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Multisimplicial chains and configuration spaces

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Abstract

We define an E_∞-coalgebra structure on the chains of multisimplicial sets. Our primary focus is on the surjection chain complexes of McClure-Smith, for which we construct a zig-zag of complexity preserving quasi-isomorphisms of E_∞-coalgebras relating them to both the singular chains on configuration spaces and the Barratt–Eccles chain complexes.

Keywords Multisimplicial sets · Configuration spaces · E_∞-algebras

Mathematics Subject Classification 18N50 · 55U15 · 18N70 · 55R80 · 18N40 · 18G31

1 Introduction

The cochain complex of a simplicial set is equipped with the classical Alexander–Whitney product defining the ring structure in cohomology. This cochain level structure has several explicit extensions to an E_∞-algebra [5, 17, 26] encoding commutativity and associativity up to coherent homotopies. The importance of E_∞-algebras in homotopy theory is well known. For example, Mandell showed that finite type nilpotent spaces are weakly equivalent if and only if their singular cochains are quasi-isomorphic as E_∞-algebras [16]. Our first objective is to define a natural product

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together with an E_∞-algebra extension on the cochains of multisimplicial sets [11]. These are generalizations of simplicial sets which are useful for concrete computations since they can model homotopy types using fewer cells. For example, the proof of the non-formality of the cochain algebra of planar configuration spaces [30] used a simplicial model and the Alexander–Whitney product on its cochains. By using a multisimplicial model and the product defined here, these computations become simpler and faster, paving the way for extending this result to higher dimensions.

Multisimplicial sets are contravariant functors from products of the simplex category $\triangle$ to Set. Explicitly, for any positive integer k the category $\mathsf{mSet}^{(k)}$ of k-fold multisimplicial sets is the presheaf category $\mathsf{Fun}((\triangle^{\mathrm{op}})^{\times k}, \mathsf{Set})$. There is a notion of geometric realization for multisimplicial sets, which results in a CW complex having, for each non-degenerate multisimplex, a cell modeled on a product of geometric simplices $\Delta^{n_1} \times \cdots \times \Delta^{n_k}$. We are interested in modeling homotopy types algebraically, for which we consider the composition of the geometric realization and the functor of cellular chains C. This composition defines $\mathrm{N}\colon \mathsf{mSet}^{(k)} \to \mathsf{Ch}$, the functor of (normalized) chains. In §2.5 we define a lift of N to the category of E_∞-coalgebras, and, consequently, a lift of the functor of cochains to the category of E_∞-algebras. We do so using the finitely presented E_∞-prop introduced in [17] and its monoidal properties. Specifically, using the isomorphism

$$\mathrm{C}(\Delta^{n_1} \times \cdots \times \Delta^{n_k}) \cong \mathrm{C}(\Delta^{n_1}) \otimes \cdots \otimes \mathrm{C}(\Delta^{n_k}),$$

we extend the image of the prop generators constructed in [17] from the chains of standard simplices to those of standard multisimplices. These generators are the Alexander–Whitney coproduct, the augmentation map, and an algebraic version of the join product. The resulting E_∞-coalgebra structure generalizes those defined in [5, 17, 26] for simplicial chains and in [14] for cubical chains. As an application, we study the Steenrod construction for multisimplicial chains in §2.7 emphasizing the explicit nature of our construction.

Let us now focus on the relationship between multisimplicial and simplicial theories. The restriction to the image of the diagonal inclusion $\triangle^{\mathrm{op}} \to (\triangle^{\mathrm{op}})^{\times k}$ of any k-fold multisimplicial set X defines its associated diagonal simplicial set X^{D}. There is a natural homeomorphism of realizations $|X| \cong |X^{\mathrm{D}}|$ [28]. Under this homeomorphism the cells of $|X^{\mathrm{D}}|$ arise from those of $|X|$ through subdivision, a procedure described algebraically by the Eilenberg–Zilber quasi-isomorphism

$$\mathrm{EZ}\colon\ \mathrm{N}(X) \to \mathrm{N}(X^{\mathrm{D}}).$$

The functor induced by the diagonal restriction has a right adjoint $\mathcal{N}^{(k)}$, the multisimplicial nerve of a simplicial set. This pair of functors defines a Quillen equivalence between the model categories of k-fold multisimplicial and of simplicial sets. Furthermore, there is a natural inclusion

$$\mathrm{I}\colon\ \mathrm{N}(Y) \to \mathrm{N}(\mathcal{N}^{(k)} Y)$$

which is also a quasi-isomorphism. On one hand, the EZ map preserves the counital coalgebra structure, but it does not respect the higher E_∞-structure.[1] On the other, the map I is an E_∞-coalgebra quasi-isomorphism as proven in §3.3. We use this fact to prove in §3.4 that, for any topological space $\mathfrak{X}$, the linear map from its singular simplicial chains to its singular k-fold multisimplicial chains, given by precomposing a continuous map $(\Delta^n \to \mathfrak{X})$ with the projection $(\Delta^n \times \Delta^0 \times \cdots \times \Delta^0 \xrightarrow{\pi_1} \Delta^n)$, induces a natural quasi-isomorphism of E_∞-coalgebras.

In the second part of the paper, we use these constructions to study a multisimplicial model of the canonical filtration

$$Conf_r(\mathbb{R}^1) \subseteq Conf_r(\mathbb{R}^2) \subseteq \cdots$$

of the space $Conf_r(\mathbb{R}^\infty)$ of r distinct ordered points in $\mathbb{R}^\infty = \operatorname{colim}(\mathbb{R}^1 \subseteq \mathbb{R}^2 \subseteq \cdots)$. Concretely, for any integer r, McClure and Smith [26] introduced a chain complex $\mathcal{X}(r)$ of $\mathbb{Z}[\mathbb{S}_r]$-modules with a filtration

$$\mathcal{X}_1(r) \subseteq \mathcal{X}_2(r) \subseteq \cdots$$

and showed that $\mathcal{X}(r)$ is connected to the singular chains of $Conf_r(\mathbb{R}^\infty)$ via a zig-zag of filtration preserving $\mathbb{S}_r$-equivariant quasi-isomorphisms. Presumably it was observed by both McClure–Smith and Berger–Fresse that $\mathcal{X}(r)$ can be interpreted as the chains of an r-fold multisimplicial set $Sur(r)$, which we introduce in §4.2 with a filtration

$$Sur_1(r) \subseteq Sur_2(r) \subseteq \cdots$$

so that $\mathrm{N}\, Sur_d(r) \cong \mathcal{X}_d(r)$. There is an operad structure on $\{\mathcal{X}_d(r)\}_{r \geq 1}$ for each $d \geq 1$, but we do not focus on it since it is not induced from one at the multisimplicial level. By the constructions in §2 the complex $\mathrm{N}\, Sur(r)$ is equipped with an E_∞-coalgebra structure, which we connect to the singular chains of $Conf_r(\mathbb{R}^\infty)$ via an explicit zig-zag of filtration preserving $\mathbb{S}_r$-equivariant quasi-isomorphisms of E_∞-coalgebras.

In a similar way, Berger and Fresse [5] studied a chain complex $\mathcal{E}(r)$ of $\mathbb{Z}[\mathbb{S}_r]$-modules with a filtration

$$\mathcal{E}_1(r) \subseteq \mathcal{E}_2(r) \subseteq \cdots .$$

This complex comes from the chains on a simplicial set introduced by Barratt and Eccles [2] equipped with a filtration

$$E_1(r) \subseteq E_2(r) \subseteq \cdots$$

due to Smith [31]. (As before we disregard the operadic structure.) Since $\mathcal{E}(r)$ is induced from a simplicial set, it is endowed with an E_∞-coalgebra structure, and

[1] The Alexander–Whitney chain homotopy inverse to EZ is not a coalgebra map in general.

it is not hard to see that the zig-zag of filtration preserving $\mathbb{S}_r$-equivariant quasi-isomorphisms used to compare it to the singular chains of $Conf_r(\mathbb{R}^\infty)$ respects this higher structure. Consequently, $\mathcal{X}(r)$ and $\mathcal{E}(r)$ can be related by an explicit zig-zag of such maps.

It is desirable to have a direct map between the multisimplicial and simplicial models. Berger–Fresse constructed two such filtrations preserving $\mathbb{S}_r$-equivariant quasi-isomorphisms

$$\mathrm{TR}\colon \mathrm{N}\, E(r) \to \mathrm{N}\, Sur(r) \quad \text{and} \quad \mathrm{TC}\colon \mathrm{N}\, Sur(r) \to \mathrm{N}\, E(r).$$

The first one, introduced in [5, 1·3], is unfortunately not a coalgebra map. Therefore we will focus on the second one, which was introduced in [4]. Our contribution, presented in §4.4, is the construction of a factorization

$$\mathrm{TC}\colon \mathrm{N}\, Sur(r) \xrightarrow{\mathrm{EZ}} \mathrm{N}\, Sur(r)^{\mathrm{D}} \xrightarrow{\mathrm{N(tc)}} \mathrm{N}\, E(r),$$

up to sign, where the second map is induced from a filtration preserving $\mathbb{S}_r$-equivariant weak-equivalence of simplicial sets. Therefore, we prove that TC is a coalgebra map since EZ is one.

2 Multisimplicial algebraic topology

2.1 Multisimplicial sets

Let us consider an arbitrary positive integer k. The *k-fold multisimplex category* $\triangle^{\times k}$ is the k-fold Cartesian product of the simplex category $\triangle$. The category

$$\mathsf{mSet}^{(k)} = \mathsf{Fun}\big((\triangle^{\times k})^{\mathrm{op}}, \mathsf{Set}\big)$$

is referred to as the category of *k-fold multisimplicial sets*. We remark that $\mathsf{mSet}^{(1)}$ and $\mathsf{mSet}^{(2)}$ are the categories of simplicial and bisimplicial sets respectively. A representable k-fold multisimplicial sets is denoted by $\triangle^{n_1,\dots,n_k}$.

Explicitly, a k-fold multisimplicial set X consists of a collection of sets

$$X_{m_1,\dots,m_k} = X\big([m_1] \times \cdots \times [m_k]\big)$$

indexed by k-tuples of non-negative integers $(m_1, \dots, m_k)$ together with *face maps*

$$\mathrm{d}_i^j : X_{m_1,\dots,m_j,\dots,m_k} \to X_{m_1,\dots,m_j-1,\dots,m_k}$$

and *degeneracy map*

$$\mathrm{s}_i^j : X_{m_1,\dots,m_j,\dots,m_k} \to X_{m_1,\dots,m_j+1,\dots,m_k}$$

for $1 \leq j \leq k$ and $0 \leq i \leq m_j$ such that, referring to j as the *direction* of these maps, two of them satisfy the simplicial identities when they have the same direction and commute when they do not. An element of $X_{m_1,\dots,m_k}$ is called an $(m_1, \dots, m_k)$-*multisimplex* and it is said to be *degenerate* if it is in the image of a degeneracy map.

2.2 Geometric realization

We will use the following model of the topological simplex:

$$\Delta^n = \left\{(t_1, \dots, t_n) \in [0,1]^n \mid t_1 \geq \cdots \geq t_n\right\}$$

with

$$\delta_i(t_1, \dots, t_n) = \begin{cases} (1, t_1, \dots, t_n) & i = 0, \\ (t_1, \dots, t_i, t_i, \dots, t_n) & 0 < i < n, \\ (t_1, \dots, t_n, 0) & i = n, \end{cases}$$

and

$$\sigma_i(t_1, \dots, t_n) = (t_1, \dots, \widehat{t_i}, \dots, t_n).$$

The *geometric realization* functor

$$|-|\colon \mathsf{mSet}^{(k)} \to \mathsf{Top}$$

is the Yoneda extension of the functor defined on representable objects by

$$|\Delta^{n_1,\dots,n_k}| = \Delta^{n_1} \times \cdots \times \Delta^{n_k}.$$

Explicitly, for a k-fold multisimplicial set X we have

$$|X| \cong \coprod \Delta^{n_1} \times \cdots \times \Delta^{n_k} \times X_{n_1,\dots,n_k} /\!\sim$$

where

$$(\vec{t}_1, \dots, \vec{t}_j, \dots, \vec{t}_k, \mathrm{d}_i^j(x)) \sim (\vec{t}_1, \dots, \delta_i(\vec{t}_j), \dots, \vec{t}_k, x),$$
$$(\vec{t}_1, \dots, \vec{t}_j, \dots, \vec{t}_k, \mathrm{s}_i^j(x)) \sim (\vec{t}_1, \dots, \sigma_i(\vec{t}_j), \dots, \vec{t}_k, x),$$

which equips $|X|$ with a canonical cellular structure.

The geometric realization functor has a right adjoint

$$\mathrm{Sing}^{(k)}\colon \mathsf{Top} \to \mathsf{mSet}^{(k)}$$

defined on a topological space $\mathfrak{X}$, as usual, by the expression

$$\mathrm{Sing}^{(k)}(\mathfrak{X})_{n_1,\dots,n_k} = \mathsf{Top}(\Delta^{n_1} \times \cdots \times \Delta^{n_k}, \mathfrak{X}).$$

2.3 Algebraic realization

The functor of *chains*

$$\mathrm{N}\colon \mathsf{mSet}^{(k)} \to \mathsf{Ch},$$

is the Yoneda extension of the functor defined on representable objects by

$$\mathrm{N}\left(\triangle^{n_1,\dots,n_k}\right) = \mathrm{N}(\triangle^{n_1}) \otimes \cdots \otimes \mathrm{N}(\triangle^{n_k}).$$

It is naturally isomorphic to the composition of the geometric realization functor and the functor of cellular chains with respect to the canonical cellular structure.

Explicitly, for a k-fold multisimplicial set X the $\mathbb{k}$-module $\mathrm{N}(X)_n$ is freely generated by the non-degenerate $(n_1, \dots, n_k)$-multisimplices with $n_1 + \cdots + n_k = n$. The differential $\partial\colon\ \mathrm{N}(X)_n \to \mathrm{N}(X)_{n-1}$ is given on one such basis element by

$$\partial(x) = \sum_{j=1}^{k} \sum_{\ell_j=1}^{n_j} (-1)^{n_1+\cdots+n_{j-1}+\ell_j}\ \mathrm{d}^{j}_{\ell_j}(x).$$

For any topological space $\mathfrak{X}$ the chain complex $\mathrm{N}\,\mathrm{Sing}^{(k)}(\mathfrak{X})$ is denoted $\mathrm{S}^{(k)}(\mathfrak{X})$ and referred to as the *k-fold singular chains* of $\mathfrak{X}$.

2.4 Coalgebra structure

A *counital coalgebra* structure on a chain complex C is a pair of chain maps $\Delta\colon C \to C \otimes C$ and $\epsilon\colon C \to \mathbb{k}$ satisfying

$$(\mathsf{id} \otimes \epsilon) \circ \Delta = \mathsf{id} = (\epsilon \otimes \mathsf{id}) \circ \Delta\,.$$

The tensor product of two counital coalgebras C and C' is itself a counital coalgebra with structure maps given by

$$C \otimes C' \xrightarrow{\Delta \otimes \Delta'} (C \otimes C) \otimes (C' \otimes C') \xrightarrow{\tau} (C \otimes C') \otimes (C \otimes C'),$$
$$C \otimes C' \xrightarrow{\epsilon \otimes \epsilon'} \mathbb{k} \otimes \mathbb{k} \xrightarrow{\cong} \mathbb{k},$$

where τ transposes the second and third factors.

For each $n \in \mathbb{N}$, the complex $\mathrm{N}(\triangle^n)$ is naturally equipped with a counital coalgebra structure defined by:

$$\Delta\left([v_0, \ldots, v_m]\right) = \sum_{i=0}^{m} [v_0, \ldots, v_i] \otimes [v_i, \ldots, v_m],$$
$$\epsilon\left([v_0, \ldots, v_q]\right) = \begin{cases} 1 & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

We will refer to it as the *Alexander–Whitney structure*.

Using the tensor product structure, we deduce a natural counital coalgebra structure on the chains of representable multisimplicial sets

$$\mathrm{N}\left(\triangle^{n_1,\ldots,n_k}\right) = \mathrm{N}(\triangle^{n_1}) \otimes \cdots \otimes \mathrm{N}(\triangle^{n_k})$$

and, via a Yoneda extension, one on the chains of general multisimplicial sets.

Explicitly, for a k-fold multisimplicial set X and $(m_1, \ldots, m_k)$-multisimplex x let

$$\mathfrak{I}_{m_1,\ldots,m_k} = \{(i_1, \ldots, i_k) \mid 0 \le i_j \le m_j, \ \forall j = 1, \ldots, k\},$$

then

$$\Delta(x) = \sum_{I \in \mathfrak{I}_{k,x}} (-1)^{\sum_{1\le l<h\le k} i_h(m_l - i_l)}\, x\rfloor_{(i_1,\ldots,i_k)} \otimes {}_{(m_1-i_1,\ldots,m_k-i_k)}\lfloor x$$

where the *front* $(i_1, \ldots, i_k)$*-face* of x is the multisimplex

$$x\rfloor_{(i_1,\ldots,i_k)} = X(F_{i_1}, \ldots, F_{i_k})(x) \in X_{i_1,\ldots,i_k}$$

with $F_{i_j} : [i_j] \to [n_j]$ defined by $F_{i_j}(h) = h$, and the *back* $(i_1, \ldots, i_k)$*-face* of x is the multisimplex

$${}_{(i_1,\ldots,i_k)}\lfloor x = X(B_{i_1}, \ldots, B_{i_k})(x) \in X_{i_1,\ldots,i_k}$$

with $B_{i_j} : [i_j] \to [n_j]$ defined by $B_j(h) = h + m_j - i_j$.

2.5 E_∞-extension

An $\mathcal{M}$*-bialgebra* is a counital coalgebra (C, Δ, ϵ) together with a degree 1 linear map $*\colon C \otimes C \to C$ satisfying

$$\partial\,(c_1 * c_2) - \partial\, c_1 * c_2 + (-1)^{|c_1|} c_1 * \partial\, c_2 = \epsilon\,(c_1)c_2 - \epsilon\,(c_2)c_1,$$
$$\epsilon\,(c_1 * c_2) = 0,$$

for all $c_1, c_2 \in C$. As proven in [17], the collection of all maps $\{C \to C^{\otimes r}\}_{r \in \mathbb{N}}$ generated by Δ, ϵ and $*$ make C into an E_∞-coalgebra, that is to say, a coalgebra over certain operad $\mathrm{U}\mathcal{M}$ that is a cofibrant resolution of the terminal operad.

As proven in [25], the counital coalgebra structure on the tensor product of two $\mathcal{M}$-bialgebras C and C' can be naturally extended to an $\mathcal{M}$-bialgebra structure using

$$(C \otimes C') \otimes (C \otimes C') \xrightarrow{\tau} C \otimes C \otimes C' \otimes C' \xrightarrow{\epsilon \otimes \mathrm{id} \otimes * + * \otimes \mathrm{id} \otimes \epsilon} C \otimes C'.$$

For any integer n, the *join product* $*\colon \mathrm{N}(\triangle^n)^{\otimes 2} \to \mathrm{N}(\triangle^n)$ is the natural degree 1 linear map defined by

$$\big[v_0, \dots, v_p\big] * \big[v_{p+1}, \dots, v_q\big] = \begin{cases} (-1)^p \operatorname{sign}(\pi) \big[v_{\pi(0)}, \dots, v_{\pi(q)}\big] & \text{if } v_i \neq v_j \text{ for } i \neq j, \\ 0 & \text{if not,} \end{cases}$$

where π is the permutation that orders the vertices. It is proven in [17] that on the chains of representable simplicial sets the Alexander–Whitney structure together with the join product make $\mathrm{N}(\triangle^n)$ into a natural $\mathcal{M}$-bialgebra and, consequently, a natural E_∞-coalgebra. We mention that this structure is induced by one present at the level of geometric realizations [21].

Using the tensor product structure, we deduce a natural $\mathcal{M}$-bialgebra structure on the chains of representable multisimplicial sets

$$\mathrm{N}\left(\triangle^{n_1, \dots, n_k}\right) = \mathrm{N}(\triangle^{n_1}) \otimes \cdots \otimes \mathrm{N}(\triangle^{n_k}),$$

and consequently, a natural E_∞-coalgebra structure, which extends along the Yoneda inclusion to the chains on any multisimplicial set X.

Explicitly, for two basis elements of $\mathrm{N}\left(\triangle^{n_1, \dots, n_k}\right)$ we have

$$(x_1 \otimes \cdots \otimes x_n) * (y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^{n} x_{<i}\, \epsilon(y_{<i}) \otimes x_i * y_i \otimes \epsilon(x_{>i})\, y_{>i},$$

where, with the convention $x_{<1} = x_{>n} = 1 \in \Bbbk$,

$$\begin{aligned} z_{<i} &= z_1 \otimes \cdots \otimes z_{i-1}, \\ z_{>i} &= z_{i+1} \otimes \cdots \otimes z_n. \end{aligned}$$

We remark that since the category of $\mathcal{M}$-bialgebras is not cocomplete, we do not necessarily have an $\mathcal{M}$-bialgebra structure on $\mathrm{N}(X)$ for a general multisimplicial set X. An example for which such structure does not exist is given by one such X whose geometric realization consists of just two points.

2.6 Cubical comparison

Since the complex of chains of the k-fold multisimplicial set $\triangle^1 \times \cdots \times \triangle^1$ is isomorphic to the chains on the standard cubical set $\square^k$, it is natural to compare the

$\mathcal{M}$-bialgebra structure defined here with that presented in [14], defining an E_∞-structure on cubical chains. As counital coalgebras $\mathrm{N}(\triangle^1 \times \cdots \times \triangle^1)$ and $\mathrm{N}(\square^k)$ are isomorphic, and, denoting the product of the $\mathcal{M}$-bialgebra defined there by $\widetilde{*}$, we have

$$x \,\widetilde{*}\, y = (-1)^{|x|} x * y$$

under this chain isomorphism. The sign convention used here is more natural, used for example to endow Adams' cobar construction with the structure of a monoidal E_∞-coalgebra [25].

2.7 Steenrod construction

In [32], Steenrod introduced natural operations on the mod 2 cohomology of spaces, the celebrated *Steenrod squares*

$$\begin{aligned} \mathrm{Sq}^k : \mathrm{H}^{-n} &\longrightarrow \mathrm{H}^{-n-k} \\ [\alpha] &\mapsto \big[(\alpha \otimes \alpha)\Delta_{n-k}\big], \end{aligned}$$

via an explicit construction of natural linear maps $\Delta_i \colon \mathrm{N}(X) \to \mathrm{N}(X) \otimes \mathrm{N}(X)$ for any simplicial set X, satisfying up to signs the following homological relations

$$\partial \circ \Delta_i + \Delta_i \circ \partial = (1 + T)\Delta_{i-1},$$

with the convention $\Delta_{-1} = 0$. These so-called *cup-i coproducts* appear to be fundamental, as they are axiomatically characterized [22] and induce the nerve of strict infinity categories [18]. A description of cup-i coproducts for multisimplicial sets can be deduced from our E_∞-coalgebra structure. As presented in [23], it is given recursively by

$$\begin{aligned} \Delta_0 &= \Delta, \\ \Delta_i &= (* \otimes \mathsf{id}) \circ (23) \circ (\Delta_{i-1} \otimes \mathsf{id}) \circ \Delta. \end{aligned}$$

Steenrod also introduced operations on the mod p cohomology of spaces when p is an odd prime [33, 34]. To define these effectively, generalization of the cup-i coproducts were introduced in [13]. After the present work, these so-called *cup-(p, i) coproducts* are defined on multisimplicial chains, and their formulas are explicit enough to be implemented in the computer algebra system[2] `ComCH` [20], where constructions of Cartan and Adem coboundaries [8, 9, 19] for multisimplicial sets can also be found.

[2] https://comch.readthedocs.io/en/latest/

3 Comparison with the simplicial theory

We will use sSet to denote the category of 1-fold multisimplicial sets $\mathsf{mSet}^{(1)}$ referring to its objects and morphisms as simplicial sets and simplicial morphisms as usual.

3.1 Diagonal simplicial set

For any $k \in \mathbb{N}$, the diagonal

$$\triangle^{\mathrm{op}} \xrightarrow{\mathrm{D}} (\triangle^{\mathrm{op}})^{\times k} \xrightarrow{\cong} (\triangle^{\times k})^{\mathrm{op}}$$

induces a functor

$$(-)^{\mathrm{D}} \colon \mathsf{mSet}^{(k)} \to \mathsf{sSet}$$

explicitly defined on a k-fold multisimplicial set X by

$$X^{\mathrm{D}}_m = X_{m,\dots,m}, \qquad \mathrm{d}_i = \mathrm{d}^1_i \circ \cdots \circ \mathrm{d}^k_i, \qquad \mathrm{s}_i = \mathrm{s}^1_i \circ \cdots \circ \mathrm{s}^k_i \,.$$

It is straightforward to verify that

$$\left(\triangle^{n_1,\dots,n_k}\right)^{\mathrm{D}} \cong \triangle^{n_1} \times \cdots \times \triangle^{n_k}$$

as simplicial sets.

The functor $(-)^{\mathrm{D}} \colon \mathsf{mSet}^{(k)} \to \mathsf{sSet}$ admits a right adjoint $\mathcal{N}^{(k)} \colon \mathsf{sSet} \to \mathsf{mSet}^{(k)}$, defined, as usual, by the expression

$$\mathcal{N}^{(k)}(Y)_{m_1,\dots,m_k} = \mathsf{sSet}\big(\triangle^{m_1} \times \cdots \times \triangle^{m_k},\ Y\big).$$

These functors define a Quillen equivalence. A proof of this fact can be given using [15, Proposition 1.6.8] or adapting that in [24, Proposition 1.2].

3.2 Eilenberg–Zilber map

Recall that an $(n_1, \dots, n_k)$-*shuffle* σ is a permutation in $\mathbb{S}_n$ satisfying

$$\begin{gathered}
\sigma(1) < \cdots < \sigma(n_1), \\
\sigma(n_1+1) < \cdots < \sigma(n_1+n_2), \\
\vdots \\
\sigma(n-n_k+1) < \cdots < \sigma(n),
\end{gathered}$$

where $n = n_1 + \cdots + n_k$. We denote the set of such permutations by $\mathsf{Sh}(n_1, \ldots, n_k)$. For any $\sigma \in \mathsf{Sh}(n_1, \ldots, n_k)$ the inclusion

$$\mathrm{i}_\sigma \colon \Delta^n \to \Delta^{n_1} \times \cdots \times \Delta^{n_k}$$

is defined by the assignment

$$(x_1, \ldots, x_n) \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

If e is the identity permutation, we denote i_e simply as i. The set $\{\, \mathrm{i}_\sigma \mid \sigma \in \mathsf{Sh}(n_1, \ldots, n_k)\}$ defines a triangulation of $\Delta^{n_1} \times \cdots \times \Delta^{n_k}$ making it isomorphic, in the category of cellular spaces, to the geometric realization of the simplicial set $\Delta^{n_1} \times \cdots \times \Delta^{n_k}$. Using this identification, the identity map induces a cellular map

$$\mathfrak{ez} \colon \Delta^{n_1} \times \cdots \times \Delta^{n_k} \to \left|\Delta^{n_1} \times \cdots \times \Delta^{n_k}\right|,$$

whose induced chain map

$$\mathrm{EZ} \colon \mathrm{N}(\Delta^{n_1, \ldots, n_k}) \to \mathrm{N}\left(\Delta^{n_1} \times \cdots \times \Delta^{n_k}\right)$$

agrees, under the natural identifications, with the traditional Eilenberg–Zilber map.

For a multisimplicial set X, the induced chain map $\mathrm{EZ} \colon \mathrm{N}(X) \to \mathrm{N}(X^{\mathrm{D}})$ is explicitly given on an $(n_1, \ldots, n_k)$-multisimplex x by

$$\mathrm{EZ}(x) = \sum_{\sigma \in \mathsf{Sh}(n_1, \ldots, n_k)} \mathrm{sign}(\sigma)\, X(\sigma_1, \ldots, \sigma_k)(x)$$

where, for $\ell \in \{1, \ldots, k\}$, the morphisms $\sigma_\ell \colon [n] \to [n_\ell]$ are defined by the following property: for each $j \in \{0, \ldots, n-1\}$ there is exactly one $\ell \in \{1, \ldots, k\}$ such that $\sigma_\ell(j+1) = \sigma_\ell(j) + 1$ and $\sigma_i(j+1) = \sigma_i(j)$ for all $i \neq \ell$.

Geometrically this collection $(\sigma_1, \ldots, \sigma_k)$ represents a sequence of $(a_1 + \cdots + a_k)$ moves in a lattice of integral points with k-coordinates, starting at the origin and moving in a single direction at each stage until the point $(a_1, \ldots, a_k)$ is reached. The associated permutation sends $j+1$ to $a_1 + \cdots + a_{\ell-1} + \sigma_\ell(j+1)$.

Theorem 3.2.1 *For every multisimplicial set X the map* $\mathrm{EZ} \colon \mathrm{N}(X) \to \mathrm{N}(X^{\mathrm{D}})$ *is a quasi-isomorphism of counital coalgebras.*

Proof The fact that EZ is quasi-isomorphism follows from it being induced from a subdivision map. To prove it is a coalgebra map, it suffices to assume X is a representable multisimplicial sets $\Delta^{n_1, \ldots, n_k}$. In this case $\mathrm{N}(\Delta^{n_1, \ldots, n_k}) = \mathrm{N}(\Delta^{n}_1) \otimes \cdots \otimes \mathrm{N}(\Delta^{n}_k)$, $\mathrm{N}((\Delta^{n_1, \ldots, n_k})^{\mathrm{D}}) \cong \mathrm{N}(\Delta^{n_1} \times \cdots \times \Delta^{n_k})$ and EZ agrees with the usual Eilenberg–Zilber map, which is known to be a coalgebra map [10, (17.6)]. $\square$

We remark that the Eilenberg–Zilber map is not a morphism of E_∞-coalgebras. For example, as shown in [14, §5.4], we have

$$\Delta_1 \circ \mathrm{EZ}\big([0,1] \otimes [0,1]\big) \neq \mathrm{EZ}^{\otimes 2} \circ \Delta_1\big([0,1] \otimes [0,1]\big).$$

3.3 Canonical inclusion

We might consider an analogue of the traditional Alexander–Whitney map, a natural chain homotopy inverse to the usual Eilenberg–Zilber map. Unfortunately, this map does not preserve coalgebra structures. In this subsection we introduce an alternative quasi-isomorphism that preserves E_∞-coalgebra structures. Let Y be a simplicial set and n an integer. Consider the function $Y_n \to \mathcal{N}^{(k)}Y_{n,0,\dots,0}$ sending a simplex with characteristic map $\zeta : \triangle^n \to Y$ to the composition

$$\triangle^n \times \triangle^0 \times \cdots \times \triangle^0 \xrightarrow{\pi_1} \triangle^n \xrightarrow{\zeta_y} Y.$$

These functions induce a chain map

$$\mathrm{I}\colon\ \mathrm{N}(Y) \to \mathrm{N}(\mathcal{N}^{(k)}Y)$$

and we have the following.

Theorem 3.3.1 *The canonical inclusion* $\mathrm{I}\colon\ \mathrm{N}(Y) \to \mathrm{N}(\mathcal{N}^{(k)}Y)$ *is a quasi-isomorphism of* E_∞*-coalgebras for any simplicial set* Y.

Proof The structure-preserving properties of this map are immediate. It remains to be shown that it induces a homology isomorphism. Consider the composition of quasi-isomorphisms

$$\mathrm{N}(\mathcal{N}^{(k)}Y) \xrightarrow{\mathrm{EZ}} \mathrm{N}\left((\mathcal{N}^{(k)}Y)^{\mathrm{D}}\right) \to \mathrm{N}(Y)$$

where the second map is induced by the counit of the adjunction. We will now verify that it is left inverse to I. Consider a simplex y with characteristic map $\zeta : \triangle^n \to Y$. The multisimplex $\mathrm{I}(y)$ is given by the simplicial map $\triangle^n \times \triangle^0 \times \cdots \times \triangle^0 \xrightarrow{\pi_1} \triangle^n \xrightarrow{\zeta} Y$. Since the only $(n, 0, \dots, 0)$-shuffle is the identity, the simplex $\mathrm{EZ} \circ \mathrm{I}(y)$ is the simplicial map

$$\zeta \circ \pi_1 : \triangle^n \times \triangle^n \times \cdots \times \triangle^n \to Y.$$

Finally, the image of this simplex under the counit is the evaluation of $[n] \times \cdots \times [n]$ on $\zeta \circ \pi_1$ which gives $\zeta[n] = y$ as claimed. □

3.4 Singular chains

Theorem 3.4.1 *Let* $\mathfrak{X}$ *be a topological space. The chain map*

$$\mathrm{S}(\mathfrak{X}) \to \mathrm{S}^{(k)}(\mathfrak{X}),$$

defined by precomposing a continuous map $(\Delta^n \to \mathfrak{X})$ *with the projection*

$$\Delta^n \times \Delta^0 \times \cdots \times \Delta^0 \xrightarrow{\pi_1} \Delta^n,$$

is a quasi-isomorphism of E_∞-coalgebras.

Proof This map factors as the composition of two quasi-isomorphisms of E_∞-coalgebras. The first is $\mathrm{I}\colon \mathrm{S}(\mathfrak{X}) \to \mathrm{N}(\mathcal{N}^{(k)}\,\mathrm{Sing}(\mathfrak{X}))$, which was studied in §3.3. The second is induced by a multisimplicial isomorphism

$$\mathcal{N}^{(k)}\,\mathrm{Sing}(\mathfrak{X}) \to \mathrm{Sing}^{(k)}(\mathfrak{X})$$

defined as follows. Using the adjunction of §2.2, any simplicial map $\triangle^{n_1}\times\cdots\times\triangle^{n_k} \to \mathrm{Sing}(\mathfrak{X})$ corresponds canonically to a continuous map $|\triangle^{n_1} \times \cdots \times \triangle^{n_k} \to \mathfrak{X}|$, which precomposing with $\mathfrak{e}_\mathfrak{Z}$ gives a continuous map $\Delta^{n_1} \times \cdots \times \Delta^{n_k} \to \mathfrak{X}$. It is not hard to see that every such map arises this way since $\mathfrak{e}_\mathfrak{Z}$ is a homeomorphism. □

4 Models of configuration spaces

We are interested in modeling algebraically the $\mathbb{S}_r$-equivariant homotopy type of the space of configurations of r labeled and distinct points in Euclidean d-dimensional space. Multisimplicial sets can be used to provide an explicit chain complex model with a small number of generators, which, using the E_∞-structure defined in this paper, retains all homotopical information by Mandell's theorem [16].

In the initial subsection, we revisit Berger's method for identifying spaces that are homotopy equivalent to Euclidean configuration spaces. This approach utilizes a filtration indexed by a *complete graph poset*. In the second subsection, we construct the multisimplicial model and show that is equipped with such a filtration. In the third subsection, we recall the construction of the simplicial Barratt–Eccles model and show that is equipped with a similar filtration. In the fourth subsection, we relate the multisimplicial and simplicial chain models by an explicit map. In the last subsection, we give some examples of the sizes of the two models, showing that the multisimplicial is smaller.

4.1 Recognition of configuration spaces

Let $Conf_r(\mathbb{R}^d)$ denote the configuration space of r-tuples of pairwise disjoint vectors in $\mathbb{R}^d$. This space is equipped with a free action of the symmetric group $\mathbb{S}_r$ of permutations of $\{1,\ldots,r\}$ swapping elements of a r-tuple.

Definition 4.1.1 A *complete graph* on r vertices is a pair (μ,σ) with μ a collection of non-negative integers μ_{ij} for all $1 \le i < j \le r$, and σ is an ordering of $\{1,\ldots,r\}$. We write σ_{ij} for the restriction of the ordering σ to the set $\{i,j\}$. Graphically (μ,σ) is a simple directed graph in the edge corresponding to $i < j$ directed according to σ_{ij} and labeled by μ_{ij}. Please consult Fig. 1 for an example. Let us denote the set of complete graphs with r vertices by $\mathcal{K}(r)$ equipped with the poset structure

$$(\mu,\sigma) \le (\nu,\tau) \iff \forall i,j\ (\mu_{ij} < \nu_{ij}) \text{ or } (\mu_{ij},\sigma_{ij}) = (\nu_{ij},\tau_{ij})$$

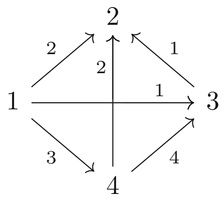

Fig. 1 A complete graph on 4 vertices with ordering $\sigma = (1432)$ and $\mu = (\mu_{12}, \mu_{13}, \mu_{14}, \mu_{23}, \mu_{24}, \mu_{34}) = (2, 1, 3, 1, 2, 4)$

for each pair $i < j$. It is equipped with an exhaustive filtration by subposets

$$\mathcal{K}_1(r) \subset \mathcal{K}_2(r) \subset \cdots$$

where $\mathcal{K}_d(r)$ consists of those graphs with $\max(\mu_{ij}) < d$.

Definition 4.1.2 For a given poset A, a cellular A-decomposition of a topological space $\mathfrak{X}$ is a family of subspaces $\{\mathfrak{X}_a\}_{a \in A}$ such that:

i. $a \leq b$ implies $\mathfrak{X}_a \subseteq \mathfrak{X}_b$;
ii. $\operatorname{colim}_{a \in A} \mathfrak{X}_a = \mathfrak{X}$;
iii. $\mathfrak{X}_a$ is contractible for each a;
iv. $\bigcup_{a<b} \mathfrak{X}_a \subset \mathfrak{X}_b$ is a closed cofibration.

The relevance of this notion is the well-known fact that if a topological space $\mathfrak{X}$ admits a *cellular A-decomposition*, then the natural maps

$$\mathfrak{X} = \operatorname*{colim}_A \mathfrak{X}_a \leftarrow \operatorname*{hocolim}_A \mathfrak{X}_a \rightarrow |A| \tag{1}$$

are cellular homotopy equivalences. Please consult [3, §1.7] for a proof.

Let $\mathcal{C}_d(r)$ be the space of r little d-dimensional cubes, which is equipped with an equivariant homotopy equivalence to $Conf_r(\mathbb{R}^d)$ picking the center of cubes. Brun and others in [6] show that $\mathcal{C}_d(r)$ has a cellular $\mathcal{K}_d^{\text{ex}}(r)$-decomposition $\{\mathcal{C}_a\}$, where $\mathcal{K}_d^{\text{ex}}(r)$ is a poset containing the poset $\mathcal{K}_d(r)$ and the inclusion of posets induces an equivariant homotopy equivalence on realizations. For detailed proof, we refer to [7]. Combining these results we have

Proposition 4.1.3 *If a space $\mathfrak{X}$ has a cellular $\mathcal{K}_d(r)$-decomposition, then $\mathfrak{X}$ is equal to* $\operatorname{colim}_{\mathcal{K}_d(r)} \mathfrak{X}_a$ *and*

$$\begin{array}{ccccccc} \operatorname*{colim}_{\mathcal{K}_d(r)} \mathfrak{X}_a \leftarrow \operatorname*{hocolim}_{\mathcal{K}_d(r)} \mathfrak{X}_a \rightarrow & |\mathcal{K}_d(r)| & & & & \\ & \downarrow & & & & \\ & |\mathcal{K}_d^{\text{ex}}(r)| & \leftarrow & \operatorname*{hocolim}_{\mathcal{K}_d^{\text{ex}}(r)} \mathcal{C}_\alpha & \rightarrow \mathcal{C}_d(r) \rightarrow & Conf_r(\mathbb{R}^d) \end{array}$$

is a zig-zag of equivariant homotopy equivalences.

Definition 4.1.4 Let X be a multisimplicial (or simplicial) set. A $\mathcal{K}(r)$-*filtration* of X is a family of (multi)simplicial subsets $\{X_a\}$ indexed by $a \in \mathcal{K}(r)$ so that

1. $a \leq b$ implies $X_a \subseteq X_b$,
2. $|X_a|$ is a cellular $\mathcal{K}(r)$-decomposition of the realization $|X|$.

In particular this implies that $X = \operatorname{colim}_{a \in \mathcal{K}(r)} X_a$. Let $X_d = \operatorname{colim}_{a \in \mathcal{K}_d(r)} X_a$. There is a nested sequence

$$X_1 \subset X_2 \subset \cdots$$

For a given (multi)simplex $x \in X$ we will refer to $\min \{d \mid x \in X_d\}$ as the *complexity* of x.

4.2 Multisimplicial model

We define for each positive integer r a multisimplicial set $Sur(r)$ equipped with a $\mathcal{K}(r)$-filtration. The functor of chains applied to the nested sequence

$$Sur_1(r) \subset Sur_2(r) \subset \cdots$$

will recover the algebraic models

$$\chi_1(r) \subset \chi_2(r) \subset \cdots$$

of configuration spaces developed by McClure–Smith [27].

Spaces Y_0^r homeomorphic to $|Sur(r)|$ were studied in the work of McClure–Smith [26]. The homeomorphism between Y_0^r and $|Sur(r)|$ is described explicitly in the appendix of [29].

Let $Sur(r)$ be the k-fold multisimplicial set that has as $(m_1, \dots, m_r)$-multisimplices the surjective maps

$$f : \{1, \dots, m + r\} \to \{1, \dots, r\},$$

where $m = m_1 + \cdots + m_r$, satisfying that the cardinality of $f^{-1}(\ell)$ is m_ℓ for each $\ell \in \{1, \dots, r\}$. We represent this multisimplex by the sequence $f(1) \cdots f(m+r)$. The face and degeneracy maps d_j^ℓ and s_j^ℓ act on it by respectively removing and doubling the $(j + 1)^{\text{th}}$ occurrence of ℓ in the sequence.

Next we define a $\mathcal{K}(r)$-filtration on $Sur(r)$. For $i < j$, let f_{ij} be the subsequence of $f(1) \cdots f(m + r)$ obtained by omitting all occurrences of elements different from i and j. For $(\mu, \sigma) \in \mathcal{K}(r)$ we say that $f \in Sur(r)_{(\mu,\sigma)}$ if for each $i < j$, either i and j alternate strictly less than μ_{ij} times in the sequence f_{ij}, or they do so exactly μ_{ij} times and the ordering formed by the first occurrences of i and j in f_{ij} agrees with σ_{ij}.

The surjection f has complexity d or less if the alternation number of each f_{ij} is less than $d+1$, i.e., if the non-degenerate dimension of f_{ij} in $Sur(2)$ is d or less for each $i<j$. We notice that the action of $\mathbb{S}_r$ on $Sur(r)$ preserves the nested sequence

$$Sur_1(r) \subset Sur_2(r) \subset \cdots .$$

For the proof that $|Sur(r)|$ has indeed an induced cellular $\mathcal{K}(r)$-decomposition we refer to Lemma 14.8 in [27]. Applying the functor of singular chains to the zig-zag of Proposition 4.1.3 produces a zig-zag of equivariant quasi-isomorphisms of $\mathrm{U}\mathcal{M}$-coalgebras connecting $\mathrm{S}\,|Sur_d(r)|$ and $\mathrm{S}\,Conf_r(\mathbb{R}^d)$. We can extend it using the following zig-zag of maps of the same kind $\mathrm{S}\,|Sur_d(r)| \cong \mathrm{S}\,|Sur_d(r)^D| \to \mathrm{N}\,(Sur_d(r)^D) \to \mathrm{N}\,(\mathcal{N}^{(r)}(Sur_d(r)^D)) \leftarrow \mathrm{N}\,Sur_d(r)$. The first map is induced by the homeomorphism $||Sur_d(r) \cong ||Sur_d(r)^D$, the second by the unit of the Quillen equivalence between simplicial sets and topological spaces, the third is the comparison map of §3.3, and last one is induced by the unit of the Quillen equivalence between multisimplicial sets and simplicial sets.

As announced in the introduction, this construction relates the chains on the multisimplicial model of configuration space and its singular chains via an explicit zig-zag of equivariant quasi-isomorphisms of E_∞-coalgebras.

4.3 Simplicial model

We will recall the definition of the Barratt–Eccles simplicial set $E(r)$ for $r \in \mathbb{N}$ which comes equipped with a $\mathcal{K}(r)$-filtration. Applying the functor of chains to the nested sequence

$$E_1(r) \subset E_2(r) \subset \cdots$$

will provide the algebraic models

$$\mathcal{E}_1(r) \subset \mathcal{E}_2(r) \subset \cdots$$

of configuration spaces studied by Berger and Fresse in [5].

The n-simplices of $E(r)$ are tuples of $n+1$ elements of the symmetric group $\mathbb{S}_r$. Its face and degeneracy maps are defined by removing and doubling elements respectively. There is an operad structure on these simplicial sets, but we do not consider it here.

Next we recall a $\mathcal{K}(r)$-filtration on $E(r)$. For $i<j$ and σ in $\mathbb{S}_r$ let σ_{ij} be the associated permutation in $\mathbb{S}_2$. Given $(\mu, \sigma) \in \mathcal{K}(r)$ then an element $w = (w_0, \dots w_n) \in E(r)_n$, $w \in E(r)_{(\mu,\sigma)}$ if for each $i<j$, the cardinality of $\{\ell \mid (w_\ell)_{ij} \neq (w_{\ell+1})_{ij}\}$ is either less than μ_{ij} or equal to it and $(w_0)_{ij} = \sigma_{ij}$.

In particular w has complexity d or less if for each $i<j$ the non-degenerate dimension of $w_{ij} = ((w_0)_{ij}, \dots, (w_n)_{ij})$ in $E(2)$ is d or less for all $i<j$. We notice that the action of $\mathbb{S}_r$ on $E(r)$ preserves the nested sequence

$$E_1(r) \subset E_2(r) \subset \cdots .$$

For a proof that this is a $\mathcal{K}(r)$-filtration we refer to Example 2.8 in [3]. Please consult [3, 12, 31] for more details.

Applying the functor of singular chains to the zig-zag of Proposition 4.1.3 produces a zig-zag of equivariant quasi-isomorphisms of $\mathrm{U}\mathcal{M}$-coalgebras connecting $\mathrm{S}\,|E_d(r)|$ and $\mathrm{S}\,Conf_r(\mathbb{R}^d)$. Using the unit of the Quillen equivalence extends this zig-zag to one relating $\mathrm{N}\,E_d(r)$ and $\mathrm{S}\,Conf_r(\mathbb{R}^d)$, which can be combined with the zig-zag constructed in the previous subsection. As announced in the introduction, this construction relates the chains on the multisimplicial model of configuration space and those the simplicial model via an explicit zig-zag of equivariant quasi-isomorphisms of E_∞-coalgebras.

4.4 Table completion

It is desirable to have a direct $\mathbb{S}_r$-equivariant quasi-isomorphism between these algebraic models. Two filtration preserving quasi-isomorphisms were constructed by Berger–Fresse

$$\mathrm{TR}\colon\ \mathrm{N}\,E(r) \to \mathrm{N}\,Sur(r) \quad \text{and} \quad \mathrm{TC}\colon\ \mathrm{N}\,Sur(r) \to \mathrm{N}\,E(r).$$

The first one, introduced in [5, 1·3], is not a coalgebra map, as the reader familiar with its definition can easily verify. We will focus on the second one which was introduced in [4] and termed *table completion*. We will construct a factorization up to signs

$$\mathrm{TC}\colon\ \mathrm{N}\,Sur(r) \xrightarrow{\mathrm{EZ}} \mathrm{N}\,Sur(r)^{\mathrm{D}} \xrightarrow{\mathrm{N(tc)}} \mathrm{N}\,E(r),$$

where the second map is induced from a simplicial map defined below. This factorization proves that TC is a coalgebra map since both factors are. We warn the reader that since EZ does not respect the E_∞-coalgebra structure, neither does TC. For example, we have

$$\Delta_1 \circ \mathrm{TC}\,\big(12312\big) \neq \mathrm{TC}^{\otimes 2} \circ \Delta_1\big(12312\big).$$

Let us give a more explicit description of the simplicial set $Sur(r)^{\mathrm{D}}$. Each m-simplex corresponds to a map

$$f : \{1, \dots, rm + r\} \to \{1, \dots, r\},$$

satisfying that the cardinality of $f^{-1}(\ell)$ is $m+1$ for each $\ell \in \{1, \dots, r\}$. We represent this simplex by the sequence $f(1) \cdots f(rm+r)$. The i^{th} face and degeneracy maps act on it respectively by removing or doubling the i^{th} occurrence of each $\ell \in \{1, \dots, r\}$ in $f(1) \cdots f(rm + r)$.

The restriction to the diagonal defines a $\mathcal{K}(r)$-filtration of $Sur(r)^{\mathrm{D}}$ and a nested sequence

$$Sur_1(r)^{\mathrm{D}} \subset Sur_2(r)^{\mathrm{D}} \subset \cdots$$

on $Sur(r)^{\mathrm{D}}$ that is preserved by the action of $\mathbb{S}_r$ on $Sur(r)^{\mathrm{D}}$. In terms of cellular $\mathcal{K}(r)$-decompositions, given $(\mu, \sigma) \in \mathcal{K}(r)$ then $f \in Sur(r)^{\mathrm{D}}_{(\mu,\sigma)}$ if for each $i < j$, either i and j alternate strictly less than μ_{ij} times in the sequence f_{ij}, or they do so exactly μ_{ij} times and the ordering formed by the first occurrences of i and j in f_{ij} agrees with σ_{ij}.

Since the complexity of an element is unchanged by degeneracy maps, it can easily be seen that $\mathrm{EZ}\colon \mathrm{N}\, Sur(r) \to \mathrm{N}\, Sur(r)^{\mathrm{D}}$ preserves $\mathcal{K}(r)$-filtrations.

Let us now define the simplicial map tc. For f as above, let

$$\mathrm{tc}(f) = (\sigma_0, \dots, \sigma_m)$$

with σ_j represented by the subsequence of f containing the $(j+1)^{\mathrm{st}}$ occurrence of each $\ell \in \{1, \dots, r\}$. For example, we have

$$\mathrm{tc}(122333112) = (123, 231, 312).$$

For every surjection $f \in Sur(r)_{m_1,\dots,m_r}$, as described in [3, §1.2.2], there is a collection of elements $(k_0, \dots, k_{m-1})$, called *caesuras*, where $m = m_1 + \cdots + m_r$ and $k_j \in \{1, \dots r\}$, $j \in \{0, \dots, m-1\}$. The caesuras are the elements of the sequence representing f which are not the last occurrence of a value $k = 1, \dots, r$. The caesuras of f define a collection of morphisms $\pi_\ell\colon [m] \to [m_\ell]$, $\ell \in \{1, \dots, r\}$ such that for $j \in \{0, \dots, m-1\}$, $\pi_{k_j}(j+1) = \pi_{k_j}(j) + 1$ and $\pi_\ell(j+1) = \pi_\ell(j)$ for all $\ell \neq k_j$. We can interpret these morphisms geometrically, as in §3.2. Then the collection $(\pi_1, \dots, \pi_r)$ defines a permutation $\vartheta_f \in \mathbb{S}_m$ such that $\vartheta_f(j+1) = a_1 + \cdots + a_{k_j - 1} + \pi_{k_j}(j+1)$, for $j \in \{0, \dots, m-1\}$.

Theorem 4.4.1 *The simplicial map* $\mathrm{tc}\colon Sur(r)^{\mathrm{D}} \to E(r)$ *satisfies*

$$\mathrm{TC}(f) = sign(\vartheta_f)(\mathrm{N}(\mathrm{tc}) \circ \mathrm{EZ})(f)$$

and induces a weak equivalence

$$\mathrm{tc}_d\colon Sur_d(r)^{\mathrm{D}} \to E_d(r)$$

for every $r, d \in \mathbb{N}$.

Before providing the proof of this theorem we briefly recall the definition of TC from [4]. For each $f \in Sur(r)_{m_1,\dots,m_r}$ there is a certain map of simplicial sets

$$\tau_f : \triangle^{m_1} \times \cdots \times \triangle^{m_r} \to E(r).$$

Observe that a map into $E(r)$ is completely determined by its restriction to the 0-simplexes. The 0-simplexes of $\triangle^{m_1} \times \cdots \times \triangle^{m_r}$ are r-tuples $(n_1, \dots, n_r)$ such that $0 \le n_i \le m_i$, for $i = 1, \dots, r$. The map τ_f is determined by the requirement that the permutation $\tau_f(n_1, \dots, n_r)$ is the subsequence of f where one picks the $(n_i+1)^{st}$ entry of the value i, for each $i = 1, \dots, r$.

As seen in §3.2 each maximal non-degenerate simplex

$$\sigma \in (\Delta^{m_1} \times \cdots \times \Delta^{m_r})_m$$

is determined by a $(m_1, \ldots, m_r)$-shuffle, and equivalently by a sequence $k_i \in \{1, \ldots, r\}$, $i = 0, \ldots, m-1$. The *fundamental simplex* of f is the maximal non-degenerate simplex corresponding to the sequence $(k_0, \ldots, k_{m-1})$ of the caesuras of f. The homomorphism $TC(f)$ is defined as the sum, with signs, of the simplexes $\tau_f(\sigma)$, where σ runs over all maximal non-degenerate simplices of the domain. The sign of each summand is positive if and only if the natural orientation of the corresponding simplex agrees with that of the fundamental simplex.

Proof of Theorem 4.4.1 It is clear that tc is a simplicial map. Let

$$\hat{f} : \Delta^{m_1, \ldots, m_r} \to Sur$$

be the map sending the only non-degenerate $(m_1, \ldots, m_r)$-multisimplex s to f. We claim that

$$\tau_f = tc \circ \hat{f}^{\mathrm{D}},$$

where we identify

$$(\Delta^{m_1, \ldots, m_r})^{\mathrm{D}} \cong \Delta^{m_1} \times \cdots \times \Delta^{m_r}.$$

Namely $\hat{f}^{\mathrm{D}}$ coincides with τ_f on 0-simplexes and tc_0 is the identity. Observe that $EZ(s)$ is the signed sum of all maximal non-degenerate simplices σ where the sign takes care of the orientation. By naturality $\mathrm{N}(\hat{f}^{\mathrm{D}})$ sends $\mathrm{EZ}(s)$ to $\mathrm{EZ}(f)$ and so

$$\mathrm{TC}(f) = \pm \mathrm{N}(\tau_f)(\mathrm{EZ}(s)) = \pm \mathrm{N}(\mathrm{tc})(\mathrm{N}(\hat{f}^{\mathrm{D}})(\mathrm{EZ}(s)) = \pm \mathrm{N}(\mathrm{tc})(\mathrm{EZ}(f)).$$

Comparing with the definition of TC we see that the sign in the equation is exactly the sign of θ_f.

We now prove the filtration compatibility of tc. Let $f \in Sur(r)^{\mathrm{D}}$ be an m-simplex and denote

$$\mathrm{tc}(f) = w = (w_0, \ldots, w_m) \in E(r)_m.$$

Suppose that in f_{ij} i and j alternate μ_{ij} times, and in w_{ij} i and j alternate μ'_{ij} times. We will prove that $\mu'_{ij} \leq \mu_{ij}$.

Suppose without loss of generality that i occurs before j in f_{ij}, that starts with i repeated h times, followed by j repeated l times, and then by i again (or terminating). If the sequence terminates then $h = l = k$ and $\mu_{ij} = \mu'_{ij} = 0$. For $h < l$ let $\bar{f}_{ij}$ be the subsequence of f_{ij} obtained taking out the first h values of i and of j. Let $\bar{\mu}_{ij}$ be the number of variations in $\bar{f}_{ij}$ and $\bar{\mu}'_{ij}$ the number of variations in $\mathrm{tc}(\bar{f}_{ij})$.

Then $\bar{f}_{ij}$ starts with j, $\mu_{i,j} = 1 + \bar{\mu}_{ij}$, and $\mu'_{ij} = 1 + \bar{\mu}'_{ij}$.

If $h \geq l$ and $l < r$ let $\bar{f}_{ij}$ be the subsequence of f_{ij} obtained taking out the first l values of i and j. Then $\bar{f}_{ij}$ starts with i, $\mu_{i,j} = 1 + \bar{\mu}_{ij}$, and $\mu'_{ij} = \bar{\mu}'_{ij}$. By induction on the length of sequences, we obtain that $\mu'_{ij} \leq \mu_{ij}$. Moreover, tc is compatible with orderings: the first occurrences of i and j form the ordering $(w_1)_{ij}$ that is the first permutation of $\mathrm{tc}(f_{ij})$. This concludes the proof of filtration compatibility.

Regarding the maps $\mathrm{tc}_d \colon Sur_d(r)^{\mathrm{D}} \to E(r)_d$ we can express source and target as (homotopy) colimits of contractible simplicial sets along the poset of complete graphs, so we have a commutative diagram

$$\begin{array}{ccc} hocolim_{\mathcal{K}_d(r)} Sur_{(\mu,\sigma)}(r)^{\mathrm{D}} & \longrightarrow & hocolim_{\mathcal{K}_d(r)} E(r)_{(\mu,\sigma)} \\ \downarrow & & \downarrow \\ Sur_d(r)^{\mathrm{D}} & \longrightarrow & E(r)_d \end{array}$$

Where the vertical and top arrows are weak equivalences, and so the bottom map is a weak equivalence. □

We remark that, as informed to us by the referee, the tc map appears in [1, Example 3.10(b)] in the context of the simplicial condensation of the lattice path operad.

4.5 Counting generators

We would like to stress that the number of non-degenerate multisimplices in $Sur_d(r)$ is much smaller than the number of non-degenerate simplices in $E_d(r)$. For example,

$$P_{\chi}^{2,4}(x) = 24(1 + 6x + 10x^2 + 5x^3)$$
$$P_{\mathcal{E}}^{2,4}(x) = 24(1 + 23x + 104x^2 + 196x^3 + 184x^4 + 86x^5 + 16x^6)$$

and

$$P_{\chi}^{3,3}(x) = 6(1 + 3x + 7x^2 + 9x^3 + 6x^4 + x^5)$$
$$P_{\mathcal{E}}^{3,3}(x) = 6(1 + 5x + 25x^2 + 60x^3 + 70x^4 + 38x^5 + 8x^6)$$

where

$$P_{\chi}^{d,r}(x) = \sum_n \mathrm{rank}(\chi_d(r)_n) \cdot x^n,$$
$$P_{\mathcal{E}}^{d,r}(x) = \sum_n \mathrm{rank}(\mathcal{E}_d(r)_n) \cdot x^n.$$

This makes the multisimplicial approach substantially more efficient than the simplicial one when performing computations. A calculation of obstruction to formality similar to that in [30] took a full day with the simplicial model, and few seconds with the multisimplicial model.

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