Arab. J. Math. (2015) 4:231–253
DOI 10.1007/s40065-014-0114-5

Arabian Journal of Mathematics

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Generalized functions beyond distributions

Received: 6 February 2014 / Accepted: 11 July 2014 / Published online: 6 August 2014

Abstract Ultrafunctions are a particular class of functions defined on a non-Archimedean field $\mathbb{R}^* \supset \mathbb{R}$. They have been introduced and studied in some previous works (Benci, Adv Nonlinear Stud 13:461–486, 2013; Benci and Luperi Baglini, EJDE, Conf 21:11–21, 2014; Benci, Basic Properties of ultrafunctions, to appear in the WNDE2012 Conference Proceedings, arXiv:1302.7156, 2014). In this paper we introduce a modified notion of ultrafunction and discuss systematically the properties that this modification allows. In particular, we will concentrate on the definition and the properties of the operators of derivation and integration of ultrafunctions.

Mathematics Subject Classification 26E30 · 26E35 · 46F30

الملخص

فوق الدوال صف خاص من الدوال معرفة على حقل غير أرخميدي $\mathbb{R} \subset \mathbb{R}^*$. تم تقديم هذه الدوال ودراستها في بعض الأعمال السابقة ([1]، [2]،[3]). نقدم في هذه الورقة مفهوما معدلا لفوق الدوال، وندرس بشكل منهجي الخواص التي يسمح بها هذا التعديل. سنركز بشكل خاص على تعريف وخواص مؤثري الاشتقاق والتكامل لفوق الدوال.

1 Introduction

In some recent papers the notion of ultrafunction has been introduced and studied [1,8,9]. Ultrafunctions are a particular class of functions defined on a non-Archimedean field $\mathbb{R}^* \supset \mathbb{R}$. We recall that a non-Archimedean field is an ordered field which contains infinite and infinitesimal numbers. In general, as we showed in our previous works, when working with ultrafunctions we associate to any continuous function $f : \mathbb{R}^N \to \mathbb{R}$ an ultrafunction $\widetilde{f} : (\mathbb{R}^*)^N \to \mathbb{R}^*$ which extends f; more exactly, to any vector space of functions $V(\Omega) \subseteq L^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ we associate a space of ultrafunctions $\widetilde{V}(\Omega)$. The spaces of ultrafunctions are much larger

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than the corrispective spaces of functions, and they have much more "compactness"[1]: these two properties ensure that in the spaces of ultrafunctions we can find solutions to functional equations which do not have any solutions among the real functions or the distributions.

In [9] we studied the basic properties of ultrafunctions. One property that is missing, in general, is the "locality": the extensions of operators that are local[2] on $V(\Omega)$ may not be local on $\widetilde{V}(\Omega)$. This problem is related to the properties of a particular basis of the spaces of ultrafunctions, called "Delta basis" (see [8,9]). The elements of a Delta basis are called Delta ultrafunctions and, in some precise sense, they are an analog of the Delta distributions. More precisely, given a point $a \in \mathbb{R}^*$, the Delta ultrafunction centered in a [denoted by $\delta_a(x)$] is the unique ultrafunction such that, for every ultrafunction $u(x)$, we have[3]

$$\int^* u(x)\delta_a(x)\mathrm{d}x = u(a).$$

It would be useful for applications to have an orthonormal Delta basis, namely a Delta basis $\{\delta_a(x)\}_{a\in\Sigma}$ such that, for every $a, b \in \Sigma$, $\int^* \delta_a(x)\delta_b(x)\mathrm{d}x = \delta_{a,b}$; unfortunately, this seems to be impossible.

The main aim of this paper is to show how to modify the constructions exposed in [9] (that will be recalled) to avoid such unwanted issues. We will show how to construct spaces of ultrafunctions that have "good local properties" and that have Delta bases $\{\delta_a(x)\}_{a\in\Sigma}$ that are "almost orthogonal" where, by saying that a Delta basis is "almost orthogonal", we mean the following: for every $a, b \in \Sigma$, if $|a - b|$ is not infinitesimal[4] then $\int^* \delta_a(x)\delta_b(x)\mathrm{d}x = 0$.

We will also discuss a few other properties of ultrafunctions that were missing in the previous approach but that hold in this new context.

The techniques on which the notion of ultrafunction is based are related to non-Archimedean mathematics (NAM) and to nonstandard analysis (NSA). In particular, the most important notion that we use is that of Λ-limit (see [1,8,9]). In this paper this notion will be considered known; however, for sake of completeness, we will recall its basic properties in the Appendix.

1.1 Notations

If X is a set then

- $\mathcal{P}(X)$ denotes the power set of X and $\mathcal{P}_{fin}(X)$ the family of finite subsets of X;
- $\mathfrak{F}(X, Y)$ denotes the set of all functions from X to Y and $\mathfrak{F}(\mathbb{R}^N) = \mathfrak{F}(\mathbb{R}^N, \mathbb{R})$.

Let Ω be a subset of $\mathbb{R}^N$: then

- $\mathcal{C}(\Omega)$ denotes the set of continuous functions defined on $\Omega \subset \mathbb{R}^N$;
- $\mathcal{C}_0(\Omega)$ denotes the set of continuous functions in $\mathcal{C}(\Omega)$ having compact support in Ω;
- $\mathcal{C}^k(\Omega)$ denotes the set of functions defined on $\Omega \subset \mathbb{R}^N$ which have continuous derivatives up to the order k;
- $\mathcal{C}^k_0(\Omega)$ denotes the set of functions in $\mathcal{C}^k(\Omega)$ having compact support;
- $\mathcal{C}^1_\sharp(\mathbb{R})$ denotes the set of functions f of class $\mathcal{C}^1(\Omega)$ except than on a discrete set $\Gamma \subset \mathbb{R}$ and such that, for any $\gamma \in \Gamma$, the limits
$$\lim_{x\to\gamma^\pm} f(x)$$
exist and are finite;
- $\mathcal{D}(\Omega)$ denotes the set of the infinitely differentiable functions with compact support defined on $\Omega \subset \mathbb{R}^N$; $\mathcal{D}'(\Omega)$ denotes the topological dual of $\mathcal{D}(\Omega)$, namely the set of distributions on Ω;
- if $A \subset \mathbb{R}^N$ is a set, then χ_A denotes the characteristic function of A;
- for any $\xi \in (\mathbb{R}^N)^*$, $\rho \in \mathbb{R}^*$, we set $\mathfrak{B}_\rho(\xi) = \{x \in (\mathbb{R}^N)^* : \ |x - \xi| < \rho\}$;

[1] In this context "compactness" means that the space of ultrafunctions is a space of hyperfinite dimension so, by transfer, it satisfies many properties of finite-dimensional vector spaces. This is an useful property for applications (see, e.g., [1] and [8]).

[2] By local operator we mean any operator $F : V(\Omega) \to V(\Omega)$ such that $\mathrm{supp}(F(f)) \subseteq \mathrm{supp}(f)\ \forall f \in V(\Omega)$.

[3] $\int^* : L^1(\mathbb{R})^* \to \mathbb{C}^*$ is an extension of the integral $\int : L^1(\mathbb{R}) \to \mathbb{C}$.

[4] We recall that an element x of a non-Archimedean superreal ordered field $\mathbb{K} \supset \mathbb{R}$ is infinitesimal if $|x| < r$ for every $r \in \mathbb{R}_{>0}$.

- $\mathfrak{mon}(x) = \{y \in (\mathbb{R}^N)^* : x \sim y\}$, where $x \sim y$ is defined in Definition 9.4;
- $\mathfrak{gal}(0) = \{y \in (\mathbb{R}^N \setminus \{0\})^* : x^{-1} \not\sim 0\} \cup \{0\}$;
- $\forall^{a.e.}\ x \in X$ means "for almost every $x \in X''$;
- if $a, b \in \mathbb{R}^*$, then
  - $[a, b]_{\mathbb{R}^*} = \{x \in \mathbb{R}^* : a \leq x \leq b\}$;
  - $(a, b)_{\mathbb{R}^*} = \{x \in \mathbb{R}^* : a < x < b\}$;
  - $]a, b[= [a, b]_{\mathbb{R}^*} \setminus (\mathfrak{mon}(a) \cup \mathfrak{mon}(b))$.

2 Definition of ultrafunctions

Following the same approach of [1,8,9], we want the space of ultrafunctions to be an hyperfinite-dimensional vector space. An easy way[5] to obtain this property is the following. Let $\mathfrak{X} = \mathcal{P}_{fin}(\mathfrak{F}(\mathbb{R}, \mathbb{R}))$. Given $\lambda \in \mathfrak{X}$, we set $V_\lambda = Span(\{f \mid f \in \lambda\})$.

Definition 2.1 An internal function

$$u = \lim_{\lambda \uparrow \Lambda} u_\lambda \in \mathfrak{F}(\mathbb{R})^*$$

is an ultrafunction if, for every $\lambda \in \mathfrak{X}$, $u_\lambda \in V_\lambda$. The space of ultrafunctions will be denoted by $\mathfrak{F}_\Lambda(\mathbb{R})$. With some abuse of notation the restrictions of ultrafunctions to internal subset of $\mathbb{R}^*$ will still be called ultrafunctions.

In particular, we have that

$$\mathfrak{F}_\Lambda(\mathbb{R}) = \lim_{\lambda \uparrow \Lambda} V_\lambda$$

so, being a Λ-limit of finite-dimensional vector spaces, the vector space of ultrafunctions has hyperfinite dimension. Moreover, given any vector space of functions $W \subset \mathfrak{F}(\mathbb{R})$, we can define the space of ultrafunctions generated by W as follows:

$$W_\Lambda = W^* \cap \mathfrak{F}_\Lambda(\mathbb{R}).$$

Let us observe that

$$W_\Lambda = \lim_{\lambda \uparrow \Lambda} W_\lambda,$$

where for every $\lambda \in \mathfrak{X}$ we pose $W_\lambda = V_\lambda \cap W$.

The space of ultrafunctions $\mathfrak{F}_\Lambda(\mathbb{R})$ is too large for applications. We want to have a smaller space $V_\Lambda(\mathbb{R}) \subset \mathfrak{F}_\Lambda(\mathbb{R})$ which satisfies suitable properties for applications. There are three kinds of properties that we would like ultrafunctions to satisfy:

- "Compactness": we want the space of ultrafunctions to be hyperfinite dimensional so to have the "nice" properties of finite-dimensional vector spaces; moreover, we want the support of all ultrafunctions to be contained in a common compact set.
- Extension of spaces: the space of ultrafunction should contains some standard spaces useful for applications as subspaces like, e.g., the vector space generated by the characteristic functions of intervals and $\mathcal{C}^1(\mathbb{R})$. Moreover, it should be possible to associate an ultrafunctions to every distribution (or, at least, to every compactly supported distribution).
- Extension of operators: we want to be able to extend to ultrafunctions many operators between standard functions preserving, when it is possible, the locality of the operators. In particular, the operator that we are most interested in is the derivative.

[5] We note that the construction presented here is not the unique way to construct a space of ultrafunctions. In any case, with the present construction it is easy to apply a modification of the Faedo–Galerkin method to solve some particular differential equation, see [8] for details.

The reader should be aware that this list of properties is subjective and motivated mainly by the fact that all these properties are easily obtainable with our construction and (as we will show in some planned future papers) they are sufficient to study some applications of ultrafunctions to PDE. By no means we intend our list to be complete or suitable for all possible applications.

Desideratum 2.2 There is an infinite number β such that if $u(x) \in V_\Lambda(\mathbb{R})$ then $u(x) = 0$ for $|x| > \beta$ and $u(x) \in L^\infty(\mathbb{R})^*$.

Desideratum 2.2 states that the ultrafunctions have an uniform compact support and are bounded in $\mathbb{R}^*$. From these conditions it follows that, if $u(x) \in V_\Lambda(\mathbb{R})$, then $u(x) \in L^p(\mathbb{R})^*$ for every p; in particular, $u(x)$ is summable and is in $L^2(\mathbb{R})^*$. So $V_\Lambda(\mathbb{R}) \subset L^2(\mathbb{R})^*$, and this allows to give to $V_\Lambda(\mathbb{R})$ the euclidean structure and the norm induced by $L^2(\mathbb{R})^*$.

Desideratum 2.3 $V_\Lambda(\mathbb{R}) \subset F_\sharp(\mathbb{R})^*$, where

$$F_\sharp(\mathbb{R}) = \left\{ u \in L^1_{loc} \;\middle|\; u(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} u(y)\, dy \right\}.$$

This request, which may seem strange at first sight, will allow to associate to every point $a \in [-\beta, \beta]$ a delta (or Dirac) ultrafunction centered in a, namely an ultrafunction $\delta_a(x)$ such that, for every ultrafunction $u(x)$, we have

$$\int^* u(x)\delta_a(x)\mathrm{d}x = u(a).$$

Desideratum 2.4 If $f \in \mathcal{C}^1(\mathbb{R})$, and $a, b \in \mathbb{R}$, then $(f \cdot \chi_{[a,b]})^* \in V_\Lambda(\mathbb{R})$.

Desideratum 2.4 is introduced because, for many applications, it is important to have both the characteristic functions of intervals and the $\mathcal{C}^1(\mathbb{R})$ functions in the function space that we consider. Moreover, we will show that from Desideratum 2.4 it follows that the delta functions have compact support concentrated around their center: in fact we will show that, $\forall a \in \mathfrak{gal}(0)$, $\mathfrak{supp}(\delta_a) \subset \mathfrak{mon}(a)$.

However, it would be nice to have the previous property in the following more general fashion:

Desideratum 2.5 $\forall a \in [-\beta, \beta]$, $\mathfrak{supp}(\delta_a) \subset \mathfrak{mon}(a)$.

Our next desideratum is the following:

Desideratum 2.6 There exists a linear map $\widetilde{(\cdot)} : [L^1_{loc}(\Omega)]^* \to V_\Lambda(\mathbb{R})$ such that $\forall f \in [L^1_{loc}(\Omega)]^*$, $\forall v \in V_\Lambda(\mathbb{R})$, we have

$$\int^* f v \mathrm{d}x = \int^* \widetilde{f} v \mathrm{d}x.$$

Desideratum 2.6 will be used to construct an embedding of $\mathcal{C}^{-\infty}(\mathbb{R})$ into our space of ultrafunctions. The relations between ultrafunctions, distributions and Schwartz impossibility result are precised in [10], where it is constructed an algebra of ultrafunctions in which all distributions can be embedded. Let us note that, in general, we do not require the space of ultrafunctions to be an algebra: in fact, the multiplication uv of two ultrafunctions u, v is defined, since ultrafunctions are internal functions, but in general we do not have (and for many applications, we do not need to require) that $uv \in V_\Lambda(\mathbb{R})$.

Desideratum 2.7 There exists a map $D : V_\Lambda(\mathbb{R}) \to V_\Lambda(\mathbb{R})$ such that

- $\forall f \in \mathcal{C}^1(\mathbb{R}),\ \forall x \in \mathbb{R},\ D\widetilde{f}(x) = \widetilde{f'}(x)$;
- $\forall u, v \in V_\Lambda(\mathbb{R}),\ \int_{-\beta}^{\beta} Du(x)v(x)\mathrm{d}x = -\int_{-\beta}^{\beta} u(x)Dv(x)\mathrm{d}x + [u(x)v(x)]_{-\beta}^{\beta}$;
- $D\widetilde{1} = 0$;
- $D\chi_{[a,b]} = \delta_a - \delta_b$.

Desideratum 2.7 simply states that it is possible to define a derivative on $V_\Lambda(\mathbb{R})$ which satisfies a few expected properties. Let us note that the second property presented in this desideratum is a weak form of the Leibniz rule. In fact our derivative will not satisfy Leibniz rule (in general); nevertheless once again we note that this weak form is sufficient for many applications (see [10] for a discussion on this point).

In the next sections, we show how to construct a space that satisfies all the Desideratum that we presented.

3 Construction of a canonical space of ultrafunctions

We want to consider a special subset of $\mathfrak{F}_\Lambda(\mathbb{R})$. Let β be an infinite number; we set

$$\Gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_\ell\} \subset \mathbb{R}^*,$$

where $l \in \mathbb{N}^*$, $\gamma_0 = -\beta;\ \gamma_\ell = \beta$ and for $j = 0, 1, \ldots, \ell - 1$, we require that

$$0 < \gamma_{j+1} - \gamma_j < \eta$$

where η is an infinitesimal number. Moreover, it is useful to assume[6] that $\mathbb{R} \subseteq \Gamma$.

For $j = 0, 1, \ldots, \ell - 1$, we set

$$\mathbb{I}_j := (\gamma_j, \gamma_{j+1})_{\mathbb{R}^*}.$$

For every $a, b \in \Gamma$ we denote by $\chi_{[a,b]}(x)$ the characteristic function of $[a, b]_{\mathbb{R}^*}$ defined in a slightly different way:

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in (a, b)_{\mathbb{R}^*} \\ 0 & \text{if } x \notin [a, b]_{\mathbb{R}^*} \\ \frac{1}{2} & \text{if } x = a, b;\ a \neq -\beta;\ b \neq \beta \\ 1 & \text{if } x = a = -\beta \\ 1 & \text{if } x = b = \beta \end{cases}; \tag{1}$$

For every $j = 0, 1, \ldots, \ell - 1$, we set

$$\chi_j(x) = \chi_{\mathbb{I}_j}(x).$$

The set of functions

$$\mathfrak{G} = \{\sum_{j=0}^{\ell-1} c_j \chi_j(x) \mid c_j \in \mathbb{R}^*\}$$

will be referred to as the set of **grid functions**.

Definition 3.1 We denote by $V_\Lambda(\mathbb{R})$ the space of ultrafunctions

$$u : [-\beta, \beta]_{\mathbb{R}^*} \to \mathbb{R}^*$$

which can be represented as follows:

$$u(x) = \sum_{j=0}^{\ell-1} v_j(x) \chi_j(x)$$

where, $\forall j \in J,\ v_j(x) \in \mathcal{C}^1_\Lambda(\mathbb{R})$. We will refer to $V_\Lambda(\mathbb{R})$ as the canonical space of ultrafunctions.

Let us note that the space of ultrafunctions depends on the particular choices of β, l and Γ. Depending on the applications that one wants to study it might be useful to assume some further properties for these quantities, e.g., to assume that β is a multiple of 2π, that l is odd/even or that Γ contains some fixed infinite number. In any case all the properties that we will present in this paper do not depend on particular choices of β, l and Γ. Moreover, if $\Omega \subseteq \mathbb{R}$ is open, we pose

$$V_\Lambda(\Omega) = \{u_{1_{\Omega^*}} \mid u \in V_\Lambda(\mathbb{R})\}.$$

Nevertheless, we will mainly study $V_\Lambda(\mathbb{R})$.

Proposition 3.2 *The elements of $V_\Lambda(\mathbb{R})$ are restriction to $[-\beta, \beta]_{\mathbb{R}^*}$ of ultrafunctions.*

[6] We can always assume this property if we work in a $\mathfrak{c}^+$-saturated hyperextensions.

Proof Let $u(x)=\sum_{j=0}^{\ell-1} v_j(x)\chi_j(x)$, let $\ell=\lim_{\lambda\uparrow\Lambda}\ell_\lambda$, $\chi_j(x)=\lim_{\lambda\uparrow\Lambda}\chi_{j,\lambda}(x)$ and $v_j(x)=\lim_{\lambda\uparrow\Lambda} v_{j,\lambda}$. Then

$$u(x)=\lim_{\lambda\uparrow\Lambda}\sum_{j=0}^{\ell_\lambda-1} v_{j,\lambda}(x)\chi_{j,\lambda}(x),$$

so it is an ultrafunction. □

Proposition 3.3 $V_\Lambda(\mathbb{R})$ *is an hyperfinite-dimensional vector space, and* $\dim(V_\Lambda(\mathbb{R}))\le\ell\cdot\dim\mathcal{C}^1_\Lambda(\mathbb{R})$.

Proof $\mathcal{C}^1_\Lambda(\mathbb{R})$ is hyperfinite dimensional because it is an internal vector subspace of the hyperfinite-dimensional vector space $\mathfrak{F}_\Lambda(\mathbb{R})$. If

$$B=\{v_i(x)\mid i\le\dim(\mathcal{C}^1_\Lambda(\mathbb{R}))\}$$

is a basis for $\mathcal{C}^1_\Lambda(\mathbb{R})$, the set

$$B_V=\{v_i(x)\chi_j(x)\mid v_i\in B,\ j=0,\dots,\ell\}$$

is a set of generators for $V_\Lambda(\mathbb{R})$, and its cardinality is $\ell\cdot\dim\mathcal{C}^1_\Lambda(\mathbb{R})$. So $\dim(V_\Lambda(\mathbb{R}))\le\ell\cdot\dim\mathcal{C}^1_\Lambda(\mathbb{R})$. □

Since $V_\Lambda(\mathbb{R})\subset[L^2(\mathbb{R})]^*$, it can be equipped with the following scalar product

$$(u,v)=\int^* u(x)\overline{v(x)}\,\mathrm{d}x,$$

where $\int^*$ is the natural extension of the Lebesgue integral considered as a functional

$$\int: L^1(\Omega)\to\mathbb{C}.$$

The norm of a (canonical) ultrafunction will be given by

$$\|u\|=\left(\int^*|u(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

Canonical ultrafunctions have a few interesting properties:

Proposition 3.4 *The following properties hold:*

(i) *If* $f\in\mathcal{C}^1(\mathbb{R})$ *then* $f^*\cdot\chi_{[-\beta,\beta]_{\mathbb{R}^*}}\in V_\Lambda(\mathbb{R})$;
(ii) *if* $u\in V_\Lambda(\mathbb{R})$ *and* $a,b\in\Gamma$, *then* $u\cdot\chi_{[a,b]_{\mathbb{R}^*}}\in V_\Lambda(\mathbb{R})$;
(iii) *if* $u\in V_\Lambda(\mathbb{R})$ *then for* $j=1,\dots,\ell-1$ *the limits*

$$(\lim_{x\to\gamma_j^\pm})^*u(x)$$

are well defined and we set

$$u(\gamma_j^+):=\left(\lim_{x\to\gamma_j^+}\right)^*u(x);\ u(\gamma_j^-):=(\lim_{x\to\gamma_j^-})^*u(x);\tag{2}$$

(iv) *if* $u\in V_\Lambda(\mathbb{R})$ *then for* $j=0$ *the limit*

$$\left(\lim_{x\to\gamma_0^+}\right)^*u(x)$$

is well defined and for $j=l$ *the limit*

$$\left(\lim_{x\to\gamma_l^-}\right)^*u(x)$$

is well defined.

(v) *if, for every* $j = 0, \ldots, \ell - 1$ *we set*

$$V(\mathbb{I}_j) := \{u(x)\chi_j(x) \mid u(x) \in \mathcal{C}^1_\Lambda(\mathbb{R})\},$$

then, for $k \neq j$, $V(\mathbb{I}_j)$ *and* $V(\mathbb{I}_k)$ *are orthogonal;*

(vi) $V_\Lambda(\mathbb{R})$ *can be split in orthogonal spaces as follows:*

$$V_\Lambda(\mathbb{R}) = \bigoplus_{j=0}^{\ell-1} V(\mathbb{I}_j).$$

Proof (i) If $f \in \mathcal{C}^1(\mathbb{R})$, then $f^* \in \mathcal{C}^1_\Lambda(\mathbb{R})$, and

$$f^* \cdot \chi_{[-\beta,\beta]_{\mathbb{R}^*}} = \sum_{j=0}^{\ell-1} f^*(x)\chi_j(x) \in \mathfrak{U}(\mathbb{R}).$$

(ii) It follows by the definition of $\chi_{[a,b]}(x)$.

(iii) If $u(x) = \sum_{j=0}^{\ell-1} u_j(x)\chi_j(x)$, then

$$u(\gamma_0^-) = (\lim_{x\to\gamma_j^-})^* u_{j-1}(x)$$

and

$$u(\gamma_j^+) = (\lim_{x\to\gamma_j^+})^* u_j(x)$$

and these limits exist because u_{j-1}, u_j are continuous on $\overline{\mathbb{I}_{j-1}}, \overline{\mathbb{I}_j}$, respectively.

(iv) The same as in (i).

(v) This is immediate since, if $j \neq k$, if $u \in V(\mathbb{I}_j)$ and $v \in V(\mathbb{I}_k)$ then the supports of u and v are disjoint.

(vi) Having proved (iii), it remains only to prove that $\bigoplus_{j=0}^{\ell} V(\mathbb{I}_j)$ generates all $V_\Lambda(\mathbb{R})$, and this is clear because, if $u(x) = \sum_{j=0}^{\ell} u_j(x)\chi_j(x)$ then, for every $j = 0, \ldots, \ell - 1$, $u_j(x)\chi_j(x) \in V(\mathbb{I}_j)$.

□

Definition 3.5 A basis $\{e_{j,k} : j = 0, \ldots, \ell - 1,\ k = 1, \ldots, s_j\}$ for $V_\Lambda(\mathbb{R})$ is called **split basis** if, for every $j = 0, \ldots, \ell - 1$, $\{e_{j,k}\}_{k=1}^{s_j}$ is a basis for $V(\mathbb{I}_j)$.

4 Delta and Sigma basis

Following the approach presented in [9], in this section we introduce two particular bases for $V_\Lambda(\mathbb{R})$ and study their main properties. We start by defining the *Delta ultrafunctions*. To do this, it is useful to observe that the value of an ultrafunction u for γ_j, $j = 1, \ldots, \ell - 1$, can be defined as follows:

$$u(\gamma_j) = \frac{u(\gamma_j^+) + u(\gamma_j^-)}{2}$$

where $u(x^+)$, $u(x^-)$ are defined by (2). The fact that this definition makes sense follows by points (iii) and (iv) in Proposition 3.4. Obviously, this choice is conventional, and other linear combinations of the limits could be used to define the value of an ultrafunction in a point γ_j, $j = 1, \ldots, \ell - 1$, and all the results that follow would be repeatable (with the natural modifications due to the change in the definition of $u(\gamma_j)$). Nevertheless, in some sense our definition is "natural", since it gives to $u(\gamma_j)$ the intermediate value between the limits $u(\gamma_j^+)$ and $u(\gamma_j^-)$. Moreover, we pose

$$u(\gamma_0) = u(-\beta) = u^+(-\beta);\ \ u(\gamma_\ell) = u(\beta) = u^-(\beta).$$

These observations are relevant in the following definition:

Definition 4.1 Given a number $q \in [-\beta, \beta]$ we denote by $\delta_q(x)$ an ultrafunction in $V_\Lambda(\mathbb{R})$ such that

$$\forall v \in V_\Lambda(\mathbb{R}), \ \int^* v(x)\delta_q(x)\mathrm{d}x = v(q). \tag{3}$$

$\delta_q(x)$ is called the Delta (or Dirac) ultrafunction concentrated in q.

Let us see the main properties of the Delta ultrafunctions:

Theorem 4.2 *We have the following properties:*

(i) *for every* $q \in [-\beta, \beta]$ *there exists an unique Delta ultrafunction concentrated in* q;
(ii) *for every* $a, \ b \in [-\beta, \beta]$ $\delta_a(b) = \delta_b(a)$;
(iii) $\|\delta_q\|^2 = \delta_q(q)$.

Proof (i) Let $\{e_{j,k} : j = 0, \ldots, \ell - 1, \ k = 1, \ldots, s_j\}$ be an orthogonal split basis of $V_\Lambda(\mathbb{R})$ (see Definition 3.5). If $q \in \mathbb{I}_j$ we pose

$$\delta_q(x) = \sum_{k=1}^{s_j} e_{j,k}(q)e_{j,k}(x).$$

For every $i \neq j$, for every $v \in V(\mathbb{I}_i)$ we have $\int^* v(x)\delta_q(x)\mathrm{d}x = 0 = v(q)$. If $v \in V(\mathbb{I}_j)$, $v = \sum_{k=1}^{s_j} v_k e_{j,k}(x)$ we have

$$\begin{aligned}
&\int^* v(x)\delta_q(x)\mathrm{d}x \\
&= \int^* (\sum_{k=1}^{s_j} e_{j,k}(q)e_{j,k}(x))(\sum_{k=1}^{s_j} v_k e_{j,k}(x))\mathrm{d}x = \sum_{k=1}^{s_j} \int^* e_{j,k}(q)e_{j,k}(x)v_k e_{j,k}(x) \\
&= \sum_{k=1}^{s_j} e_{j,k}(q)v_k = v(q).
\end{aligned}$$

If $q = \gamma_0$ we pose

$$\delta_q(x) = \sum_{k=1}^{s_0} e^+_{j,k}(q)e_{j,k}(x)$$

and if $q = \gamma_\ell$ we pose

$$\delta_q(x) = \sum_{k=1}^{s_{\ell-1}} e^-_{j,k}(q)e_{j,k}(x).$$

The verification that these definitions are well posed is equal to the one carried out for $q \in \mathbb{I}_j$.
If $q = \gamma_j, \ j \neq 0, \ell$ we set

$$\delta_q(x) = \frac{1}{2}\left(\sum_{k=1}^{s_{j-1}} e^-_{j-1,k}(q)e_{j-1,k}(x) + \sum_{k=1}^{s_j} e^+_{j,k}(q)e_{j,k}(x)\right).$$

Then

$$\begin{aligned}
&\int^{*} v(x)\delta_q(x)\mathrm{d}x\\
&= \frac{1}{2}\left(\int^{*}_{[\gamma_{j-1},\gamma_j]} v(x)\left(\sum_{k=1}^{s_j} e^{-}_{j-1,k}(q)e_{j-1,k}(x)\right)\mathrm{d}x + \int^{*}_{[\gamma_j,\gamma_{j+1}]} v(x)\left(\sum_{k=1}^{s_j} e^{+}_{j,k}(q)e_{j,k}(x)\right)\mathrm{d}x\right)\\
&= \frac{1}{2}\left(\int^{*}_{[\gamma_{j-1},\gamma_j]} v^{-}(x)\left(\sum_{k=1}^{s_j} e^{-}_{j-1,k}(q)e_{j-1,k}(x)\right)\mathrm{d}x + \int^{*}_{[\gamma_j,\gamma_{j+1}]} v^{+}(x)\left(\sum_{k=1}^{s_j} e^{+}_{j,k}(q)e_{j,k}(x)\right)\mathrm{d}x\right)\\
&= \frac{1}{2}[v^{-}(\gamma_j)+v^{+}(\gamma_j)] = v(\gamma_j).
\end{aligned}$$

The Delta function in q is unique: if $f_q(x)$ is another Delta ultrafunction centered in q then for every $y \in [-\beta,\beta]$ we have:

$$\delta_q(y) - f_q(y) = \int^{*} (\delta_q(x) - f_q(x))\delta_y(x)\mathrm{d}x = \delta_y(q) - \delta_y(q) = 0$$

and hence (i) $\delta_q(y) = f_q(y)$ for every $y \in (-\beta,\beta)$.
(ii) $\delta_a(b) = \int^{*} \delta_a(x)\delta_b(x)\,\mathrm{d}x = \delta_b(a)$.
(iii) $\|\delta_q\|^2 = \int^{*} \delta_q(x)\delta_q(x) = \delta_q(q)$. □

A consequence of the previous proof is that, for every $\gamma_j \in \Gamma \setminus \{-\beta,\beta\}$, it is possible to define three delta functions centered in γ_j, namely $\delta^{-}_{\gamma_j}(x)$, $\delta^{+}_{\gamma_j}(x)$ and $\delta_{\gamma_j}(x)$, which satisfy the following properties: $\forall v \in V_\Lambda(\mathbb{R})$, we have

$$\begin{aligned}
&\int^{*} v(x)\delta^{-}_{\gamma_j}(x)\mathrm{d}x = v^{-}(\gamma_j);\\
&\int^{*} v(x)\delta^{+}_{\gamma_j}(x)\mathrm{d}x = v^{+}(\gamma_j);\\
&\int^{*} v(x)\delta_{\gamma_j}(x)\mathrm{d}x = v(\gamma_j).
\end{aligned} \tag{4}$$

Moreover, it is immediate to prove that the conditions in (4) characterize uniquely the functions $\delta^{-}_{\gamma_j}(x)$, $\delta^{+}_{\gamma_j}(x)$ and $\delta_{\gamma_j}(x)$. So we will consider (4) as a definition for $\delta^{-}_{\gamma_j}(x)$, $\delta^{+}_{\gamma_j}(x)$ and $\delta_{\gamma_j}(x)$.

Definition 4.3 A Delta basis $\{\delta_a(x)\}_{a\in\Sigma}$ ($\Sigma \subset [-\beta,\beta]$) is a basis for $V_\Lambda(\mathbb{R})$ whose elements are Delta ultrafunctions. Its dual basis $\{\sigma_a(x)\}_{a\in\Sigma}$ is called Sigma basis. We recall that, by definition of dual basis, for every $a, b \in \Sigma$ the equation

$$\int^{*} \delta_a(x)\sigma_b(x)\mathrm{d}x = \delta_{ab} \tag{5}$$

holds. A set $A \subset [-\beta,\beta]$ is called set of independent points if $\{\delta_a(x)\}_{a\in A}$ is a basis

The existence of a Delta basis is a consequence of the following fact:

Remark 4.4 The set $\{\delta_a(x) | a \in [-\beta,\beta]\}$ generates $V_\Lambda(\mathbb{R})$. In fact, let $G(\mathbb{R})$ be the vector space generated by the set $\{\delta_a(x) \mid a \in [-\beta,\beta]\}$ and suppose that $G(\mathbb{R})$ is properly included in $V_\Lambda(\mathbb{R})$. Then, the orthogonal $G(\mathbb{R})^{\perp}$ of $G(\mathbb{R})$ in $V_\Lambda(\mathbb{R})$ contains a function $f \neq 0$. But, since $f \in G(\mathbb{R})^{\perp}$, for every $a \in [-\beta,\beta]$ we have

$$f(a) = \int^{*} f(x)\delta_a(x)\mathrm{d}x = 0,$$

so $f_{1_{[-\beta,\beta]}} = 0$ and this is absurd. Thus, the set $\{\delta_a(x) \mid a \in (-\beta,\beta)\}$ generates $V_\Lambda(\mathbb{R})$, hence it contains a basis.

Let us see some properties of Delta and Sigma bases (which, in this new context, are slightly different from the one presented in [9]):

Theorem 4.5 *A Delta basis* $\{\delta_q(x)\}_{q\in\Sigma}$ *and its dual basis* $\{\sigma_q(x)\}_{q\in\Sigma}$ *satisfy the following properties:*

(i) *if* $u \in V_\Lambda(\mathbb{R})$ *then*

$$u(x) = \sum_{q\in\Sigma}\left(\int^* \sigma_q(\xi)u(\xi)d\xi\right)\delta_q(x);$$

(ii) *if* $u \in V_\Lambda(\mathbb{R})$ *then*

$$u(x) = \sum_{q\in\Sigma} u(q)\sigma_q(x); \tag{6}$$

(iii) *if two ultrafunctions* u *and* v *coincide on a set of independent points then they are equal;*
(iv) *if* Σ *is a set of independent points and* $a, b \in \Sigma$ *then* $\sigma_a(b) = \delta_{ab}$;
(v) *for every* $q \in [-\beta, \beta]$, $\sigma_q(x)$ *exists and is unique;*
(vi) *for every* $q \in [-\beta, \beta]$ *if* $q \in \mathbb{I}_j$ *then* $\mathfrak{supp}(\delta_q(x)) \subset \overline{\mathbb{I}_j}$ *and* $\mathfrak{supp}(\sigma_q(x)) \subset \overline{\mathbb{I}_j}$;
(vii) *for every* $\gamma_j \in \Gamma \setminus \{\gamma_0, \gamma_\ell\}$, $\mathfrak{supp}(\delta_{\gamma_j}(x)) \subset \overline{\mathbb{I}_{j-1} \cup \mathbb{I}_j}$ *and* $\mathfrak{supp}(\sigma_{\gamma_j}(x)) \subset \overline{\mathbb{I}_{j-1} \cup \mathbb{I}_j}$;
(viii) $\mathfrak{supp}(\delta_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\mathfrak{supp}(\sigma_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\mathfrak{supp}(\delta_{\gamma_\ell}(x)) \subset \overline{\mathbb{I}_\ell}$ *and* $\mathfrak{supp}(\sigma_{\gamma_\ell}(x)) \subset \overline{\mathbb{I}_{\ell-1}}$;
(ix) *for every* $q \in [-\beta, \beta]$, $\mathfrak{supp}(\delta_q(x)) \subset \mathfrak{mon}(q)$ *and* $\mathfrak{supp}(\sigma_q(x)) \subset \mathfrak{mon}(q)$.

Proof (i) It is an immediate consequence of the definition of dual basis.
(ii) Since $\{\delta_q(x)\}_{q\in\Sigma}$ is the dual basis of $\{\sigma_q(x)\}_{q\in\Sigma}$ we have that

$$u(x) = \sum_{q\in\Sigma}\left(\int \delta_q(\xi)u(\xi)d\xi\right)\sigma_q(x) = \sum_{q\in\Sigma} u(q)\sigma_q(x).$$

(iii) It follows directly from the previous point.
(iv) If follows directly by Eq. (5).
(v) Given any point $q \in [-\beta, \beta]$ clearly there is a Delta basis $\{\delta_a(x)\}_{a\in\Sigma}$ with $q \in \Sigma$. Then $\sigma_q(x)$ can be defined by mean of the basis $\{\delta_a(x)\}_{a\in\Sigma}$. We have to prove that, given another Delta basis $\{\delta_a(x)\}_{a\in\Sigma'}$ with $q \in \Sigma'$, the corresponding $\sigma'_q(x)$ is equal to $\sigma_q(x)$. Using (ii), with $u(x) = \sigma'_q(x)$, we have that

$$\sigma'_q(x) = \sum_{a\in\Sigma} \sigma'_q(a)\sigma_a(x).$$

Then, by (iv), it follows that $\sigma'_q(x) = \sigma_q(x)$. So $\sigma_q(x)$ exists and is unique.
(vi) As we proved in Theorem 4.2, if $q \in \mathbb{I}_j$ then δ_q is an element of $V(\mathbb{I}_j)$, so $\mathfrak{supp}(\delta_q(x)) \subset \overline{\mathbb{I}_j}$. Now $\delta_q \in V(\mathbb{I}_j)$, so there is a corrispective function $\sigma_q \in V(\mathbb{I}_j)$ which is the sigma function centered in q. If we extend this function to $[-\beta, \beta]$ by posing $\sigma_q(x) = 0$ for $x \notin \mathbb{I}_j$ we obtain, by uniqueness, exactly the sigma function centered in q in $V_\Lambda(\mathbb{R})$. And $\mathfrak{supp}(\sigma_q(x)) \subset \overline{\mathbb{I}_j}$.
(vii) In Theorem 4.2 we proved that δ_{γ_j} is an element in $V(\mathbb{I}_j) \cup V(\mathbb{I}_{j+1})$, so $\mathfrak{supp}(\delta_{\gamma_j}(x)) \subset \overline{\mathbb{I}_{j-1} \cup \mathbb{I}_j}$. Now we can consider its corrispective sigma function $\sigma_{\gamma_j} \in V(\mathbb{I}_j) \cup V(\mathbb{I}_{j+1})$. If we extend this function to $V_\Lambda(\mathbb{R})$ by posing $\sigma_{\gamma_j}(x) = 0$ for $x \notin \mathbb{I}_j \cup \mathbb{I}_{j+1}$, we obtain the sigma function in $V_\Lambda(\mathbb{R})$ centered in γ_j. And, by construction, $\mathfrak{supp}(\sigma_{\gamma_j}(x)) \subset \overline{\mathbb{I}_j \cup \mathbb{I}_{j+1}}$.
(viii) In Theorem 4.2 we proved that δ_{γ_0} is an element in $V(\mathbb{I}_0)$ and δ_{γ_ℓ} is in $V(\mathbb{I}_{\ell-1})$, and that the same property holds for the corrispective σ functions that can be proved as in (vi). So $\mathfrak{supp}(\delta_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\mathfrak{supp}(\sigma_{\gamma_0}(x)) \subset \overline{\mathbb{I}_0}$, $\mathfrak{supp}(\delta_{\gamma_{\ell+1}}(x)) \subset \overline{\mathbb{I}_\ell}$ and $\mathfrak{supp}(\sigma_{\gamma_\ell}(x)) \subset \overline{\mathbb{I}_{\ell-1}}$.
(ix) It is a straightforward consequence of (vi) and (vii), since for every $j \in J$ we have $\mathbb{I}_j \cup \mathbb{I}_{j+1} \subset \mathfrak{mon}(q)$. □

5 Canonical extension of functions

We start by defining a map

$$\widetilde{(\cdot)} : [L^1_{loc}(\mathbb{R})]^* \to V_\Lambda(\mathbb{R})$$

which will be very useful in the extension of functions and distributions.

Definition 5.1 If $u \in [L^1_{loc}(\mathbb{R})]^*$, $\widetilde{u}$ denotes the unique ultrafunction such that

$$\forall v \in V_\Lambda(\mathbb{R}),\ \int^* \widetilde{u}(x)v(x)\mathrm{d}x = \int^* u(x)v(x)\mathrm{d}x.$$

Remark 5.2 Notice that, if $u \in \left[L^2(\mathbb{R})\right]^*$, then $\widetilde{u} = P_V u$ where

$$P_V : [L^2(\mathbb{R})]^* \to V_\Lambda(\mathbb{R})$$

is the orthogonal projection.

The following theorem proves that $\widetilde{u}$ exists and is unique for every $u \in L^1_{loc}(\mathbb{R})$ by showing the explicit expressions of $\widetilde{u}(x)$ in terms of sigma and delta bases:

Theorem 5.3 *If* $u \in \left[L^1_{loc}(\mathbb{R})\right]^*$ *then*

$$\widetilde{u}(x) = \sum_{q\in\Sigma}\left[\int u(\xi)\delta_q(\xi)d\xi\right]\sigma_q(x) \tag{7}$$

$$= \sum_{q\in\Sigma}\left[\int u(\xi)\sigma_q(\xi)d\xi\right]\delta_q(x). \tag{8}$$

Proof It is sufficient to prove that

$$\forall v \in V_\Lambda(\mathbb{R}),\ \int \sum_{q\in\Sigma}\left[\int u(\xi)\delta_q(\xi)d\xi\right]\sigma_q(x)v(x)\mathrm{d}x = \int u(\xi)v(\xi)d\xi.$$

We have that $v(x) = \sum_{q\in\Sigma} v_q\delta_q(x)$ with $v_q = \int \sigma_q(x)v(x)\mathrm{d}x$; then

$$\int \sum_{q\in\Sigma}\left[\int u(\xi)\delta_q(\xi)d\xi\right]\sigma_q(x)v(x)\mathrm{d}x = \sum_{q\in\Sigma}\left(\int u(\xi)\delta_q(\xi)d\xi\right)\left(\int \sigma_q(x)v(x)\mathrm{d}x\right)$$

$$= \sum_{q\in\Sigma}\left(\int u(\xi)\delta_q(\xi)d\xi\right)v_q = \int u(\xi)\left[\sum_{q\in\Sigma} v_q\delta_q(\xi)\right]d\xi = \int u(\xi)v(\xi)d\xi.$$

The other equalities can be proved similarly. So $\widetilde{u}$ exists and is unique for every $u \in [L^1_{loc}(\mathbb{R})]^*$. □

In particular, if $f \in L^1_{loc}(\mathbb{R})$, the function $\widetilde{f^*}$ is uniquely defined. From now on we will simplify the notation just writing $\widetilde{f}$.

Example 5.4 Take $|x|^{-1/2} \in L^1_{loc}(-1, 1)$, then

$$\widetilde{|x|^{-1/2}} = \sum_{q\in\Sigma}\left(\int^* |\xi|^{-1/2}\delta_q(\xi)d\xi\right)\sigma_q(x)$$

makes sense for every $x \in \mathbb{R}^*$; in particular

$$(\widetilde{|x|^{-1/2}})_{x=0} = \int^* |x|^{-1/2}\delta_0(x)\mathrm{d}x,$$

and it is easy to check that this is an infinite number. Notice that the ultrafunction $\widetilde{|x|^{-1/2}}$ is different from $(|x|^{-1/2})^*$ since the latter is not defined for $x = 0$ (and they also differ for $|x| > \beta$).

Now we want to show some interesting relations between $\widetilde{f}$ and f^*. More precisely we are interested in the following question.

Take $f \in L^1_{loc}(\mathbb{R})$ and $\Omega \subset \mathbb{R}$, which are the conditions that ensure the following:

$$\forall x \in \Omega^*, \ \widetilde{f}(x) = f^*(x)? \tag{Q}$$

Notice that the above equality must be intended for almost every x.

Lemma 5.5 *Let $\Omega \subset \mathbb{R}$ be an open set and let $f \in L^1_{loc}(\mathbb{R})$. Then*

$$\forall^{a.e.} x \in \Omega \ \ f(x) = 0 \Leftrightarrow \forall x \in \Omega^* \cap [-\beta, \beta] \ \ \widetilde{f}(x) = 0.$$

Proof We recall that, by (7),

$$\widetilde{f}(x) = \sum_{q \in \Sigma} \left[\int f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(x).$$

If $\forall^{a.e.} x \in \Omega \ f(x) = 0$ then by Leibnitz principle we have that $\forall^{a.e.} x \in \Omega^* \ f^*(x) = 0$, so that $\forall x \in \Omega^* \cap [-\beta, \beta] \ \widetilde{f}(x) = 0$ follows by (7). Conversely, since $f^*(x)$ is an ultrafunction for every $g \in \mathcal{C}^\infty_0(\Omega)$, we have

$$\begin{aligned} 0 = \int_{\Omega^*} \widetilde{f}(x) g^*(x) \mathrm{d}x &= \int_{\Omega^*} f^*(x) g^*(x) \\ &= \int_{\Omega} f(x) g(x) \mathrm{d}x, \end{aligned}$$

so $f(x) = 0 \ \forall^{a.e.} x \in \Omega$. □

Corollary 5.6 *Let $\Omega \subset \mathbb{R}$ be an open set and let $f, g \in L^1_{loc}(\mathbb{R})$, then*

$$\forall^{a.e.} x \in \Omega \ \ f(x) = g(x) \Leftrightarrow \forall x \in \Omega^* \cap [-\beta, \beta] \ \ \widetilde{f}(x) = \widetilde{g}(x).$$

Proof We apply the previous theorem to the function $h(x) = f(x) - g(x)$ and use that the operation $f \to \widetilde{f}$ is linear. □

Theorem 5.7 *Let $\Omega \subset \mathbb{R}$ be an open bounded set, let $f \in L^1_{loc}(\mathbb{R})$, if $f_{\restriction\Omega} \in \mathcal{C}^1(\Omega)$ then*

$$\forall x \in \Omega^* \cap [-\beta, \beta] \ \ \widetilde{f}(x) = f^*(x).$$

Proof Let $\{\delta_a(x)\}_{a \in \Sigma}$ be a Delta basis, let $y \in \Omega^*$ and let $y \in \mathbb{I}_j$. Since, by Theorem (4.5), for every $q \in \Sigma$ with $q \notin \mathbb{I}_j \ \sigma_q(y) = 0$, by (7) we deduce that

$$\widetilde{f}(y) = \sum_{q \in \Sigma \cap \mathbb{I}_j} \left[\int_{\mathbb{I}_j} f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(y).$$

Now let $g_j(x)$ be the function such that

$$g_j(x) = \begin{cases} f^*(x) & \text{if } x \in \mathbb{I}_j \,; \\ 0 & \text{otherwise.} \end{cases}$$

Since $f_{\restriction\Omega} \in \mathcal{C}^1(\Omega)$ then $g_j(x)$ is an ultrafunction. By construction, we have that $g_j(y) = \widetilde{f}(y)$ since, by (6),

$$\begin{aligned} g_j(y) = \sum_{q \in \Sigma} g_j(q) \sigma_q(y) &= \sum_{q \in \Sigma \cap \mathbb{I}_j} \left[\int_{\mathbb{I}_j} g_j(\xi) \delta_q(\xi) d\xi \right] \sigma_q(y) \\ &= \sum_{q \in \Sigma \cap \mathbb{I}_j} \left[\int_{\mathbb{I}_j} f^*(\xi) \delta_q(\xi) d\xi \right] \sigma_q(y) = \widetilde{f}(y). \end{aligned}$$

But, by definition, $g_j(y) = f^*(y)$; hence, we deduce that $f^*(y) = \widetilde{f}(y)$. □

Example 5.8 If $f(x) = 1$, then

$$\widetilde{1}(x) = \begin{cases} 1 & if\ x \in [-\beta, \beta]_{\mathbb{R}^*}; \\ 0 & if\ x \notin [-\beta, \beta]_{\mathbb{R}^*}. \end{cases}$$

By Theorem 5.7 and the above example, we get:

Corollary 5.9 *Let $f \in \mathcal{C}^1(\Omega)$, then,*

$$\widetilde{f} = f^* \cdot \widetilde{1}$$

By Theorem 5.7, given a function $f(x) \in \mathcal{C}^1(\mathbb{R})$ we have that $\widetilde{f}(x)$ extends $f(x)$ to $[-\beta, \beta]_{\mathbb{R}^*}$. $\widetilde{f}(x)$ will be called the **canonical extension of** $f(x)$. With some abuse of notation, $\widetilde{f}(x)$ will be called the "canonical extension of $f(x)$" even when $f(x) \in L^1_{loc}(\mathbb{R})$.

Example 5.10 If we consider the Example 5.4, by Theorem 5.7, we have that

$$\forall a \in [-\beta, \beta] \backslash \mathfrak{mon}(0),\ (\widetilde{|x|^{-1/2}})_{x=a} = (|x|^{-1/2})^*_{x=a} = |a|^{-1/2}.$$

Example 5.11 For a fixed $k \in \mathbb{R}$, the function e^{ikx} defines a unique ultrafunction $\widetilde{e^{ikx}}$. Notice that $\widetilde{e^{ikx}}$ is different from the natural extension of e^{ikx} even if

$$\forall x \in \mathfrak{gal}(0),\ \widetilde{e^{ikx}} = e^{ikx}.$$

6 Derivative

Definition 6.1 For every ultrafunction $u \in V_\Lambda(\mathbb{R})$, the derivative $Du(x)$ of $u(x)$ is the ultrafunction defined by the following formula:

$$Du(x) = P_V u'(x) + \sum_{j=1}^{\ell-1} \triangle u(\gamma_j)\delta_{\gamma_j}(x), \tag{9}$$

where $P_V u'(x)$ denotes the orthogonal projection of $u'(x)$ on $V_\Lambda(\mathbb{R})$ w.r.t. the L^2 scalar product and, for every $j = 1, \ldots, l-1$,

$$\triangle u(\gamma_j) = u^+(\gamma_j) - u^-(\gamma_j).$$

Let us observe that the derivative operator is almost-local in the following sense: if $u(x) \in V_\Lambda(\mathbb{R})$ is an ultrafunction and there are $0 < i < j < l-1 \in J$ such that $supp(u(x)) \subseteq \bigcup_{i \le k \le j} \mathbb{I}_k$, then $supp(D(u(x))) \subseteq \bigcup_{i-1 \le k \le j+1} \mathbb{I}_k$, so the support of the derivative of $u(x)$ can be larger than the support of $u(x)$ but only of an infinitesimal.

Theorem 6.2 *For every $u, v \in V_\Lambda(\mathbb{R})$ the following equality holds:*

$$\int Du(x)v(x)\,\mathrm{d}x = -\int u(x)Dv(x)\,\mathrm{d}x + [u(x)v(x)]_{-\beta}^{\beta}. \tag{10}$$

Proof We have:

$$\begin{aligned}
&\int (Du(x)v(x) + u(x)Dv(x))\mathrm{d}x \\
&= \int \left(P_V u'(x) + \sum_{j=1}^{\ell-1} \triangle u(\gamma_j)\delta_{\gamma_j}(x) \right) v(x)\mathrm{d}x + \int \left(P_V v'(x) + \sum_{j=1}^{\ell-1} \triangle v(\gamma_j)\delta_{\gamma_j}(x) \right) u(x)\mathrm{d}x \\
&= \int \left[P_V u'(x)v(x) + u(x)P_V v'(x) \right] \mathrm{d}x
\end{aligned}$$

$$+\int\left[\left(\sum_{j=0}^{\ell-1}\triangle u(\gamma_j)\delta_{\gamma_j}(x)\right)v(x)+\left(\sum_{j=0}^{\ell-1}\triangle v(\gamma_j)\delta_{\gamma_j}(x)\right)u(x)\right]\mathrm{d}x$$

$$=\int\left[P_V u'(x)v(x)+u(x)P_V v'(x)\right]\mathrm{d}x+\sum_{j=1}^{\ell-1}\left[\triangle u(\gamma_j)v(\gamma_j)+\triangle v(\gamma_j)u(\gamma_j)\right].$$

Now let us compute the two terms of the sum separately; the first one:

$$\int\left[P_V u'(x)v(x)+u(x)P_V v'(x)\right]\mathrm{d}x=\sum_{j=0}^{\ell-1}\int_{\gamma_j}^{\gamma_{j+1}}\left[P_V u'(x)v(x)+u(x)P_V v'(x)\right]\mathrm{d}x$$

$$=\sum_{j=0}^{\ell-1}\int_{\gamma_j}^{\gamma_{j+1}}\left[u'(x)v(x)+u(x)v'(x)\right]\mathrm{d}x=\sum_{j=0}^{\ell-1}\int_{\gamma_j}^{\gamma_{j+1}}(u(x)v(x))'\,\mathrm{d}x$$

$$=\sum_{j=0}^{\ell-1}\left[u^-(\gamma_{j+1})v^-(\gamma_{j+1})-u^+(\gamma_j)v^+(\gamma_j)\right].$$

The second one:

$$\sum_{j=1}^{\ell-1}\left[\triangle u(\gamma_j)v(\gamma_j)+\triangle v(\gamma_j)u(\gamma_j)\right]$$

$$=\sum_{j=1}^{\ell-1}\left((u^+(\gamma_j)-u^-(\gamma_j))\left(\frac{v^+(\gamma_j)+v^-(\gamma_j)}{2}\right)+(v^+(\gamma_j)-v^-(\gamma_j))\left(\frac{u^+(\gamma_j)+u^-(\gamma_j)}{2}\right)\right)$$

$$=\sum_{j=1}^{\ell-1}(u^+(\gamma_j)v^+(\gamma_j)-u^-(\gamma_j)v^-(\gamma_j)).$$

Thus

$$\int\left[P_V u'v(x)+u(x)P_V v'\right]\mathrm{d}x+\sum_{j=0}^{\ell-1}\left(\triangle u(\gamma_j)v(\gamma_j)+\triangle v(\gamma_j)u(\gamma_j)\right)$$

$$=\sum_{j=0}^{\ell-1}\left(u^-(\gamma_{j+1})v^-(\gamma_{j+1})-u^+(\gamma_j)v^+(\gamma_j)\right)+\sum_{j=1}^{\ell-1}(u^+(\gamma_j)v^+(\gamma_j)-u^-(\gamma_j)v^-(\gamma_j)).$$

But $\sum_{j=0}^{\ell-1}\left(u^-(\gamma_{j+1})v^-(\gamma_{j+1})-u^+(\gamma_j)v^+(\gamma_j)\right) = -u(-\beta)v(-\beta) + u(\beta)v(\beta) + \sum_{j=1}^{\ell-1}\left(u^-(\gamma_j)v^-(\gamma_j)\right.$ $\left.-u^+(\gamma_j)v^+(\gamma_j)\right)$, hence

$$\sum_{j=0}^{\ell-1}\left(u^-(\gamma_{j+1})v^-(\gamma_{j+1})-u^+(\gamma_j)v^+(\gamma_j)\right)+\sum_{j=1}^{\ell-1}(u^+(\gamma_j)v^+(\gamma_j)-u^-(\gamma_j)v^-(\gamma_j))$$
$$=u(\beta)v(\beta)-u(-\beta)v(-\beta).$$

□

Remark 6.3 The generalized derivative

$$D:V_\Lambda(\mathbb{R})\to V_\Lambda(\mathbb{R})$$

is a linear operator, as can be directly derived by (9). Moreover for every ultrafunction $u\in V_\Lambda(\mathbb{R})\cap\mathcal{C}^1(\mathbb{R})^*$ we have that

$$Du(x)=\widetilde{u'(x)}, \tag{11}$$

since in this case $\triangle u(\gamma_j) = 0$ for every $j = 1, \ldots, l-1$. In particular, if $f \in \mathcal{C}^2(\mathbb{R})$ then $\forall x \in [-\beta, \beta]$

$$Df^*(x) = \left(f'\right)^*(x), \tag{12}$$

because in this case $\left(f'\right)^*(x) \in V_\Lambda(\mathbb{R})$, so $P_V\left(f'\right)^* = \left(f'\right)^*$.

Remark 6.4 Notice that by (11) and (12) we have that $\forall f \in \mathcal{C}^1(\mathbb{R})$ and $\forall x \in \mathbb{R}$

$$D\widetilde{f}(x) = \widetilde{f'(x)} \sim f'(x)$$

and $\forall f \in \mathcal{C}^2(\mathbb{R})$ and $\forall x \in \mathbb{R}$

$$D\widetilde{f}(x) = f'(x).$$

In this sense, D extends the usual derivative to all ultrafunctions and to all the points in $\mathbb{R}^*$.

Example 1 By (10) we have that

$$D\widetilde{1} = 0. \tag{13}$$

If $u(x) = \widetilde{x}$ then

$$D\widetilde{x} = \widetilde{1}.$$

Example 2 If $a \neq -\beta, b \neq \beta$ and $u(x) = \chi_{[a,b]}(x)$, then

$$D\chi_{[a,b]} = \delta_a - \delta_b.$$

Example 3 If $a = -\beta, b \neq \beta$ and $u(x) = \chi_{[a,b]}(x)$, then

$$D\chi_{[a,b]} = -\delta_b,$$

and if $a \neq -\beta, b = \beta$ and $u(x) = \chi_{[a,b]}(x)$, then

$$D\chi_{[a,b]} = \delta_a.$$

Example 4 if $u(x) = w(x)\chi_{[a,b]}(x)$ with $a, b \in \Gamma \backslash \{-\beta, \beta\}$, then by (9)

$$u(x)' = P_V w'(x)\chi_{[a,b]}(x) + w(a)\delta_a(x) - w(b)\delta_b(x).$$

7 Definite integral

Since every ultrafunction is an internal function, the definite integral is well defined:

$$\int_a^b u(x)\mathrm{d}x := \left(\int_a^b\right)^* u(x)\mathrm{d}x.$$

Let us observe that, for every $a, b \in \Gamma$, the characteristic function $\chi_{[a,b]}$ of $[a, b]$ in the usual sense and the characteristic function $\chi_{[a,b]_{\mathbb{R}^*}}$ of $[a, b]$ in the sense of ultrafunctions are different (at most) only in the points a and b. In particular, for every ultrafunction $u(x)$ we have

$$\int_a^b u(x)\mathrm{d}x = \int^* u(x)\chi_{[a,b]}(x)\mathrm{d}x = \int^* u(x)\chi_{[a,b]_{\mathbb{R}^*}}(x)\mathrm{d}x.$$

This observation is important to prove the following theorem:

Corollary 7.1 (Fundamental Theorem of Calculus) *If* $a, b \in \Gamma$, *then*

$$\int_a^b Du(x)\mathrm{d}x = u(b) - u(a).$$

Proof We have:

$$\int_a^b Du(x)\mathrm{d}x = \int^* Du(x)\chi_{[a,b]}(x)\mathrm{d}x$$
$$= \int^* Du(x)\chi_{[a,b]_{\mathbb{R}^*}}(x)\mathrm{d}x = -\int u(x)D\chi_{[a,b]_{\mathbb{R}^*}}(x)\mathrm{d}x + \left[u(x)\chi_{[a,b]_{\mathbb{R}^*}}\right]_{-\beta}^{\beta}.$$

Now if $a \neq -\beta, b \neq \beta$ we have $\left[u(x)\chi_{[a,b]_{\mathbb{R}^*}}\right]_{-\beta}^{\beta} = 0$ and $D\chi_{[a,b]_{\mathbb{R}^*}}(x) = \delta_a - \delta_b$, so

$$-\int u(x)D\chi_{[a,b]_{\mathbb{R}^*}}(x)\mathrm{d}x = -\int u(x)(\delta_a - \delta_b)\mathrm{d}x$$
$$= u(b) - u(a).$$

If $a = -\beta, b \neq \beta$ we have $\left[u(x)\chi_{[a,b]_{\mathbb{R}^*}}\right]_{-\beta}^{\beta} = -u(-\beta)$ and $D\chi_{[a,b]}(x) = -\delta_b$, so

$$-\int u(x)D\chi_{[a,b]_{\mathbb{R}^*}}(x)\mathrm{d}x - u(-\beta) = -\int u(x)(-\delta_b)\mathrm{d}x - u(-\beta)$$
$$= u(b) - u(-\beta) = u(b) - u(a).$$

The case $a \neq -\beta, b = \beta$ can be proved similarly. If $a = -\beta, b = \beta$ then

$$\int Du(x)\chi_{[-\beta,\beta]}(x)\mathrm{d}x = \int Du(x)\widetilde{1}\mathrm{d}x$$
$$= -\int u(x)D\widetilde{1}\mathrm{d}x + [u(x)]_{-\beta}^{\beta} = u(\beta) - u(-\beta),$$

since $D\widetilde{1} = 0$. □

We assumed that $\mathbb{R} \subset \Gamma$; thus if $f \in \mathcal{C}^1(\mathbb{R})$ we have that, $\forall a, b \in \mathbb{R}$,

$$\int_a^b D\widetilde{f}(x)\mathrm{d}x = f(b) - f(a).$$

A question that arises is: does it hold, for ultrafunctions, some kind of "rule of integration by parts for continuous functions", at least for the points in Γ? E.g., is it true that, if $u, v \in V_\Lambda(\mathbb{R})$ and $a, b \in \Gamma$, then

$$\int_a^b Du(x)v(x)\ \mathrm{d}x = -\int_a^b u(x)Dv(x)\ \mathrm{d}x + [u(x)v(x)]_a^b? \tag{14}$$

The answer is no, as a simple computation shows. Nevertheless, we have the following:

Proposition 7.2 *Let* $u, v \in V_\Lambda(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})^*$, *and* $\gamma_n < \gamma_m \in \Gamma$. *Then*

$$\int_{\gamma_n}^{\gamma_m} Du(x)v(x)\ \mathrm{d}x = -\int_{\gamma_n}^{\gamma_m} u(x)Dv(x)\ \mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n).$$

Proof By (11), since $u, v \in V_\Lambda(\mathbb{R}) \cap \mathcal{C}^1(\mathbb{R})^*$ then $Du = \widetilde{u'}$ and $Dv = \widetilde{v'}$. Moreover, since $V_\Lambda(\mathbb{R}) = \bigoplus_{j=0}^{l-1} V(\mathbb{I}_j)$, if for every $j = 0, \ldots, l-1$ we denote by P_j the orthogonal projection on $\mathbb{I}_j$ we have

$$P_V u'(x) = \sum_{j=0}^{l-1} P_j(u'(x)).$$

Now, if $m = n + 1$, since u and v are continuous we have

$$\begin{aligned}
&\int_{\gamma_n}^{\gamma_m} Du(x)v(x)\,\mathrm{d}x = \int_{\gamma_n}^{\gamma_m} P_V u'(x)v(x)\,\mathrm{d}x \\
&= \int_{\gamma_n}^{\gamma_m} P_n u'(x)v(x)\,\mathrm{d}x = \int_{\gamma_n}^{\gamma_m} u'(x)v(x)\,\mathrm{d}x \\
&= -\int_{\gamma_n}^{\gamma_m} u(x)v'(x)\mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) \\
&= -\int_{\gamma_n}^{\gamma_m} u(x)P_j v'(x)\mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) \\
&= -\int_{\gamma_n}^{\gamma_m} u(x)Dv(x)\mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n).
\end{aligned}$$

In the general case,

$$\begin{aligned}
&\int_{\gamma_n}^{\gamma_m} Du(x)v(x)\,\mathrm{d}x = \sum_{i=n}^{m-1} \int_{\gamma_i}^{\gamma_{i+1}} Du(x)v(x)\,\mathrm{d}x \\
&= \sum_{i=n}^{m-1} \left[-\int_{\gamma_i}^{\gamma_{i+1}} u(x)Dv(x)\mathrm{d}x + u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i) \right],
\end{aligned}$$

and since u, v are continuous we have

$$\begin{aligned}
&\sum_{i=n}^{m-1} \left[-\int_{\gamma_i}^{\gamma_{i+1}} u(x)Dv(x)\mathrm{d}x + u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i) \right] \\
&= \sum_{i=n}^{m-1} \left[-\int_{\gamma_i}^{\gamma_{i+1}} u(x)Dv(x)\mathrm{d}x \right] + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) \\
&= -\int_{\gamma_n}^{\gamma_m} u(x)Dv(x)\,\mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n).
\end{aligned}$$

□

The previous proposition is, in general, false if at least one between u, v is not in $\mathcal{C}^1(\mathbb{R})^*$.

Example: Let $0 = \gamma_i$, and let us consider the ultrafunction

$$u(x) = \begin{cases} 1 & x < 0; \\ 0 & x \in \mathbb{I}_i; \\ 1 & x \in \mathbb{I}_{i+1}; \\ 0 & x > \gamma_{i+2}. \end{cases}$$

Let $v(x) = x$. Then

$$\begin{aligned}
&\int_{\gamma_i}^{\gamma_{i+2}} Du(x)v(x)\,\mathrm{d}x = \int_{\gamma_i}^{\gamma_{i+2}} \left[-\delta_{\gamma_i}^+(x) + \delta_{\gamma_{i+1}}(x) - \delta_{\gamma_{i+2}}^-(x) \right] v(x)\mathrm{d}x \\
&= -\gamma_i + \gamma_{i+1} - \gamma_{i+2} \neq -\int_{\gamma_i}^{\gamma_{i+2}} u(x)Dv(x)\,\mathrm{d}x + u^-(\gamma_{i+2})v^-(\gamma_{i+2}) - u^+(\gamma_i)v^+(\gamma_i) \\
&= \gamma_{i+1} - \gamma_{i+2} + \gamma_{i+2} = \gamma_{i+1}.
\end{aligned}$$

Just for sake of completeness, we now show how to obtain a relaxed version of (14) by considering a different notion of derivative on $V_\Lambda(\mathbb{R})$. The relaxed version of (14) is the following: since the functions in

$V_\Lambda(\mathbb{R})$ are piecewise $\mathcal{C}^1$ functions, does it hold, for ultrafunctions, an analog of the rule of integration by parts for piecewise $\mathcal{C}^1$ functions? Namely, is it true that, if $u, v \in V_\Lambda(\mathbb{R})$ and $\gamma_n < \gamma_m \in \Gamma$, then

$$\int_{\gamma_n}^{\gamma_m} Du(x)v(x)\,\mathrm{d}x = -\int_{\gamma_n}^{\gamma_m} u(x)Dv(x)\,\mathrm{d}x + \sum_{i=n}^{m-1}\left[u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i)\right]? \tag{15}$$

With the operator D the answer is no. But there is a different linear operator that actually satisfies (15):

Definition 7.3 We denote by $D_2u(x)$ the linear operator such that, for every $u \in V_\Lambda(\mathbb{R})$, we have

$$D_2u(x) = P_V(u'(x)).$$

Since $V_\Lambda(\mathbb{R}) = \bigoplus_{j=0}^{l-1} V(\mathbb{I}_j)$, if we denote by P_j the orthogonal projection on $\mathbb{I}_j$, we have

$$D_2u(x) = P_V u'(x) = \sum_{j=0}^{l-1} P_j(u'(x)).$$

Moreover we have that if $u(x)$ is continuous in γ_j, γ_{j+1}, then

$$Du(x) = D_2u(x)$$

on $\mathbb{I}_j$. In particular, if $u(x)$ is continuous in $[-\beta, \beta]$ then

$$Du(x) = D_2u(x).$$

This new linear operator is what we need to obtain the generalization to $V_\Lambda(\mathbb{R})$ of the rule of integration by parts for piecewise continuous functions:

Theorem 7.4 (Integration by parts for piecewise $\mathcal{C}^1$functions) *For every $u, v \in V_\Lambda(\mathbb{R})$ and $\gamma_n < \gamma_m \in \Gamma$ we have*

$$\int_{\gamma_n}^{\gamma_m} D_2u(x)v(x)\,\mathrm{d}x = -\int_{\gamma_n}^{\gamma_m} u(x)D_2v(x)\,\mathrm{d}x + \sum_{i=n}^{m-1}\left[u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i)\right].$$

Proof If $m = n + 1$ then

$$\begin{aligned}\int_{\gamma_n}^{\gamma_m} D_2u(x)v(x)\,\mathrm{d}x &= \int_{\gamma_n}^{\gamma_m} u'(x)v(x)\,\mathrm{d}x \\ &= -\int_{\gamma_n}^{\gamma_m} u(x)v'(x)\mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n) \\ &= -\int_{\gamma_n}^{\gamma_m} u(x)D_2v(x)\mathrm{d}x + u^-(\gamma_m)v^-(\gamma_m) - u^+(\gamma_n)v^+(\gamma_n).\end{aligned}$$

In the general case we have

$$\begin{aligned}\int_{\gamma_n}^{\gamma_m} D_2u(x)v(x)\,\mathrm{d}x &= \sum_{i=n}^{m-1}\int_{\gamma_i}^{\gamma_{i+1}} D_2u(x)v(x)\,\mathrm{d}x \\ &= \sum_{i=n}^{m-1}\left(-\int_{\gamma_i}^{\gamma_{i+1}} u(x)D_2v(x)\mathrm{d}x + u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i)\right) \\ &= -\int_{\gamma_n}^{\gamma_m} u(x)D_2v(x)\,\mathrm{d}x + \sum_{i=n}^{m-1}\left[u^-(\gamma_{i+1})v^-(\gamma_{i+1}) - u^+(\gamma_i)v^+(\gamma_i)\right].\end{aligned}$$

□

In particular, since $D_2\widetilde{1} = 0$, it is immediate to prove that the following holds:

Corollary 7.5 (Fundamental Theorem of Calculus for piecewise continuous functions) *For every* $u \in V_\Lambda(\mathbb{R})$ *and* $\gamma_n < \gamma_m \in \Gamma$ *we have*

$$\int_{\gamma_n}^{\gamma_m} D_2 u(x)\mathrm{d}x = \sum_{i=n}^{m-1} \left[u^-(\gamma_{i+1}) - u^+(\gamma_i)\right].$$

Of course, the derivative D_2 has also many drawbacks, e.g., for every grid function g we have $D_2(g) = 0$. So in the following we will only consider the derivative D.

8 Ultrafunctions and distributions

In this section we briefly explain how to associate an ultrafunction to every distribution $T \in \mathcal{C}^{-\infty}(\mathbb{R})$, where

$$\mathcal{C}^{-\infty}(\mathbb{R}) = \{T \in \mathcal{D}'(\mathbb{R}) \mid \exists k \in \mathbb{N}, \exists f \in \mathcal{C}^0(\mathbb{R}) \text{ such that } T = \partial^k f\}.$$

Note that, by definition, if $T \in \mathcal{C}^{-\infty}(\mathbb{R})$ then there exists a natural number k and a function $f \in \mathcal{C}^1(\mathbb{R})$ such that:

$$T = \partial^k f. \tag{16}$$

So it is natural to introduce the following definition:

Definition 8.1 Given a distribution $T \in \mathcal{C}^{-\infty}(\mathbb{R})$, let k be the minimum natural number such that there exists $f \in \mathcal{C}^1(\mathbb{R})$ with $T = \partial^k f$. We denote by $\widetilde{T}$ the ultrafunction

$$\widetilde{T}(x) = D^k f^*.$$

$\widetilde{T}$ will be called the ultrafunction associated with the distribution T.

Proposition 8.2 *For every distribution* $T \in \mathcal{C}^{-\infty}(\mathbb{R})$, *for every test function* $\varphi \in \mathcal{D}(\mathbb{R})$ *we have*

$$\int^* \widetilde{T}(x)\varphi^*(x)\mathrm{d}x = \langle T, \varphi\rangle.$$

Proof Let us suppose that $T = \partial^k f$, where k, f are given as in Definition 8.1. Then, by (10), since $\varphi^*(\beta) = \varphi^*(-\beta) = 0$, we have that

$$\begin{aligned}\int^* \widetilde{T}(x)\varphi^*(x)\mathrm{d}x &= \int^* D^k f^*(x)\varphi^*(x)\mathrm{d}x = (-1)^k \int^* f^*(x)\partial^k\varphi^*(x)\mathrm{d}x \\ &= \left[(-1)^k \int f(x)\partial^k\varphi(x)\mathrm{d}x\right]^* = \langle T, \varphi\rangle^* = \langle T, \varphi\rangle.\end{aligned}$$

□

In [10] we showed that, actually, it is possible to define an embedding of the whole space of distributions in a particular algebra of ultrafunctions. Our embedding preserves the pointwise moltiplication of $\mathcal{C}^1$ functions; we do not contrast with Schwartz impossibility result because the derivative of ultrafunctions satisfies only a weak form of the Leibniz rule.

Acknowledgments The authors would like to thank the referee for his careful reading of the paper. Lorenzo Luperi Baglini was supported by grant P25311-N25 of the Austrian Science Fund FWF.

9 Appendix: Λ-theory

In this section we present the basic notions of non-Archimedean mathematics and of non-standard analysis following a method inspired by [4] (see also [1] and [8]).

Non-Archimedean fields

Here, we recall the basic definitions and facts regarding non-Archimedean fields. In the following, $\mathbb{K}$ will denote an ordered field. We recall that such a field contains (a copy of) the rational numbers. Its elements will be called numbers.

Definition 9.1 Let $\mathbb{K}$ be an ordered field. Let $\xi \in \mathbb{K}$. We say that:

- ξ is infinitesimal if, for all positive $n \in \mathbb{N}$, $|\xi| < \frac{1}{n}$;
- ξ is finite if there exists $n \in \mathbb{N}$ such as $|\xi| < n$;
- ξ is infinite if, for all $n \in \mathbb{N}$, $|\xi| > n$ (equivalently, if ξ is not finite).

Definition 9.2 An ordered field $\mathbb{K}$ is called non-Archimedean if it contains an infinitesimal $\xi \neq 0$.

It is easily seen that all infinitesimal are finite, that the inverse of an infinite number is a nonzero infinitesimal number, and that the inverse of a nonzero infinitesimal number is infinite.

Definition 9.3 A superreal field is an ordered field $\mathbb{K}$ that properly extends $\mathbb{R}$.

It is easy to show, due to the completeness of $\mathbb{R}$, that there are nonzero infinitesimal numbers and infinite numbers in any superreal field. Infinitesimal numbers can be used to formalize a new notion of "closeness":

Definition 9.4 We say that two numbers $\xi, \zeta \in \mathbb{K}$ are infinitely close if $\xi - \zeta$ is infinitesimal. In this case, we write $\xi \sim \zeta$.

Clearly, the relation "$\sim$" of infinite closeness is an equivalence relation.

Theorem 9.5 *If $\mathbb{K}$ is a superreal field, every finite number $\xi \in \mathbb{K}$ is infinitely close to a unique real number $r \sim \xi$, called the **shadow** or the **standard part** of ξ.*

Given a finite number ξ, we denote its shadow as $sh(\xi)$, and we put $sh(\xi) = +\infty$ ($sh(\xi) = -\infty$) if $\xi \in \mathbb{K}$ is a positive (negative) infinite number.

Definition 9.6 Let $\mathbb{K}$ be a superreal field, and $\xi \in \mathbb{K}$ a number. The monad of ξ is the set of all numbers that are infinitely close to it:

$$\mathfrak{mon}(\xi) = \{\zeta \in \mathbb{K} : \xi \sim \zeta\},$$

and the galaxy of ξ is the set of all numbers that are finitely close to it:

$$\mathfrak{gal}(\xi) = \{\zeta \in \mathbb{K} : \xi - \zeta \text{ is finite}\}$$

By definition, it follows that the set of infinitesimal numbers is $\mathfrak{mon}(0)$ and that the set of finite numbers is $\mathfrak{gal}(0)$.

The Λ-limit

In this section we will introduce a superreal field $\mathbb{K}$ and will analyze its main properties by mean of the Λ-theory (see also [1,8]).

We set

$$\mathfrak{X} = \mathcal{P}_{fin}(\mathfrak{F}(\mathbb{R}, \mathbb{R}));$$

we will refer to $\mathfrak{X}$ as the "parameter space". Clearly $(\mathfrak{X}, \subset)$ is a directed set and, as usual, a function $\varphi : \mathfrak{X} \to E$ will be called *net* (with values in E). So nets are a natural generalization of the notion of sequence, and the $\Lambda-$limit is, in some sense, a generalization of the notion of "limit" of a sequence.

We present axiomatically the notion of Λ-limit:

Axioms of the Λ-limit

- (Λ-1) **Existence Axiom.** *There is a superreal field* $\mathbb{K} \supset \mathbb{R}$ *such that every net* $\varphi : \mathfrak{X} \to \mathbb{R}$ *has a unique limit* $L \in \mathbb{K}$ *(called the* "Λ-limit" *of* φ.*) The* Λ*-limit of* φ *will be denoted as*

$$L = \lim_{\lambda \uparrow \Lambda} \varphi(\lambda).$$

Moreover we assume that every $\xi \in \mathbb{K}$ *is the* Λ*-limit of some real function* $\varphi : \mathfrak{X} \to \mathbb{R}$.
- (Λ-2) **Real numbers axiom**. *If* $\varphi(\lambda)$ *is eventually constant, namely* $\exists \lambda_0 \in \mathfrak{X}, r \in \mathbb{R}$ such that $\forall \lambda \supset \lambda_0,\ \varphi(\lambda) = r$, *then*

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = r.$$

- (Λ-3) **Sum and product Axiom**. *For all* $\varphi, \psi : \mathfrak{X} \to \mathbb{R}$*:*

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) + \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) + \psi(\lambda))\,;$$
$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) \cdot \lim_{\lambda \uparrow \Lambda} \psi(\lambda) = \lim_{\lambda \uparrow \Lambda} (\varphi(\lambda) \cdot \psi(\lambda))\,.$$

Theorem 9.7 *The set of axioms* {*(Λ-1),(Λ-2),(Λ-3)*} *is consistent.*

In this paper the word "limit" will be used and should be intended, only as a "suggestive" terminology for our constructions.[7] It is possible to show[8] that the Λ-limit is related to a notion of convergence (to be more precise, it is the projection of a topological limit), but we will not adopt this topological point of view in this paper.

Theorem 9.7 is proved in [1] and in [9]. The Λ-limit can be extended to more general nets; to this aim, we recall that the superstructure on $\mathbb{R}$ is defined as follows:

$$\mathbb{U} = \bigcup_{n=0}^{\infty} \mathbb{U}_n$$

where $\mathbb{U}_n$ is defined by induction as follows:

$$\mathbb{U}_0 = \mathbb{R};$$
$$\mathbb{U}_{n+1} = \mathbb{U}_n \cup \mathcal{P}(\mathbb{U}_n).$$

Here $\mathcal{P}(E)$ denotes the power set of E. Identifying the couples with the Kuratowski pairs and the functions and the relations with their graphs, it follows that $\mathbb{U}$ contains almost every usual mathematical object.

We can extend the definition of the Λ-limit to any bounded net of mathematical objects in $\mathbb{U}$ (a net $\varphi : \mathfrak{X} \to \mathbb{U}$ is bounded if there exists n such that $\forall \lambda \in \mathfrak{X}, \varphi(\lambda) \in \mathbb{U}_n$). To this aim, consider a net

$$\varphi : \mathfrak{X} \to \mathbb{U}_n. \tag{17}$$

We will define $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$ by induction on n. For $n = 0$, $\lim_{\lambda \uparrow \Lambda} \varphi(\lambda)$ is defined by the axioms (Λ- 1),(Λ-2),(Λ-3); so by induction we may assume that the limit is defined for $n-1$ and define it for the net (17) as follows:

$$\lim_{\lambda \uparrow \Lambda} \varphi(\lambda) = \left\{ \lim_{\lambda \uparrow \Lambda} \psi(\lambda) \mid \psi : \mathfrak{X} \to \mathbb{U}_{n-1} \text{ and } \forall \lambda \in \mathfrak{X},\ \psi(\lambda) \in \varphi(\lambda) \right\}.$$

Definition 9.8 A mathematical entity (number, set, function or relation) which is the Λ-limit of a net is called **internal**.

We explicitate the definition of Λ-limit for nets of subsets of $\mathbb{R}$ and functions in $\mathfrak{F}(\mathbb{R}, \mathbb{R})$:

[7] In fact, it is easy to prove that $\mathbb{K}$ is isomorphic to $\mathbb{R}^{\mathfrak{X}}/\sim$, where we set $\varphi_1 \sim \varphi_2 \Leftrightarrow \lim_{\lambda \uparrow \Lambda} \varphi_1(\lambda) = \lim_{\lambda \uparrow \Lambda} \varphi_2(\lambda)$. This identification can be used to prove that, in what follows, all our definitions do not depend on the choice of representatives.

[8] We plan to do this in a forthcoming paper.

Definition 9.9 The **natural extension** of a set $E \subset \mathbb{R}$ is given by

$$E^* := \lim_{\lambda\uparrow\Lambda} c_E(\lambda) = \left\{ \lim_{\lambda\uparrow\Lambda} \psi(\lambda) \mid \psi(\lambda) \in E \right\}$$

where $c_E(\lambda)$ is the net identically equal to E.

This definition, combined with axiom (Λ-1), entails that

$$\mathbb{K} = \mathbb{R}^*.$$

Since a function f can be identified with its graph then the natural extension of a function is defined by the above definition. Moreover we have the following result:

Theorem 9.10 *The* natural extension *of a function*

$$f : E \to F$$

is a function

$$f^* : E^* \to F^*$$

and for every net $\varphi : \mathfrak{X} \to E$, *and every function* $f : E \to F$, *we have that*

$$\lim_{\lambda\uparrow\Lambda} f(\varphi(\lambda)) = f^* \left(\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \right).$$

The following theorem is a fundamental tool in using the Λ-limit:

Theorem 9.11 ***(Leibniz Principle)*** *Let* $\mathcal{R}$ *be a relation in* $\mathbb{U}_n$ *for some* $n \geq 0$ *and let* $\varphi, \psi : \mathfrak{X} \to \mathbb{U}_n$. *If*

$$\forall\lambda \in \mathfrak{X},\ \varphi(\lambda)\mathcal{R}\psi(\lambda)$$

then

$$\left(\lim_{\lambda\uparrow\Lambda} \varphi(\lambda) \right) \mathcal{R}^* \left(\lim_{\lambda\uparrow\Lambda} \psi(\lambda) \right).$$

When $\mathcal{R}$ is $\in$ or $=$ we will not use the symbol $*$ to denote their extensions, since their meaning is unaltered in universe constructed over $\mathbb{R}^*$. To give an example of how Leibniz Principle can be used to prove facts about internal entities, let us prove that the set

$$\mathbb{R}^\circ := \lim_{\lambda\uparrow\Lambda} (\mathbb{R} \cap \lambda)$$

has the maximum and the minimum: for every $\lambda \in \Lambda$ let M_λ, m_λ be, respectively, the maximum and the minimum of $\mathbb{R} \cap \lambda$ (that exists because $\mathbb{R} \cap \lambda$ is finite).[9] Then

$$M = \lim_{\lambda\uparrow\Lambda} M_\lambda$$

and

$$m = \lim_{\lambda\uparrow\Lambda} m_\lambda$$

are, respectively, the maximum and the minimum of $\mathbb{R}^\circ$. In fact, let, e.g., $\xi \in \mathbb{R}^\circ$. Let $\xi = \lim_{\lambda\uparrow\Lambda} \xi_\lambda$. For every λ we have $m_\lambda \leq \xi_\lambda \leq M_\lambda$, so we can apply Leibniz Principle and we get

$$m \leq \xi \leq M \quad \forall\xi \in \mathbb{R}^\circ.$$

[9] Here we are identifying constant functions in $\mathfrak{F}(\mathbb{R}, \mathbb{R})$ with real numbers, so $\mathbb{R} \cap \lambda$ has to be intended as the set of constants c such that the function with constant value c is in λ.

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