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The annihilating-submodule graph of modules over commutative rings II

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Abstract Let M be a module over a commutative ring R. The annihilating-submodule graph of M, denoted by $\mathrm{AG}(M)$, is a simple graph in which a non-zero submodule N of M is a vertex if and only if there exists a non-zero proper submodule K of M such that $NK = (0)$, where NK, the product of N and K, is denoted by $(N : M)(K : M)M$ and two distinct vertices N and K are adjacent if and only if $NK = (0)$. This graph is a submodule version of the annihilating-ideal graph. We prove that if $\mathrm{AG}(M)$ is a tree, then either $\mathrm{AG}(M)$ is a star graph or a path of order 4 and in the latter case $M \cong F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule. Moreover, we prove that if M is a cyclic module with at least three minimal prime submodules, then $gr(\mathrm{AG}(M)) = 3$ and for every cyclic module M, $cl(\mathrm{AG}(M)) \geq |\mathrm{Min}(M)|$.

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الملخص

ليكن M حلقيًا على حلقة إبدالية R. بيان الحلقي الجزئي المُعدِم من M، الذي يرمز له بـ $AG(M)$، بيان بسيط يكون فيه نصف الحلقيّ غير الصفري N من M رأسًا إذا وُجِدَ وفقط إذا وُجِدَ حلقي جزئي غير صفري K من M بحيث يكون $NK = (0)$، حيث يعرَّف NK، جداء N وَ K، على أنه $(N{:}M)(K{:}M)M$، ويكون الرأسان المختلفان N وَ K متجاورين إذا كان وفقط إذا كان $NK = (0)$. يعد هذا البيان نسخة الحلقي الجزئي من بيان المثالي المُعدِم. نثبت أنه إذا كان $AG(M)$ شجرة فإن $AG(M)$ بيان نجمة أو طريق من الرتبة 4 وأنه في هذه الحالة الأخيرة $M \cong F \times S$، حيث F حلقيّ بسيط وَ S حلقيّ له حلقيّ جزئيّ غير صفريّ وحيد. بالإضافة إلى ذلك، نثبت أنه إذا كان M حلقيًا دوريًا له على الأقل ثلاث حلقيات جزئية أولية دنيا، فإن $gr(AG(M)) = 3$ ولكل حلقي جزئي M يكون $cl(AG(M)) \geq |Min(M)|$.

1 Introduction

Throughout this paper, R is a commutative ring with a non-zero identity and M is a unital R-module. By $N \leq M$ (resp., $N < M$) we mean that N is a submodule (resp., proper submodule) of M.

Define $(N :_R M)$ or simply $(N : M) = \{r \in R| \, rM \subseteq N\}$ for any $N \leq M$. We denote $((0) : M)$ by $\mathrm{Ann}_R(M)$ or simply $\mathrm{Ann}(M)$. M is said to be faithful if $\mathrm{Ann}(M) = (0)$.

Let $N, K \leq M$. Then, the product of N and K, denoted by NK, is defined by $(N : M)(K : M)M$ (see [6]).

There are many papers on assigning graphs to rings or modules (see, for example, [4,7,10,11]). The annihilating-ideal graph $\mathrm{AG}(R)$ was introduced and studied in [11]. $\mathrm{AG}(R)$ is a graph whose vertices are

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ideals of R with non-zero annihilators and in which two vertices I and J are adjacent if and only if $IJ = (0)$. Later, it was modified and further studied by many authors (see [1–3]).

In [7,8], we generalized the above idea to submodules of M and defined the (undirected) graph $\mathrm{AG}(M)$, called *the annihilating-submodule graph*, with vertices $V(\mathrm{AG}(M)) = \{N \leq M \mid$ there exists $(0) \neq K < M$ with $NK = (0)\}$. In this graph, distinct vertices $N, L \in V(\mathrm{AG}(M))$ are adjacent if and only if $NL = (0)$. Let $\mathrm{AG}(M)^*$ be the subgraph of $\mathrm{AG}(M)$ with vertices $V(\mathrm{AG}(M)^*) = \{N < M$ with $(N : M) \neq \mathrm{Ann}(M) \mid$ there exists a submodule $K < M$ with $(K : M) \neq \mathrm{Ann}(M)$ and $NK = (0)\}$. Note that M is a vertex of $\mathrm{AG}(M)$ if and only if there exists a non-zero proper submodule N of M with $(N : M) = \mathrm{Ann}(M)$ if and only if every non-zero submodule of M is a vertex of $\mathrm{AG}(M)$.

In this work, we continue our study in [7,8] and we generalize some results related to annihilating-ideal graph obtained in [1–3] for annihilating-submodule graph.

A prime submodule of M is a submodule $P \neq M$, such that whenever $re \in P$ for some $r \in R$ and $e \in M$, we have $r \in (P : M)$ or $e \in P$ [14].

The prime radical $\mathrm{rad}_M(N)$ or simply $\mathrm{rad}(N)$ is defined to be the intersection of all prime submodules of M containing N, and in case N is not contained in any prime submodule, $\mathrm{rad}_M(N)$ is defined to be M [14].

The notations $Z(R)$, $\mathrm{Nil}(R)$, and $\mathrm{Min}(M)$ will denote the set of all zero-divisors, the set of all nilpotent elements of R, and the set of all minimal prime submodules of M, respectively. In addition, $Z_R(M)$ or simply $Z(M)$, the set of zero divisors on M, is the set $\{r \in R \mid rm = 0$ for some $0 \neq m \in M\}$.

A clique of a graph is a complete subgraph and the supremum of the sizes of cliques in G, denoted by $cl(G)$, is called the clique number of G. Let $\chi(G)$ denote the chromatic number of the graph G, that is, the minimal number of colors needed to color the vertices of G, so that no two adjacent vertices have the same color. Obviously $\chi(G) \geq cl(G)$.

In Sect. 2, we prove that if $\mathrm{AG}(M)$ is a tree, then either $\mathrm{AG}(M)$ is a star graph or is the path P_4 and in this case, $M \cong F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule (see Theorem 2.7). Next, we study the bipartite annihilating-submodule graphs of modules over Artinian rings (see Theorem 2.8). In Sect. 3, we study coloring of the annihilating-submodule graph and investigate the interplay between $\chi(\mathrm{AG}(M))$, $cl(\mathrm{AG}(M))$, and $\mathrm{Min}(M)$ (see Theorems 3.5 and 3.8). In Corollary 3.7, we prove that if M is a cyclic module with at least three minimal prime submodules, then $gr(\mathrm{AG}(M)) = 3$ and for every cyclic module M, $cl(\mathrm{AG}(M)) \geq |\mathrm{Min}(M)|$.

Let us introduce some graphical notions and denotations that are used in what follows: a graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a non-empty set of vertices, $V(G)$, a set $E(G)$ of edges, and an incident function ψ_G that associates an unordered pair of distinct vertices with each edge. The edge e joins x and y if $\psi_G(e) = \{x, y\}$, and we say x and y are adjacent. A path in graph G is a finite sequence of vertices $\{x_0, x_1, \ldots, x_n\}$, where x_{i-1} and x_i are adjacent for each $1 \leq i \leq n$ and we denote $x_{i-1} - x_i$ for existing an edge between x_{i-1} and x_i.

A graph H is a subgraph of G, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A bipartite graph is a graph whose vertices can be divided into two disjoint sets U and V, such that every edge connects a vertex in U to one in V; that is, U and V are each independent sets and complete bipartite graph on n and m vertices, denoted by $K_{n,m}$, where V and U are of size n and m, respectively, and $E(G)$ connects every vertex in V with all vertices in U. Note that a graph $K_{1,m}$ is called a star graph and the vertex in the singleton partition is called the center of the graph. For some $U \subseteq V(G)$, we denote by $V(U)$, the set of all vertices of $G \backslash U$ adjacent to at least one vertex of U. For every vertex $v \in V(G)$, the size of $V(v)$ is denoted by $d(v)$. If all the vertices of G have the same degree k, then G is called k-regular, or simply regular. An independent set is a subset of the vertices of a graph, such that no vertices are adjacent. We denote by P_n and C_n, a path and a cycle of order n, respectively. Let G and G' be two graphs. A graph homomorphism from G to G' is a mapping $\phi : V(G) \longrightarrow V(G')$, such that for every edge $\{u, v\}$ of G, $\{\phi(u), \phi(v)\}$ is an edge of G'. A retract of G is a subgraph H of G, such that there exists a homomorphism $\phi : G \longrightarrow H$ such that $\phi(x) = x$, for every vertex x of H. The homomorphism ϕ is called the retract (graph) homomorphism (see [12]).

2 Cycles in the annihilating-submodule graphs

An ideal $I \leq R$ is said to be nil if I consist of nilpotent elements.

Proposition 2.1 *Suppose that e is an idempotent element of R. We have the following statements.*

(a) *$R = R_1 \times R_2$, where $R_1 = eR$ and $R_2 = (1 - e)R$.*

(b) *$M = M_1 \times M_2$, where $M_1 = eM$ and $M_2 = (1-e)M$.*
(c) *For every submodule N of M, $N = N_1 \times N_2$ such that N_1 is an R_1-submodule M_1, N_2 is an R_2-submodule M_2, and $(N :_R M) = (N_1 :_{R_1} M_1) \times (N_2 :_{R_2} M_2)$.*
(d) *For submodules N and K of M, $NK = N_1K_1 \times N_2K_2$ such that $N = N_1 \times N_2$ and $K = K_1 \times K_2$.*
(e) *Prime submodules of M are $P \times M_2$ and $M_1 \times Q$, where P and Q are prime submodules of M_1 and M_2, respectively.*

Proof This is clear. □

We need the following lemmas.

Lemma 2.2 [5, Proposition 7.6] *Let $R_1, R_2, \ldots, R_n$ be non-zero ideals of R. Then, the following statements are equivalent:*

(a) *$R = R_1 \times \cdots \times R_n$;*
(b) *As an abelian group, R is the direct sum of $R_1, \ldots, R_n$;*
(c) *There exist pairwise orthogonal idempotents $e_1, \ldots, e_n$ with $1 = e_1 + \cdots + e_n$, and $R_i = Re_i$, $i = 1, \ldots, n$.*

Lemma 2.3 [13, Theorem 21.28] *Let I be a nil ideal in R and $u \in R$ be such that $u + I$ is an idempotent in R/I. Then, there exists an idempotent e in uR such that $e - u \in I$.*

Lemma 2.4 [8, Lemma 2.4] *Let N be a minimal submodule of M and let $\text{Ann}(M)$ be a nil ideal. Then, we have $N^2 = (0)$ or $N = eM$ for some idempotent $e \in R$.*

Proposition 2.5 *Let M be a finitely generated R-module such that $R/\text{Ann}(M)$ is Artinian. Then, every non-zero proper submodule N of M is a vertex in $\text{AG}(M)$.*

Proof Let N be a non-zero submodule of M. Therefore, there exists a maximal submodule K of M, such that $N \subseteq K$. Hence, we have $(0 :_M (K : M)) \subseteq (0 :_M (N : M))$. Since $R/\text{Ann}(M)$ is an Artinian ring, $(K : M)$ is a minimal prime ideal containing $\text{Ann}(M)$. Thus, $(K : M) \in Ass(M)$. It follows that $(K : M) = (0 : m)$ for some $0 \neq m \in M$. Therefore, $N(Rm) = (0)$, as desired. □

Lemma 2.6 *Let $M = M_1 \times M_2$, where $M_1 = eM$, $M_2 = (1-e)M$, and e $(e \neq 0, 1)$ is an idempotent element of R. If $\text{AG}(M)$ is a triangle-free graph, then one of the following statements holds.*

(a) *Both M_1 and M_2 are prime R-modules.*
(b) *One M_i is a prime module for $i = 1, 2$ and the other one is a module with a unique non-trivial submodule.*

Moreover, $\text{AG}(M)$ has no cycle if and only if either $M = F \times S$ or $M = F \times D$, where F is a simple module, S is a module with a unique non-trivial submodule, and D is a prime module.

Proof If none of M_1 and M_2 is a prime module, then there exist $r \in R_i$ ($R_1 = Re$ and $R_2 = R(1-e)$), $0 \neq m_i \in M_i$ with $r_i m_i = 0$, and $r_i \notin \text{Ann}_{R_i}(M_i)$ for $i = 1, 2$. Therefore, $r_1M_1 \times (0)$, $(0) \times r_2M_2$, and $R_1m_1 \times R_2m_2$ form a triangle in $\text{AG}(M)$, a contradiction. Thus, without loss of generality, one can assume that M_1 is a prime module. We prove that $\text{AG}(M_2)$ has at most one vertex. On the contrary suppose that $\{N, K\}$ is an edge of $\text{AG}(M_2)$. Therefore, $M_1 \times (0)$, $(0) \times N$, and $(0) \times K$ form a triangle, a contradiction. If $\text{AG}(M_2)$ has no vertex, then M_2 is a prime module and so part (a) occurs. If $\text{AG}(M_2)$ has exactly one vertex, then by [7, Theorem 3.6] and Proposition 2.5, we obtain part (b). Now, suppose that $\text{AG}(M)$ has no cycle. If none of M_1 and M_2 is a simple module, then choose non-trivial submodules N_i in M_i for some $i = 1, 2$. Therefore, $N_1 \times (0)$, $(0) \times N_2$, $M_1 \times (0)$, and $(0) \times M_2$ form a cycle, a contradiction. The converse is trivial. □

Theorem 2.7 *If $\text{AG}(M)$ is a tree, then either $\text{AG}(M)$ is a star graph or $\text{AG}(M) \cong P_4$. Moreover, $\text{AG}(M) \cong P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule.*

Proof If M is a vertex of $\text{AG}(M)$, then there exists only one vertex N such that $\text{Ann}(M) = (N : M)$ and since $\text{AG}(M)^*$ is an empty subgraph, $\text{AG}(M)$ is a star graph. Therefore, we may assume that M is not a vertex of $\text{AG}(M)$. Suppose that $\text{AG}(M)$ is not a star graph. Then, $\text{AG}(M)$ has at least four vertices. Obviously, there are two adjacent vertices N and K of $\text{AG}(M)$, such that $|V(N)\backslash\{K\}| \geq 1$ and $|V(K)\backslash\{N\}| \geq 1$. Let $V(N)\backslash\{K\} = \{N_i\}_{i \in \Lambda}$ and $V(K)\backslash\{N\} = \{K_j\}_{j \in \Gamma}$. Since $\text{AG}(M)$ is a tree, we have $V(N) \cap V(K) = \emptyset$. By [7, Theorem 3.4], $\text{diam}(\text{AG}(M)) \leq 3$. So every edge of $\text{AG}(M)$ is of the form $\{N, K\}$, $\{N, N_i\}$ or $\{K, K_j\}$, for some $i \in \Lambda$ and $j \in \Gamma$. Now, consider the following claims:

Claim 1 Either $N^2 = (0)$ or $K^2 = (0)$. Pick $p \in \Lambda$ and $q \in \Gamma$. Since $\text{AG}(M)$ is a tree, N_pK_q is a vertex of $\text{AG}(M)$. If $N_pK_q = N_u$, for some $u \in \Lambda$, then $KN_u = (0)$, a contradiction. If $N_pK_q = K_v$, for some $v \in \Gamma$,

then $NK_v = (0)$, a contradiction. If $N_pK_q = N$ or $N_pK_q = K$, then $N^2 = (0)$ or $K^2 = (0)$, respectively, and the claim is proved.
Here, without loss of generality, we suppose that $N^2 = (0)$. Clearly, $(N : M)M \nsubseteq K$ and $(K : M)M \nsubseteq N$.

Claim 2 Our claim is to show that N is a minimal submodule of M and $K^2 \neq (0)$. To see that, first, we show that for every $0 \neq m \in N$, $Rm = N$. Assume that $0 \neq m \in N$ and $Rm \neq N$. If $Rm = K$, then $K \subseteq N$, a contradiction. Thus $Rm \neq K$, and the induced subgraph of $\mathrm{AG}(M)$ on N, K, and Rm is K_3, a contradiction. Therefore, $Rm = N$. This implies that N is a minimal submodule of M. Now, if $K^2 = (0)$, then we obtain the induced subgraph on N, K, and $(N : M)M + (K : M)M$ is K_3, a contradiction. Thus, $K^2 \neq (0)$, as desired.

Claim 3 For every $i \in \Lambda$ and every $j \in \Gamma$, $N_i \cap K_j = N$. Let $i \in \Lambda$ and $j \in \Gamma$. Since $N_i \cap K_j$ is a vertex and $N(N_i \cap K_j) = K(N_i \cap K_j) = (0)$, either $N_i \cap K_j = N$ or $N_i \cap K_j = K$. If $N_i \cap K_j = K$, then $K^2 = (0)$, a contradiction. Hence, $N_i \cap K_j = N$ and the claim is proved.

Claim 4 We complete the claim by showing that M has exactly two minimal submodules N and K. Let L be a non-zero submodule properly contained in K. Since $NL \subseteq NK = (0)$, either $L = N$ or $L = N_i$ for some $i \in \Lambda$. Thus, by Claim 3, $N \subseteq L \subseteq K$, a contradiction. Hence, K is a minimal submodule of M. Suppose that L' is another minimal submodule of M. Since N and K both are minimal submodules, we deduce that $NL' = KL' = (0)$, a contradiction. Therefore, the claim is proved.

Now by Claims 2 and 4, $K^2 \neq (0)$ and K is a minimal submodule of M. Then, by Lemma 2.4, $K = eM$ for some idempotent $e \in R$. Now, we have $M \cong eM \times (1-e)M$. By Lemma 2.6, we deduce that either $M = F \times S$ and $\mathrm{AG}(M) \cong P_4$ or $R = F \times D$ and $\mathrm{AG}(M)$ is a star graph. Conversely, we assume that $M = F \times S$. Then, $\mathrm{AG}(M)$ has exactly four vertices $(0) \times S$, $F \times (0)$, $(0) \times N$, and $F \times N$. Thus, $\mathrm{AG}(M) \cong P_4$ with the vertices $(0) \times S$, $F \times (0)$, $(0) \times N$, and $F \times N$. □

Theorem 2.8 *Let R be an Artinian ring and $\mathrm{AG}(M)$ is a bipartite graph. Then, either $\mathrm{AG}(M)$ is a star graph or $\mathrm{AG}(M) \cong P_4$. Moreover, $\mathrm{AG}(M) \cong P_4$ if and only if $M = F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule.*

Proof First, suppose that R is not a local ring. Hence, by [9, Theorem 8.9], $R = R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring for $i = 1, \ldots, n$. By Lemma 2.2 and Proposition 2.1, since $\mathrm{AG}(M)$ is a bipartite graph, we have $n = 2$ and $M \cong M_1 \times M_2$. If M_1 is a prime module, then it is easy to see that M_1 is a vector space over $R/\mathrm{Ann}(M_1)$ and so is a semisimple R-module. Hence, by Lemma 2.6 and Theorem 2.7, we deduce that either $\mathrm{AG}(M)$ is isomorphic to P_2 or P_4. Now, we assume that R is an Artinian local ring. Let m be the unique maximal ideal of R and k be a natural number such that $m^kM = (0)$ and $m^{k-1}M \neq (0)$. Clearly, $m^{k-1}M$ is adjacent to every other vertex of $\mathrm{AG}(M)$ and, therefore, $\mathrm{AG}(M)$ is a star graph. □

Proposition 2.9 *Assume that $\mathrm{Ann}(M)$ is a nil ideal of R.*

(a) *If $\mathrm{AG}(M)$ is a finite bipartite graph, then either $\mathrm{AG}(M)$ is a star graph or $\mathrm{AG}(M) \cong P_4$.*
(b) *If $\mathrm{AG}(M)$ is a regular graph of finite degree, then $\mathrm{AG}(M)$ is a complete graph.*

Proof (a) If M is a vertex of $\mathrm{AG}(M)$, then $\mathrm{AG}(M)$ has only one vertex N, such that $\mathrm{Ann}(M) = (N : M)$ and since $\mathrm{AG}(M)^*$ is an empty subgraph, $\mathrm{AG}(M)$ is a star graph. Thus, we may assume that M is not a vertex of $\mathrm{AG}(M)$, and hence, by [7, Theorem 3.3], M is not a prime module. Therefore, [7, Theorem 3.6] follows that $R/\mathrm{Ann}(M)$ is an Artinian ring. If $(R/\mathrm{Ann}(M), m/\mathrm{Ann}(M))$ is a local ring, then there exists a natural number k, such that $m^kM = (0)$ and $m^{k-1}M \neq (0)$. Clearly, $m^{k-1}M$ is adjacent to every other vertex of $\mathrm{AG}(M)$ and, therefore, $\mathrm{AG}(M)$ is a star graph. Otherwise, by [9, Theorem 8.9] and Lemma 2.2, there exist pairwise orthogonal idempotents modulo $\mathrm{Ann}(M)$. By Lemma 2.3, it is easy to see that $M \cong eM \times (1-e)M$, where e is an idempotent element of R and Lemma 2.6 implies that $\mathrm{AG}(M)$ is a star graph or $\mathrm{AG}(M) \cong P_4$.

(b) If M is a vertex of $\mathrm{AG}(M)$, since $\mathrm{AG}(M)$ is a regular graph, $\mathrm{AG}(M)$ is a complete graph. Hence, we may assume that M is not a vertex of $\mathrm{AG}(M)$. Thus, M is not a prime module, and hence, $rm = 0$, such that $0 \neq m \in M$, $r \notin \mathrm{Ann}(M)$. It is easy to see that $(rM)(0 :_M r) = (0)$. If the set of R-submodules of rM (resp., $(0 :_M r)$)) is infinite, then $(0 :_M r)$ (resp., rM) has infinite degree, a contradiction. Thus, rM and $(0 :_M r)$ have finite length. Since $rM \cong M/(0 :_M r)$, M has finite length, so that $R/\mathrm{Ann}(M)$ is an Artinian ring. As in the proof of part (a), $M \cong M_1 \times M_2$. If M_1 has one non-trivial submodule N, then $\deg((0) \times M_2) > \deg(N \times M_2)$ and this contradicts the regularity of $\mathrm{AG}(M)$. Hence, M_1 is a simple module. Similarly, M_2 is a simple module. Therefore, $\mathrm{AG}(M) \cong K_2$. Now, suppose that

$(R/\text{Ann}(M), m/\text{Ann}(M))$ is an Artinian local ring. Now, as we have seen in part (a), there exists a natural number k, such that $m^{k-1}M$ is adjacent to all other vertices and we deduce that $\text{AG}(M)$ is a complete graph. □

Let S be a multiplicatively closed subset of R. A non-empty subset S^* of M is said to be S-closed if $se \in S^*$ for every $s \in S$ and $e \in S^*$. An S-closed subset S^* is said to be saturated if the following condition is satisfied: whenever $ae \in S^*$ for $a \in R$ and $e \in M$, then $a \in S$ and $e \in S^*$.

We need the following result due to Chin-Pi Lu.

Theorem 2.10 [16, Theorem 4.7] *Let $M = Rm$ be a cyclic module. Let S^* be an S-closed subset of M relative to a multiplicatively closed subset S of R, and N a submodule of M maximal in $M \backslash S^*$. If S^* is saturated, the ideal $(N : M)$ is maximal in $R \backslash S$, so that N is prime in M.*

Theorem 2.11 *If M is a cyclic module,* $\text{Ann}(M)$ *is a nil ideal, and* $|\text{Min}(M)| \geq 3$*, then* $\text{AG}(M)$ *contains a cycle.*

Proof If $\text{AG}(M)$ is a tree, then by Theorem 2.7, either $\text{AG}(M)$ is a star graph or $M \cong F \times S$, where F is a simple module and S has a unique non-trivial submodule. The latter case is impossible, because $|\text{Min}(F \times S)| = 2$. Suppose that $\text{AG}(M)$ is a star graph and N is the center of star. Clearly, one can assume that N is a minimal submodule of M. If $N^2 \neq (0)$, then by Lemma 2.4, there exists an idempotent $e \in R$ such that $N = eM$, so that $M \cong eM \times (1-e)M$. Now, by Proposition 2.1 and Lemma 2.6, we conclude that $|\text{Min}(M)| = 2$, a contradiction. Hence, $N^2 = 0$. Thus, one may assume that $N = Rm$ and $(Rm)^2 = (0)$. Suppose that P_1 and P_2 are two distinct minimal prime submodules of M. Since $(Rm)^2 = (0)$, we have $(Rm : M)^2 \subseteq \text{Ann}(M) \subseteq (P_i : M)$, $i = 1, 2$. So $(Rm : M)M = Rm \subseteq P_i$, $i = 1, 2$. Hence, $m \in P_i$, $i = 1, 2$. Choose $z \in (P_1 : M) \backslash (P_2 : M)$ and set $S_1 = \{1, z, z^2, \ldots\}$, $S_2 = M \backslash P_1$, and $S^* = S_1 S_2$. If $0 \notin S^*$, then $\Sigma = \{N < M |\ N \cap S^* = \emptyset\}$ is not empty. Then, Σ has a maximal element, say N. Hence, by Theorem 2.10, N is a prime submodule of M. Since $N \subseteq P_1$, we have $N = P_1$, a contradiction because $z \notin (N : M)$. So $0 \in S^*$. Therefore, there exists positive integer k and $m' \in S_2$, such that $z^k m' = 0$. Now, consider the submodules (m), (m'), and $z^k M$. It is clear that $(m) \neq (m')$ and $(m) \neq z^k M$. If $(m) = z^k M$, then $z \in (P_2 : M)$, a contradiction. Thus (m), (m'), and $z^k M$ form a triangle in $\text{AG}(M)$, a contradiction. Hence, $\text{AG}(M)$ contains a cycle. □

Theorem 2.12 *Suppose that M is a cyclic module,* $\text{rad}_M(0) \neq (0)$*, and* $\text{Ann}(M)$ *is a nil ideal. If* $|\text{Min}(M)| = 2$*, then either* $\text{AG}(M)$ *contains a cycle or* $\text{AG}(M) \cong P_4$.

Proof A similar argument to the proof of Theorem 2.11 shows that either $\text{AG}(M)$ contains a cycle or $M \cong F \times S$, where F is a simple module and S is a module with a unique non-trivial submodule. The latter case implies that $\text{AG}(M) \cong P_4$ (note that $\text{rad}_{F \times D}(0) = (0)$, where F is a simple module and D is a prime module). □

The radical of I, defined as the intersection of all prime ideals containing I, is denoted by $\sqrt{I}$. Before stating the next theorem, we recall that if M is a finitely generated module, then $\sqrt{(Q : M)} = (\text{rad}(Q) : M)$, where $Q < M$ (see [18, Theorem 4.4]). In addition, we know that if M is a finitely generated module, then for every prime ideal p of R with $p \supseteq \text{Ann}(M)$, there exists a prime submodule P of M, such that $(P : M) = p$ (see [15, Theorem 2]).

Theorem 2.13 *Assume that M is a finitely generated module,* $\text{Ann}(M)$ *is a nil ideal, and* $|\text{Min}(M)| = 1$*. If* $\text{AG}(M)$ *is a triangle-free graph, then* $\text{AG}(M)$ *is a star graph.*

Proof Suppose first that P is the unique minimal prime submodule of M. Since M is not a vertex of $\text{AG}(M)$, $Z(M) \neq (0)$. Therefore, there exist non-zero elements $r \in R$ and $m \in M$, such that $rm = 0$. It is easy to see that rM and Rm are vertices of $\text{AG}(M)$, because $(rM)(Rm) = (0)$. Since $\text{AG}(M)$ is triangle-free, Rm or rM is a minimal submodule of M. Without loss of generality, we can assume that Rm is a minimal submodule of M, so that $(Rm)^2 = (0)$ (if rM is a minimal submodule of M, then there exists $0 \neq m' \in M$ such that $rM = Rm'$). We claim that Rm is the unique minimal submodule of M. On the contrary, suppose that K is another minimal submodule of M. So either $K^2 = K$ or $K^2 = (0)$. If $K^2 = K$, then by Lemma 2.4, $K = eM$ for some idempotent element $e \in R$ and hence, $M \cong eM \times (1-e)M$. This implies that $|\text{Min}(M)| > 1$, a contradiction. If $K^2 = (0)$, then we have $C_3 = K - (K : M)M + (Rm : M)M - Rm - K$, a contradiction. Therefore, Rm is the unique minimal submodule of M. Let $V_1 = V(Rm)$, $V_2 = V(\text{AG}(M)) \backslash V_1$, $A = \{K \in V_1 | Rm \subseteq K\}$, $B = V_1 \backslash A$, and $C = V_2 \backslash \{Rm\}$. We prove that $\text{AG}(M)$ is a bipartite graph with parts V_1 and V_2. We may assume that V_1 is an independent set because AG(M) is triangle-free. We claim that one end of every edge of AG(M) is adjacent to Rm and another end contains Rm. To prove this, suppose that $\{N, K\}$ is an edge of AG(M) and $Rm \neq N$, $Rm \neq K$. Since $N(Rm) \subseteq Rm$, by the minimality of Rm, either $N(Rm) = (0)$ or $Rm \subseteq N$. The latter case follows that $K(Rm) = (0)$. If $N(Rm) = (0)$, then $K(Rm) \neq (0)$ and hence $Rm \subseteq K$. So, our plain is proved. This gives that V_2 is an independent set and $V(C) \subseteq V_1$. Since every vertex of A contains Rm and AG(M) is triangle-free, all vertices in A are just adjacent to Rm and so by [7, Theorem 3.4], $V(C) \subseteq B$. Since one end of every edge is adjacent to Rm and another end contains Rm, we also deduce that every vertex of C contains Rm and so every vertex of $A \cup V_2$ contains Rm. Note that if $Rm = P$, then one end of each edge of AG(M) is contained in Rm, and since Rm is a minimal submodule of M, AG(M) is a star graph with center $Rm = P$. Now, suppose that $P \neq Rm$. We claim that $P \in A$. Since $Rm \subseteq P$, it suffices to show that $(Rm)P = (0)$. To see this, let $r \in (P : M)$. We prove that $rm = 0$. Clearly, $(Rrm) \subseteq Rm$. If $rm = 0$, then we are done. Thus $Rrm = Rm$ and so $m = rsm$ for some $s \in R$. We have $m(1 - rs) = 0$. By [15, Theorem 2], we have $\mathrm{Nil}(R) = (P : M)$ (note that $\sqrt{\mathrm{Ann}(M)} = (\mathrm{rad}(0) : M) = (P : M)$). Therefore, $1 - rs$ is unit, a contradiction, as required. Since $N(C) \subseteq B$, if $B = \emptyset$, then $C = \emptyset$ and, therefore, AG(M) is a star graph with center Rm. It remains to show that $B = \emptyset$. Suppose that $K \in B$ and consider the vertex $K \cap P$ of AG(M). Since every vertex of $A \cup V_2$ contains Rm, yields $K \cap P \in B$. Pick $0 \neq m' \in K \cap P$. Since AG(M) is triangle-free, one can find an element $m'' \in Rm'$ such that Rm'' is a minimal submodule of M and $(Rm'')^2 = (0)$. Since Rm is the unique minimal submodule of M, we have $Rm = Rm'' \subseteq Rm'$. Thus $Rm \subseteq K \cap P$, a contradiction. So $B = \emptyset$ and we are done. Hence, AG(M) is a star graph whose center is Rm, as desired. □

Corollary 2.14 *Assume that M is a finitely generated module,* Ann(M) *is a nil ideal, and* $|\mathrm{Min}(M)| = 1$. *If* AG(M) *is a bipartite graph, then* AG(M) *is a star graph.*

3 On the coloring of the annihilating-submodule graphs

We recall that $N < M$ is said to be a semiprime submodule of M if for every ideal I of R and every submodule K of M, $I^2K \subseteq N$ implies that $IK \subseteq N$. Furthermore, M is called a semiprime module if $(0) \subseteq M$ is a semiprime submodule. Every intersection of prime submodules is a semiprime submodule (see [20]).

Theorem 3.1 *Let S be a multiplicatively closed subset of R containing no zero-divisors on finitely generated module M. Then, $cl(\mathrm{AG}(M_S)) \leq cl(\mathrm{AG}(M))$. Moreover,* AG$(M_S)$ *is a retract of* AG(M) *if M is a semiprime module. In particular, $cl(\mathrm{AG}(M_S)) = cl(\mathrm{AG}(M))$, whenever M is a semiprime module.*

Proof Consider a vertex map $\phi : V(\mathrm{AG}(M)) \longrightarrow V(\mathrm{AG}(M_S))$, $N \longrightarrow N_S$. Clearly, $N_S \neq K_S$ implies $N \neq K$ and $NK = (0)$ if and only if $N_SK_S = (0)$. Thus, ϕ is surjective, and hence, $cl(\mathrm{AG}(M_S)) \leq cl(\mathrm{AG}(M))$. In what follows, we assume that M a semiprime module. If $N \neq K$ and $NK = (0)$, then we show that $N_S \neq K_S$. Without loss of generality, we can assume that M is not a vertex of AG(M), and On the contrary, suppose that $N_S = K_S$. Then, $N_S^2 = N_SK_S = (NK)_S = (0)$ and so $N^2 = (0)$, a contradiction. This shows that the map ϕ is a graph homomorphism. Now, for any vertex N_S of AG(M_S), we can choice the fixed vertex N of AG(M). Then, ϕ is a retract (graph) homomorphism which clearly implies that $cl(\mathrm{AG}(M_S)) = cl(\mathrm{AG}(M))$ under the assumption. □

Corollary 3.2 *If M is a finitely generated semiprime module, then $cl(\mathrm{AG}(T(M)) = cl(\mathrm{AG}(M))$, where $T = R\backslash Z(M)$.*

Since the chromatic number $\chi(G)$ of a graph G is the least positive integer r, such that there exists a retract homomorphism $\psi : G \longrightarrow K_r$, the following corollaries follow directly from the proof of Theorem 3.1.

Corollary 3.3 *Let S be a multiplicatively closed subset of R containing no zero-divisors on finitely generated module M. Then, $\chi(\mathrm{AG}(M_S)) \leq \chi(\mathrm{AG}(M))$. Moreover, if M is a semiprime module, then $\chi(\mathrm{AG}(M_S)) = \chi(\mathrm{AG}(M))$.*

Corollary 3.4 *If M is a finitely generated semiprime module, then $\chi(\mathrm{AG}(T(M)) = \chi(\mathrm{AG}(M))$, where $T = R\backslash Z(M)$.*

Eben Matlis in [17, Proposition 1.5] proved that if $\{p_1, \ldots, p_n\}$ is a finite set of distinct minimal prime ideals of R and $S = R \backslash \cup_{i=1}^{n} p_i$, then $R_{p_1} \times \cdots \times R_{p_n} \cong R_S$. In [19], this result was generalized to finitely generated multiplication modules. In Theorem 3.6, we use this generalization for a cyclic module.

Theorem 3.5 [19, Theorem 3.11] *Let* $\{P_1, \ldots, P_n\}$ *be a finite set of distinct minimal prime submodules of finitely generated multiplication module* M *and* $S = R\backslash \cup_{i=1}^{n} (P_i : M)$. *Then,* $M_{p_1} \times \cdots \times M_{p_n} \cong M_S$, *where* $p_i = (P_i : M)$ *for* $1 \leq i \leq n$.

Theorem 3.6 *Let* M *be a cyclic module and* $\{P_1, \ldots, P_n\}$ *be a finite set of distinct minimal prime submodules of* M. *Then, there exists a clique of size* n.

Proof Let M be a cyclic module and $S = R\backslash \cup_{i=1}^{n} p_i$, where $p_i = (P_i : M)$ for $1 \leq i \leq n$. Then, since M is a multiplication module, by Theorem 3.5, there exists an isomorphism $\phi : M_{p_1} \times \cdots \times M_{p_n} \longrightarrow M_S$. Let $M = Rm$, $e_i = (0, \ldots, 0, m/1, \ldots, 0, \ldots, 0)$ and $\phi(e_i) = n_i/t_i$, where $m \in M$, $1 \leq i \leq n$, and $m/1$ is in the ith position of e_i. Consider the principal submodules $N_i = (n_i/t_i) = (n_i/1)$ in the module M_S. By Lemma 2.2 and Proposition 2.1, the product of submodules $(0) \times \cdots \times (0) \times (m/1)R_{p_i} \times (0) \times \cdots \times (0)$ and $(0) \times \cdots \times (0) \times (m/1)R_{p_j} \times (0) \times \cdots \times (0)$ are zero, $i \neq j$. Since ϕ is an isomorphism, there exists $t_{ij} \in S$, such that $t_{ij} r_i n_j = 0$, for every $i, j, 1 \leq i < j \leq n$, where $n_i = r_i m$ for some $r_i \in R$. Let $t = \Pi_{1 \leq i < j \leq n} t_{ij}$. We show that $\{(tn_1), \ldots, (tn_n)\}$ is a clique of size n in $\mathrm{AG}(M)$. For every $i, j, 1 \leq i < j \leq n$, $(Rtn_i)(Rtn_j) = (Rtn_j : M)Rtn_i = (Rtn_j : M)tr_i M = tr_i Rtn_j = (0)$. Since $(tn_i)_S = (n_i/1) = N_i$, we deduce that (tn_i) are distinct non-trivial submodules of M. □

Corollary 3.7 *For every cyclic module* M, $cl(\mathrm{AG}(M)) \geq |\mathrm{Min}(M)|$ *and if* $|\mathrm{Min}(M)| \geq 3$, *then* $gr(\mathrm{AG}(M)) = 3$.

Theorem 3.8 *Let* M *be a cyclic module and* $\mathrm{rad}_M(0) = (0)$. *Then,* $\chi(\mathrm{AG}(M)) = cl(\mathrm{AG}(M)) = |\mathrm{Min}(M)|$.

Proof If $|\mathrm{Min}(M)| = \infty$, then by Corollary 3.7, there is nothing to prove. Thus, suppose that $|\mathrm{Min}(M)| = \{P_1, \ldots, P_n\}$, for some positive integer n. Let $p_i = (P_i : M)$ and $S = R\backslash \cup_{i=1}^{n} p_i$. By Theorem 3.5, we have $M_{p_1} \times \cdots \times M_{p_n} \cong M_S$. Clearly, $cl(\mathrm{AG}(M_S)) \geq n$. Now, we show that $\chi(\mathrm{AG}(M_S)) \leq n$. By [15, Corollary 3], $P_i R_{p_i}$ is the only prime submodule of M and since $\mathrm{rad}_M(0) = (0)$, every M_{p_i} is a simple R_{p_i}-module. Define the map $C : V(\mathrm{AG}(M_S)) \longrightarrow \{1, 2, \ldots, n\}$ by $C(N_1 \times \cdots \times N_n) = \min\{i |\ N_i \neq (0)\}$. Since each M_{p_i} is a simple module, C is a proper vertex coloring of $\mathrm{AG}(M_S)$. Thus $\chi(\mathrm{AG}(M_S)) \leq n$ and so $\chi(\mathrm{AG}(M_S)) = cl(\mathrm{AG}(M_S)) = n$. Since $\mathrm{rad}_M(0) = (0)$, it is easy to see that $S \cap Z(M) = \emptyset$. Now, by Theorem 3.1 and Corollary 3.3, we obtain the desired. □

Theorem 3.9 *For every module* M, $cl(\mathrm{AG}(M)) = 2$ *if and only if* $\chi(\mathrm{AG}(M)) = 2$. *In particular,* $\mathrm{AG}(M)$ *is bipartite if and only if* $\mathrm{AG}(M)$ *is triangle-free.*

Proof For the first assertion, we use the same technique in [3, Theorem 13]. Let $cl(\mathrm{AG}(M)) = 2$. On the contrary assume that $\mathrm{AG}(M)$ is not bipartite. Therefore, $\mathrm{AG}(M)$ contains an odd cycle. Suppose that $C := N_1 - N_2 - \cdots - N_{2k+1} - N_1$ be a shortest odd cycle in $\mathrm{AG}(M)$ for some natural number k. Clearly, $k \geq 2$. Since C is a shortest odd cycle in $\mathrm{AG}(M)$, $N_3 N_{2k+1}$ is a vertex. Now, consider the vertices N_1, N_2, and $N_3 N_{2k+1}$. If $N_1 = N_3 N_{2k+1}$, then $N_4 N_1 = (0)$. This implies that $N_1 - N_4 - \cdots - N_{2k+1} - N_1$ is an odd cycle, a contradiction. Thus, $N_1 \neq N_3 N_{2k+1}$. If $N_2 = N_3 N_{2k+1}$, then we have $C_3 = N_2 - N_3 - N_4 - N_2$, again a contradiction. Hence, $N_2 \neq N_3 N_{2k+1}$. It is easy to check N_1, N_2, and $N_3 N_{2k+1}$ form a triangle in $\mathrm{AG}(M)$, a contradiction. The converse is clear. In particular, we note that empty graphs and the isolated vertex graphs are bipartite graphs. □

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