



The universal fibration with fibre X in rational homotopy theory

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Abstract

Let X be a simply connected space with finite-dimensional rational homotopy groups. Let $p_\infty: UE \rightarrow \text{Baut}_1(X)$ be the universal fibration of simply connected spaces with fibre X . We give a DG Lie algebra model for the evaluation map $\omega: \text{aut}_1(\text{Baut}_1(X_\mathbb{Q})) \rightarrow \text{Baut}_1(X_\mathbb{Q})$ expressed in terms of derivations of the relative Sullivan model of p_∞ . We deduce formulas for the rational Gottlieb group and for the evaluation subgroups of the classifying space $\text{Baut}_1(X_\mathbb{Q})$ as a consequence. We also prove that $\mathbb{C}P_\mathbb{Q}^n$ cannot be realized as $\text{Baut}_1(X_\mathbb{Q})$ for $n \leq 4$ and X with finite-dimensional rational homotopy groups.

Keywords Classifying space for fibrations · Evaluation map · Rationalization · Derivations · Minimal model

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1 Introduction

Given a simply connected CW complex X of finite type, let $\text{aut}_1(X)$ denote the space of self-maps of X homotopic to the identity map. The group-like space $\text{aut}_1(X)$ has a classifying space $\text{Baut}_1(X)$. The space $\text{Baut}_1(X)$ appears as the base space of the universal example $p_\infty: UE \rightarrow \text{Baut}_1(X)$ of a fibration of simply connected CW complexes with fibre of the homotopy type of X [2,9,13].

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The classifying space $Baut_1(X)$ offers a computational challenge in homotopy theory. When X is a finite complex, $Baut_1(X)$ is of CW type (albeit, generally infinite) and satisfies the localization identity $Baut_1(X_P) \simeq Baut_1(X)_P$ for any collection of primes by work of May [9,10]. In rational homotopy theory, models for $Baut_1(X_{\mathbb{Q}})$ are due to Sullivan, Schlessinger-Stasheff and Tanré [12,14,15]. The study of the classifying space using these models is an area of continued activity (see, e.g., [8,16,17]).

We say a space X is π -finite if X is a simply connected CW complex and $\dim \pi_*(X_{\mathbb{Q}}) < \infty$. A π -finite space X has a finitely generated Sullivan minimal model $\wedge(V; d)$. If X is a π -finite space then $Baut_1(X_{\mathbb{Q}})$ is one also (Proposition 2.3, below). Consequently, we may iterate the classifying space construction for π -finite rational spaces. Our first result here describes the passage from $Baut_1(X_{\mathbb{Q}})$ to $aut_1(Baut_1(X_{\mathbb{Q}}))$ in the setting of derivations of Sullivan models. We describe this result briefly now, with fuller definitions in Sect. 2.

The relative Sullivan model for the universal fibration $p_{\infty} : UE \rightarrow Baut_1(X)$ with fibre X a π -finite space is an inclusion of DG algebras. We write this model throughout as:

$$\wedge(Z; d_{\infty}) \rightarrow (\wedge Z \otimes \wedge V; D_{\infty}).$$

Let $Der(\wedge V; d)$ denote the DG Lie algebra of derivations of $\wedge(V; d)$ and write $Der_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty})$ for the derivations of $\wedge Z \otimes \wedge V$ vanishing on $\wedge Z$. We will assume derivation spaces are connected. Thus we restrict $Der^1(\wedge V; d)$ to the cycles $Z_1(Der(\wedge V; d))$ and set $Der^n(\wedge V; d) = 0$ for $n \leq 0$. We do the same for $Der_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty})$. Define a DG Lie algebra map

$$P_* : Der_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty}) \rightarrow Der(\wedge V; d)$$

by restricting a derivation θ to $\wedge V$ and composing with the projection $P : \wedge Z \otimes \wedge V \rightarrow \wedge V$.

Sullivan showed the DG Lie algebra $Der(\wedge V; d)$ gives a model for the classifying space ([14, Sect. 7], see Theorem 2.2, below). We extend Sullivan’s result to the following:

Theorem 1 *Let X be a π -finite space. The map*

$$P_* : Der_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty}) \rightarrow Der(\wedge V; d)$$

is a Quillen model for $\tilde{\omega} : \widetilde{aut}_1(Baut_1(X_{\mathbb{Q}})) \rightarrow Baut_1(X_{\mathbb{Q}})$, the universal cover of the evaluation map.

We mention two consequences of Theorem 1. First we deduce an interesting feature of the derivations of the relative Sullivan model $\wedge(Z; d_{\infty}) \rightarrow (\wedge Z \otimes \wedge V; D_{\infty})$ for X a π -finite space.

Corollary 1.1 *The DG Lie algebra $Der_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty})$ satisfies:*

- (i) $H_*(Der_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty}))$ is an abelian Lie algebra

(ii) *There are vector space isomorphisms for $n \geq 1$*

$$H_n(\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty)) \cong H_{n+1}(\text{Der}(\wedge Z; d))$$

We also deduce a formula for the n th *Gottlieb group* of the classifying space $\text{Baut}_1(X)$. Recall the subgroup $G_n(Y)$ of $\pi_n(Y)$ is the image of the map induced on homotopy groups by the evaluation map: $G_n(Y) = \text{im}\{\omega_n^* : \pi_n(\text{aut}_1(Y)) \rightarrow \pi_n(Y)\}$.

Corollary 1.2 *Let X be a π -finite space. Then*

$$G_{n+1}(\text{Baut}_1(X_\mathbb{Q})) \cong \text{im}\{H(P_*) : H_n(\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty)) \rightarrow H_n(\text{Der} \wedge (V; d))\}$$

for $n \geq 1$.

Corollary 1.2 leads to an obstruction theory for Gottlieb elements of the classifying space (Proposition 3.1, below). More generally, we obtain a description of the poset of evaluation subgroups $G_*(\xi; X_\mathbb{Q}) \subseteq \pi_*(\text{Baut}_1(X_\mathbb{Q}))$ parameterized by fibrations ξ with fibre $X_\mathbb{Q}$. We give some examples and results on this poset in Sect. 3, complementing work of Yamaguchi in [17].

We also prove a non-realization result for the classifying space.

Theorem 2 *There is no simply connected, π -finite space X such that*

$$\mathbb{C}P_\mathbb{Q}^n \simeq \text{Baut}_1(X_\mathbb{Q})$$

for $n = 2, 3, 4$.

Theorem 2 extends [7, Th. 2] for the case $n = 2$. The case $n = 3$ was recently obtained, independently, in [16].

The paper is organized as follows. In Sect. 2, we introduce our notation and recall some results on the rational homotopy theory of the space $\text{Baut}_1(X)$ and of the monoid $\text{aut}_1(p)$ of fibrewise self-equivalences of a fibration p . We prove Theorem 1 using these results together with an identity from [1] that connects these spaces. Section 3 contains our results on the evaluation subgroups of the classifying space. We prove Theorem 2 in Sect. 4.

2 Derivations and fibrewise self-equivalences

May’s localization equivalence $\text{Baut}_1(X)_\mathbb{Q} \simeq \text{Baut}_1(X_\mathbb{Q})$ for X finite [10, Th. 4.1] implies one may study the rationalization of the classifying space using algebraic models with this restriction. We are interested here in the space $\text{aut}_1(\text{Baut}_1(X))$. We cannot expect to have $\text{aut}_1(\text{Baut}_1(X_\mathbb{Q})) \simeq \text{aut}_1(\text{Baut}_1(X))_\mathbb{Q}$ even for X finite, since $\text{Baut}_1(X)$ is generally of infinite CW type. Thus in what follows, we state our main results for rationalized spaces $X_\mathbb{Q}$ for which the various constructions can be made algebraically.

We establish notation for working in rational homotopy theory. Our overriding reference for this material is [3]. Let X be simply connected and and CW complex

of finite type. A *Sullivan model* for X is a DG algebra $\wedge(V; d)$ freely generated by the graded space V with differential d satisfying the nilpotence condition ([3, p. 138]) and such that there is a quasi-isomorphism $\wedge(V; d) \rightarrow A_{PL}(X)$ with the latter the de Rham algebra of rational differential forms on X [3, p. 122]. A Sullivan model for a map $f: X \rightarrow Y$ is a map of Sullivan models making the diagram of quasi-isomorphisms with $A_{PL}(f): A_{PL}(Y) \rightarrow A_{PL}(X)$ commute (see [3, Ch.23]). A Sullivan model $\wedge(V; d)$ for X is the *Sullivan minimal model* if the differential d is decomposable. The homotopy type of $X_{\mathbb{Q}}$ is completely determined by a Sullivan minimal model $\wedge(V; d)$.

A fibration $p: E \rightarrow B$ of simply connected spaces with fibre X has a *relative Sullivan model* which is an inclusion $\wedge(W; \hat{d}) \rightarrow (\wedge W \otimes \wedge V; D)$ of DG algebras in which $\wedge(W; \hat{d})$ is a Sullivan minimal model for the base B . The differential satisfies $D(w) = \hat{d}(w)$ for $w \in W$ while $D(v) - d(v) \in \wedge^+ W \cdot (\wedge W \otimes \wedge V)$ for $v \in V$. The differential D is not generally decomposable but the DG algebra $(\wedge W \otimes \wedge V; D)$ is a Sullivan model for the total space E ([3, Ch.14]).

Quillen’s framework for rational homotopy theory is the category of connected DG Lie algebras. An object here is a pair (L, ∂) with $L = \bigoplus_{n \geq 1} L_n$ equipped with a homogenous bracket and differential ∂ lowering degree by one [3, p. 383]. The commutative cochains functor may be applied to a DG Lie algebra (L, ∂) to obtain a Sullivan algebra $C^*(L, \partial) = \wedge(sL; d = d_0 + d_{[-1, \cdot]})$ [3, Lem.23.1]. Here sL is the graded vector space suspension of L with d_0 the dual to ∂ and $d_{[-1, \cdot]}$ induced by the bracket in L . A DG Lie algebra L, ∂ is a *Quillen model* for X if $C^*(L, \partial)$ is a Sullivan model for X . In this case, we have an isomorphism $\pi_*(\Omega X) \otimes \mathbb{Q} \cong H_*(L, \partial)$. The Quillen model for a map $f: X \rightarrow Y$ is a DG Lie algebra map $\psi: L_X \rightarrow L_Y$ such that the induced map $C^*(\psi): C^*(L_Y, \partial_Y) \rightarrow C^*(L_X, \partial_X)$ gives a commutative diagram with quasi-isomorphisms to the de Rham forms as for Sullivan models.

Beginning with a Sullivan minimal model $\wedge(V; d)$ we obtain the DG Lie algebra $\text{Der}(\wedge V; d)$ defined as follows: In degree n , $\text{Der}^n(\wedge V; d)$ consists of linear self-maps θ of $\wedge V$ reducing degrees by n , $\theta(\wedge V)^m \subseteq (\wedge V)^{m-n}$, and satisfying the derivation law $\theta(\chi_1 \chi_2) = \theta(\chi_1) \chi_2 + (-1)^{n|\chi_1|} \chi_1 \theta(\chi_2)$ for $\chi_1, \chi_2 \in \wedge V$. The bracket of two derivations is defined by the rule $[\theta_1, \theta_2] = \theta_1 \circ \theta_2 - (-1)^{|\theta_1||\theta_2|} \theta_2 \circ \theta_1$. The differential δ is given by $\delta(\theta) = [d, \theta]$ for $\theta \in \text{Der}(\wedge V)$. As we will only consider connected DG Lie algebras, we restrict in degree 1 to those θ with $\delta(\theta) = 0$. To ease notation, we write $\text{Der}(\wedge V; d) = \text{Der}(\wedge V), \delta$ for the connected DG Lie algebra. Sullivan’s original result on the classifying space is the following:

Theorem 2.1 [14, Sect. 7] *Let X be simply connected and of finite type with Sullivan minimal model $\wedge(V; d)$. There is an isomorphism of graded Lie algebras*

$$\pi_*(\Omega \text{Baut}_1(X_{\mathbb{Q}})) \cong H_*(\text{Der}(\wedge V; d)). \quad \square$$

Theorem 2.1 strengthens to the following statement by the work of several authors:

Theorem 2.2 *Let X be simply connected and of finite type. Then $\text{Der}(\wedge V; d)$ is a Quillen model for $\text{Baut}_1(X_{\mathbb{Q}})$.*

Proof Schlessinger-Stasheff and Tanré constructed a Quillen model for $Baut_1(X_{\mathbb{Q}})$ written $cl(L_X; \partial_X)$ (see [15, Cor.7.4(4)]). Gatsinzi [5, Th. 1] constructed a quasi-isomorphism from $Der(\wedge V; d)$ to $cl(L_*(\wedge V; d))$ where $L_*(_)$ is the Quillen functor from DG algebras to DG Lie algebras. \square

Given a graded vector space V , write $\max(V) = \max\{n \mid V^n \neq 0\}$. We have

Proposition 2.3 *Let X be simply connected and π -finite. Then $Baut_1(X_{\mathbb{Q}})$ is π -finite and, we may construct its classifying space $Baut_1(Baut_1(X_{\mathbb{Q}}))$. Further, if $N = \max(\pi_*(X_{\mathbb{Q}}))$, then*

$$(i) \max(\pi_*(Baut_1(X_{\mathbb{Q}}))) = N - 1 \quad \text{and} \quad (ii) \pi_{N-1}(Baut_1(X_{\mathbb{Q}})) \cong \pi_N(X_{\mathbb{Q}}).$$

Proof Parts (i) and (ii) are direct consequence of Theorem 2.1 (cf. [8, Pro.2.2]). We note that if $V \cong \pi_*(X_{\mathbb{Q}})$ then $H_*(Der(\wedge V; d))$ is a sub-quotient of $\text{Hom}(V, \wedge V)$ and so finite-dimensional. Thus $\pi_*(Baut_1(X_{\mathbb{Q}}))$ is finite-dimensional by Theorem 2.2. Let $\wedge(Z; d_{\infty})$ denote the Sullivan minimal model for $C^*(Der(\wedge V; d))$. Then $Baut_1(Baut_1(X_{\mathbb{Q}}))$ is the rational space with Quillen model $Der(\wedge Z; d_{\infty})$. Finally, note that the spatial realization of a finitely generated Sullivan model is a CW complex [3, p. 247–248].

Next we consider the monoid of fibrewise equivalences. Given a fibration $p: E \rightarrow B$ set

$$\text{aut}_1(p) = \{f: E \rightarrow E \mid p \circ f = f, f \simeq 1_E\} \subseteq \text{map}(E, E).$$

Let $Baut_1(p)$ denote the classifying space for this monoid. The main result of [1], specialized to universal covers, is the following identity:

Theorem 2.4 [1, Th. 4.1] *Let $p: E \rightarrow B$ be a fibration of simply connected CW complexes with fibre X . There is a weak homotopy equivalence*

$$Baut_1(p) \simeq_w \widetilde{\text{map}}(B, Baut_1(X); h)$$

where the latter space is the universal cover of the function space component of the classifying map $h: B \rightarrow Baut_1(X)$ for the fibration p . \square

Sullivan’s result, Theorem 2.1 above, extends to an identification for the monoid $\text{aut}_1(p)$ by the main result of [4]. We recall this result now. Given $p: E \rightarrow B$ with relative Sullivan model $\wedge(W; \hat{d}) \rightarrow (\wedge W \otimes \wedge V; D)$, define $Der_{\wedge W}(\wedge W \otimes \wedge V; D)$ to be the sub-DG Lie algebra of $Der(\wedge W \otimes \wedge V; D)$ obtained by restricting to derivations θ with $\theta(W) = 0$. The differential δ is the restriction of the differential for $Der(\wedge W \otimes \wedge V; D)$. We continue to restrict, in degree 1, to the kernel of δ . We have:

Theorem 2.5 [4, Th. 4.1] *Let $p: E \rightarrow B$ be a fibration of simply connected CW complexes with fibre X and $p_{\mathbb{Q}}: E_{\mathbb{Q}} \rightarrow B_{\mathbb{Q}}$ the rationalization of p . There is a natural isomorphism of graded Lie algebras in positive degrees:*

$$\pi_*(\text{aut}_1(p_{\mathbb{Q}})) \cong H_*(\text{Der}_{\wedge W}(\wedge W \otimes \wedge V; D)).$$

□

The identification given in Theorem 2.5 is natural with respect to maps induced by pull-backs of fibrations [17, Pro.1.5]. A map $f : B' \rightarrow B$ into the base B of a fibration $p : E \rightarrow B$ with fibre X induces a multiplicative map $\text{aut}_1(p) \rightarrow \text{aut}_1(p')$ where p' is the pull-back. Then f induces the map of derivation spaces

$$f_* : \text{Der}_{\wedge W}(\wedge W \otimes \wedge V; D) \rightarrow \text{Der}_{\wedge W'}(\wedge W' \otimes \wedge V; D')$$

obtained by composing a derivation with $f^* \otimes 1$ where $f^* : \wedge(W; \hat{d}) \rightarrow \wedge(W'; \hat{d}')$ is a Sullivan model of f . In particular, the inclusion of the base-point in B induces the DG algebra map $P_* : \text{Der}_{\wedge W}(\wedge W \otimes \wedge V; D) \rightarrow \text{Der}(\wedge V; d)$ given by $P_*(\theta) = P \circ \theta$ with $P : \wedge W \otimes \wedge V \rightarrow \wedge V$ the projection. The map P_* is the subject of Theorem 1 which we prove now.

Proof of Theorem 1 Let X be simply connected, π -finite and of finite type. Let $\wedge(Z; d_{\infty}) \rightarrow (\wedge Z \otimes \wedge V; D_{\infty})$ be the relative model for the universal fibration with fibre X . Recall we are to prove the map

$$P_* : \text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty}) \rightarrow \text{Der}(\wedge V; d)$$

is a Quillen model for $\tilde{\omega} : \widetilde{\text{aut}}_1(\text{Baut}_1(X_{\mathbb{Q}})) \rightarrow \text{Baut}_1(X_{\mathbb{Q}})$. Since $\widetilde{\text{aut}}_1(\text{Baut}_1(X_{\mathbb{Q}}))$ is an H-space, a Quillen model for this space is just a DG Lie algebra with the correct homotopy groups. Applying Theorem 2.4 to the identity map, we obtain a weak equivalence:

$$\widetilde{\text{aut}}_1(\text{Baut}_1(X_{\mathbb{Q}})) = \widetilde{\text{map}}(\text{Baut}_1(X_{\mathbb{Q}}), \text{Baut}_1(X_{\mathbb{Q}}); 1) \simeq_w \text{Baut}_1((p_{\infty})_{\mathbb{Q}}).$$

Applying Theorem 2.5, we deduce that

$$\pi_*(\widetilde{\text{aut}}_1(\text{Baut}_1(X_{\mathbb{Q}}))) \cong H_*(\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty})),$$

as needed.

Finally, Theorem 2.2 and the naturality of the identification in Theorem 2.5, mentioned above, gives that P_* is a Quillen model for $\tilde{\omega}$. □

Proof of Corollary 1.1 Since $\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty})$ is a Quillen model for the H-space $\text{aut}_1(p_{\infty})$ it has vanishing brackets in homology. The isomorphism in Corollary 1.1 (ii) follows from the chain of isomorphisms:

$$\begin{aligned} H_n(\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_{\infty})) &\cong \pi_n(\text{aut}_1((p_{\infty})_{\mathbb{Q}})) \\ &\cong \pi_n(\Omega \text{Baut}_1((p_{\infty})_{\mathbb{Q}})) \\ &\cong \pi_{n+1}(\text{aut}_1(\text{Baut}_1(X_{\mathbb{Q}}))) \\ &\cong H_{n+1}(\text{Der}(\wedge Z; d_{\infty})). \end{aligned}$$

□

Corollary 1.2 follows directly from Theorem 1 and the definition of the Gottlieb group.

We next give a partial description of the differential D_∞ in terms of derivations. For any relative model $\wedge(W; d) \rightarrow (\wedge W \otimes \wedge V; D)$, the minimality condition for D implies for each $v \in V$ we have $D(v) = d(v) + \sum_{i=1}^s \theta_{w_i}(v)w_i + \Phi(v)$ where $\Phi(v)$ is in the ideal $(\wedge^+ W \cdot \wedge^+ W)$ of $\wedge W \otimes \wedge V$. The linear maps $\theta_{w_i} : \wedge V \rightarrow \wedge V$ are of degree $|w_i| - 1$ and extend to degree $|w_i| - 1$ cycles of $\text{Der}(\wedge V; d)$. The map

$$w_i \mapsto \langle \theta_{w_i} \rangle : W^{|w_i|} \rightarrow H_{|w_i|-1}(\text{Der}(\wedge V; d))$$

corresponds to the map induced on rational homotopy groups by the classifying map $h : B \rightarrow \text{Baut}_1(X)$ [7, Th. 3.2]. For the universal fibration with fibre X , the map $Z^* \rightarrow H_{*-1}(\text{Der}(\wedge V; d))$ is thus an isomorphism. Writing $H_*(\text{Der}(\wedge V; d)) = \langle \theta_1, \dots, \theta_n \rangle$ in a homogeneous basis, we conclude there is a corresponding basis $Z = \langle z_1, \dots, z_n \rangle$ with $|z_i| = |\theta_i| + 1$ such that for $v \in V$

$$D_\infty(v) = d(v) + \sum_{i=1}^n \theta_i(v)z_i + \Phi(v) \text{ for } \Phi(v) \in (\wedge^+ W \cdot \wedge^+ W) \cdot \wedge W \otimes \wedge V. \quad (1)$$

We use this description of D_∞ in the following simple example illustrating Corollary 1.1. In what follows, we write (v, P) for the derivation obtained by sending $v \in V$ to $P \in \wedge V$ (or $P \in \wedge Z \otimes \wedge V$) and vanishing on a complementary subspace of V . We write $v^* = (v, 1)$. Note that $|(v, P)| = |v| - |P|$.

Example 2.6 Let $X = S^3 \times \mathbb{C}P^2$. Write the Sullivan minimal model for X as $\wedge(x_2, y_3, z_5; d)$ with subscripts indicating degree and differential given by $d(x) = d(y) = 0, d(z) = x^3$. We see $H_*(\text{Der} \wedge (V; d)) = \langle (y, x), (z, y), y^*, (z, x), z^* \rangle$. with one non-trivial bracket $z^* = [(z, y), y^*]$. We then compute $C^*(\text{Der}(\wedge V; d))$ and obtain a (minimal) model for $\text{Baut}_1(X_\mathbb{Q})$ of the form $\wedge(Z; d_\infty) = \wedge(a, b, u, v, w; d_\infty)$ with $|a| = 2, |b| = 3, |u| = |v| = 4, |w| = 6$ and $d_\infty(a) = d_\infty(b) = d_\infty(c) = d_\infty(u) = d_\infty(v) = 0$ and $d_\infty(w) = bu$. Using (1), we see that universal fibration has relative Sullivan model

$$\wedge(a, b, u, v, w; d_\infty) \rightarrow (\wedge(a, b, u, v, w) \otimes \wedge(x, y, z); D_\infty)$$

with $D_\infty = d_\infty$ on $\wedge(a, b, u, v, w)$, $D_\infty(z) = w + vx + by + x^3$, $D_\infty(y) = u + ax$ and $D_\infty(x) = 0$. Computing homology groups gives an illustration of Corollary 1.1: Note also that $H_*(\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty))$ is abelian.

Theorem 1 implies a formula for a Quillen model for the universal cover of the monoid $\text{aut}_1^*(\text{Baut}_1(X_\mathbb{Q}))$ of basepoint-preserving automorphisms of the classifying space. Write

$$\overline{\text{Der}}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty) = \{ \theta \in \text{Der}_{\wedge W}(\wedge W \otimes \wedge V) \mid \theta(v) \subset \wedge^+ W \cdot (\wedge W \otimes \wedge V) \}$$

with the induced differential.

$H_*(\text{Der}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty))$		$H_*(\text{Der}(\wedge V; d))$	
Degree	Derivation classes	Degree	Derivation classes
5	z^*	6	w^*
4		5	
3	$(z, x), (z, a)$	4	$(w, a), v^*$
2		3	
1	$(z, u) - (z, ax), (z, a^2),$ $(z, d) - (y, a), (z, x^2)$	2	$a^*, (w, a^2), (w, a),$ (w, v)

Corollary 2.7 *Let X be a π -finite space. Then $\overline{\text{Der}}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty)$ is a Quillen model for $\widetilde{\text{aut}}_1^*(\text{Baut}_1(X_\mathbb{Q}))$.*

Proof The isomorphism of graded Lie algebras

$$\pi_*(\text{aut}_1^*(\text{Baut}_1(X_\mathbb{Q}))) \cong H_*(\overline{\text{Der}}_{\wedge Z}(\wedge Z \otimes \wedge V; D_\infty))$$

is a consequence of Theorem 1 and the 5-lemma applied to the long exact homotopy sequence of the evaluation fibration

$$\text{aut}_1^*(\text{Baut}_1(X_\mathbb{Q})) \rightarrow \text{aut}_1(\text{Baut}_1(X_\mathbb{Q})) \rightarrow \text{Baut}_1(X_\mathbb{Q}).$$

□

3 The evaluation subgroups of the classifying space

The Gottlieb group $G_*(X)$ plays a central role in the theory of fibrations, as it corresponds to the universal image of connecting homomorphisms for fibrations with fibre X [6, Th. 2.]. The rational Gottlieb groups $G_*(X_\mathbb{Q})$ are the subject of a well-known structure theorem in rational homotopy theory. For X a finite complex, $G_{\text{even}}(X_\mathbb{Q}) = 0$ and $\dim G_{\text{odd}}(X_\mathbb{Q}) \leq \text{cat}(X_\mathbb{Q})$ [3, Pro.29.8]. The significance of the Gottlieb group of the classifying space is less clear. We give some examples and results here to suggest the rational Gottlieb group and, more generally, the rational evaluation subgroups of the classifying space offer interesting invariants of the homotopy theory of fibrations.

We begin with a description of $G_*(\text{Baut}_1(X_\mathbb{Q}))$ in terms of derivations, assuming the identification: $G_n(\text{Baut}_1(X_\mathbb{Q})) \subseteq \pi_n(\text{Baut}_1(X_\mathbb{Q})) \cong H_{n-1}(\text{Der}(\wedge V; d))$.

Theorem 3.1 *A cycle $\theta \in \text{Der}^{n-1}(\wedge V; d)$ represents an element of $G_n(\text{Baut}_1(X_\mathbb{Q}))$ if and only if θ extends to a cycle $\hat{\theta}$ in $\text{Der}_{\wedge W}^{n-1}(\wedge W \otimes \wedge V; D)$ for every relative model $\wedge(W; \hat{d}) \rightarrow (\wedge W \otimes \wedge V; D)$.*

Proof A fibration ξ pulled back from the universal gives a factorization of monoids of fibrewise equivalences: $\text{aut}_1(p_\infty) \rightarrow \text{aut}_1(p) \rightarrow \text{aut}_1(X)$. The result now follows from Theorems 1 and 2.5.

Theorem 3.1 roughly implies that, the more ample the fibrations with fibre $X_\mathbb{Q}$, the fewer Gottlieb elements in $H_*(\text{Der}(\wedge V; d))$. When X is an H-space, fibrations with fibre X are abundant and we have:

Theorem 3.2 *Let X be a simply connected π -finite space with $X_{\mathbb{Q}}$ an H -space. Then $G_n(\text{Baut}_1(X_{\mathbb{Q}})) = 0$ for $n > N - 1$ and $G_{N-1}(\text{Baut}_1(X_{\mathbb{Q}})) \cong \pi_N(X_{\mathbb{Q}})$ where $N = \max(\pi_*(X_{\mathbb{Q}}))$.*

Proof The Sullivan minimal model for X has trivial differential. The differential δ for $\text{Der}(\wedge V; 0)$ is trivial as well. Let $\theta \in \text{Der}_n(\wedge V; 0)$ be a derivation. Suppose $\theta(x) \neq 0$ for some $x \in V^n$ with $n < N$. Take w to have degree $N - |x| + 1$ and set $D(v) = wx$ with D vanishing on a complementary subspace to $\langle v \rangle$ in V . For (ii), we choose an element $y \in V$ appearing in $\theta(x)$. We then let $|w| = |y| + 1$ and set $D(y) = z$ with D vanishing on a complementary subspace to $\langle y \rangle$ in V . In both cases, we see that θ does not extend to a cycle of $\text{Der}_{\wedge(w)}(\wedge(w) \otimes \wedge V; D)$, as needed. \square

We note that Theorem 3.2 can be proved easily from the various models for $\text{Baut}_1(X)$. We may extend the argument above to give the following:

Theorem 3.3 *Let X be a π -finite rational H -space and Y any π -finite space. Suppose $\max(\pi_*(X_{\mathbb{Q}})) < \max(\pi_*(Y_{\mathbb{Q}}))$. Then*

$$G_*(\text{Baut}_1(X_{\mathbb{Q}} \times Y_{\mathbb{Q}})) \subseteq G_*(\text{Baut}_1(Y_{\mathbb{Q}})).$$

Proof Write the Sullivan minimal model for X as $\wedge(V; 0)$ and Y as $\wedge(W'; d')$. Suppose $\theta \in \text{Der}_n(\wedge V \otimes \wedge W')$ is a cycle derivation satisfying either (i) $\theta(z) \in (\wedge V)^+ \cdot (\wedge V \otimes \wedge W')$ for some $z \in W'$ or (ii) $\theta(x) \neq 0$ for some $x \in V$. Define a relative model of the form $\wedge(w; 0) \rightarrow (\wedge(w) \otimes \wedge V \otimes \wedge W'; D)$ where the degree of w depends on the case. For (i) we pick $v \in V$ where $v \in V$ appears in $\theta(z)$. Extending $v = v_1$ to a basis of V we set $D(v_1) = w$ and $D(v_i) = 0$ for $i > 1$ with $D = d'$ on W' . For (ii), choose $z \in W'$ of maximal degree and set $D(z) = xw + d'(z)$. In either case, we see θ does not extend to a cycle of $\text{Der}_{\wedge(w)}(\wedge(w) \otimes \wedge V \otimes \wedge Z; D)$. \square

At the other extreme from H -spaces, in terms of admitting fibrations with a given fibre, are the F_0 -spaces by which we mean finite complexes X which are π -finite and satisfy $H^{\text{odd}}(X; \mathbb{Q}) = 0$. The Halperin Conjecture for F_0 -spaces asserts that $\text{Der}(H^*(X; \mathbb{Q})) = 0$ for all F_0 -spaces. The conjecture has been affirmed in many cases (see [3, Prob. 1, p. 516]).

Theorem 3.4 *Let X be an F_0 -space satisfying $\text{Der}(H^*(X; \mathbb{Q})) = 0$. Then*

$$G_*(\text{Baut}_1(X_{\mathbb{Q}})) = \pi_*(\text{Baut}_1(X_{\mathbb{Q}})).$$

Proof By [11, Pro.2.6], $\text{Baut}_1(X_{\mathbb{Q}})$ is an H -space and so the evaluation map $\omega: \text{aut}_1(\text{Baut}_1(X_{\mathbb{Q}})) \rightarrow \text{Baut}_1(X_{\mathbb{Q}})$ has a section given by left multiplication. \square

We turn to the evaluation subgroups of the classifying space. Let $\text{EF}(X)$ denote the set of fibre-homotopy equivalence classes of fibrations ξ with fibre X . The set $\text{EF}(X)$ is partially ordered by the relation induced by pull-backs. That is, we define $\xi \leq \xi'$ if ξ is fibre homotopy equivalent to the pullback of ξ' . Fixing a base space B , let $\text{EF}(X; B)$ denote the sub-poset consisting of fibrations ξ over B with fibre X . By the classification theory [2,9,13], the assignment: $h \mapsto \xi = h^{-1}(p_{\infty})$ induces a natural

bijection $[B, \text{Baut}_1(X)] \cong \text{EF}(X; B)$. By naturality, if $[B, \text{Baut}_1(X)]$ has the partial order corresponding to factorization of maps, i.e., $h \leq h'$ if there exists $f: B \rightarrow B$ with $h = h' \circ f$, the above identification is then an isomorphism of posets.

For any space Y , the Gottlieb group $G_*(Y)$ is the initial object of a poset under inclusion of subgroups of $\pi_*(Y)$ called the *evaluation subgroups* of Y . Let $h: B \rightarrow Y$ be any map and write $\omega: \text{map}(B, Y; h) \rightarrow Y$ for the evaluation map for the component of the function space. Define

$$G_n(Y; B, h) = \text{im}\{\omega_{\#}: \pi_n(\text{map}(B, Y; h)) \rightarrow \pi_n(Y)\} \subseteq \pi_n(Y).$$

Given maps $h: B \rightarrow Y$ and $h': B' \rightarrow Y$, we see a factorization $h = h' \circ f$ for $f: B \rightarrow B'$ implies the reverse inclusion $G_*(Y; B', h') \subseteq G_*(Y; B, h)$ of evaluation subgroups.

When $Y = \text{Baut}_1(X)$, the evaluation subgroups are parametrized by equivalence classes of fibrations ξ with fibre X . Write

$$G_n(\xi; X) = G_n(\text{Baut}_1(X); B, h) \subseteq \pi_n(\text{Baut}_1(X))$$

where $h: B \rightarrow \text{Baut}_1(X)$ is the classifying map. The assignment $\xi \mapsto G_*(\xi; X)$ from the poset $\text{EF}(X)$ to the evaluation subgroups of $\text{Baut}_1(X)$, partially ordered by inclusion, is order-reversing.

In [17], Yamaguchi introduced a related poset $G_*^{\xi}(X)$ of the Gottlieb group $G_*(X)$. Yamaguchi’s groups are recovered, with a shift in degrees, as images:

$$G_n^{\xi}(X) = \text{im}\{\Gamma: G_{n+1}(\xi; X) \rightarrow \pi_n(X)\} \subseteq G_n(X)$$

where Γ is the restriction of $\omega_{\#}: \pi_n(\text{aut}_1(X)) \rightarrow \pi_n(X)$ pre-composed with the isomorphism $\pi_{n+1}(\text{Baut}_1(X)) \cong \pi_n(\text{aut}_1(X))$. We have the identifications:

Theorem 3.5 *Let X be π -finite and ξ be a fibration of simply connected spaces with fibre X with relative Sullivan model $(\wedge W; \hat{d}) \rightarrow (\wedge W \otimes \wedge V; D)$. Then*

$$G_{n+1}(\xi_{\mathbb{Q}}; X_{\mathbb{Q}}) \cong \text{im}\{H(P_*) : H_n(\text{Der}_{\wedge W}(\wedge W \otimes \wedge V; D)) \rightarrow H_n(\text{Der}(\wedge V; d))\}$$

$$G_n^{\xi_{\mathbb{Q}}}(X_{\mathbb{Q}}) \cong \text{im}\{\varepsilon^* \circ H(P_*) : H_n(\text{Der}_{\wedge W}(\wedge W \otimes \wedge V; D)) \rightarrow \text{Hom}(V^n; \mathbb{Q})\}$$

with P_* induced by composition with the projection $P: \wedge W \otimes \wedge V \rightarrow \wedge V$ and ε^* by composition with an augmentation $\varepsilon: \wedge V \rightarrow \mathbb{Q}$.

Proof The first result follows from Theorem 2.5 and the naturality of this identification. The second result is [17, Th. 1.4]. □

We give some examples and results concerning the poset $G_*(\xi; X_{\mathbb{Q}})$. Given a set A , write $\mathcal{P}(A) = \mathcal{P}(A), \subseteq$ for the power set of partially ordered by inclusion. We will make use of the order-preserving bijection $\mathcal{P}(\{1, \dots, n\}) \cong \mathbb{Z}_2^n$, where the latter set has the cartesian product partial order.

Example 3.6 Let $X = S^3 \times S^5 \times S^7$. We show that the poset $G_*(\xi; X_{\mathbb{Q}})$ is isomorphic to the power set $\mathcal{P}(1, 2, 3, 4)$. Write the Sullivan minimal model for X as $\wedge(V; d) = \wedge(x_3, y_5, z_7; 0)$ with subscripts denoting degrees. Then

$$H_*(\text{Der}(\wedge V; 0)) = \text{Der}_*(\wedge V) = \langle z^*, y^*, (z, x), x^*, (z, y), (y, x) \rangle.$$

As our base space, we take $B = \text{Baut}_1(X_{\mathbb{Q}})$ which has Sullivan minimal model $\wedge(W; d) = \wedge(w_1, w_2, w_3, w_4, w_5, w_6; d)$ with $|w_1| = 4, |w_2| = 6, |w_3| = 5, |w_4| = 3, |w_5| = 8, |w_6| = 8$ with $d(w_i) = 0$ for $i = 1, \dots, 4, d(w_5) = -w_3w_1$ and $d(w_6) = -w_4w_2$. We obtain a family of relative Sullivan models:

$$\xi_{(q_1, q_2, q_3, q_4)}: \wedge(W; d) \rightarrow (\wedge W \otimes \wedge V; D)$$

by setting $D(x) = q_1w_1, D(y) = q_2w_2, D(z) = q_3w_3x + q_4w_4y + q_1q_3w_5 + q_2q_4w_6$ for $q_i = 0$ or 1 . The order-reversing map $(q_1, q_2, q_3, q_4) \mapsto G_*(\xi_{(q_1, q_2, q_3, q_4)}; X_{\mathbb{Q}})$ then gives a bijection from \mathbb{Z}_2^4 to the set of distinct evaluation subgroups $G_*(\xi; X_{\mathbb{Q}})$. For in any relative model $\wedge(W; \hat{d}) \rightarrow (\wedge W \otimes V; D)$, if $D(x) \neq 0$ then (z, x) and (y, x) are both non-cycles. On the other hand, $D(x) = 0$ implies y^* and (y, x) are either both non-cycles or both are cycles depending on the occurrence or non-occurrence of a non-zero term wy in $D(z)$.

Following Yamaguchi [17, Def.1.12], define the *depth* of the poset $G_*(\xi; X)$ over a base space B , written $\text{depth}_B G_*(\xi; X)$, to be the number n in the longest proper chain of subgroups

$$G_*(\xi_0; X) \subsetneq \dots \subsetneq G_*(\xi_n; X)$$

with each ξ_i a fibration over B with fibre X . Example 3.6 gives $\text{depth} = 4$ for $G_*(\xi; X_{\mathbb{Q}})$ over $\text{Baut}_1(X_{\mathbb{Q}})$. Here is one maximal chain:

(q_1, q_2, q_3, q_4)	$(1, 1, 1, 1)$	$(0, 1, 1, 1)$	$(0, 0, 1, 1)$	$(0, 0, 0, 1)$	$(0, 0, 0, 0)$
$G_*(\xi_{(q_1, q_2, q_3, q_4)}; X_{\mathbb{Q}})$	z^*	$z^*, (z, x)$	$z^*, (z, x), (z, y)$	$z^*, (z, x), (z, y), y^*$	$z^*, (z, x), (z, y), x^*, y^*, (y, x)$

For a finite H-space X and any space B , by [17, Ex.5.2]:

$$\text{depth}_B G_*^{\xi}(X_{\mathbb{Q}}) = \dim(\pi_*(X_{\mathbb{Q}})) - \dim(\pi_N(X_{\mathbb{Q}}))$$

where $N = \max(\pi_*(X_{\mathbb{Q}}))$. Example 3.6 thus implies a strict inequality:

$$2 = \text{depth}_B(G_*^{\xi}(X_{\mathbb{Q}})) < \text{depth}_B(G_*(\xi; X_{\mathbb{Q}})) = 4$$

with $B = \text{Baut}_1(X_{\mathbb{Q}})$. In fact, we can deduce that

Proposition 3.7 *Given any $M > 0$ there exists a π -finite H-space X and a base space B such that*

$$\text{depth}_B(G_*(\xi; X_{\mathbb{Q}})) > \text{depth}_B(G_*^{\xi}(X_{\mathbb{Q}})) + M.$$

Proof Given spaces X and Y , the product fibration implies the relation:

$$\text{depth}_B(G_*(\xi; X \times Y)) \geq \text{depth}_B(G_*(\xi; X)) + \text{depth}_B(G_*(\xi; Y))$$

(see [17, Lem.1.13]). In particular,

$$\text{depth}_B(G_*(\xi; X^m_{\mathbb{Q}})) \geq 4m \text{ while } \text{depth}_B(G_*^{\xi}(X^m_{\mathbb{Q}})) = 2m$$

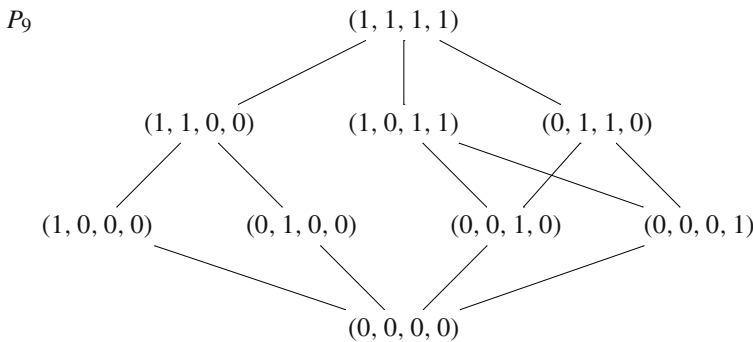
with $X = S^3 \times S^5 \times S^7$ as in Example 3.6 and $B = \text{Baut}_1(X_{\mathbb{Q}})$. □

Changing the degree of just one generator in Example 3.6 gives a more complicated example:

Example 3.8 Let $X = S^3 \times S^5 \times S^9$ with Sullivan model $\wedge(x_3, y_5, z_9; 0)$. Then

$$\text{Der}_*(\wedge V) = \langle z^*, y^*, x^*, (z, x), (y, x), (z, y), (z, xy) \rangle.$$

We show the full poset of evaluation subgroups $G_*(\xi; X_{\mathbb{Q}})$ of $\pi_*(\text{Baut}_1(X_{\mathbb{Q}}))$ is isomorphic to $P_9 \times \mathbb{Z}_2$ where $P_9 \subseteq \mathbb{Z}_2^4$ has Hasse diagram:



We explain this briefly. Let $\wedge(W; d) \rightarrow (\wedge W \otimes \wedge V; D)$ be a relative Sullivan model. Then $z^* \in G_*(\xi; X_{\mathbb{Q}})$ automatically. Let $(a_1, a_2, a_3, a_4) \in \mathbb{Z}_2^4$ record the membership status of the derivations $y^*, x^*, (z, x), (y, x)$ in $G_*(\xi; X_{\mathbb{Q}})$ in this order. We claim the vectors representing realizable subsets of $G_*(\xi; X_{\mathbb{Q}})$ correspond to P_9 . Suppose $a_1 = 0, a_2 = 1$ so that y^* is not a δ -cycle and x^* is one. Then $D(z)$ has a term involving y alone which implies (y, x) is a non-cycle ($a_4 = 0$). However, (z, x) is unconstrained as (z, x) is a cycle exactly when $D(x) = 0$. On the other hand, when $a_1 = a_2 = 0$ we can suppose $D(z)$ has a term xy and neither x nor y appear elsewhere in the image of D . Such a term does not obstruct (y, x) from being

a cycle since $x^2 = 0$. Then, in this case, (y, x) and (z, x) are both cycles exactly when $D(x) = 0$ and so a_3 and a_4 are unconstrained. The allowable vectors with $a_1 = 0$ are thus $(0, 1, 1, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 1)$, $(0, 0, 1, 0)$, $(0, 0, 0, 0)$. When $a_1 = 1$, the only constraint is that $a_3 = a_4$ and we obtain the other four vectors in P_9 , namely $(1, 1, 1, 1)$, $(1, 0, 1, 1)$, $(1, 1, 0, 0)$, $(1, 0, 0, 0)$.

Now observe that (z, y) is a cycle precisely when $D(y) = 0$. The vanishing or non-vanishing of $D(y)$ can be achieved independently of the terms in D that affect the membership of $y^*, x^*, (z, x), (y, x)$. Also (z, xy) is a cycle exactly when both (z, x) and (z, y) are cycles. It remains to check that all sets described can be realized as $G_*(\xi; X_{\mathbb{Q}})$ for some ξ . This is straightforward if laborious.

In Example 3.8, the depth of $G_*(\xi; X_{\mathbb{Q}})$ over the classifying space $Baut_1(X_{\mathbb{Q}})$ can be seen to be 3 while the depth of the full poset of evaluation subgroups is 4. The following result implies the maximal depth of $G_*(\xi; X_{\mathbb{Q}})$ over all base spaces is the depth of the full poset $G_*(\xi; X_{\mathbb{Q}})$.

Theorem 3.9 *Let X be a π -finite space. Then there exists a base space B such that the depth of $G_*(\xi; X_{\mathbb{Q}})$ over B equals the length of the longest chain in the poset $G_*(\xi; X_{\mathbb{Q}})$.*

Proof Let $\xi_0, \xi_1, \dots, \xi_n$ be fibrations with fibre X giving a maximal chain of evaluation subgroups. Writing $p_i: E_i \rightarrow B_i$ for ξ_i we set $B = B_0 \times \dots \times B_n$. Let ξ'_i denote the fibration $p'_i: E_i \rightarrow B$ given by the composition of p_i with the inclusion $B_i \rightarrow B$. Then we see $G_*(\xi_i; X_{\mathbb{Q}}) = G_*(\xi'_i; X_{\mathbb{Q}})$. □

When X is an F_0 -space with $\text{Der}(H^*(X; \mathbb{Q})) = 0$ the poset $G_*(\xi; X_{\mathbb{Q}})$ is trivial. For in this case, $Baut_1(X_{\mathbb{Q}})$ is an H-space and so the evaluation map

$$\omega: \text{map}(B, Baut_1(X_{\mathbb{Q}}); h_{\mathbb{Q}}) \rightarrow Baut_1(X_{\mathbb{Q}})$$

has a section. It follows that $G_*(\xi; X_{\mathbb{Q}}) = \pi_*(Baut_1(X_{\mathbb{Q}}))$ for all ξ . We give an example mixing even and odd spheres

Example 3.10 Let $X = S^3 \times S^4 \times S^6 \times S^9$. We show the poset $G_*(\xi; X_{\mathbb{Q}})$ is isomorphic to \mathbb{Z}_2^4 . Write the minimal model for X as $\wedge(x_3, u_4, t_6, v_7, y_9, z_{11}; d)$ with subscripts indicating degrees and $d(u) = d(w) = d(y) = 0, d(v) = u^2, d(z) = t^2$. In this case:

$$H_*(\text{Der}(\wedge V; d)) = \langle z^*, y^*, (z, x), (z, u), v^*, (y, u), (y, t)(z, xu), x^*, (y, xu), (z, y), (z, ut), (v, t) \rangle.$$

We also have $G_*(Baut_1(X_{\mathbb{Q}})) = \langle z^*, v^*, (z, u), (v, x) \rangle$. Thus $G_*(\xi_{\mathbb{Q}}; X_{\mathbb{Q}})$ contains these cycles for any ξ . The inclusion or exclusion of $y^*, (z, y), x^*, (z, x)$ in $G_*(\xi; X_{\mathbb{Q}})$ gives the poset \mathbb{Z}_2^4 . The status of $(y, x), (y, u), (y, t), (z, xu), (y, xu)$, as regards membership in $G_*(\xi; X_{\mathbb{Q}})$, depends on the status of these four. Precisely, (y, x) and (y, xu) are cycles exactly when both y^* and (z, x) are cycles, (y, u) and (y, t) are cycles exactly when y^* is a cycle, and (z, xu) is a cycle exactly when (z, x) is one.

We conclude this section by observing that the depth of the poset $\text{EF}(X_{\mathbb{Q}})$ of all fibre homotopy equivalence classes of fibrations with fibre $X_{\mathbb{Q}}$ is bounded below by the depth of the poset of evaluation subgroups of $\text{Baut}_1(X_{\mathbb{Q}})$. The difference can be made arbitrarily large.

Proposition 3.11 *Given any $M > 0$ there exists a π -finite space X such that*

$$\text{depth EF}(X_{\mathbb{Q}}) \geq \text{depth } G_*(\xi_{\mathbb{Q}}; X_{\mathbb{Q}}) + M.$$

Proof Let $m = M + 2$ and consider $X = \mathbb{C}P^m$. Then $\text{Baut}_1(X_{\mathbb{Q}})$ is an H-space and it is direct to compute that $\dim(\pi_*(\text{Baut}_1(X_{\mathbb{Q}}))) = m - 1$. Write $\pi_*(\text{Baut}_1(X_{\mathbb{Q}})) = \langle x_1, \dots, x_{m-1} \rangle$ in a basis. We may factor the trivial self-map of $\text{Baut}_1(X_{\mathbb{Q}})$ as a composition $h_1 \circ \dots \circ h_{m-1}$ such that the $m - 1$ compositions $H_k = h_1 \circ \dots \circ h_k$ for $k = 1, \dots, n$ are not homotopic. We do this by defining h_k to be the map with $(h_k)_\#(x_i) = x_i$ for $i = 1, \dots, k$ and $(h_k)_\#(x_j) = 0$ for $j > k$. We conclude that $\text{depth}(\text{EF}(X_{\mathbb{Q}})) \geq M$ while $\text{depth}(G_*(\xi_{\mathbb{Q}}; X_{\mathbb{Q}})) = 0$. \square

4 A non-realization result for the classifying space

An open question in rational homotopy theory asks:

Question 4.1 [3, p. 519] *Is every simply connected rational homotopy type $Y_{\mathbb{Q}}$ realized as a classifying space in the sense that $Y_{\mathbb{Q}} \simeq \text{Baut}_1(X_{\mathbb{Q}})$ for some simply connected space X ?*

In [8], we proved certain rational homotopy types, including $\mathbb{C}P^2_{\mathbb{Q}}$, could not be realized if X is restricted to be a π -finite space. Thus to realize these rational types as a classifying space requires X with infinite-dimensional rational homotopy. In this section, we describe the relative Sullivan model of the universal fibration under the assumption that there is a space X with $\text{Baut}_1(X_{\mathbb{Q}}) \simeq \mathbb{C}P^n_{\mathbb{Q}}$ (Proposition 4.2). We apply this description to prove Theorem 2, that $\mathbb{C}P^n_{\mathbb{Q}}$ cannot be realized as $\text{Baut}_1(X_{\mathbb{Q}})$ for any π -finite X and $n = 2, 3$ or 4 .

Write the minimal Sullivan model for $\mathbb{C}P^n$ as $\wedge(u_2, v_{2n+1}; \hat{d})$ with $\hat{d}(v) = u^{n+1}$. Suppose first that X is a simply connected space with minimal model $\wedge(V; d)$. A fibration $X \rightarrow E \rightarrow \mathbb{C}P^n$ has relative Sullivan model of the form: $\wedge(u, v; \hat{d}) \rightarrow (\wedge(u, v) \otimes \wedge V; D)$. Let $\chi \in \wedge V$ and use the minimality condition for D to write:

$$D(\chi) = d(\chi) + u\theta_u(\chi) + u^2\theta_{u^2}(\chi) + \dots + v\theta_v(\chi) + vu\theta_{vu}(\chi) + vu^2\theta_{vu^2}(\chi) + \dots$$

The maps $\theta_{u^k}, \theta_{v u^k}$ extend to derivations of $\wedge V$ of degrees $2k - 1$ and $2(n + k) + 5$, respectively. Taking $D^2 = 0$ and equating terms with like powers in the generators gives a sequence of relations involving brackets and differentials amongst these derivations. In particular, we have that $\delta(\theta_u) = \delta(\theta_v) = 0$. Any set of derivations satisfying these identities gives a rational fibration with fibre $X_{\mathbb{Q}}$. We have:

Proposition 4.2 *Suppose there exists a simply connected space X with Sullivan model $\wedge(V; d)$ such that $Baut_1(X_{\mathbb{Q}}) \simeq \mathbb{C}P_{\mathbb{Q}}^n$. Then there exists a relative Sullivan model $\wedge(u, v; \hat{d}) \rightarrow (\wedge(u, v) \otimes \wedge V; D_{\infty})$ as above with $\theta_u \in \text{Der}^1(\wedge V; d)$ and $\theta_v \in \text{Der}^{2n}(\wedge V; d)$ non-bounding δ -cycles. Conversely, any relative model $\wedge(u, v; \hat{d}) \rightarrow (\wedge(u, v) \otimes \wedge V; D)$ with θ_u non-bounding is a relative Sullivan model for the universal fibration with fibre X and so, in this case, θ_v is automatically non-bounding.*

Proof By the description of the differential D_{∞} given in (1) we see that θ_u and θ_v represent the non-trivial classes in $\pi_*(\Omega \mathbb{C}P_{\mathbb{Q}}^n \cong H_*(\text{Der}(\wedge V; d)))$. Now suppose we are given a relative Sullivan model $\wedge(u, v; \hat{d}) \rightarrow (\wedge(u, v) \otimes \wedge V; D)$ with θ_u not a δ -boundary. The corresponding rational fibration has classifying map $h: \mathbb{C}P_{\mathbb{Q}}^n \rightarrow Baut_1(X_{\mathbb{Q}}) \simeq \mathbb{C}P_{\mathbb{Q}}^n$. Since θ_u is not a δ -boundary, h induces an isomorphism on degree two homotopy groups, again by (1). It follows that h is a homotopy equivalence and the given relative Sullivan model is fibre homotopy equivalent to that of the universal. \square

For the remainder of the paper, we suppose X is π -finite with $Baut_1(X_{\mathbb{Q}}) \simeq \mathbb{C}P_{\mathbb{Q}}^n$. Write $\wedge(V; d)$ for the Sullivan minimal model for X . Then $V^{2n} \cong \mathbb{Q}$ and $V^q = 0$ for $q > 2n$ by Proposition 2.3. Let $\wedge(u, v; \hat{d}) \rightarrow (\wedge(u, v) \otimes \wedge V; D_{\infty})$ denote the relative Sullivan model for the universal fibration with fibre X . For degree reasons, the only possible non-vanishing derivations are $\theta_u, \theta_{u^2}, \dots, \theta_{u^n}, \theta_v$ of degrees $1, 3, 5, \dots, 2n - 1$ and $2n$, respectively, where the last two are linear maps: $\theta_{u^n}: V^{2n-1} \rightarrow \mathbb{Q}$ and $\theta_v: V^{2n} \rightarrow \mathbb{Q}$. The identities arising from the equation $D_{\infty}^2 = 0$ are as follows:

$$\begin{aligned}
 u\text{-terms} \quad & \delta(\theta_u) = 0 \\
 u^k\text{-terms} \quad & \delta(\theta_{u^k}) = \sum_{i+j=k, i \leq j} [\theta_i, \theta_j] \text{ for } k = 2, \dots, n + 1 \\
 v\text{-terms} \quad & \delta(\theta_v) = 0
 \end{aligned}
 \tag{2}$$

Proposition 4.2 may be refined, in this case, to the following:

Lemma 4.3 *In the relative Sullivan model $\wedge(u, v; \hat{d}) \rightarrow (\wedge(u, v) \otimes \wedge V; D_{\infty})$ the derivation $\theta_u \in \text{Der}^1(\wedge V)$ is not a δ -boundary and $\theta_v \neq 0$. Conversely, any collection $\theta_u, \theta_{u^2}, \dots, \theta_{u^n}, \theta_v$ satisfying the identities (2) with θ_u not a δ -boundary is a relative Sullivan model for the universal fibration with fibre X . Consequently, $\theta_v \neq 0$. \square*

We show that altering θ_u by a boundary yields a compatible collection of derivations:

Lemma 4.4 *Let $\varphi \in \text{Der}^2(\wedge V)$. There is a relative Sullivan model for the universal fibration with fibre X with derivations given by $\theta'_u, \theta'_{u^2}, \dots, \theta'_{u^n}, \theta'_v$ with*

$$\theta'_u = \theta_u + \delta(\varphi).$$

Proof Since θ'_u is a cycle in $\text{Der}^1(\wedge V)$ and $H_*(\text{Der}(\wedge V; d))$ is concentrated in degrees 1 and $2n$, the derivation cycle $2[\theta'_u, \theta'_u] \in \text{Der}^2(\wedge V)$ must be a δ -boundary. Thus we

can choose $\theta'_{u^2} \in \text{Der}^3(\wedge V)$ with $\delta(\theta'_{u^2}) = 2[\theta'_u, \theta'_u]$. Next observe $[\theta'_u, \theta'_{u^2}]$ is a δ -cycle and so a δ -boundary. Thus we can find $\theta'_{u^3} \in \text{Der}^5(\wedge V)$ with $\delta(\theta'_{u^3}) = [\theta'_u, \theta'_{u^2}]$. Continuing in this manner, we obtain a collection θ'_{u^k} for $k = 1, \dots, n$ satisfying all but the last identity in (2). Finally, set $\theta'_v = \sum_{i+j=n, i \leq j} [\theta'_{u^i}, \theta'_{u^j}]$. By Proposition 4.3, these derivations give a relative Sullivan model for the universal fibration. \square

Regarding the differential d , we have a quadratic pairing:

Lemma 4.5 *Let $y \in V^{2n} \cong \mathbb{Q}$ be nontrivial. Given a basis $\{z_1, \dots, z_n\}$ for V^{2n-1} there is a corresponding basis $\{x_1, \dots, x_n\}$ for V^2 so that*

$$d(y) = x_1z_1 + x_2z_2 + \dots + x_nz_n + \text{terms not involving any } z_j.$$

Proof The derivations z_j^* in $\text{Der}^{2n-1}(\wedge V)$ cannot be cycles for it is not possible for these derivations to be boundaries. Thus each z_j must appear in $d(y)$ and we have a pairing as above. If there is some $x \in V^2$ not in the span of $\{x_1, \dots, x_n\}$ then (y, x) is a non-bounding cycle of degree $2n - 2$, a contradiction. \square

The quadratic part of $d(y)$ also has terms involving elements of V^3 and V^{2n-2} :

Lemma 4.6 *Given $w \in V^3$ there is $\bar{w} \in V^{2n-2}$ such that $w\bar{w}$ appears in $d(y)$ and \bar{w} does not appear in other terms of $d(y)$.*

Proof Write $d(w) = \sum_{i=1}^n q_i x_i x'_i$ for some $x'_i \in V^2$. Define $\theta \in \text{Der}^{2n-3}(\wedge V)$ by the formula

$$\theta = (y, w) - \sum_{i=1}^n q_i (z_i, x'_i).$$

We see $\delta(\theta) = 0$ and so $\theta = \delta(\alpha)$ for some $\alpha \in \text{Der}^{2n-2}(\wedge V)$. Then $\delta(\alpha(y)) = -\alpha(d(y)) = w$ implies $\alpha = \bar{w}^* + \alpha'$ for some $\bar{w} \in V^{2n-2}$, $\alpha'(V^4) = 0$. Further we must have the term $w\bar{w}$ with \bar{w} as specified. \square

We apply the preceding to deduce:

Lemma 4.7 *In the relative Sullivan model for the universal fibration with fibre X , we may assume that $\theta_u(y)$ decomposable in $\wedge V$ for $y \in V^{2n}$ nontrivial.*

Proof Suppose $\theta_u(y) = z + \chi$ for some $z \in V^{2n-1}$ and χ decomposable. Taking $z = z_1$ and extending to a basis, we set $\theta'_u = \theta_u - \delta(x_1^*)$ with $x_1 \in V^2$ as in Lemma 4.5. Then $\theta'_u(y)$ is decomposable. Now apply Lemma 4.4 to obtain a compatible collection with θ'_u for the relative model of the universal fibration. \square

Regarding θ_{u^2} , we have:

Lemma 4.8 *If $\theta_u(y)$ is decomposable for $y \in V^{2n}$ nontrivial, then θ_{u^2} vanishes on V^3 .*

Proof Suppose $w \in V^3$ satisfies $\theta_{u^2}(w) = 1$. Then $D_\infty(w) = u^2 + \theta_u(w)u + d(w)$. Consider the term $w\bar{w}$ occurring in $d(y)$ with $\bar{w} \in V^{2n-2}$ from Lemma 4.8. This term occurs as a summand of $D_\infty(y)$. Applying D_∞ again gives a summand $u^2\bar{w}$ in $D_\infty^2(y)$. We claim that this term cannot be cancelled. For note, for degree reasons, $u\bar{w}$ can only occur in $D_\infty(z)$ for $z \in V^{2n-1}$. Since $\theta_u(y)$ is indecomposable we cannot have a corresponding term uz in $D_\infty(y)$. \square

We apply these results to prove there is no π -finite X with $Baut_1(X_\mathbb{Q}) \simeq \mathbb{C}P_\mathbb{Q}^n$ for $n = 2, 3, 4$

Proof of Theorem 2 By Lemma 4.7, we may assume $\theta_u(V^{2n}) \subseteq \wedge^+ V \cdot \wedge^+ V$. By Lemma 4.8, this implies $\theta_{u^2}(V^3) = 0$. The formulas for $\theta_v: V^{2n} \rightarrow \mathbb{Q}$ given in Eq. (2) for the cases $n = 2, 3, 4$ are as follows.

$$\begin{aligned} n = 2 : \theta_v &= [\theta_u, \theta_{u^2}] \\ n = 3 : \theta_v &= [\theta_u, \theta_{u^3}] + 2[\theta_{u^2}, \theta_{u^2}] \\ n = 4 : \theta_v &= [\theta_u, \theta_{u^4}] + [\theta_{u^2}, \theta_{u^3}] \end{aligned}$$

Let $y \in V^{2n}$. Then $\theta_u(y)$ decomposable implies $[\theta_u, \theta_{u^n}](y) = 0$ in each case. Also, $\theta_{u^2}(V^3) = 0$ implies $[\theta_{u^2}, \theta_{u^n}](y) = 0$. Thus, in all three cases, $\theta_v = 0$, contradicting Lemma 4.3. \square

References

- Booth, Peter, Heath, Philip, Morgan, Chris, Piccinini, Renzo: H -spaces of self-equivalences of fibrations and bundles. Proc. Lond. Math. Soc. (3) **49**(1), 111–127 (1984)
- Dold, Albrecht: Halbexakte Homotopiefunktoren. Lecture Notes in Mathematics, vol. 12. Springer-Verlag, Berlin (1966)
- Félix, Yves, Halperin, Stephen, Thomas, Jean-Claude: Rational Homotopy Theory, Graduate Texts in Mathematics, vol. 205. Springer-Verlag, New York (2001)
- Félix, Yves, Lupton, Gregory, Smith, Samuel B.: The rational homotopy type of the space of self-equivalences of a fibration. Homol. Homotopy Appl. **12**(2), 371–400 (2010)
- Gatsinzi, J.-B.: The homotopy Lie algebra of classifying spaces. J. Pure Appl. Algebra **120**(3), 281–289 (1997)
- Gottlieb, D.H.: Evaluation subgroups of homotopy groups. Am. J. Math. **91**, 729–756 (1969)
- Gregory Lupton and Samuel Bruce Smith: The evaluation subgroup of a fibre inclusion. Topol. Appl. **154**(6), 1107–1118 (2007)
- Gregory Lupton and Samuel Bruce Smith: Realizing spaces as classifying spaces. Proc. Am. Math. Soc. **144**(8), 3619–3633 (2016)
- May, J.Peter: Classifying spaces and fibrations. Mem. Am. Math. Soc. **1**, 155, xiii+98 (1975)
- May, J.Peter: Fibrewise localization and completion. Trans. Am. Math. Soc. **258**(1), 127–146 (1980)
- Meier, W.: Rational universal fibrations and flag manifolds. Math. Ann. **258**(3), 329–340 (1981)
- Schlessinger, Mike, Stasheff, James: Deformation theory and rational homotopy type, preprint
- Stasheff, James: A classification theorem for fibre spaces. Topology **2**, 239–246 (1963)
- Sullivan, Dennis: Infinitesimal computations in topology. Inst. Hautes Études Sci. Publ. Math. (1977) **47**, 269–331 (1978)
- Tanré, Daniel: Homotopie rationnelle: modèles de Chen, Quillen, Sullivan. Lecture Notes in Mathematics, vol. 1025. Springer-Verlag, Berlin (1983)
- Xie, Sang, Liu, Jian, Liu, Xiugui: Spaces realized and non-realized as classifying spaces, preprint

17. Yamaguchi, Toshihiro: A fibre-restricted Gottlieb group and its rational realization problem. *Topology Appl.* **196**, (2015)

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