Struwe [37,38].

In this paper, we revisit this problem to give new existence as well as multiplicity results, extending previous known ones.

To state our results we need to introduce some notations and assumptions.

We denote by $G(a, .)$ the Green's function of the conformal Laplacian L_{g_0} with pole at a and by A_a the value of its regular part, evaluated at a.

Let $0 < K \in C^2(M^4)$ be a positive function, defined on the manifold $(M^4, g_0$. We say that the function K satisfies the condition $(\mathbf{H_0})$, if K has only nondegenerate critical points and for each critical point y, there holds

$$\frac{-\Delta K(y)}{3K(y)} - 2A_y \neq 0.$$

Denoting $\mathcal{K}$ the set of critical point of K, we set

$$\mathcal{K}^+ := \left\{ y \in \mathcal{K};\ \frac{-\Delta K(y)}{3K(y)} - 2A_y > 0 \right\}.$$

To each p-tuple $\tau_p := (y_1, \ldots, y_p) \in (\mathcal{K}^+)^p$, we associate a Matrix $M(\tau_p) = (M_{ij})$ defined by

$$\begin{aligned} M_{ii} &= \frac{-\Delta K(y_i)}{3K(y_i)^2} - 2\frac{A_{y_i}}{K(y_i)}, \\ M_{ij} &= \frac{-2G(y_i, y_j)}{\sqrt{K(y_i)K(y_j)}} \quad \text{for } i \neq j. \end{aligned} \tag{1.1}$$

We denote by $\rho(\tau_p)$ the least eigenvalue of $M(\tau_p)$ and we say that a function K satisfies the condition $(\mathbf{H_1})$ if for every $\tau_p \in (\mathcal{K}^+)^p$, we have that $\rho(\tau_p) \neq 0$.

We set

$$\mathcal{F}_\infty := \left\{\tau_p = (y_1, \ldots, y_p) \in (\mathcal{K}+)^p;\ \rho(\tau_p) > 0\right\} \tag{1.2}$$

and define an index $\iota : \mathcal{F}_\infty \to \mathbb{Z}$ defined by

$$\iota(\tau_p) := p - 1 + \sum_{i=1}^{p}(4 - m(K, y_i)),$$

where $m(K, y_i)$ denotes the Morse index of K at its critical point y_i.

Now we state our main result.

Theorem 1.1 *Let $0 < K \in C^2(M^4)$ be a positive function satisfying the conditions (H_0) and (H_1).*

If there exists $k \in \mathbb{N}$ such that

1.
$$\sum_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \le k-1} (-1)^{\iota(\tau_p)} \neq 1,$$

2.
$$\forall \tau_p \in \mathcal{F}_\infty,\ \iota(\tau_p) \neq k$$

Then there exists a solution w to the problem (P_K) such that:

$$Morse(w) \le k,$$

where $Morse(w)$ is the Morse index of w, defined as the dimension of the space of negativity of the linearized operator:

$$\mathcal{L}_w(\varphi) := L_{g_0}(\varphi) - 3Kw^2\varphi.$$

Moreover, for generic K, it holds

$$\#\mathcal{N}_k \ge \left|1 - \sum_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \le k-1} (-1)^{\iota(\tau_p)}\right|,$$

where $\mathcal{N}_k$ denotes the set of solutions of (P_K) having their Morse indices less or equal k.

Please observe that, taking in the above k to be $l_\# + 1$, where $l_\#$ is the maximal index of the elements of $\mathcal{F}_\infty$, the second assumption is trivially satisfied. Therefore, in this case, we have the following Corollary, which recovers previous existence results, see [14,16,29].

Corollary 1.2 *Let $0 < K \in C^2(M^4)$ be a positive function satisfying the conditions (H_0) and (H_1).*

If

$$\sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{\iota(\tau_p)} \neq 1,$$

Then the problem (P_K) has at least one solution.

Moreover, for generic K, it holds

$$\#\mathcal{S} \ge \left|1 - \sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{\iota(\tau_p)}\right|,$$

where $\mathcal{S}$ denotes the set of solutions of (P_K).

We point out that the dimension four is crucial in the above existence results. Indeed on manifolds of dimension $n \geq 5$ and under the nondegeneracy assumptions (H_0) and (H_1) the above sum is always equal to 1.

We notice that the main new contribution of Theorem 1.1 is that we address here the case where the total sum in the above corollary equals 1, but a partial one is different from 1. The main issue being the possibility to use such an information to prove existence of solution to the problem (P_K). To understand the difficulty in addressing such a case, we give, following Li [29], a new interpretation of the above counting formula in terms of Leray–Schauder degree of the solutions. Indeed Y. Y. Li proved[1] that, under the assumption of Corollary 1.2, there exists $R > 0$ such that the all solutions of (P_K) remain, for $\alpha \in (0, 1)$ in

$$\Omega_R := \left\{ u \in C^{2,\alpha}; \frac{1}{R} < u < R, \|u\|_{C^{2,\alpha}} < R \right\}.$$

Hence the Leray–Schauder degree $\deg(v - L^{-1}(K\, v^3)), \Omega_R, 0)$ is well defined. Moreover, it turns out that:

$$\deg(v - L^{-1}(K\, v^3)), \Omega_R, 0) = 1 - \sum_{\tau_p \in \mathcal{F}_\infty} (-1)^{\iota(\tau_p)}.$$

Therefore, considering the case where the counting formula in Corollary 1.2 equals 1, amounts to consider the case where the Leray–Schauder degree equals zero.

Besides the degree interpretation of the counting formula, another interpretation of the fact that the above sum is different from one, is that the topological contribution of the *critical points at infinity* to the level sets of the associated Euler–Lagrange functional is not trivial. In view of such an interpretation, the above question can be formulated as follows: what happens if the total contribution is trivial, but some critical points at infinity induce a difference of topology. Can we still use such a local topological information to prove existence of solution?

With respect to the above question, Theorem 1.1 gives a sufficient condition to be able to derive from such a local information, an existence as well as a multiplicity result together with information on the Morse index of the obtained solution. At the end of this paper, see, we give a more general condition. Since this condition involves the critical points at infinity of the variational problem, we have postponed its statement to this end of the paper.

As pointed out above, our result does not only give existence results, but also, under generic conditions, gives a lower bound on the number of solutions of (P_K). Such a result is reminiscent to the celebrated Morse Theorem, which states that, the number of critical points of a Morse function defined on a compact manifold, is lower bounded in terms of the topology of the underlying manifold. Our result can be seen as some sort of *Morse Inequality at Infinity*. Indeed it gives a lower bound on the number of metrics with prescribed curvature in terms of the *topology at infinity*, that is the one induced by the *critical point at infinity*.

The remainder of this paper is organized as follows. In Sect. 2 we set up the variational problem, its critical points at Infinity are characterized in Sect. 3. Section 4 is devoted to the proof of the main result Theorem 1.1 while we give in Sect. 5 a more general statement than Theorem 1.1.

2 Variational structure and the lack of compactness

In this section we recall the functional setting, its variational structure and its main features. Problem (P_K) has a variational structure. The Euler–Lagrange functional is

$$J(u) = \frac{\int_M L_{g_0} u\, u}{\left(\int_M K|u|^4\right)^{1/2}} \tag{2.1}$$

defined on $H^1(M, \mathbb{R}) \setminus \{0\}$ equipped with the norm

$$\|u\|^2 = \int_M L_{g_0} u\, u.$$

[1] Actually Y. Y. Li proved the a priori estimates on $\mathbb{S}^4$ but the proof extends virtually to four dimensional manifolds.

We denote by Σ the unit sphere of $H^1(M, \mathbb{R})$ and we set $\Sigma^+ = \{u \in \Sigma : u \geq 0\}$. The Palais–Smale condition fails to be satisfied for J on Σ^+. To characterize the sequences failing the Palais–Smale condition, we need to introduce some notations.

Given $a \in M$, we choose a conformal metric

$$g_a := u_a^2 g$$

such that u_a depends smoothly on a. Let x be a conformal normal coordinate centered at a and $\varrho > 0$ uniform independent of a such that x is well defined on $B_{2\varrho}(a)$.

We set

$$\delta_{a,\lambda} := \frac{\lambda}{1 + \lambda^2 |x - a|^2}, \quad x \in B_\varrho(a),\ \lambda > 0,$$

and

$$\hat{\delta}_{a,\lambda}(x) := u_a(x)\, \omega_a(x)\, \delta_{a,\lambda}(x),$$

where ω_a is a cutoff function such that:

$$\omega_a(x) = 1 \quad \text{on } B_\varrho(a), \qquad \omega_a(x) = 0 \quad \text{on } M \backslash B_{2\varrho}(a)$$

we define $\varphi_{a,\lambda}$ to be the solution of

$$L_{g_0} \varphi_{a,\lambda} = 8 \hat{\delta}^3_{a,\lambda}.$$

We define now the set of potential critical points at infinity associated to the functional J.

For $\varepsilon > 0$ and $p \in \mathbb{N}^*$, let us define

$$V(p, \varepsilon) = \Bigg\{ u \in \Sigma / \exists\, a_i \in M, \lambda_i > \varepsilon^{-1}, \alpha_i > 0 \quad \text{for } i = 1, \ldots, p \text{ s.t.}$$
$$\left\| u - \sum_{i=1}^{p} \alpha_i \varphi_i \right\| < \varepsilon, \ \left| \frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 \right| < \varepsilon, \quad \text{and } \varepsilon_{ij} < \varepsilon \Bigg\}$$

where $\varphi_i = \varphi_{(a_i, \lambda_i)}$ and $\varepsilon_{ij} = (\lambda_i/\lambda_j + \lambda_j/\lambda_i + \lambda_i \lambda_j d(a_i, a_j)^2)^{-1}$.

For w a solution of (P_K) we also define $V(p, \varepsilon, w)$ as

$$\left\{ u \in \Sigma / \exists\, \alpha_0 > 0 \text{ s.t. } u - \alpha_0 w \in V(p, \varepsilon) \text{ and } |\alpha_0^2 J(u)^2 - 1| < \varepsilon \right\}. \tag{2.2}$$

The failure of the Palais–Smale condition can be described as follows.

Proposition 2.1 [13,14,16,17,39,40] *Let $(u_j) \in \Sigma^+$ be a sequence such that $\nabla J(u_j)$ tends to zero and $J(u_j)$ is bounded. Then, there exist an integer $p \in \mathbb{N}^*$, a sequence $\varepsilon_j > 0$, ε_j tends to zero, and an extracted subsequence of u_j's, again denoted u_j, such that $u_j \in V(p, \varepsilon_j, w)$ where w is zero or a solution of (P_K).*

We consider the following minimization problem for $u \in V(p, \varepsilon)$ with ε small

$$\min_{\alpha_i > 0,\, \lambda_i > 0,\, a_i \in \mathbb{S}^n} \left\| u - \sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} \right\|_{H^1}. \tag{2.3}$$

We then have the following parametrization of the set $V(p, \varepsilon)$.

Proposition 2.2 [9,12,14] *For any $p \in \mathbb{N}^*$, there is $\varepsilon_p > 0$ such that if $\varepsilon < \varepsilon_p$ and $u \in V(p, \varepsilon)$, the minimization problem (2.3) has a unique solution (up to permutation). In particular, we can write $u \in V(p, \varepsilon)$ as follows*

$$u = \sum_{i=1}^{p} \bar{\alpha}_i \varphi_{(\bar{a}_i, \bar{\lambda}_i)} + v,$$

where $(\bar{\alpha}_1, \ldots, \bar{\alpha}_p, \bar{a}_1, \ldots, \bar{a}_p, \bar{\lambda}_1, \ldots, \bar{\lambda}_p)$ *is the solution of* (2.3) *and* $v \in H^1(\mathbb{S}^n)$ *such that*

$$(V_0) \quad \|v\| \leq \varepsilon, \qquad (v, \psi) = 0 \quad \text{for } \psi \in \bigcup_{i \leq p,\, j \leq n} \left\{ \varphi_i, \frac{\partial \varphi_i}{\partial \lambda_i}, \frac{\partial \varphi_i}{\partial (a_i)^j} \right\},$$

where $(a_i)^j$ *denotes the* j*th component of* a_i *and* $(.,.)$ *is the inner scalar associated to the norm* $\|.\|$.

In the following we will say that $v \in (V_0)$ if v satisfies (V_0).

Following A. Bahri one performs the following finite dimensional reduction:

Proposition 2.3 [9] *There exists a* C^1 *map which, to each* $(\alpha_1, \ldots, \alpha_p, a_1, \ldots, a_p, \lambda_1, \ldots, \lambda_p)$ *such that* $\sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} \in V(p, \varepsilon)$ *with small* ε, *associates* $\overline{v} = \overline{v}_{(\alpha_i, a_i, \lambda_i)}$ *satisfying*

$$J\left(\sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} + \overline{v}\right) = \min_{v \in (V_0)} J\left(\sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} + v\right).$$

Moreover, there exists $c > 0$ *such that the following holds*

$$\|\overline{v}\| \leq c\left(\sum_{i \leq p}\left(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2}\right) + \sum_{k \neq r} \varepsilon_{kr} (\mathrm{Log}(\varepsilon_{kr}^{-1}))^{1/2}\right).$$

Let w be a non degenerate solution of (P_K). The following proposition defines a parametrization of the set $V(p, \varepsilon, w)$.

Proposition 2.4 [10] *There is* $\varepsilon_0 > 0$ *such that if* $\varepsilon \leq \varepsilon_0$ *and* $u \in V(p, \varepsilon, w)$, *then the problem*

$$\min_{\alpha_i > 0,\, \lambda_i > 0,\, a_i \in M,\, h \in T_w(W_u(w))} \left\| u - \sum_{i=1}^{p} \alpha_i \varphi_{(a_i, \lambda_i)} - \alpha_0 (w + h) \right\|$$

has a unique solution $(\overline{\alpha}, \overline{\lambda}, \overline{a}, \overline{h})$. *Thus, we write* u *as follows*:

$$u = \sum_{i=1}^{p} \overline{\alpha}_i \varphi_{(\overline{a}_i, \overline{\lambda}_i)} + \overline{\alpha}_0 (w + \overline{h}) + v,$$

where v *belongs to* $H^1(M) \cap T_w(W_s(w))$ *and it satisfies* (V_0), $T_w(W_u(w))$ *and* $T_w(W_s(w))$ *are the tangent spaces at* w *to the unstable and stable manifolds of* w.

3 Critical points at infinity of the variational problem

Following A. Bahri we set the following definitions and notations.

Definition 3.1 *A critical point at infinity* of J on Σ^+ is a limit of a flow line $u(s)$ of the equation:

$$\begin{cases} \frac{\partial u}{\partial s} = -\nabla J(u) \\ u(0) = u_0 \end{cases}$$

such that $u(s)$ remains in $V(p, \varepsilon(s), w)$ for $s \geq s_0$.

Here w is either zero or a solution of (P_K) and $\varepsilon(s)$ is some function tending to zero when $s \to \infty$. Using Proposition 2.4, $u(s)$ can be written as:

$$u(s) = \sum_{i=1}^{p} \alpha_i(s)\, \varphi_{(a_i(s), \lambda_i(s))} + \alpha_0(s)(w + h(s)) + v(s).$$

Denoting $a_i := \lim_{s\to\infty} a_i(s)$ and $\alpha_i = \lim_{s\to\infty} \alpha_i(s)$, we denote by

$$(a_1, \ldots, a_p, w)_\infty \quad \text{or} \quad \sum_{i=1}^{p} \alpha_i \, \varphi_{(a_i,\infty)} + \alpha_0 w$$

such a critical point at infinity. If $w \neq 0$ it is called of *w-type*.

We first point out that, as is proven in [16], given a function K a C^2 positive function satisfying the condition of Theorem 1.1 and w a solution of (P_K). Then for each $p \in \mathbb{N}$, there are no critical point or critical point at infinity of J in the set $V(p, \varepsilon, w)$. The reason is that the term coming from the contribution of w is strong enough to bring down any flow line.

Moreover, it follows from [14,16], that the critical points at infinity are in one-to-one correspondence with the elements of the set $\mathcal{F}_\infty$ defined in (1.2). that is a critical point at infinity corresponds to $\tau_p := (y_1, \ldots, y_p) \in (\mathcal{K}^+)^p$ such that the related Matrix $M(\tau_p)$ defined in (1.1) is positive definite. Such a critical point at infinity will be denoted by $\tau_p^\infty := (y_1, \ldots, y_p)_\infty$.

Like an usual critical point, it is associated to any *critical point at infinity* x_∞ of the problem (P_K), which are combination of classical critical points with a 1-dimensional asymptote, stable and unstable manifolds, $W_s^\infty(x_\infty)$ and $W_u^\infty(x_\infty)$. These manifolds can be easily described once a Morse type reduction is performed, see [10,14]. The stable manifold is, as usual, defined to be the set of points attracted by the asymptote. The unstable one is a shadow object, which is the limit of $W_u(x_\lambda)$, x_λ being the critical point of the reduced problem and $W_u(x_\lambda)$ its associated unstable manifolds. Indeed the flow in this case splits the variable λ from the other variables near x_∞.

Next, following A. Bahri [10] we extend the definition of domination of critical points to "critical points at Infinity".

Definition 3.2 z_∞ is said to be dominated by another critical point at infinity z'_∞ if

$$W_u(z'_\infty) \cap W_s(z_\infty) \neq \emptyset.$$

If we assume that the intersection is transverse, then we obtain

$$\operatorname{index}(z'_\infty) \geq \operatorname{index}(z_\infty) + 1.$$

4 Proof of the main result

This section is devoted to the proof of the main result of this paper, Theorem 1.1.

Proof of Theorem 1.1 Setting

$$l_\# := \sup\{\iota(\tau_p);\ \tau_p \in \mathcal{F}_\infty\}$$

For $l \in \{0, \ldots, l_\#\}$ we define the following sets:

$$X_l^\infty := \bigcup_{\tau_p \in \mathcal{F}_\infty;\, \iota(\tau_p) \leq l} \overline{W_s^\infty(\tau_p^\infty)}, \tag{4.1}$$

where $W_s^\infty(\tau_p^\infty)$ denotes the stable manifold associated to the critical point at infinity τ_p^∞ and

$$C(X_l^\infty) := \{t\, u \, + \, (1-t)\,(y_0)_\infty,\ t \in [0,1],\ u \in X_l^\infty\}, \tag{4.2}$$

where y_0 is a global maximum of K on the manifold M^4.

By a theorem of Bahri–Rabinowitz [13], it follows that:

$$\overline{W_s^\infty(\tau_p^\infty)} = W_s^\infty(\tau_p^\infty) \bigcup \bigcup_{x_\infty < \tau_p^\infty} W_s^\infty(x_\infty) \bigcup \bigcup_{w < \tau_p^\infty} W_u(w),$$

where x_∞ is a critical point at infinity dominated by τ_p^∞ and w is a solution of (P_K) dominated by τ_p^∞. By transversality arguments we assume that the index of x_∞ and the Morse index of w are no bigger than l. Hence

$$X_l^\infty = \bigcup_{\iota(\tau_p)\le l} W_s^\infty(\tau_p^\infty) \bigcup_{w<\tau_p^\infty} W_u(w).$$

It follows that X_k^∞ is a stratified set of top dimension $\le l$. Without loss of generality we may assume it to be equal to l. Hence $C(X_k^\infty)$ is also a stratified set of top dimension $l+1$.

Now we use the gradient flow of $-\nabla J$ to deform $C(X_k^\infty)$. We first notice that without loss of generality we may assume the gradient flow is of Morse-Smale type, that is, we may assume that the intersection of its stable and unstable manifolds are transversal, see [1]. Hence we can assume that the deformation avoids all critical as well as critical points at Infinity having their Morse indices greater than $l+2$. It follows then by a Theorem of Bahri and Rabinowitz [13], that $C(X_k^\infty)$ retracts by deformation on the set

$$U := X_l^\infty \bigcup \bigcup_{\iota(x_\infty)=l+1} W_u^\infty(x_\infty) \bigcup \bigcup_{w<\tau_p^\infty} W_u(w). \tag{4.3}$$

Now taking $l=k-1$ and using that by assumption of Theorem 1.1, there are no critical point at infinity with index k, we derive that $C(X_k^\infty)$ retracts by deformation onto

$$Z_k^\infty := X_k^\infty \cup \cup_{w;\nabla J(w)=0;w \text{ dominated by } C(X_k^\infty)} W_u(w). \tag{4.4}$$

Now observe that, it follows from the above deformation retract that the problem (P_K) has necessary a solution w with $m(w)\le k$. Otherwise it follows from (4.4) that

$$1 = \chi(Z_k^\infty) = \sum_{\tau_p\in\mathcal{F}_\infty;\iota(\tau_p)\le k-1} (-1)^{\iota(\tau_p)},$$

where χ denotes the Euler Characteristic. Such an equality contradicts the assumption 2 of the theorem.

Now for generic K, it follows from the Sard–Smale Theorem [32], that all solutions of (P_K) are nondegenerate solutions, in the sense that their associated linearized operator does not admit zero as an eigenvalue. See details in [36].

We derive now from (4.4), taking the Euler Characteristic of both sides that:

$$1 = \chi(Z_k^\infty) = \sum_{\tau_p\in\mathcal{F}_\infty;\iota(\tau_p)\le k-1} (-1)^{\iota(\tau_p)} + \sum_{w<X_k^\infty;\nabla J(w)=0} (-1)^{m(w)}.$$

It follows then that

$$\left|1 - \sum_{\tau_p\in\mathcal{F}_\infty;\iota(\tau_p)\le k-1} (-1)^{\iota(\tau_p)}\right| \le \sum_{w;\nabla J(w)=0,m(w)\le k} (-1)^{m(w)} \le \#\mathcal{N}_k,$$

where $\mathcal{N}_k$ denotes the set of solutions of (P_K) having their Morse indices $\le k$. □

5 A general existence result

In this last section of this paper, we give a generalization of Theorem 1.1. Namely instead of assuming that there are no critical point at infinity of index k, we assume that the intersection number modulo 2, between the suspension of the complex at infinity of order k, $C(X_k^\infty)$ and the stable manifold of all critical points at infinity of index $k+1$ is equal to zero. More precisely, for $\tau_p\in\mathcal{F}_\infty$ such that $\iota(\tau_p) = k$, we define the following intersection number:

$$\mu_k(\tau_p) := C(X_{k-1}^\infty)\cdot W_s^\infty(\tau_p^\infty)\,(\text{mod}2).$$

Observe that this intersection number is well defined since we may assume by transversality that:

$$\partial C(X_k^\infty) \bigcap W_s^\infty(\tau_p^\infty) = \emptyset.$$

indeed

$$\dim(\partial C(X_k^\infty)) = k - 1, \quad \text{while } \dim(W_s^\infty(\tau_p^\infty)) = 4 - k.$$

We are now ready to state the following existence result:

Theorem 5.1 *Let $0 < K \in C^2(M^4)$ be a positive function satisfying the conditions (H_0) and (H_1). If there exists $k \in \mathbb{N}$ such that*

1.
$$\sum_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \leq k-1} (-1)^{\iota(\tau_p)} \neq 1,$$

2.
$$\forall \tau_p \in \mathcal{F}_\infty, \text{ such that } \iota(\tau_p) = k, \text{ there holds } \mu_k(\tau_p) = 0.$$

Then there exists a solution w of the problem (P_K) such that:

$$Morse(w) \leq k,$$

where $Morse(w)$ is the Morse index of w.

Moreover, for generic K, it holds

$$\#\mathcal{N}_k \geq \left| 1 - \sum_{\tau_p \in \mathcal{F}_\infty; \iota(\tau_p) \leq k-1} (-1)^{\iota(\tau_p)} \right|,$$

where $\mathcal{N}_k$ denotes the set of solution of (P_K) having their Morse indices less or equal k.

Proof The proof going along with the proof of Theorem 1.1, we will sketch the differences. Keeping the notation of the proof of Theorem 1.1, we observe that, since

$$\forall \tau_p \in \mathcal{F}_\infty, \text{ such that } \iota(\tau_p) = k, \text{ there holds } \mu_k(\tau_p) = 0,$$

we may assume that the deformation of C_k^∞ along any pseudogradient flow of $-J$, avoids all critical points at infinity having their Morse indices equal to k. It follows then from (4.3) that $C(X_k^\infty)$ retracts by deformation onto

$$Z_k^\infty := X_k^\infty \bigcup \bigcup_{w; \nabla J(w)=0; w \text{ dominated by } C(X_k^\infty)} W_u(y). \tag{5.1}$$

Now the remainder of the proof is identical to the proof of Theorem 1.1. □

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