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Approximation by rational functions in Smirnov classes with variable exponent

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Abstract In this article, we investigate the direct problem of approximation theory in the variable exponent Smirnov classes of analytic functions, defined on a doubly connected domain bounded by two Dini-smooth curves.

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في هذه الورقة نبحث المسألة المباشرة في نظرية التقريب في صفوف سميرنوف للدوال التحليلية متغيرة الأس المعرفة على منطقة ثنائية الترابط محاطة بمنحنيين ديني أملسين.

1 Introduction

Generally, variable exponent Lebesgue spaces are a natural generalization of the classical Lebesgue spaces L_p, $1 < p < \infty$, replacing the constant p with a function $p(.)$. The direct and inverse theorems of approximation theory in the variable exponent Smirnov classes of analytic functions, defined on the simply connected domains with Dini-smooth boundaries, were obtained by Israfilov and Testici [7,8].

In this work, rational approximation problem in variable exponent Smirnov classes of functions defined on a doubly connected domain is investigated.

2 Basic definitions and some notations

Suppose that G is an arbitrary doubly connected domain in the complex plane $\mathbb{C}$, bounded by two rectifiable Jordan curves L_1 and L_2. Without loss of generality, we may assume that the closed curve L_2 is in the closed curve L_1 and $0 \in \text{int}L_2$. Let $G_1^0 := \text{int}L_1$, $G_1^\infty := extL_1$, $G_2^0 = \text{int}L_2$, $G_2^\infty := extL_2$, $D := \{w \in \mathbb{C} : |w| < 1\}$, $D^- := \{w \in \mathbb{C} : |w| > 1\}$ and $\gamma_0 := \partial D := \{w \in \mathbb{C} : |w| = 1\}$.

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We denote by $w = \phi(t)$ $(w = \phi_1(t))$ the conformal mapping of G_1^∞ (G_2^0) onto domain D^- which satisfies the conditions

$$\phi(\infty) = \infty, \quad \lim_{t\to\infty} \frac{\phi(t)}{t} > 0, \qquad \left(\phi_1(0) = \infty, \quad \lim_{t\to 0} t\phi_1(t) > 0\right),$$

and let ψ and ψ_1 be the inverse mappings of ϕ and ϕ_1, respectively.

Throughout this paper, we assume that the letters $c_1, c_2, \ldots$ always remain to denote positive constants that may different at each occurrence.

Definition 2.1 Let Γ be some rectifiable Jordan curve, $p(.) : \Gamma \to [1, \infty)$ be some Lebesgue measurable function. By $L^{p(.)}(\Gamma)$, we denote the class of all Lebesgue measurable functions f, such that

$$I^{p(.)}(f) := \int_\Gamma |f(\varsigma)|^{p(\varsigma)} |\mathrm{d}\varsigma| < \infty.$$

$L^{p(.)}(\Gamma)$ becomes a Banach space with respect to the norm

$$\|f\|_{L^{p(.)}(\Gamma)} := \inf\left\{\lambda > 0 : I^{p(.)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

Let $\mathscr{F}$ be some Jordan rectifiable curve $\Gamma \subset \mathbb{C}$ or the segment $[0, 2\pi]$ and let $|\mathscr{F}|$ denote the Lebesgue measure of $\mathscr{F}$. We define the classes of functions $\mathscr{P}(\mathscr{F})$, $\mathscr{P}^{\log}(\mathscr{F})$ and $\mathscr{P}_0^{\log}(\mathscr{F})$ as

$$\mathscr{P}(\mathscr{F}) := \left\{p : 1 \leqslant p^- := \operatorname*{ess\,inf}_{t\in\mathscr{F}} p(t) \leqslant p^+ := \operatorname*{ess\,sup}_{t\in\mathscr{F}} p(t) < \infty\right\},$$

$$\mathscr{P}^{\log}(\mathscr{F}) := \left\{p \in \mathscr{P}(\mathscr{F}) : \exists c > 0, \quad \forall t_1, t_2 \in \mathscr{F} : |p(t_1) - p(t_2)| \ln\left(\frac{|\mathscr{F}|}{|t_1 - t_2|}\right) \leqslant c\right\},$$

$$\mathscr{P}_0^{\log}(\mathscr{F}) := \left\{p \in \mathscr{P}^{\log}(\mathscr{F}) : p^- > 1\right\}.$$

Detailed information on variable exponent Lebesgue space can be found in the books [1,2].

Definition 2.2 Let a finite simply connected domain U with the rectifiable Jordan curve boundary Γ in the complex plane $\mathbb{C}$ be given, and let Γ_r be the image of circle $\{w \in \mathbb{C} : |w| = r, \ 0 < r < 1\}$ under some conformal mapping of D onto U. By $E^1(U)$, we denote the class of analytic functions f in U which satisfy

$$\int_{\Gamma_r} |f(t)|\, |\mathrm{d}t| < \infty$$

uniformly in r.

It is known that every function of class $E^1(U)$ has nontangential boundary values almost everywhere on Γ and the boundary function belongs to $L^1(\Gamma)$ [3, pp. 438–453].

Definition 2.3 Let a finite simply connected domain U with the rectifiable Jordan curve boundary Γ in the complex plane $\mathbb{C}$ be given, and let $p(.) \in \mathscr{P}_0^{\log}(\Gamma)$. The variable exponent Smirnov class of analytic functions is defined as:

$$E^{p(.)}(U) := \left\{f \in E^1(U) : f \in L^{p(.)}(\Gamma)\right\}.$$

Definition 2.4 Let $L = L_1 \cup L_2^-$ and $p(.) \in \mathscr{P}_0^{\log}(L)$. The variable exponent Smirnov class with respect to the doubly connected domain G is defined as:

$$E^{p(.)}(G) := \left\{f \in E^1(G) : f \in L^{p(.)}(L)\right\}.$$

For $f \in E^{p(.)}(G)$, the norm $E^{p(.)}(G)$ can be defined as:

$$\|f\|_{E^{p(.)}(G)} := \|f\|_{L^{p(.)}(L)}.$$

Definition 2.5 We define the modulus of continuity of a function $g \in L^{p(.)}(\gamma_0)$ by the relation

$$\Omega(g,\delta)_{p(.)} := \sup_{0<\theta\leqslant\delta} \|g(.) - \sigma_\theta g(.)\|_{L^{p(.)}(\gamma_0)},$$

where $\sigma_\theta g(w) := \frac{1}{\theta}\int_0^\theta g(we^{it})\mathrm{d}t,\ w \in \gamma_0,\ 0 < \theta < \pi.$

Definition 2.6 Let Γ be a rectifiable Jordan curve in the complex plane $\mathbb{C}$. For a given $t \in \Gamma$ and $f \in L^1(\Gamma)$, the operator defined by

$$S_\Gamma(f)(t) := \lim_{\epsilon\to 0^+} \frac{1}{2\pi i}\int_{\Gamma\cap\{\varsigma:\,|\varsigma-t|>\epsilon\}} \frac{f(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma$$

is called the Cauchy singular operator.

Definition 2.7 A smooth Jordan curve Γ is called Dini-smooth, if

$$\int_0^\delta \frac{\Omega(\sigma,s)}{s}ds < \infty, \quad \delta > 0,$$

where $\sigma(s)$ is the angle, between the tangent line of Γ and the positive real axis expressed as a function of arclength s, with the modulus of continuity $\Omega(\sigma,s)$.

Kokilashvili and Samko proved in [11] that, if Γ is a Dini-smooth curve, then the operator S_Γ is bounded in $L^{p(.)}(\Gamma)$ with $p(.) \in \mathscr{P}_0^{\log}(\Gamma)$, i.e., there exists a positive constant c_1 such the following inequality holds for any $f \in L^{p(.)}(\Gamma)$

$$\|S_\Gamma(f)\|_{L^{p(.)}(\Gamma)} \leqslant c_1\|f\|_{L^{p(.)}(\Gamma)}. \tag{1}$$

To prove our main theorem, we need the following lemma. It can be found in [3, p. 431].

Lemma 2.8 *Let $f \in L^1(\Gamma)$. Then, the functions $f^+ : int\Gamma \to \mathbb{C}$ and $f^- : ext\Gamma \to \mathbb{C}$ defined by*

$$f^+(t) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma,\ t \in int\Gamma, \quad f^-(t) := \frac{1}{2\pi i}\int_\Gamma \frac{f(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma,\ t \in ext\Gamma$$

are analytic in $int\Gamma$ and $ext\Gamma$, respectively, and satisfy the following formulas $f^-(\infty) = 0$:

$$f^+(t) = S_\Gamma f(t) + \frac{1}{2}f(t), \quad f^-(t) = S_\Gamma f(t) - \frac{1}{2}f(t),$$
$$f(t) = f^+(t) - f^-(t)$$

a.e. on Γ.

The level lines of the domains G_1^0 and G_2^0 are defined for $r, R > 1$ by

$$C_r := \{t : |\phi(t)| = r\}, \quad C_R := \{t : |\phi_1(t)| = R\}.$$

The Faber polynomials $\Phi_k(t)$ of degree k are defined by the relation

$$\frac{\psi'(w)}{\psi(w) - t} = \sum_{k=0}^\infty \frac{\Phi_k(t)}{w^{k+1}}, \quad t \in G_1^0, \quad w \in D^-,$$

and have the following integral representations [12]:

If $t \in \mathrm{int}C_r$, then

$$\Phi_k(t) = \frac{1}{2\pi i}\int_{C_r} \frac{\phi^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma = \frac{1}{2\pi i}\int_{|w|=r} \frac{\psi'(w)w^k}{\psi(w) - t}\mathrm{d}w. \tag{2}$$

And for $t \in \mathrm{ext}C_r$, we have

$$\Phi_k(t) = \phi^k(t) + \frac{1}{2\pi i}\int_{C_r} \frac{\phi^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma. \tag{3}$$

Similarly, the Faber polynomials $\widetilde{\Phi}_k(1/t)$ of degree k with respect to $1/z$ are defined by the relation

$$\frac{\psi_1'(w)}{\psi_1(w)-t}=\sum_{k=0}^{\infty}\frac{\widetilde{\Phi}_k(1/t)}{w^{k+1}},\quad t\in G_2^{\infty},\quad w\in D^-,$$

and satisfy the following relations:

If $t\in \mathrm{int}C_R$, then

$$\widetilde{\Phi}_k(1/t)=\phi_1^k(t)-\frac{1}{2\pi i}\int_{C_R}\frac{\phi_1^k(\varsigma)}{\varsigma-t}\mathrm{d}\varsigma. \tag{4}$$

And in case $t\in \mathrm{ext}C_R$, we obtain

$$\widetilde{\Phi}_k(1/t)=-\frac{1}{2\pi i}\int_{C_R}\frac{\phi_1^k(\varsigma)}{\varsigma-t}\mathrm{d}\varsigma=-\frac{1}{2\pi i}\int_{|w|=R}\frac{\psi_1'(w)w^k}{\psi_1(w)-t}\mathrm{d}w. \tag{5}$$

If $f(t)$ is a function in $E^1(G)$, then $f(t)$ has the following formula [13]:

$$f(t)=\sum_{k=0}^{\infty}a_k\Phi_k(t)+\sum_{k=1}^{\infty}\widetilde{a}_k\widetilde{\Phi}_k(1/t), \tag{6}$$

where

$$a_k=\frac{1}{2\pi i}\int_{|w|=r_1}\frac{f(\psi(w))}{w^{k+1}}\mathrm{d}w,\quad 1<r_1<r,\quad k=0,1,2,\ldots,$$

and

$$\widetilde{a}_k=\frac{1}{2\pi i}\int_{|w|=R_1}\frac{f(\psi_1(w))}{w^{k+1}}\mathrm{d}w,\quad 1<R_1<R. \tag{7}$$

In case if G is an annulus domain, then the series Eq. (6) becomes the Laurent series for the function $f(t)$. Taking the first n terms of the series Eq. (6), we obtain the rational function

$$R_n(f,t):=\sum_{k=0}^{n}a_k\Phi_k(t)+\sum_{k=1}^{n}\widetilde{a}_k\widetilde{\Phi}_k(1/t). \tag{8}$$

For large values of n and if $f\in E^{p(.)}(G)$, we will prove that such a rational function $R_n(f,t)$ approximated the function $f(t)$ arbitrarily closely.

If L_1 and L_2 are Dini-smooth, then by [15, pp. 321–456], it follows that

$$0<c_2\le|\psi'(w)|\le c_3<\infty,\quad 0<c_4\le|\psi_1'(w)|\le c_5<\infty, \tag{9}$$

where c_2, c_3, c_4 and c_5 are positive constants.

Let L_i $(i=1,2)$ be a Dini-smooth curve, we define the following functions $f_0=f\circ\psi$ for $f\in L^{p(.)}(L_1)$ with $p\in\mathscr{P}^{\log}(L_1)$, $f_1=f\circ\psi_1$ for $f\in L^{p(.)}(L_2)$ with $p\in\mathscr{P}^{\log}(L_2)$, $p_0=p\circ\psi$ for $p\in\mathscr{P}^{\log}(L_1)$ and $p_1=p\circ\psi_1$ for $p\in\mathscr{P}^{\log}(L_2)$.

From [7], it follows that $f_0\in L^{p_0(.)}(\gamma_0)$ with $p_0\in\mathscr{P}_0^{\log}(\gamma_0)$ and $f_1\in L^{p_1(.)}(\gamma_0)$ with $p_1\in\mathscr{P}_0^{\log}(\gamma_0)$. Further that we obtain $f_0^+\in E^{p_0(.)}(D)$, $f_0^-\in^{p_0(.)}(D^-)$, $f_1^+\in E^{p_1(.)}(D)$ and $f_1^+\in E^{p_1(.)}(D^-)$ such that $f_0^-(\infty)=\infty$, $f_1^-(\infty)=0$ and the following relations hold a. e. on γ_0

$$f_0(t)=f_0^+(t)-f_0^-(t), \tag{10}$$

$$f_1(t)=f_1^+(t)-f_1^-(t). \tag{11}$$

The following lemma was proved in [7].

Lemma 2.9 *Let $g \in E^{p(.)}(D)$ with $p \in \mathscr{P}_0^{\log}(\gamma_0)$. If $\sum_{k=0}^{n} a_k w^k$ is the n th partial sum of the Taylor series of g at the origin, then the following estimate*

$$\|g(w) - \sum_{k=0}^{n} a_k w^k\|_{L^{p(.)}(\gamma_0)} \leqslant c_6\, \Omega(g, 1/n)_{p(.)}$$

holds, where c_6 is a positive constant.

In the literature, there are sufficiently wide investigations relating to the approximation problems in the simply connected domains. For example, the problems of approximation theory for Smirnov classes with variable exponent, weighted Smirnov classes, weighted Smirnov Orlicz classes and weighted rearrangement invariant Smirnov classes were studied in [4–8]. But the approximation problems in the doubly connected domains were not investigated sufficiently wide.

In this work, we study the direct theorem of approximation theory in the variable exponent Smirnov classes, defined in the doubly connected domains bounded by two Dini-smooth curves.

Similar problems in Smirnov classes, Smirnov Orlicz classes and weighted rearrangement invariant Smirnov classes were obtained in [9,10,14].

3 The main result

Our main result is given in the following theorem.

Theorem 3.1 *Let G be a finite doubly connected domain with the Dini-smooth boundary, $L = L_1 \cup L_2^-$, $L^{p(.)}(L)$ be a Lebesgue space with variable exponent $p \in \mathscr{P}_0^{\log}(L)$. If f is a function in $E^{p(.)}(G)$, then for every $n \in \mathbb{N}$ the estimate*

$$\|f - R_n(f, .)\|_{E^{p(.)}(G)} \leqslant c_7 \left[\Omega(f_0, 1/n)_{p_0(.)} + \Omega(f_1, 1/n)_{p_1(.)}\right]$$

holds, where c_7 is a positive constant and $R_n(f, .)$ is the rational function defined by Eq. (8).

Proof Let $f \in E^{p(.)}(G)$, then $f_0 \in L^{p_0(.)}(\gamma_0)$, $f_1 \in L^{p_1(.)}(\gamma_0)$ and putting $\phi(\varsigma)$ and $\phi_1(\varsigma)$ in place of w in Eqs. (10) and (11), respectively, we obtain

$$f(\varsigma) = f_0^+(\phi(\varsigma)) - f_0^-(\phi(\varsigma)), \quad \varsigma \in L_1, \tag{12}$$

$$f(\varsigma) = f_1^+(\phi_1(\varsigma)) - f_1^-(\phi_1(\varsigma)), \quad \varsigma \in L_2. \tag{13}$$

We suppose that $t \in ext L_1$, then using the relation (3), we have

$$\sum_{k=0}^{n} a_k \Phi_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma,$$

and by the relation Eq. (12)

$$\sum_{k=0}^{n} a_k \Phi_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\varsigma) - f_0^+(\phi(\varsigma))}{\varsigma - t}\mathrm{d}\varsigma$$
$$+\frac{1}{2\pi i}\int_{L_1} \frac{f(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma + \frac{1}{2\pi i}\int_{L_1} \frac{f_0^-(\phi(\varsigma))}{\varsigma - t}\mathrm{d}\varsigma.$$

Since $f_0^-(\phi(\varsigma)) \in E^{p_0(.)}(G_1^\infty)$, we get

$$\frac{1}{2\pi i}\int_{L_1} \frac{f_0^-(\phi(\varsigma))}{\varsigma - t}\mathrm{d}\varsigma = -f_0^-(\phi(t)).$$

So, we reach the following relation:

$$\sum_{k=0}^{n} a_k \Phi_k(t) = \sum_{k=0}^{n} a_k \phi^k(t) + \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k\phi^k(\varsigma) - f_0^+(\phi(\varsigma))}{\varsigma - t}\mathrm{d}\varsigma + \frac{1}{2\pi i}\int_{L_1}\frac{f(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma - f_0^-(\phi(t)). \tag{14}$$

Now for $t \in \mathrm{ext}L_2$, and using the relation Eqs. (5) and (13), we obtain

$$\sum_{k=1}^{n} \widetilde{a}_k \widetilde{\Phi}_k(1/t) = -\frac{1}{2\pi i}\int_{L_2}\frac{\sum_{k=1}^{n}\widetilde{a}_k\phi_1^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma = \frac{1}{2\pi i}\int_{L_2}\frac{f_1^+(\phi_1(\varsigma)) - \sum_{k=0}^{n}\widetilde{a}_k\phi_1^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma - \frac{1}{2\pi i}\int_{L_2}\frac{f(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma. \tag{15}$$

Since $extL_1 \subset extL_2$, the relation Eqs. (14) and (15) are valid for $t \in \mathrm{ext}L_1$, and give

$$\sum_{k=0}^{n} a_k \Phi_k(t) + \sum_{k=1}^{n}\widetilde{a}_k\widetilde{\Phi}_k(1/t) = \sum_{k=0}^{n} a_k\phi^k(t) - f_0^-(\phi(t)) - \frac{1}{2\pi i}\int_{L_1}\frac{f_0^+(\phi(\varsigma)) - \sum_{k=0}^{n} a_k\phi^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma - \frac{1}{2\pi i}\int_{L_2}\frac{f_1^+(\phi_1(\varsigma)) - \sum_{k=0}^{n}\widetilde{a}_k\phi_1^k(\varsigma)}{\varsigma - t}\mathrm{d}\varsigma.$$

Limiting as $t \to z \in L_1$ along non-tangential path outside L_1 for almost every $z \in L_1$, we get

$$f(z) - \sum_{k=0}^{n} a_k\Phi_k(z) - \sum_{k=1}^{n}\widetilde{a}_k\widetilde{\Phi}_k(1/z) = f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k\phi^k(z) + \frac{1}{2}\left(f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k\phi^k(z)\right) + S_{L_1}\left(f_0^+(\phi(z)) - \sum_{k=0}^{n} a_k\phi^k(z)\right) - \frac{1}{2\pi i}\int_{L_2}\frac{f_1^+(\phi_1(\varsigma)) - \sum_{k=1}^{n}\widetilde{a}_k\phi_1^k(\varsigma)}{\varsigma - z}\mathrm{d}\varsigma. \tag{16}$$

Using Eq. (16), Minkowski's inequality and the relation Eq. (1), we have

$$\|f - R_n(f,.)\|_{L^{p(.)}(L_1)} \leqslant c_8\,\|f_0^+(w) - \sum_{k=0}^{n} a_k w^k\|_{L^{p_0(.)}(\gamma_0)} + c_9\,\|f_1^+(w) - \sum_{k=0}^{n}\widetilde{a}_k w^k\|_{L^{p_1(.)}(\gamma_0)}. \tag{17}$$

From the relation Eq. (17), and using Lemma 2.9, we get

$$\|f - R_n(f,.)\|_{L^{p(.)}(L_1)} \leqslant c_{10}\left[\Omega(f_0, 1/n)_{p_0(.)} + \Omega(f_1, 1/n)_{p_1(.)}\right]. \tag{18}$$

For $t' \in \mathrm{int}L_2$, by the relation Eqs. (3) and (13), we get

$$\sum_{k=1}^{n}\widetilde{a}_k\widetilde{\Phi}_k(1/t') = \sum_{k=1}^{n}\widetilde{a}_k\phi_1^k(t') - \frac{1}{2\pi i}\int_{L_2}\frac{\sum_{k=0}^{n}\widetilde{a}_k\phi_1^k(\varsigma)}{\varsigma - t'}\mathrm{d}\varsigma = \sum_{k=1}^{n}\widetilde{a}_k\phi_1^k(t') - \frac{1}{2\pi i}\int_{L_2}\frac{\sum_{k=0}^{n}\widetilde{a}_k\phi_1^k(\varsigma) - f_1^+(\phi_1(\varsigma))}{\varsigma - t'}\mathrm{d}\varsigma - \frac{1}{2\pi i}\int_{L_2}\frac{f(\varsigma)}{\varsigma - t'}\mathrm{d}\varsigma - f_1^-(\phi_1(t')). \tag{19}$$

And for $t' \in \mathrm{int}L_1$, from (2) and (12), we have

$$\sum_{k=1}^{n} a_k \Phi_k(t') = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=1}^{n} a_k \phi^k(\varsigma)}{\varsigma - t'}\mathrm{d}\varsigma$$
$$= \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=1}^{n} a_k \phi^k(\varsigma) - f_0^+(\phi(\varsigma))}{\varsigma - t'}\mathrm{d}\varsigma + \frac{1}{2\pi i}\int_{L_1} \frac{f(\varsigma)}{\varsigma - t'}\mathrm{d}\varsigma. \tag{20}$$

Since $\mathrm{int}L_2 \subset \mathrm{int}L_1$, the relation Eqs. (19) and (20) are valid for $t' \in \mathrm{int}L_2$, and give

$$\sum_{k=0}^{n} a_k \Phi_k(t') + \sum_{k=1}^{n} \widetilde{a}_k \widetilde{\Phi}_k(1/t') = \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\varsigma) - f_0^+(\phi(\varsigma))}{\varsigma - t'}\mathrm{d}\varsigma$$
$$-\frac{1}{2\pi i}\int_{L_2} \frac{\sum_{k=1}^{n} \widetilde{a}_k \phi_1^k(\varsigma) - f_1^+(\phi_1(\varsigma))}{\varsigma - t'}\mathrm{d}\varsigma$$
$$-f_1^-(\phi_1(t')) + \sum_{k=1}^{n} \widetilde{a}_k \phi_1^k(t').$$

Limiting as $t' \to z \in L_2$ along non-tangential path inside L_2 for almost every $z \in L_2$, we get

$$f(z) - \sum_{k=0}^{n} a_k \Phi_k(z) - \sum_{k=1}^{n} \widetilde{a}_k \widetilde{\Phi}_k(1/z) = f_1^+(\phi_1(z)) - \frac{1}{2}\left(\sum_{k=1}^{n} \widetilde{a}_k \phi_1^k(z) - f_1^+(\phi_1(z))\right)$$
$$- S_{L_2}\left(\sum_{k=1}^{n} \widetilde{a}_k \phi_1^k(z) - f_1^+(\phi_1(z))\right)$$
$$- \frac{1}{2\pi i}\int_{L_1} \frac{\sum_{k=0}^{n} a_k \phi^k(\varsigma) - f_0^+(\phi(\varsigma))}{\varsigma - z}\mathrm{d}\varsigma. \tag{21}$$

Using Eq. (21), Minkowski's inequality, and the relation Eq. (1), we obtain

$$\|f - R_n(f,.)\|_{L^{p(.)}(L_2)} \leqslant c_{11}\, \|f_1^+(w) - \sum_{k=1}^{n} \widetilde{a}_k w^k\|_{L^{p_1(.)}(\gamma_0)}$$
$$+ c_{12}\, \|f_0^+(w) - \sum_{k=0}^{n} a_k w^k\|_{L^{p_0(.)}(\gamma_0)}. \tag{22}$$

From the relation Eq. (22), and using Lemma 2.9, we get

$$\|f - R_n(f,.)\|_{L^{p(.)}(L_2)} \leqslant c_{13}\left[\Omega(f_0, 1/n)_{p_0(.)} + \Omega(f_1, 1/n)_{p_1(.)}\right]. \tag{23}$$

Since $L = L_1 \cup L_2^-$, and $f \in E^{p(.)}(G)$, we get

$$\|f - R_n(f,.)\|_{L^{p(.)}(L)} \leqslant \|f - R_n(f,.)\|_{L^{p(.)}(L_1)} + \|f - R_n(f,.)\|_{L^{p(.)}(L_2)}.$$

Then taking into account the relation Eqs. (18) and (23), we reach

$$\|f - R_n(f,.)\|_{E^{p(.)}(G)} \leqslant c_7\left[\Omega(f_0, 1/n)_{p_0(.)} + \Omega(f_1, 1/n)_{p_1(.)}\right].$$

Thus, the theorem is proved. □

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