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Equivariant formality of isotropic torus actions

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Abstract Considering the potential equivariant formality of the left action of a connected Lie group K on the homogeneous space G/K, we arrive through a sequence of reductions at the case G is compact and simply-connected and K is a torus. We then classify all pairs (G, S) such that G is compact connected Lie and the embedded circular subgroup S acts equivariantly formally on G/S. In the process we provide what seems to be the first published proof of the structure (known to Leray and Koszul) of the cohomology rings

1 Introduction

A natural request of a continuous group action $G \times X \longrightarrow X$ is that it be *equivariantly formal*, meaning the fiber inclusion in the Borel fibration $X \to X_G \to BG$ induces a surjection $H^*_G(X; \mathbb{Q}) \twoheadrightarrow H^*(X; \mathbb{Q})$ of Borel equivariant cohomology upon singular cohomology. While the term was only coined in 1997 by Goresky, Kottwitz, and MacPherson [28], the condition had already been alighted upon by Borel in Chapter XII of his Seminar [7]. This condition makes available a comparatively tractable computation of $H^*_G(X; \mathbb{Q})$ in terms of G-orbits of dimensions zero and one in the case there are only finitely many of each, as well as, by definition, guaranteeing all classes of $H^*(X; \mathbb{Q})$ have equivariant extensions in $H^*_G(X; \mathbb{Q})$, to which, for example, the localization theorems of Berline–Vergne/Atiyah–Bott [3,4] and Jeffrey–Kirwan [35] can be applied.

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As any orbit of a continuous action of a Lie group G on a space X, is a homogeneous space $G/\operatorname{Stab}_G(x)$, it is natural to ask about equivariantly formal actions on such spaces. The transitive G-action is only equivariantly formal if the isotropy group $K = \operatorname{Stab}_G(x)$ is of full rank, but some restriction of this action to a subgroup H will always be equivariantly formal. For this to happen, H cannot contain a strictly larger maximal torus than K does, so that the left action of K is in some sense the "largest" action on G/K which could conceivably be equivariantly formal. Assuming that G is compact, it is known that the isotropy action of K on G/K is equivariantly formal if K is of full rank in G ([10], Proposition 1), if $H^*(G; \mathbb{Q}) \longrightarrow H^*(K; \mathbb{Q})$ is surjective ([47]. Thm. A, Cor. 4.2), or if (G, K) is a generalized symmetric pair with K connected [27], but otherwise few examples of such actions seem to be known. Nevertheless, the full-rank case has found wide application in symplectic geometry (see, e.g., the book of Ginzburg, Guillemin, and Karshon [25], in which equivariant cohomology is already mentioned in the first page of the introduction and occupies a thirty-one–page Appendix).

We show this question can be reduced to the case K is a torus. For concision, if the isotropy action of K on G/K is equivariantly formal, we call the pair (G, K) *isotropy-formal*.

Theorem 1.1 *If G is a compact Lie group, K a closed, connected subgroup, and S any torus maximal within K, then (G, K) is isotropy-formal if and only if (G, S) is.*

This result reduces the question to a study of embeddings of tori in Lie groups, an already more feasible-looking endeavor. Further, the question reduces to the case the commutator subgroup of G is simply-connected.

Theorem 1.2 *Let G be a compact, connected Lie group, K a closed, connected subgroup, $\widetilde{G}$ a finite central covering of G, and $\widetilde{K}_0$ the identity component of the preimage of K in $\widetilde{G}$. Then (G, K) is isotropy-formal if and only if $(\widetilde{G}, \widetilde{K}_0)$ is.*

The question is largely dependent on the case G where itself is simply-connected.

Theorem 1.3 *Let G be a compact, connected Lie group, G' its commutator subgroup, K a closed connected subgroup of G, and S a maximal torus in K. Write $K' = K \cap G'$ and $S' = S \cap G'$ for the intersections with G' and K'_0 and S'_0 for their respective identity components. Then (G, K) is isotropy-formal if and only if*

1. *the pair (G', K'_0) is isotropy-formal and*
2. *the inclusion $N_G(S) \hookrightarrow N_G(S'_0)$ induces an isomorphism of component groups.*

These reductions are proven in Sect. 3, with some additional partial reductions having to do with disconnected goups and general compact Hausdorff groups expounded in Appendix B. The reductions achieved, in Sect. 4, we are able to completely determine for G any compact, connected Lie group and S any circular subgroup whether (G, S) is isotropy-formal. A key condition turns out to be that there exist an element of the G conjugation by which acts as $s \longmapsto s^{-1}$ on S. We say such an element *reflects* S.

Proposition 1.4 *Let G be a compact, connected Lie group and S a circular subgroup of G. There are the following three mutually exclusive cases.*

Table 1 Reflected circles in simple Lie groups

Type of K	The circle S in K is reflected …
A_n	When the exponent multiset J satisfies $J = -J$
B_n	Always
C_n	Always
D_{2n}	Always
D_{2n+1}	If S is contained in a D_{2n} subgroup
G_2	Always
F_4	Always
E_6	If S is contained in a D_4 subgroup
E_7	Always
E_8	Always

1. The inclusion $S \longrightarrow G$ surjects in cohomology and S is not reflected in G.
2. The inclusion $S \longrightarrow G$ is trivial in cohomology and
  2a. S is reflected in G.
  2b. S is not reflected in G.

Only in the last case is (G, S) not isotropy-formal.

Reflected circles can classified entirely, and from Propositions 1.4, 4.2, 4.5, and 4.6, one assembles the following result.

Theorem 1.5 *Let G be a compact, connected Lie group and S a circular subgroup of G. If S is not contained in the commutator subgroup G' of G, then (G, S) is isotropy-formal. Otherwise, we may assume by Theorem 1.2 that G' is a product of simple Lie groups K_j. Pick for each a maximal torus containing the image S_j of $S \hookrightarrow G' \twoheadrightarrow K_j$. Then (G, S) is isotropy-formal if and only if for each K_j there is an element of the Weyl group $W(K_j)$ reflecting S_j, which is determined as laid out in Table 1.*

This table is compiled in Sect. 4.2.

Remark 1.6 (Explanatory remarks on Table 1) The notation J in the A_n case is the multiset of exponents $a_1, \ldots, a_n \in \mathbb{Z}$ such that the injection $S^1 \rightarrowtail \mathrm{U}(1)^{\oplus n} \hookrightarrow \mathrm{U}(n)$ realizing a conjugate of S as a circular subgroup of the block-diagonal maximal torus of $\mathrm{U}(n)$ is given by $z \longmapsto \mathrm{diag}(z^{a_1}, \ldots, z^{a_n})$. We write $-J$ for the multiset $\{-a_j\}_{1 \leqslant j \leqslant n}$ whose entries are the opposites of those of J; that is to say, for each $a \in \mathbb{Z}$, the element $-a$ occurs in $-J$ with the same multiplicity that a occurs in J. For example, $[-1\,0\,1] \in \mathbb{Z}^3$ meets the condition $J = -J$ and $[2\,1\,-3]$ does not. See Corollary 4.13.

In the D_{2n+1} case, S is contained in a D_{2n} subgroup just if it is conjugate into a subtorus $T^{2n} \times \{1\}$ of the standard maximal torus T^{2n+1} whose Lie algebra is the block-diagonal subspace $\mathfrak{so}(2)^{\oplus 2n+1}$ of $\mathfrak{so}(4n+2)$. See Corollary 4.16.

The condition that a circle in E_6 be contained in a D_4 subgroup manifests, within a given maximal torus T^6 of E_6, in a more intricate fashion. Precise statements are Proposition 4.20 and Remark 4.22.

Table 2 The classification for circles in $\mathrm{U}(n)$

Embedding of S	Is $\big(\mathrm{U}(n), S\big)$ isotropy-formal?
$S \not\leqslant \mathrm{SU}(n)$	Yes
$S \leqslant \mathrm{SU}(n)$ and $J = -J$	Yes
$S \leqslant \mathrm{SU}(n)$ and $J \neq -J$	No

As an example of Theorem 1.5, we can recover Shiga's characterization ([47], Prop. 4.3) of circles in the unitary group yielding isotropy-formality.

Example 1.7 If S is a circle in the unitary group $\mathrm{U}(n)$, then $\big(\mathrm{U}(n), S\big)$ is or is not isotropy-formal as indicated in Table 2.

Corollary 1.8 (anonymous referee) *Let G be a compact, connected Lie group and K a subgroup isomorphic to* $\mathrm{SO}(3)$ *or* $\mathrm{SU}(2)$. *Then (G, K) is isotropy-formal.*

Proof This follows from Theorem 1.5 because the maximal torus S^1 of K is contained in the commutator subgroup G' of G and is already reflected in K and hence *a fortiori* in G. □

Alternate proof Koszul ([39], 2.2°) and Stiefel (unpublished) showed $H^*G \longrightarrow H^*K$ is always surjective in this case (Samelson [46] derives this from the fact the Cartan 3-form given at the identity by $(u, v, w) \longmapsto B\big(u, [v, w]\big)$ is natural up to a scalar factor) so it follows ([47], Thm. A, Cor. 4.2) that (G, K) is isotropy-formal. □

A crucial step of in obtaining the key Proposition 1.4 is the following structure theorem for $H^*(G/S)$, which turns out to mildly extend a result which can be pieced together from two *Comptes Rendus* notes of Leray and Koszul, a complete proof of which seems never to have been published. In case the result may be of independent interest, we take the opportunity to provide a proof in Appendix A.

Theorem 1.9 *Let G be a compact, connected Lie group and S a circular subgroup.*

1. *If $H^1G \longrightarrow H^1S$ is surjective, then $H^*(G/S) \longrightarrow H^*G$ is injective and its image is the exterior algebra $\Lambda\hat{P}$ on the intersection $\hat{P}$ of* $\ker\big(H^*G \to H^*S\big)$ *with the graded vector space P of primtive elements of the exterior Hopf algebra $H^*G = \Lambda P$. Noncanonically, there is a $z_1 \in H^1G$ whose image spans H^1S and*

$$H^*(G/S) = \Lambda\hat{P} \cong H^*G\big/(z_1).$$

2. *If $H^1G \longrightarrow H^1S$ is zero, then the image of $H^*(G/S) \longrightarrow H^*G$ is the exterior algebra on a codimension-one subspace $\hat{P}$ of P and $P/\hat{P} \cong \mathbb{Q}z_3$ is graded in degree* 3. *The image of $H^*_S \longrightarrow H^*(G/S)$ is the subalgebra $\mathbb{Q}[s]/(s^2)$ generated by a nonzero $s \in H^2(G/S)$, and there are noncanonical isomorphisms*

$$H^*(G/S) \cong \Lambda\hat{P} \otimes \frac{\mathbb{Q}[s]}{(s^2)} \cong \frac{H^*G}{(z_3)} \otimes \frac{\mathbb{Q}[s]}{(s^2)}.$$

2 Background

Associated to a continuous action of a topological group K on a space X, ([7] IV.3.3, p. 53) is the *(Borel) equivariant cohomology* $H^*_K(X)$, the rational singular cohomology $H^*(X_K; \mathbb{Q})$ of the *homotopy quotient* ([7], Def. IV.3.1, p. 52) (or *Borel construction*)

$$_K X = X_K := \frac{EK \times X}{(ek, x) \sim (e, kx)},$$

where $EK \to BK$ is a universal principal K-bundle. Until the last appendix, all cohomology will be singular cohomology with **rational coefficients**, which will henceforth be suppressed in the notation. We write H^*_K for the coefficient ring $H^*(BK) = H^*_K(\mathrm{pt})$. Associated to the homotopy quotient is a fiber bundle $X \to {}_S X \to BS$, the *Borel fibration*. As noted in the introduction, an action of a topological group S on a space X is said to be *equivariantly formal* if the fiber inclusion $X \hookrightarrow {}_S X$ in this fibration surjects in cohomology.[1] This condition is equivalent to the spectral sequence of this bundle collapsing at the E_2 page ([25], Lem. C.24, p. 208). Given a Lie group G and closed subgroup K, we refer to the natural left K-action on the homogeneous space G/K of left cosets as the *isotropy action*. For brevity, when the isotropy action of K on G/K is equivariantly formal we call the pair (G, K) *isotropy-formal*.

Given a Lie group G, we write $Z(G)$ for its center, G' for its commutator subgroup, $G^{\mathrm{ab}} := G/G'$ for its abelianization, W_G for its Weyl group, and $N_G(K)$ and $Z_G(K)$ respectively for the normalizer and the centralizer of a subgroup K in G. If S is a torus in G, we write $N := \pi_0 N_G(S)$ for the component group of its normalizer. We write $h^\bullet(X) := \sum_{n \geqslant 0} \dim_{\mathbb{Q}} H^n X$ for the total Betti number, and denote subgroup containment by "$\leqslant$", isomorphism "$\cong$", homotopy equivalence "$\simeq$", and homeomorphism "$\approx$".

2.1 Earlier work

As noted in the introduction, the question we are interested in could be asked in the late 1950s but only received a name in the 1990s. As of the beginning of this work, there were only the three known classes of cases in the introduction and the following general results of Shiga and Takahashi.

Theorem 2.1 (Hiroo Shiga [47]) *Let G be a compact Lie group, K a closed, connected subgroup, and $N_G(K)$ the normalizer. If (G, K) is a Cartan pair and the map $H^*(G/K)^{N_G(K)} \hookrightarrow H^*(G/K) \to H^*(G)$ induced by $G \twoheadrightarrow G/K$ is injective, then K acts equivariantly formally on G/K.*

The notion of *Cartan pair* ([16], (3) on p. 70) here is not the notion due to Élie Cartan describing symmetric spaces, but an algebraic condition on the (Henri) Cartan

[1] Dating back to Hans Samelson's [45] *nicht homolog 0* and "$\not\sim$ 0", a space F has been said to be *(totally) nonhomologous to zero* in a superspace E if its inclusion induces an injection $H_* F \longrightarrow H_* E$. The inclusion has also been said to be *(totally) noncohomologous to zero* in the same event, and the condition is abbreviated variously *TNHZ*, *TNCZ*, and *n.c.z.*, notwithstanding the fact the map in cohomology is only injective if it is an isomorphism. In the present work we maintain a respectful distance from this terminology.

model for G/K described in Appendix A which amounts to the space G/K being formal in the sense of rational homotopy theory. Visually, it corresponds to the tensor factorization $E_2 = E_2^{\bullet,0} \otimes E_2^{0,\bullet}$ in the Serre spectral sequence of the Borel fibration $G \to {}_K G \to BK$ persisting to the E_∞ page. Shiga's theorem can be equivalently restated as follows.

Proposition 2.2 (Shiga) *Let G be a compact Lie group, K a closed, connected subgroup, and $N_G(K)$ the normalizer. If (G, K) is a Cartan pair and the map $H_G^* \longrightarrow (H_K^*)^{N_G(K)}$ is surjective, then K acts equivariantly formally on G/K.*

The result also has a partial converse. In a later-written but earlier-published technical report [48], Shiga and Hideo Takahashi prove a partial converse.

Theorem 2.3 (Shiga–Takahashi) *Let G be a compact group, S a toral subgroup, and $N_G(S)$ the normalizer. Suppose that S contains regular elements of G and (G, S) is a Cartan pair. Then S acts equivariantly formally on G/S if and only if and the map $H_G^* \longrightarrow (H_S^*)^{N_G(S)}$ is surjective.*

In work with Chi-Kwong Fok [15], we show that if (G, K) is isotropy-formal, then G/K must be formal, so the "Cartan pair" hypothesis is redudant. The hypothesis on regular elements is also unnecessary, and in further unpublished work [14], we show that S can also be replaced by any closed, connected subgroup K in the result. Although we do not need it in what follows, we state the strong version for here for reference.

Theorem 2.4 *Let G be a Lie group, K a closed, connected subgroup, and $N_G(K)$ the normalizer. Then (G, K) is isotropy-formal if and only if G/K is formal and $H_G^* \longrightarrow (H_K^*)^{N_G(K)}$ is surjective.*

Our trichotomy Proposition 1.4 about the case $K \cong S^1$ can actually be refactored through the Shiga–Takahashi result. Noting that the regular element condition is unneeded, and that G/S is always formal for S a circle by the classical results of Appendix A, the Shiga–Takahashi Theorem 2.3 reduces isotropy-formality of (G, S) to study of the map $H_G^* \longrightarrow H_S^*$. In this language, Proposition 1.4 can be reproven as follows: one has $N = \pi_0 N_G(S)$ either trivial or $\{\pm 1\}$. If it is trivial, then isotropy-formality is just that $H_G^* \longrightarrow H_S^*$ is surjective, which happens if and only if $H^*(G) \longrightarrow H^*(S)$ surjects ([16], 1°, p. 69) ([5], Cor., p. 139). Otherwise $N \cong \{\pm 1\}$, meaning exactly that S is reflected in G (Proposition 4.1), and N acts as $s \longmapsto \pm s$ on $\mathbb{Q}[s] \cong H^*(BS)$, so that $H^*(BS)^N = \mathbb{Q}[s^2]$; then one proves Lemma A.4 to see $H^*(BG) \longrightarrow \mathbb{Q}[s^2]$ is always surjective.

The way this is presented in Sect. 4.1, we use a well-known fixed point criterion for equivariant formality (Lemma 3.8) and a computation of the vector space dimension of the cohomology of the fixed point set due to Goertsches (Proposition 3.11). Whether reasoning through a dimension count or through Theorem 2.3, one way or another the crux of it is understanding the cohomology of the maps $S \to G \to G/S \to BS \to BG$.

3 Reductions

In this section we undertake a series of reductions that ultimately localizes most of the difficulty in determining which pairs (G, K) are isotropy-formal in the case where G is semisimple and K a torus. Two further reductions, from disconnected to connected groups and from connected compact groups to Lie groups, only go through partially and are sequestered in Appendix B.

3.1 Compact total group

Let G be connected pro-Lie group and H a closed, connected subgroup. By the Cartan–Iwasawa–Malcev theorem, there exists a maximal compact subgroup K_H of H, unique up to conjugacy ([32], Cor. 12.77), which is necessarily connected, such that there is a homeomorphism $H \approx K_H \times \mathbb{R}^\kappa$ for some cardinal κ ([32], Cor. 12.82). Likewise G contains a maximal compact subgroup K_G, which after conjugation can be chosen to contain K_H. In case G is a Lie group, at least, this yields a reduction result.

Proposition 3.1 *Suppose G is a connected Lie group and H a connected, closed subgroup, with respective compact, connected subgroups K_G and K_H, the one containing the other. Then (G, H) is isotropy-formal if and only if (K_G, K_H) is.*

Proof To identify the maps $H^*_{K_H}(K_G/K_H) \longrightarrow H^*(K_G/K_H)$ and $H^*_H(G/H) \longrightarrow H^*(G/H)$, it will be enough to see that in the commutative diagram

$$\begin{array}{ccccc} K_G/K_H & \xrightarrow{\alpha} & G/K_H & \xrightarrow{\gamma} & G/H \\ \downarrow & & \downarrow & & \downarrow \\ {}_{K_H}K_G/K_H & \xrightarrow[\beta]{} & {}_{K_H}G/K_H & \xrightarrow[\delta]{} & {}_HG/H, \end{array}$$

the horizontal maps are homotopy equivalences. A left–K_G-equivariant deformation retraction of G to K_G induces deformation retractions from G/K_H to K_G/K_H and from ${}_{K_H}G/K_H$ to ${}_{K_H}K_G/K_H$. The fibers of the bundles δ and ϵ are H/K_H and $(H/K_H) \times (H/K_H)$ respectively, both homeomorphic to Euclidean space, and G/K_H and G/H have the homotopy type of a CW complex so the long exact sequences of homotopy groups and Whitehead's theorem show δ and ϵ are homotopy equivalences. □

Remark 3.2 This proof of Proposition 3.1 depends only on homotopy equivalence, so the statement remains the same if H^* is replaced in the definition of isotropy-formality by any contravariant homotopy functor.

3.2 Toral isotropy

To reduce to toral isotropy actions, we require some well-known isomorphisms and the rarely remarked fact these isomorphisms are *natural*.

Let $\xi_0 \colon E_0 \to B_0$ be a fibration with homotopy fiber F such that $\pi_1 B_0$ acts trivially on H^*F. We can form a slice category of fibrations over ξ_0 with homotopy fiber F by taking as objects maps of fibrations $\xi \to \xi_0$ with homotopy fiber F and as morphisms between $\xi' \to \xi_0$ and $\xi \to \xi_0$ maps of fibrations $\xi' \to \xi$ making the expected triangle commute up to homotopy. Such a morphism entails a homotopy-commutative prism

$$\begin{array}{ccccc} E' & \xrightarrow[h]{} & E & \longrightarrow & E_0 \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi_0} \\ B' & \xrightarrow{\bar{h}} & B & \longrightarrow & B_0. \end{array} \tag{3.1}$$

Lemma 3.3 ([49], Cor. 4.4, p. 88) *Let $\xi_0 \colon E_0 \to B_0$ be a fibration such that the fiber inclusion $F \hookrightarrow E_0$ is surjective in cohomology and $\pi_1 B$ acts trivially on H^*B. Then the fiber inclusion of any fibration $\xi \colon E \to B$ over ξ_0 with homotopy fiber F is surjective in cohomology, and there is an H^*E_0-algebra isomorphism*

$$H^*B \underset{H^*B_0}{\otimes} H^*E_0 \xrightarrow{\sim} H^*E$$

natural in the fibration ξ over ξ_0.

We prove the result so as justify the naturality clause we will need, absent in the original.

Proof Surjectivity of $H^*E \longrightarrow H^*F$ is implied by that of $H^*E_0 \longrightarrow H^*F$ since the fiber inclusion $F \longrightarrow E_0$ factors up to homotopy as $F \to E \to E_0$. For the isomorphism, note that because of these surjections, the Serre spectral sequences of these fibrations collapse at the E_2 page. Thus the ring map $H^*B \otimes_{H^*B_0} H^*E_0 \longrightarrow H^*E$ induced by the maps in the right square of (3.1) is equivalent on the level of H^*B_0-modules to the canonical isomorphism

$$H^*B \underset{H^*B_0}{\otimes} \big(H^*B_0 \otimes H^*F\big) \xrightarrow{\sim} H^*B \otimes H^*F,$$

and so is itself an isomorphism. For naturality, note that the ring map $h^* \colon H^*E \longrightarrow H^*E'$ is completely determined its restrictions to its tensor-factors H^*B and H^*E_0 and that the commutative diagrams in cohomology induced by the left square and top triangle of (3.1) respectively imply these restrictions are $\bar{h}^* \colon H^*B \longrightarrow H^*B'$ and $\mathrm{id}_{H^*E_0}$. □

The naturality in the following lemma follows from the standard proof by noting that a K-equivariant map $X \longrightarrow Y$ yields commutative squares

$$\begin{array}{ccccc} X/S & \longrightarrow & X/N_K(S) & \longrightarrow & X/K \\ \downarrow & & \downarrow & & \downarrow \\ Y/S & \longrightarrow & Y/N_K(S) & \longrightarrow & Y/K. \end{array}$$

Lemma 3.4 ([34], Lemma III.1.1, p. 35) *Let K be a compact, connected Lie group with maximal torus S and Weyl group W, and X a free K-space. Then there is a ring isomorphism, natural in X,*

$$H^*(X/K) \xrightarrow{\sim} H^*(X/S)^W.$$

Lemma 3.5 ([34], Prop. III.1, p. 38) *Let K be a compact, connected Lie group with maximal torus S and Weyl group W. Then there are the following ring isomorphisms natural in X:*

$$H_K^*(X) \xrightarrow{\sim} H_S^*(X)^W,$$
$$H_S^* \underset{H_K^*}{\otimes} H_K^*(X) \xrightarrow{\sim} H_S^*(X).$$

Proof The first statement follows from Lemma 3.4 and the definitions. The second follows from Lemma 3.3, applied to the K/S-bundle $X_S \to X_K$ viewed as a bundle over $BS \to BK$; alternately, as W_K acts on H_S^2 as a reflection group, H_S^* is a free module over $H_K^* \cong (H_S^*)^{W_K}$ by the Chevalley–Shephard–Todd theorem ([36], p. 192) and Corollary B.3 applies. □

Corollary 3.6 *Let K be a compact, connected Lie group with maximal torus S and $X \to Y$ a K-equivariant map. Then $\varkappa_K : H_K^* Y \longrightarrow H_K^* X$ is surjective if and only if $\varkappa_S : H_S^* Y \longrightarrow H_S^* X$ is.*

Proof Lemma 3.5 identifies $\varkappa_K$ with the map of Weyl-invariants $(\varkappa_S)^W$ and $\varkappa_S$ with the base extension $\mathrm{id}_{H_S^*} \otimes_{H_K^*} \varkappa_S$. If $\varkappa_S$ is surjective, then it follows by averaging that $\varkappa_K$ is as well, since $\varkappa_S$ is W-equivariant and $|W|$ is invertible in $\mathbb{Q}$. On the other hand if $\varkappa_K$ is surjective, then since the functor $H_S^* \otimes_{H_K^*} -$ is right exact, $\varkappa_K$ is surjective as well. □

Finally, the following well-known lemma follows from the preceding ones.

Lemma 3.7 ([25], Prop. C.26, p. 207) *If K is a compact, connected Lie group and S a maximal torus, and K acts on a space X, then the action of K is equivariantly formal if and only if the restricted action of S is.*

We can now prove the promised reduction.

Theorem 1.1 *If G is a compact Lie group, K a closed, connected subgroup, and S any torus maximal within K, then (G, K) is isotropy-formal if and only if (G, S) is.*

Proof By Lemma 3.7, it is enough to show that K acts equivariantly formally on G/S if and only if it does on G/K. To do so, we may apply Corollary 3.6 to the map of right K-spaces $G \longrightarrow {}_K G$. □

3.3 The dimension criterion

Equivariant formality can be reduced to a condition on total Betti numbers.

Lemma 3.8 ([7, Prop. XII.3.4, p. 164], [26, Prop. 3.1, p. 81]) *An action of a torus S on a topological space X with finite total Betti number is equivariantly formal if and only if $h^\bullet(X) = h^\bullet(X^S)$.*

For later reference, note one inequality always holds:

Lemma 3.9 (Borel [7, IV.5.5, p. 62]) ([25, Lem. C.24]) *If a torus S acts on a topological space X with finite total Betti number, then $h^\bullet(X) \geqslant h^\bullet(X^S)$.*

Let G be a compact Lie group and S a torus in G. As the fixed point set of the left action of S on G/S is the quotient group $N_G(S)/S$ of the normalizer, we need to know when $h^\bullet(G/S) = h^\bullet\big(N_G(S)/S\big)$. The latter number is easily expressed in terms of other quantities. Recall that we denote by $Z_G(S)$ the centralizer of S in G, by W_K the Weyl group of K, and by N the component group $\pi_0 N_G(S)$.

Lemma 3.10 *Conjugation induces a natural injection $N \rightarrowtail \operatorname{Aut} S$. This induces homeomorphisms $N_G(S) \approx N \times Z_G(S)$ and $(G/S)^S = N_G(S)/S \approx N \times Z_G(S)/S$. If K is a closed, connected subgroup with maximal torus S, there is a further homeomorphism $(G/K)^S = N_G(S)K/K \approx (N/W_K) \times Z_G(S)/S$. Particularly, $h^\bullet\big((G/K)^S\big) = 2^{\operatorname{rk} G - \operatorname{rk} K} \cdot |N|/|W_K|$.*

Proof The centralizer $Z_G(S)$ is connected since it is the union of the maximal supertori of S in G. As $Z_G(S)$ is the kernel of the continuous homomorphism $n \longmapsto (x \mapsto nxn^{-1})$ from $N_G(S)$ into the discrete group $\operatorname{Aut} S \cong \operatorname{Aut} \mathbb{Z}^{\operatorname{rk} S}$, it is the identity component of $N_G(S)$. Thus $N = N_G(S)/Z_G(S)$; the homeomorphisms follow because group components are homeomorphic. As for K, one notes $\pi\colon (G/S)^S \longrightarrow (G/K)^S$ can be equivalently written as the surjection $N_G(S)/S \twoheadrightarrow N_G(S)/N_K(S) = N_G(S)/\big(N_G(S) \cap K\big) \cong N_G(S)K/K$ with fibers $nN_K(S)/S = nW_K$. It follows $(G/K)^S$ has $|N|/|W_K|$ components, each homeomorphic to $Z_G(S)/\big(Z_G(S) \cap K\big) = Z_G(S)/S$. But since $Z_G(S)/S$ is a compact, connected Lie group, $H^*\big(Z_G(S)/S\big)$ is an exterior algebra on $\operatorname{rk} G - \operatorname{rk} S$ generators by Hopf's theorem ([33], Satz I, p. 23), □

Proposition 3.11 (Goertsches–Noshari ([27, Props. 2.1, 3.1]) *Let G be a compact, connected Lie group and K a closed, connected subgroup. Write $N = \pi_0 N_G(S)$. Then (G, K) is isotropy-formal if and only if*

$$h^\bullet(G/K) \leqslant 2^{\operatorname{rk} G - \operatorname{rk} K} \cdot \frac{|N|}{|W_K|}.$$

Proof Let S be a maximal torus of K. By Lemma 3.7, we may replace the K-action on G/K with the S-action. By Lemmas 3.8 and 3.9 this action is equivariantly formal if and only if $h^\bullet(G/K) \leqslant h^\bullet\big((G/K)^S\big)$, which is $2^{\operatorname{rk} G - \operatorname{rk} K} \cdot |N|/|W_K|$ by Lemma 3.10. □

3.4 Torus–cross–simply-connected total group

The structure theorem for compact, connected Lie groups ([11], Thm. V.(8.1) & Ex. V.(8.7).6, p. 233, 238) states that each admits a finite central extension $p\colon \widetilde{G} \longrightarrow G$

such that the abelianization exact sequence $1 \to \widetilde{G}' \to \widetilde{G} \to \widetilde{G}^{\mathrm{ab}} \to 0$ splits on the level of topological groups. If the kernel of p is F, we can write $G \cong \widetilde{G}/F$. The total space $\widetilde{G}$ (but not p itself, if $A \neq 0$) is uniquely determined up to isomorphism.

In determining which toral isotropy actions are equivariantly formal, we will show we can replace G with $\widetilde{G}$ and the connected isotropy subgroup K (which we can take to be a torus) with the identity component $\widetilde{K}_0$ of its preimage $\widetilde{K} = p^{-1}K = F\widetilde{K}_0$.

Proposition 3.12 *These assumptions induce isomorphisms* $H^*(G/K) \rightarrowtail H^*(\widetilde{G}/\widetilde{K}) \xrightarrow{\sim} H^*(\widetilde{G}/\widetilde{K}_0)$.

This is a result of the following lemma and the homeomorphism $\widetilde{G}/\widetilde{K} \xrightarrow{\approx} G/K$.

Lemma 3.13 *Let* Γ *be a path-connected topological group,* F *a central subgroup, and* H *another subgroup such that* FH/H *is finite. Then the covering* $FH/H \to \Gamma/H \to \Gamma/FH$ *induces an isomorphism* $H^*(\Gamma/FH) \xrightarrow{\sim} H^*(\Gamma/H)$.

Proof As F is central, the covering action of $fH \in fH/H$ is given by $\gamma H \cdot fH = \gamma fH = f\gamma H$, left multiplication by f. But Γ being path-connected, left translation by its any element is homotopic to the identity. Thus ([30], Prop. 3G.1)

$$H^*(\Gamma/FH) \cong H^*(\Gamma/H)^{FH/H} = H^*(\Gamma/H). \qquad \square$$

The components of the normalizer are also preserved under this substitution.

Proposition 3.14 *Under the foregoing assumptions, the projection* $p\colon \widetilde{G} \longrightarrow G$ *induces an isomorphism* $N_{\widetilde{G}}(\widetilde{K}_0)/Z_{\widetilde{G}}(\widetilde{K}_0) \xrightarrow{\sim} N_G(K)/Z_G(K)$. *Particularly, if* S *is a torus,* $\pi_0 N_{\widetilde{G}}(\widetilde{S}_0) =: \widetilde{N} \cong N = \pi_0 N_G(S)$.

Proof As p is a homomorphism, it sends $N_{\widetilde{G}}(\widetilde{K}) \longrightarrow N_G(K)$. We show this restriction is surjective and the preimage of $Z_G(K)$ is $Z_{\widetilde{G}}(\widetilde{K})$. For surjectivity, given $\widetilde{w} \in p^{-1}N_G(\widetilde{K}_0)$, note $\widetilde{w}1\widetilde{w}^{-1} = 1$ and $p(\widetilde{w}\widetilde{K}_0\widetilde{w}^{-1}) = K$, so $\widetilde{w}\widetilde{K}_0\widetilde{w}^{-1} = \widetilde{K}_0$. For the preimage, note that if $\widetilde{z} \in p^{-1}Z_G(K)$, then $\widetilde{z}\widetilde{k}\widetilde{z}^{-1}\widetilde{k}^{-1} \in \ker p$ for each $\widetilde{k} \in \widetilde{K}_0$; since $\ker p$ is discrete and $\widetilde{z}1\widetilde{z}^{-1}1^{-1} = 1$, such a $\widetilde{z}$ centralizes $\widetilde{K}_0$. $\square$

These facts in hand, we conclude the proof of Theorem 1.2.

Theorem 1.2 *Let* G *be a compact, connected Lie group,* K *a closed, connected subgroup,* $\widetilde{G}$ *a finite central covering of* G, *and* $\widetilde{K}_0$ *the identity component of the preimage of* K *in* $\widetilde{G}$. *Then* (G, K) *is isotropy-formal if and only if* $(\widetilde{G}, \widetilde{K}_0)$ *is.*

Proof Let S be a maximal torus of K and $\widetilde{S}_0$ its connected lift in $\widetilde{K}$. We know from Proposition 3.11 that (G, K) is isotropy-formal if and only if

$$h^\bullet(G/K) = 2^{\operatorname{rk} G - \operatorname{rk} S}|N|/|W_K|,$$

and the analogous statement holds of $(\widetilde{G}, \widetilde{K})$. But evidently $\operatorname{rk}\widetilde{G} = \operatorname{rk} G$ and $\operatorname{rk}\widetilde{K} = \operatorname{rk} K$ and $W_K \cong W_{\widetilde{K}}$; from Proposition 3.12, we know $h^\bullet(\widetilde{G}/\widetilde{K}) = h^\bullet(G/K)$; and from Proposition 3.14, we know $\widetilde{N} \cong N$. $\square$

In what follows we can therefore replace G with a cover $\widetilde{G} = \widetilde{G}' \times \widetilde{G}^{\mathrm{ab}}$. For later, when we specialize to circles, we note the following corollary of Proposition 3.14.

Corollary 3.15 *Under these hypotheses, the torus S is reflected in G just if $\widetilde{S}$ is reflected in $\widetilde{G}$.*

3.5 Semisimple total group

In this section, G is a connected, compact Lie group, G' again its commutator subgroup, and G^{ab} its abelianization. To separate out information about G', we will need another covering lemma similar in spirit to Lemma 3.13.

Lemma 3.16 *Let Γ be a compact, connected Lie group, Ξ an abelian subgroup, and S a torus in Ξ such that Ξ/S is finite. Then the covering $\Xi/S \to \Gamma/S \to \Gamma/\Xi$ induces an isomorphism $H^*(\Gamma/S) \xrightarrow{\sim} H^*(\Gamma/\Xi)$.*

Proof As Ξ is abelian, it is contained in the centralizer $Z_\Gamma(S)$, which is path-connected, so that its right action on Γ/S is cohomologically trivial. Thus $H^*(\Gamma/\Xi) \cong H^*(\Gamma/S)^{\Xi/S} = H^*(\Gamma/S)$. □

Given subgroup H of G, the canonical short exact sequence $G' \to G \to G^{\mathrm{ab}}$ descends to a fiber bundle $G'/(G' \cap H) \to G/H \to \operatorname{coker}(H \hookrightarrow G \twoheadrightarrow G^{\mathrm{ab}})$.

Proposition 3.17 *If H is connected, this bundle has the cohomology of a trivial bundle.*

Proof Consider a finite central cover of the form $\widetilde{G} = \widetilde{G}' \times \widetilde{G}^{\mathrm{ab}}$. Let $\widetilde{H}$ be the full preimage of H in $\widetilde{G}$ and $\widetilde{H}_0$ its identity component. We will show $\widetilde{G}'/(\widetilde{G}' \cap \widetilde{H}_0) \to \widetilde{G}/\widetilde{H}_0 \to \operatorname{coker}(\widetilde{H}_0 \to \widetilde{G}^{\mathrm{ab}})$ is a trivial bundle. Then the Künneth theorem will yield the desired ring decomposition, for $\operatorname{coker}(H \to G^{\mathrm{ab}})$ and $\operatorname{coker}(\widetilde{H}_0 \to \widetilde{G}^{\mathrm{ab}})$ are tori of the same dimension, and $H^*(G/H) \cong H^*(\widetilde{G}/\widetilde{H}_0)$ by Proposition 3.12, while $\widetilde{G}'/(\widetilde{G}' \cap \widetilde{H}_0) \twoheadrightarrow \widetilde{G}'/(\widetilde{G}' \cap \widetilde{H}) \overset{\approx}{\to} G'/(G' \cap H)$ is a normal covering with covering action induced by right translation by central elements of $\widetilde{G}$, so by Proposition 3.12 again, $H^*\big(G'/(G' \cap H)\big) \cong H^*\big(\widetilde{G}'/(\widetilde{G}' \cap \widetilde{H}_0)\big)$.

The short exact sequence $\operatorname{im}(\widetilde{H}_0 \to \widetilde{G}^{\mathrm{ab}}) \to \widetilde{G}^{\mathrm{ab}} \to \operatorname{coker}(\widetilde{H}_0 \to \widetilde{G}^{\mathrm{ab}})$ of tori splits on the level of topological groups. Replacing $\widetilde{G}^{\mathrm{ab}}$ with the product in the expression $\widetilde{G} = \widetilde{G}' \times \widetilde{G}^{\mathrm{ab}}$, the projection of $\widetilde{H}_0$ to the cokernel component is trivial, so $\widetilde{G}/\widetilde{H}_0$ is the direct product of $\operatorname{coker}(\widetilde{H}_0 \to \widetilde{G}^{\mathrm{ab}})$ and $\big(\widetilde{G}' \times \operatorname{im}(\widetilde{H}_0 \to \widetilde{G}^{\mathrm{ab}})\big)/\widetilde{H}_0$. But the inclusion of $\widetilde{G}'/(\widetilde{G}' \cap \widetilde{H}_0)$ into the latter is a continuous bijection of compact Hausdorff spaces, hence a homeomorphism. □

Now we can carry through the claimed near-reduction to the semisimple case.

Theorem 1.3 *Let G be a compact, connected Lie group, G' its commutator subgroup, K a closed connected subgroup of G, and S a maximal torus in K. Write $K' = K \cap G'$ and $S' = S \cap G'$ for the intersections with G' and K'_0 and S'_0 for their respective identity components. Then (G, K) is isotropy-formal if and only if*

1. *the pair* (G', K'_0) *is isotropy-formal and*
2. *the inclusion* $N_G(S) \hookrightarrow N_G(S'_0)$ *induces an isomorphism of component groups.*

Proof Note that S'_0 is a maximal torus in K'_0, so by Theorem 1.1 it is enough to show (G, S) is isotropy-formal if and only if (G', S'_0) is and the condition on normalizers holds.

From the decomposition $G = G' \cdot Z(G)$, it follows that $N_G(\Gamma) = N_{G'}(\Gamma) \cdot Z(G)$ and $Z_G(\Gamma) = Z_{G'}(\Gamma) \cdot Z(G)$ for any subgroup Γ, so that particularly $\pi_0 N_G(S'_0) \cong \pi_0 N_{G'}(S'_0) =: N'$. As G' is normal in G, there is also a containment $N_G(S) \leqslant N_G(S'_0)$, and so an induced monomorphism $N \rightarrowtail N'$. Thus from Lemma 3.16, Borel's Lemma 3.9 for the action of S'_0 on G'/S'_0 and Lemma 3.10, we see

$$h^\bullet(G'/S') = h^\bullet(G'/S'_0) \geqslant |N'|\, 2^{\operatorname{rk} G' - \operatorname{rk} S'} \geqslant |N|\, 2^{\operatorname{rk} G' - \operatorname{rk} S'}. \tag{3.2}$$

Because rank is additive under direct products,

$$\begin{aligned}\operatorname{rk} G - \operatorname{rk} S &= \big(\operatorname{rk} G' + \operatorname{rk} Z(G)\big) - \big(\operatorname{rk} S'_0 + \operatorname{rk} \operatorname{im}(S \to G^{\mathrm{ab}})\big)\\ &= \operatorname{rk} G' - \operatorname{rk} S'_0 + \operatorname{rk} \operatorname{coker}(S \to G^{\mathrm{ab}}),\end{aligned}$$

so multiplying (3.2) by $2^{\operatorname{rk} \operatorname{coker}(S \to G^{\mathrm{ab}})}$ yields, by Proposition 3.17,

$$h^\bullet(G/S) \geqslant |N'|\, 2^{\operatorname{rk} G - \operatorname{rk} S} \geqslant |N|\, 2^{\operatorname{rk} G - \operatorname{rk} S}. \tag{3.3}$$

Proposition 3.11 states that (G, S) is isotropy-formal if and only if the inequalities (3.3) are in fact equalities, which is equivalent to (3.2) being equalities. But by Proposition 3.11 again, this can only happen if (G', S') is isotropy-formal and $N' \leftrightarrow N$. □

Remark 3.18 It can really happen that the inequality $|N| \leqslant |N'|$ is strict. For instance, let $G = A \times G'$ for $A = S^1$ and $G' = \mathrm{SU}(2)^2$, pick a circle S^1 in $\mathrm{SU}(2)$, and let T be the maximal torus $(S^1)^3$ of G and $S = \big\{(z, w, zw^{-1}) : z, w \in S^1\big\}$ a rank-two subtorus, so that $S' = S'_0 = \big\{(1, w, w^{-1}) : w \in S^1\big\}$. Then $N' = W_{\mathrm{SU}(2)} \cong \mathbb{Z}/2$ but $N = 1$.

4 Circular isotropy

Now we can tackle the case S is a circle. This section demonstrates the statements of Theorem 1.5 and Table 1 regarding equivariant formality of circle actions.

4.1 The trichotomy

Let $S \cong S^1$ be a circle subgroup of a compact, connected Lie group G.

Proposition 4.1 *Then the cardinality of* $\pi_0 N_G(S)$ *is* 2 *if* S *is reflected in* G *and* 1 *otherwise.*

Proof This follows from Lemma 3.10 since $s \longmapsto s^{-1}$ is the only nontrivial continuous automorphism of S^1. □

As $\widetilde{H}^* S^1 = H^1 S^1$ is one-dimensional, $H^* G \longrightarrow H^* S$ is either surjective or trivial.

Proposition 4.2 *The inclusion $S \hookrightarrow G$ is trivial in cohomology if and only if S is contained in the commutator subgroup G', if and only if the map induced in H^1 by $S \to G \to G^{\mathrm{ab}}$ is trivial.*

Proof Since G' is the kernel of $G \longrightarrow G^{\mathrm{ab}} =: A$, it contains S just if the composition $S \to G \to A$ is trivial. If so, then of course the map $H^1 A \longrightarrow H^1 S$ is trivial. If $S \to G \to A$ is nontrivial, then its image is a circle, so the induced map $\pi_1 S \longrightarrow \pi_1 A$ is nonzero and hence injective, and so $H^1 A \longrightarrow H^1 S$ is surjective. But this map is nontrivial just if $H^1 G \longrightarrow H^1 S$ is since $H^1 A \longrightarrow H^1 G$ is an isomorphism, as can be seen for example by using Proposition 3.12 to pass to a finite cover $\widetilde{A} \times \widetilde{G}'$ with $0 = \pi_1 \widetilde{G}' = H^1 \widetilde{G}' = H^1 G'$. □

We can now prove Proposition 1.4.

Proposition 1.4 Let G be a compact, connected Lie group and S a circular subgroup of G. There are the following three mutually exclusive cases.

1. The inclusion $S \hookrightarrow G$ surjects in cohomology and S is not reflected in G.
2. The inclusion $S \hookrightarrow G$ is trivial in cohomology and
   - 2a. S is reflected in G.
   - 2b. S is not reflected in G.

Only in the last case is (G, S) not isotropy-formal.

Proof Recall from Proposition 3.11 that (G, S) is isotropy-formal just when $h^\bullet(G/S) \leqslant |N|\, 2^{\operatorname{rk} G - \operatorname{rk} S}$. Theorem 1.9 imposes the constraint that $h^\bullet(G/S) \in \left\{\frac{1}{2} h^\bullet(G), h^\bullet(G)\right\}$ and Proposition 4.1 that $|N| \in \{1, 2\}$. By Lemma 3.9, it is impossible that both $h^\bullet(G/S) = \frac{1}{2} h^\bullet(G)$ and $|N| = 2$ simultaneously, so there are only the following three cases.

1. We have $h^\bullet(G/S) = \frac{1}{2} h^\bullet(G)$ and $|N| = 1$. The action is equivariantly formal.
2. We have $h^\bullet(G/S) = h^\bullet(G)$, and
   - 2a. $|N| = 2$. The action is equivariantly formal.
   - 2b. $|N| = 1$. The action is not equivariantly formal. □

It remains to determine when $|N| = 2$, or in other words when S is reflected in G.

4.2 Classification of reflected circles

In this section, we determine what circular subgroups S of compact, connected Lie groups G are reflected. First, we may assume S lies in some fixed maximal torus of T, since all maximal tori are conjugate and for any $g \in G$ one has $g N_G(S) g^{-1} = N_G(g S g^{-1})$. Further, we may represent reflections by Weyl group elements, in that $N \leqslant \operatorname{Aut} S$ is naturally a quotient of $N_W(S) \leqslant W$.

Lemma 4.3 ([9, Exercise IX.2.4, p. 391], [20, Lemma 9.7, p. 20]) *Let G be a compact, connected Lie group, T a maximal torus, and S a subtorus. Any automorphism of S induced from conjugation by an element of $N_G(S)$ is also induced by an element of $N_G(T) \cap N_G(S)$.*

Precisely, the inclusion $N_G(T) \cap N_G(S) \hookrightarrow N_G(S)$ induces maps

$$\frac{N_G(T)}{T} \hookleftarrow \frac{N_G(T) \cap N_G(S)}{T} \twoheadrightarrow \frac{N_G(T) \cap N_G(S)}{N_G(T) \cap Z_G(S)} \xrightarrow{\sim} \frac{N_G(S)}{Z_G(S)}.$$

Corollary 4.4 *A toral subgroup S is reflected in a compact, connected Lie group G if and only if some element of the Weyl group W of G acts as $s \longmapsto s^{-1}$ on S.*

From Corollary 3.15, we may replace G with the product $A \times G'$ of a torus A and a simply-connected Lie group G', but A is irrelevant:

Proposition 4.5 *A toral subgroup S is reflected in a compact, connected Lie group G if and only if it lies in and is reflected in the commutator subgroup G'.*

Proof Since the conjugation action of A is trivial, circles reflected by G are already reflected by G'. From Propositions 4.2 and 1.4, we know any reflected S in G is contained in G'. □

Reflectibility of a torus in a semisimple group H in turn depends only on simple factors.

Proposition 4.6 *A toral subgroup S is reflected in a product $\prod H_j$ of Lie groups if and only if each of its images S_j under the factor projections to H_j is reflected in H_j.*

Proof Since the homomorphisms $\prod H_j \twoheadrightarrow H_i$ preserve conjugacy and inversion, if $(h_j) \in \prod H_j$ reflects S, then h_j reflects S_j. On the other hand, if some $h_j \in H_j$ reflects each S_j, then (h_j) reflects $\prod S_j$, which contains S. □

We can in fact restrict attention to a single element of the Weyl group.

Proposition 4.7 *A circular subgroup S is reflected in a simple Lie group H if and only if it is reflected by the longest word w_0 in the Weyl group W of H.*

Proof If C is the closed Weyl chamber containing a given nonzero element $v \in \mathfrak{s} < \mathfrak{t}$, then $-v$ lies the "opposite" closed Weyl chamber $-C$. The orbit $W \cdot v$ meets $-C$ in exactly one point ([1], Thm. 5.16), which must be $w_0 \cdot v$ since $w_0 \cdot C = -C$, so $\mathfrak{s}$ is reflected if and only if $w_0 \cdot v = -v$. □

There is a representation-theoretic restatement of the same condition.

Corollary 4.8 *A circular subgroup S is reflected in a simple Lie group H if and only if the irreducible representation of H determined by S is self-dual.*

Proof Identify $\mathfrak{t}$ with its dual $\mathfrak{t}^\vee$ through the W-invariant inner product and let λ be an additive generator of the intersection of $\mathfrak{s}$ with the weight lattice of H. Then S is reflected if and only if $w_0 \cdot \lambda = -\lambda$. But the dual to the irreducible representation with highest weight λ is that with highest weight $-w_0 \cdot \lambda$. □

Remark 4.9 The original proof of the classification in Table 1 was unnecessarily intricate and involved a computer algebra verification at one point, and has been greatly simplified through the arguments in Proposition 4.7 and Corollary 4.8, due to Jay Taylor [51] and Chi-Kwong Fok (personal communication).

To construct Table 1 we march case by case through the Killing–Cartan classification.

Proposition 4.10 *A maximal torus T of a simple compact Lie group G whose type is one of*

$$B_n, \quad C_n, \quad D_{2n}, \quad G_2, \quad F_4, \quad E_7, \quad E_8$$

is reflected in G.

Proof The longest word w_0 acts as $-\operatorname{id}$ on the vector space $\mathfrak{t}$ carrying the defining representation of W precisely for Coxeter groups W of these types ([36], Lem. 27–2, p. 283) so T is reflected by Proposition 4.7. Alternately, but relatedly, central involutions of a Weyl group W reflect the maximal torus T ([21], Thm. 1.8) and the center of W is isomorphic to $\mathbb{Z}/2$ precisely for Coxeter groups W of these types ([21], Rmk. 1.9). ◻

In the remaining cases, the longest word $w_0 \in W$ does not act as $-\operatorname{id}$ on $\mathfrak{t}$, so more work is required.

Proposition 4.11 *A circular subgroup S is reflected in a simple Lie group H whose Weyl group has trivial center* (viz. *one of type A_n, D_{2n+1}, or E_6) if and only if there is some $w \in W$ such that $w \cdot \mathfrak{s}$ lies in the fixed point subalgebra $\mathfrak{t}^\theta$ of the Cartan subalgebra under an automorphism $\theta \in \operatorname{Aut}\mathfrak{t}$ induced by a nontrivial diagram automorphism of the Dynkin diagram of H.*

Proof From Proposition 4.7 we know S is reflected if and only if $\mathfrak{s}$ is fixed pointwise by the nontrivial automorphism $-w_0 \in \operatorname{Aut}\mathfrak{t}$. As $w_0 = \operatorname{Ad}(n_0)$ for some $n_0 \in N_H(T)$, we can extend $-w_0$ to $-\operatorname{Ad}(n_0) \in \operatorname{Aut}\mathfrak{k}$. Outer automorphisms of $\mathfrak{k}$ are induced ([24], Prop. D.40, p. 498) by graph automorphisms of the Dynkin diagram Γ of H in the sense that $(\operatorname{Aut}\mathfrak{k})/(\operatorname{Ad} H) \cong \operatorname{Aut}\Gamma$. Since W acts simply transitively on Weyl chambers, and $-w_0$ stabilizes but does not fix the positive closed Weyl chamber C, the automorphism $-\operatorname{Ad}(n_0)$ of $\mathfrak{k}$ is not inner and hence its outer isomorphism class corresponds to a nontrivial automorphism θ of Γ. This means the induced $\theta \in \operatorname{Aut}\mathfrak{t}$ is the restriction of $-\operatorname{Ad}(n_0 k) \in \operatorname{Aut}\mathfrak{k}$ for some $k \in N_H(T)$, so that θ fixes $\operatorname{Ad}(k^{-1})\mathfrak{s}$. ◻

It thus remains to find the fixed point subalgebras of nontrivial diagram automorphisms for Lie algebras of type A_n, D_{2n+1}, and E_6. In all of these proofs, we use the fact that the W-equivariant isomorphism $\mathfrak{t}^\vee \xrightarrow{\sim} \mathfrak{t}$ induced by the invariant inner product is also equivariant with respect to $\theta = -w_0$, and so identifies the fixed point subspaces $(\mathfrak{t}^\vee)^\theta$ and $\mathfrak{t}^\theta$.

Proposition 4.12 *In a Lie algebra of type A_n, a point $v \in \mathfrak{t}^\vee < \mathbb{R}^{n+1}$ of the dual Cartan algebra is fixed by the automorphism θ of Figure 1 if and only if a permutation of the coordinates of v yields $-v$.*

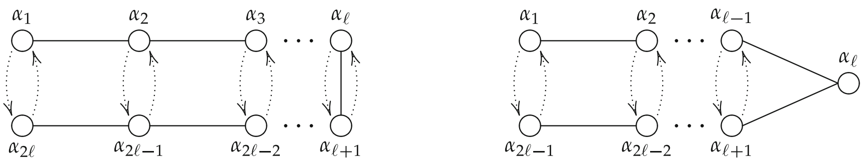

Fig. 1 The graph involution of A_n

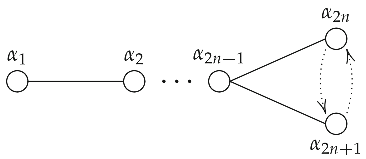

Fig. 2 The graph involution of D_{2n+1}

Proof The diagram automorphism θ acts on simple roots of A_n by exchanging $\alpha_j \longleftrightarrow \alpha_{n-j}$. The θ–fixed point subspace of $\mathfrak{t}^\vee$ is spanned by the sums $\alpha_j + \alpha_{n-j}$ and so consists of those vectors $\sum c_j\alpha_j \in \mathfrak{t}^\vee$ for which $c_j = c_{n-j}$. The α_j are usually identified with $e_j - e_{j+1} \in \mathbb{R}^{n+1}$, where $(e_\ell)_{1\leqslant \ell \leqslant n+1}$ is the standard basis and the resulting embedding $\mathfrak{t}^\vee \rightarrowtail \mathbb{R}^{n+1}$ takes

$$\sum c_j\alpha_j \longmapsto \begin{bmatrix} c_1 & (c_2 - c_1) & \cdots & (c_n - c_{n-1}) & -c_n \end{bmatrix} =: \sum v_\ell e_\ell,$$

translating the symmetry requirement $c_j = c_{n-j}$ to the antisymmetry condition $v_\ell = -v_{n+1-\ell}$. □

Corollary 4.13 *A circular subgroup S is reflected in* $\mathrm{SU}(n)$ *if and only if the exponent multiset J of the inclusion of any conjugate of S into the standard maximal torus T satisfies $J = -J$.*

Proof Let v span the tangent space $\mathfrak{s} < \mathfrak{t}$. Recalling the Weyl group $W_{A_n} = S_{n+1}$ acts on $\mathbb{R}^{n+1}$ by permuting coordinates, by Proposition 4.12 a permutation of the entries of v yields $-v$ just if some $w \in W_{A_n}$ sends v into $\mathfrak{t}^\theta$, and by Proposition 4.11, S is reflected just if this occurs. □

Remark 4.14 (*a*) The root subsystems of $A_{2\ell}$ and $A_{2\ell-1}$ fixed by θ are respectively of types B_n and C_n, corresponding to the inclusions $\mathrm{SO}(2\ell+1) \hookrightarrow \mathrm{SU}(2\ell+1)$ and $\mathrm{Sp}(n) \hookrightarrow \mathrm{SU}(2n)$ respectively induced by the ring injections $\mathbb{R} \hookrightarrow \mathbb{C}$ and $\mathbb{H} \rightarrowtail \mathbb{C}^{2\times 2}$. These subgroups are fixed points of involutive automorphisms of $\mathrm{SU}(n)$ yielding the symmetric spaces $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SU}(2n)/\mathrm{Sp}(n)$.

(*b*) In terms of the self-duality criterion Corollary 4.8, the representation τ of S on $\mathbb{C}^n$ given by restricting the defining representation of $\mathrm{SU}(n)$ to S is a direct sum $\bigoplus_{j=1}^n \rho^{\otimes a_j}$ of tensor powers of the defining representation $\rho\colon S^1 \rightarrowtail \mathrm{Aut}_{\mathbb{C}}\,\mathbb{C}$, and the dual representation $\tau^\vee = \bigoplus_{j=1}^n \rho^{\otimes(-a_j)}$, will be isomorphic to τ just if $J = -J$.

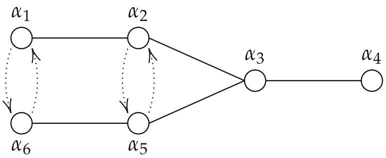

Fig. 3 The graph involution of E_6

Proposition 4.15 *In a Lie algebra of type D_{2n+1}, a point $\lambda \in \mathfrak{t}^\vee$ of the dual Cartan algebra is fixed by an automorphism of the Dynkin diagram if and only if the last coordinate of λ is zero.*

Proof The nontrivial graph automorphism θ of the Dynkin diagram of D_{2n+1}, shown in Fig. 2, fixes all simple roots except α_{2n} and α_{2n+1}, which it exchanges. The fixed point subspace of $(\mathfrak{t}^\vee)^\theta$ is spanned by $\{\alpha_j\}_{j<2n} \cup \{\alpha_{2n} + \alpha_{2n+1}\}$. The roots α_j for $j \leqslant 2n$ are usually identified with $e_j - e_{j+1} \in \mathbb{R}^{2n+1}$ and α_{2n+1} with $e_n + e_{n+1}$, where $(e_j)_{1 \leqslant j \leqslant n+1}$ is again the standard basis. The image of the composite embedding $(\mathfrak{t}^\vee)^\theta \hookrightarrow \mathfrak{t}^\vee \to \mathbb{R}^{2n+1}$ is $\mathbb{R}^{2n} \times \{0\}$ since $\alpha_{2n} + \alpha_{2n+1} = 2e_{2n}$. □

Corollary 4.16 *A circular subgroup S is reflected in* $\mathrm{Spin}(4n+2)$ *if and only if it is conjugate into a* $\mathrm{Spin}(4n)$ *subgroup.*

Proof Let v span the tangent space $\mathfrak{s} < \mathfrak{t} = \mathbb{R}^{2n+1}$. Recalling the Weyl group $W_{D_{2n+1}} = \{\pm 1\}^{2n} \rtimes S_{2n+1}$ acts on $\mathbb{R}^{2n+1}$ by permuting its coordinates and negating an even number of them, by Proposition 4.15 some entry of v is 0 just if some $w \in W_{A_n}$ sends v into $\mathfrak{t}^\theta$, and by Proposition 4.11, S is reflected just if this occurs. □

Remark 4.17 The sublattice of a D_{2n+1} lattice fixed by θ is of type B_{2n} and corresponds to a $\mathrm{Spin}(4n)$ subgroup of $\mathrm{Spin}(4n+2)$, the fixed point set of an involutive automorphism of $\mathrm{Spin}(4n+2)$ yielding the symmetric space $V_2(\mathbb{R}^{4n+2}) = \mathrm{Spin}(4n+2)/\mathrm{Spin}(4n) = \mathrm{SO}(4n+2)/\mathrm{SO}(4n)$.

Proposition 4.18 *In a Lie algebra of type E_6, a point $\lambda \in \mathfrak{t}^\vee$ of the dual Cartan algebra is fixed by the nontrivial automorphism of the Dynkin diagram if and only if it lies in a certain F_4 sublattice.*

Proof The fixed-point subspace $(\mathfrak{t}^\vee)^\theta$ of the nontrivial automorphism θ of the Dynkin diagram of E_6 depicted in Fig. 3 is spanned by $\Delta = \{\alpha_1 + \alpha_6, \alpha_2 + \alpha_5, \alpha_3, \alpha_4\}$. By assumption, we have $\alpha_i \cdot \alpha_j = -2|\alpha_i||\alpha_j|$ for adjacent α_i, α_j and $= 0$ otherwise, so Δ is a simple root system of type F_4 with $a_1 + \alpha_6$ and $\alpha_2 + \alpha_5$ long and α_3 and α_4 short. □

Proposition 4.19 *A circular subgroup S is reflected in E_6 or its universal cover $\widetilde{E}_6$ if and only if it is conjugate into a* $\mathrm{Spin}(8)$ *subgroup.*

Proof It follows from Proposition 4.11 and Proposition 4.18 that the tangent lines $\mathfrak{s}$ to reflected circles S are precisely those sent into $\mathfrak{t}^\theta$ by some $w \in W_{E_6}$. As $(\mathfrak{t}^\vee)^\theta$ is spanned by an F_4 sublattice of the E_6 root lattice, its dual $\mathfrak{t}^\theta$ is tangent to the maximal torus T^4 of an F_4 subgroup. In the series of inclusions $\mathrm{Spin}(8) < F_4 < E_6$, the first two share a maximal torus T^4, so $\mathfrak{t}^\theta$ is actually tangent to the maximal torus of a $\mathrm{Spin}(8)$. □

It may be of interest to count these four-dimensional tori.

Proposition 4.20 *Within any given maximal torus T^6 of E_6 or $\widetilde{E}_6$, there are forty-five distinct Weyl-conjugate maximal tori T^4 of* Spin(8) *subgroups, all reflected.*

Proof The Spin(8) tangent to T^4 corresponds to a D_4 sublattice of $\mathfrak{t}$ spanning $\mathfrak{t}^\theta$. Within a set of positive roots for a root system of type D_4, it is not hard to check there are precisely three spanning sets of mutually orthogonal roots, so the number of tori in question will be a third of the number of sets of four mutually orthogonal roots in the root system $\Phi(E_6)$. Any given *set* $\{\alpha, \beta, \gamma, \delta\}$ of four mutually orthogonal *positive* roots in $\Phi(E_6)$ corresponds to $\left|\{\pm 1\}^4 \rtimes S_4\right| = 384$ different mutually orthogonal *ordered quadruples* of *arbitrary* roots, so the number of tori T^4 can be obtained by dividing the number of such quadruples by $384 \cdot 3 = 1152 = \left|W_{F_4}\right|$. We will then be done if we can show W_{E_6}, which is of cardinality $51,840 = 45 \cdot 1152$, acts simply transitively on mutually orthogonal ordered quadruples $(\alpha, \beta, \gamma, \delta)$ in $\Phi(E_6)$.

For this, Carter observes ([17], Lem. 11.(i), p. 14) that W_{E_6} acts transitively on roots $\alpha \in \Phi(E_6)$, that $\mathrm{Stab}_{W_{E_6}} \alpha$ acts transitively on the A_5 subsystem of roots β orthogonal to α, and that $\mathrm{Stab}_{W_{E_6}}(\alpha, \beta)$ acts transitively on the A_3 subsystem the roots γ orthogonal to both α and β, so that W_{E_6} acts transitively on mutually orthogonal ordered triples (α, β, γ). From there we may further see $\mathrm{Stab}_{W_{E_6}}(\alpha, \beta, \gamma)$ acts transitively on the A_1 subsystem $\{\pm\delta\}$ of roots orthogonal to all of α, β, γ. That the transitivity on quadruples is simple follows, since $\left|\Phi(A_1)\right| = 2$, from repeated applications of the orbit–stabilizer theorem:

$$\underbrace{51,840}_{|W_{E_6}|} = \underbrace{720}_{|\mathrm{Stab}\,\alpha|} \cdot \underbrace{72}_{|\Phi(E_6)|} = \underbrace{24}_{|\mathrm{Stab}(\alpha,\beta)|} \cdot \underbrace{30}_{|\Phi(A_5)|} \cdot 72 = \underbrace{2}_{|\mathrm{Stab}(\alpha,\beta,\gamma)|} \cdot \underbrace{12}_{|\Phi(A_3)|} \cdot 30 \cdot 72. \quad \square$$

Remark 4.21 If we view T^4 as the maximal torus of $F_4 < E_6$, it follows from the equation $|W_{E_6}| = 45 \cdot |W_{F_4}|$ that W_{F_4} injects into W_{E_6} as the normalizer of T^4. The author is advised this result can be understood from Carter's book ([18], Sec. 13.3).

Remark 4.22 A standard system of simple roots for E_6 in $\mathbb{R}^5 \times \mathbb{R}^3$ is given ([8], Planche V, p. 260) by

$$\begin{aligned} \Delta := \Big\{ \quad \zeta &:= \tfrac{1}{2}[1 \ \ 1 \ \ 1 \ \ 1 \ \ 1;\ 1\ 1\ 1],\\ -\gamma_{12} &:= -[1 \ \ 1 \ \ 0 \ \ 0 \ \ 0;\ 0\ 0\ 0],\\ \delta_{12} &:= \ \ [1 \ {-1} \ \ 0 \ \ 0 \ \ 0;\ 0\ 0\ 0],\\ \delta_{23} &:= \ \ [0 \ \ 1 \ {-1} \ \ 0 \ \ 0;\ 0\ 0\ 0],\\ \delta_{34} &:= \ \ [0 \ \ 0 \ \ 1 \ {-1} \ \ 0;\ 0\ 0\ 0],\\ \delta_{45} &:= \ \ [0 \ \ 0 \ \ 0 \ \ 1 \ {-1};\ 0\ 0\ 0]\Big\}. \end{aligned}$$

These roots span the six-dimensional subspace $\left(\mathbb{R}^5 \times \{0\}^3\right) + \mathbb{R}\cdot[1\ 1\ 1\ 1\ 1;\ 1\ 1\ 1]$ of $\mathbb{R}^8$ and one obtains a system Φ of 72 roots obtained from permutation of the first five coordinates of

$$\zeta,\quad \gamma_{12},\quad \delta_{12},\quad \eta_{12} := \zeta - \gamma_{12},\quad \epsilon_1 := \zeta - 2\gamma_{12} + 2\delta_{12} + 3\delta_{23} + 2\delta_{34} + \delta_{45} = \tfrac{1}{2}\,[1\ {-1}\ {-1}\ \ {-1}\ \ {-1};\ 1\ \ 1\ \ 1].$$

and multiplication by ± 1. We may choose the positive roots Φ^+ to be the 36 in the union of the following 135 maximal mutually orthogonal sets:

(60) $\{\epsilon_a, \eta_{ab}, \gamma_{ac}, \delta_{de}\}$, where $\big|\{a, b, c, d, e\}\big| = 5$ and $d < e$,
(30) $\{\eta_{ab}, \eta_{cd}, \gamma_{ac}, \gamma_{bd}\}$, where $\big|\{a, b, c, d\}\big| = 4$,
(15) $\{\eta_{ab}, \eta_{cd}, \delta_{ab}, \delta_{cd}\}$, where $\big|\{a, b, c, d\}\big| = 4$ and $a < b$ and $c < d$,
(15) $\{\gamma_{ab}, \gamma_{cd}, \delta_{ab}, \delta_{cd}\}$, where $\big|\{a, b, c, d\}\big| = 4$ and $a < b$ and $c < d$,
(15) $\{\zeta, \epsilon_a, \delta_{bc}, \delta_{de}\}$, where $\big|\{a, b, c, d, e\}\big| = 5$ and $b < c$ and $d < e$.

These 135, found by brute force, form bases of the tangent spaces to the 45 tori figuring in Proposition 4.20, and each torus is reflected by the product of the four corresponding root reflections.

For example, the span $\mathbb{R}^4 \times \{0\}^4$ of $\{\gamma_{12}, \delta_{12}, \gamma_{34}, \delta_{34}\}$ meets Φ^+ in $\{\gamma_{ab}, \delta_{ab} : 1 \leqslant a < b \leqslant 4\}$. Among these, the roots orthogonal to δ_{ab} are $\{\gamma_{ab}, \gamma_{cd}, \delta_{cd}\}$ (where $\big|\{a, b, c, d\}\big| = 4$) and likewise the roots orthogonal to γ_{ab} are $\{\delta_{ab}, \gamma_{cd}, \delta_{cd}\}$, so the spanning quadruples are determined by the (three) partitions of $\{1, 2, 3, 4\}$ into pairs of pairs $\big\{\{a, b\}, \{c, d\}\big\}$.

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A. Leray and Koszul's theorem on $H^*(G/S^1)$

In order to obtain Proposition 1.4, we needed some grasp on the cohomology ring $H^*(G/S)$ of a homogeneous space G/S, for G compact connected and S a circle.

Theorem 1.7 *Let G be a compact, connected Lie group and S a circular subgroup.*

1. *If $H^1G \longrightarrow H^1S$ is surjective, then $H^*(G/S) \longrightarrow H^*G$ is injective and its image is the exterior algebra $\Lambda\hat{P}$ on the intersection $\hat{P}$ of* $\ker(H^*G \to H^*S)$ *with the graded vector space P of primtive elements of the exterior Hopf algebra $H^*G = \Lambda P$. Noncanonically, there is a $z_1 \in H^1G$ whose image spans H^1S and*

$$H^*(G/S) = \Lambda\hat{P} \cong H^*G\big/(z_1).$$

2. *If $H^1G \longrightarrow H^1S$ is zero, then the image of $H^*(G/S) \longrightarrow H^*G$ is the exterior algebra on a codimension-one subspace $\hat{P}$ of P and $P/\hat{P} \cong \mathbb{Q}z_3$ is graded in degree* 3. *The image of $H_S^* \longrightarrow H^*(G/S)$ is the subalgebra $\mathbb{Q}[s]/(s^2)$ generated by a nonzero $s \in H^2(G/S)$, and there are noncanonical isomorphisms*

$$H^*(G/S) \cong \Lambda\hat{P} \otimes \frac{\mathbb{Q}[s]}{(s^2)} \cong \frac{H^*G}{(z_3)} \otimes \frac{\mathbb{Q}[s]}{(s^2)}.$$

We found belatedly that this is a trivial generalization of long-known results. General statements on the cohomology of a homogeneous space were already available to Leray in 1946, the year after his release from prison ([41], §3, item (4)). In the second of his four *Comptes Rendus* announcements from that year ([40], bottom of p. 1421), he states the following result.[2]

Theorem A.1 (Leray 1946) *Let K be a compact, simply-connected Lie group and S a closed, one-parameter subgroup [*viz. *a circle]. Write $\pi : K \longrightarrow K/S$ for the projection. Then $H^*(K/S; \mathbb{Q})$ is generated as a commutative graded algebra by finitely many classes z_α of odd degree and one class $s \in H^2(K/S; \mathbb{Q})$, subject to the sole relation $s^{n+1} = 0$ for a certain [positive natural] n. The ring $H^*(K; \mathbb{Q})$ is freely generated as a commutative graded algebra by the classes $\pi^* z_\alpha$ and one further class $z_{2n+1} \in H^{2n+1}(K; \mathbb{Q})$.*

More explicitly, if P is a homogeneous vector space of generators for the exterior algebra $H^*K = \Lambda P$, then the image of $H^*(K/S) \longrightarrow H^*K$ is an exterior subalgebra $\Lambda\hat{P}$ on a subspace $\hat{P} \cong P/\mathbb{Q}z_{2n+1}$ of codimension 1, and lifting $\hat{P}$ back to $H^*(K/S)$ induces a $\mathbb{Q}$-algebra isomorphism

$$H^*(K/S) \cong \mathbb{Q}[s]\big/(s^{n+1}) \otimes \Lambda\hat{P}. \tag{A.1}$$

The second clause of Theorem 1.9 is clearly a refinement of this result; if one omits Leray's hypothesis K be simply-connected and admits the possibility n be 0, then so is the first clause.

The following year, Koszul published a note ([39], p. 478, display), also in the *Comptes Rendus*, regarding Poincaré polynomials for these spaces, which implies $n = 1$ in Leray's result.

Theorem A.2 (Koszul 1947) *Let K be a compact, connected Lie group and S a compact, connected* 1*-dimensional subgroup [again, a circle] such that the image of $H_1(S; \mathbb{Q}) \longrightarrow H_1(K; \mathbb{Q})$ is zero. Then the Poincaré polynomials (in the indeterminate t) of K/S and K are related by*

$$p(K)(1 + t^2) = p(K/S)(1 + t^3).$$

Koszul, unlike Leray, does include an indication of a proof, which we translate without elaboration, leaving it to the reader to decide for themself how much further detail

[2] See also Borel ([6], par. 12); only due to Borel's account are we confident "compact Lie group" was the accurate contemporary reading of Leray's *groupe bicompact*.

they require and provide it if they can. After we will provide an alternate proof of Theorem 1.9 and hence of Leray's and Koszul's theorems.

Koszul's proof A choice of K-biinvariant Riemannian metric B on K induces an isomorphism $\phi\colon v \longmapsto B(v,-)$ from the Lie algebra $\mathfrak{k}$, conceived as the space of left-invariant vector fields on K, to the space $\Omega^1 := \Omega^1(K)^K$ of left-invariant 1-forms. This allows us to define a Lie bracket on $\Omega^1(K)^K$, and to associate to S the Lie subalgebra $\Omega^{1,0} := \phi(\mathfrak{s})$ and its B-orthogonal complement $\Omega^{0,1}$. Then the differential algebra $\Omega^\bullet(K)^K$ is bigraded by $\Omega^{p,q} = \bigwedge^p \Omega^{1,0} \wedge \bigwedge^q \Omega^{0,1}$, and particularly we may consider the spectral sequence associated to the filtration by ideals $I^q = \Omega^{\bullet,\geqslant q}$. In this spectral sequence, one has

$$\begin{aligned} E_1^{0,\bullet} &\cong \Omega^\bullet(K/S)^K,\\ E_2^{0,\bullet} &\cong H^*(K/S),\\ E_\infty^{0,\bullet} &\cong \operatorname{im}\big(H^*(K/S) \longrightarrow H^*K\big). \end{aligned}$$

Observe that given any nonzero element $\alpha \in \Omega^{1,0}$, we always have $d\alpha \in I^2$. We can uniquely decompose the Cartan invariant 3-form $\omega\colon u \wedge v \wedge w \longmapsto B\big([u,v],w\big)$ on K as $\omega = \sum \omega_j$ for $\omega_j \in \Omega^{3-j,j}$. Now $d\omega = 0$ and $\omega_0 = 0$, so we have

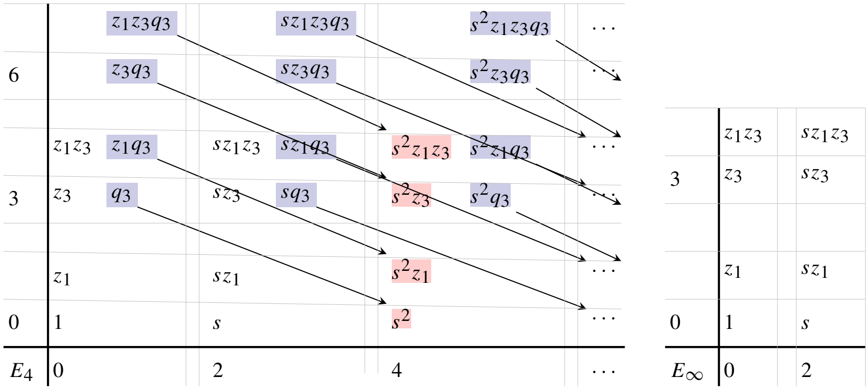
$$(d\alpha)^2 = d(\alpha \wedge d\alpha) = 3B(\alpha,\alpha)d\omega_2 = -3B(\alpha,\alpha)d\omega_3,$$

which simultaneously lies in I^4 and is the exterior derivative of an element of I^1. Thus the image of $H^*(K/S) \longrightarrow H^*K$ cannot contain the class $[\omega]$.[3] □

Before our proof, we illustrate with a representative example the features of the general case.

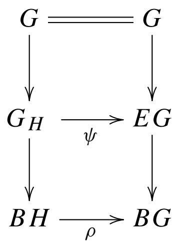
Example A.3 Let S be a circle contained in the second factor of the group $G = \mathrm{U}(2) \times \mathrm{Sp}(1)$. The cohomology of G is the exterior algebra $H^*G = \Lambda[z_1, z_3, q_3]$, where $\deg z_1 = 1$ and $\deg z_3 = \deg q_3 = 3$, and the cohomology $H_S^* = H^*(BS) = \mathbb{Q}[s]$, where $\deg s = 2$. Since $G/S = \mathrm{U}(2) \times \big(\mathrm{Sp}(1)/S\big) \approx \mathrm{U}(2) \times S^2$, we expect to find $E_\infty \cong \big(\mathbb{Q}[s]/(s^2)\big) \otimes \Lambda[z_1, z_3]$ in the Serre spectral sequence (E_r, d_r) associated to $G \to {}_SG \to BS$. Indeed, its E_2 page is the tensor product $H_S^* \otimes H^*G$. From the fact the map $H^1G \longrightarrow H^1S$ is zero it will be shown to follow that the differential d_2 is zero. Next, $E_4 = E_2$ for lacunary reasons. The differential d_4 can be shown to annihilate each of s, z_1, z_3 and take $q_3 \longmapsto s^2$.

[3] This is not made explicit by Koszul, but we have $\Omega^{\geqslant 2,\bullet} = 0$, so $\omega = \omega_2 + \omega_3$ really. If we pick a B-orthonormal basis of Ω^1 including α, and expand in terms of structure constants, we [12] get $\omega_2 = \alpha \wedge d\alpha$



Because d_4 is an antiderivation, its kernel is the subalgebra $\mathbb{Q}[s] \otimes \Lambda[z_1, z_3]$ and its image the ideal (s^2) in that subalgebra. Elements mapped to a nonzero element by d_4 are marked as blue in the diagram and elements in the image in red; the vector space spanned by these elements vanishes in E_5. Thus $E_5 = \big(\mathbb{Q}[s]/(s^2)\big) \otimes \Lambda[z_1, z_3]$. For lacunary reasons, $E_5 = E_\infty$.

We work with a general compact, connected Lie group G and closed, connected subgroup H, specializing to the desired case at the end. Because the Borel fibration $G \to G_H \to BH$ is a principal G-bundle, it admits a classifying map to BG, which can be seen to be (homotopic to) the map $BH = EG/H \to EG/G = BG$ functorially induced by the inclusion $H \hookrightarrow G$. The resulting map of principal G-bundles



induces a map (ψ_r^*) of Serre spectral sequences. Each page of the right sequence $(\widetilde{E}_r, \tilde{d}_r)$ is of tensor form, and the transgressions $\tilde{d}_{2k}\colon \widetilde{E}_{2k}^{0,2k-1} \longrightarrow \widetilde{E}_{2k}^{2k,0}$ induce ([5], Thm. 13.1) a degree-one linear isomorphism

$$PH^*G \xrightarrow{\sim} H_G^{\geqslant 1} / H_G^{\geqslant 1} H_G^{\geqslant 1}$$

between the space of primitive elements of the Hopf algebra H^*G and the space of indecomposables of the polynomial ring $H_G^* = H^*BG$, which one should think of as residues of homogeneous generators. This bijection completely determines the differentials $\tilde{d}_r$, and in turn the differentials of the left spectral sequence (E_r, d_r) are completely determined by the chain relations $\psi_r^* \tilde{d}_r = d_r \psi_r^*$. A lifting of the linear isomorphism to a degree-one linear injection $\tau\colon PH^*G \longrightarrow H^*BG$, followed by the map $\rho^*\colon H^*BG \longrightarrow H^*BH$ induces a unique derivation $d = \rho^* \circ \tau$ on the page $E_2 = H^*BH \otimes H^*G$ which vanishes on the H^*BH factor and simultaneously lifts all the differentials d_r. Borel shows ([5], Thm. 25.2) the cohomology of the resulting algebra $(H_H^* \otimes H^*G, d)$, the *Cartan algebra* is isomorphic to $H^*(G/H)$.[4] In fact, one recovers the Serre spectral sequence again as the spectral sequence induced from the Cartan algebra by the filtration induced from the grading of H_H^*.

An important feature of this CDGA is that typically some of the differentials dz of primitives $z \in PH^*G$ are "redundant" in the sense they lie in the ideal $H_G^{\geqslant 1} \cdot d(PH^*G)$ generated by positive-degree multiples of such differentials. The space $\hat{P}$ of these primitives with redundant differential is called the *Samelson space*, and if we denote its complement by $\check{P} \coloneqq PH^*G/\hat{P}$, the filtration spectral sequence induced by the grading on H_K^* shows the Cartan algebra factors as a tensor product

$$(H_H^* \otimes H^*G, d) \cong (H_H^* \otimes \Lambda \check{P}, d) \otimes (\Lambda \hat{P}, 0)$$

of CDGAs [2] ([29], Thm. 2.15.V, p. 73) ([44], Prop. 8.5.4, p. 141); moreover, viewing the filtration spectral sequence as the Serre spectral sequence of $G \to G_H \to BH$, we may identify $\Lambda \hat{P}$ with the image of $H^*(G/H) \longrightarrow H^*G$. A *pure Sullivan algebra* is a free commutative graded algebra $\mathbb{Q}[Q] \otimes \Lambda P$ on an evenly- and positively-graded rational vector space Q and an oddly- and positively-graded P equipped with a derivation d vanishing on Q such that $d^2 = 0$ and $dP < \mathbb{Q}[Q]$. For such a CDGA, a Samelson space $\hat{P}$ is similarly defined as $\{z \in P : dz \in (Q \cdot dP)\}$.

Proof of Theorem 1.9 If $H^1G \longrightarrow H^1S$ is surjective, then $H^*G \cong H^*S \otimes H^*(G/S)$ by Samelson's theorem ([45], Satz VI(b), p. 1134), yielding the first clause.

Otherwise, we compute $H^*(G/S)$ as the cohomology of the Cartan algebra $(H_S^* \otimes H^*G, d)$. Write $H_S^* = \mathbb{Q}[s]$ for $s \in H^2BS^1$. Since $\mathbb{Q}[s]$ is a graded principal ideal domain, in any homogeneous basis (z_j) of PG, all but one dz_j is a redundant generator of the ideal $(dz_j) \trianglelefteq \mathbb{Q}[s]$, so the Samelson subspace $\hat{P}$ generating $\operatorname{im}\big(H^*(G/S) \to H^*G\big)$ has dimension $\operatorname{rk} G - 1$, and hence $H^*(G/S)$ has the form claimed in (A.1) (i.e., G/S is formal in the sense of rational homotopy theory). The map $H^2BG \longrightarrow H^2BS = \mathbb{Q} \cdot s$ is conjugate through transgression isomorphisms to the map $H^1G \longrightarrow H^1S$ and hence by assumption is trivial. It follows from Proposition 4.2 that S lies in the commutator subgroup K of G and we can factor the map of interest as $H_G^* \to H_K^* \to H_S^*$. The first map is surjective since G has a finite central extension $\widetilde{G}$ of the form $\widetilde{K} \times (\widetilde{G}/\widetilde{K})$, so that $H_{\widetilde{K}}^* \cong H_K^*$ is a tensor factor of $H_{\widetilde{G}}^* \cong H_G^*$

[4] Cartan earlier arrived at the same algebra by very different methods ([16], Thm. 5, p. 216). Borel's proof can be seen in retrospect to be a consequence of a general method in rational homotopy theory ([22], Prop. 15.5,8) which converts compatible models of a fibration $E \longrightarrow B$ and of a map $\rho\colon B' \longrightarrow B$ into a model of the total space of the pullback $\rho^*E \longrightarrow B'$.

(Lemma 3.13) and we may just consider the image of $H_K^* \longrightarrow H_S^*$. By the following lemma this is (s^2), so $n = 1$ in (A.1). □

Lemma A.4 *Let K be a semisimple Lie group containing a circle S. The image of $H_K^* \longrightarrow H_S^* \cong \mathbb{Q}[s]$ contains $s^2 \in H_S^4$.*

Proof Let T be a maximal torus of K containing S. By Lemma 3.5, $H_K^* \longrightarrow H_T^*$ is an injection with image the invariant subring $(H_T^*)^W$ under the action of $W = W_K$. Write $\mathbb{R}[\mathfrak{t}]$ for the graded algebra of polynomial functions on the Lie algebra $\mathfrak{t}$ of T, assigning nonzero linear forms degree 2. Extending coefficients to $\mathbb{R}$, the Chern–Weil homomorphism ([38], Thm. 2.4) and Chevalley restriction theorem ([19], §IV) translate the sequence

$$H_K^* \xrightarrow{\sim} (H_T^*)^W \hookrightarrow H_T^* \twoheadrightarrow H_S^*$$

into

$$\mathbb{R}[\mathfrak{k}]^K \underset{\sim}{\xrightarrow{\text{rest}}} \mathbb{R}[\mathfrak{t}]^W \hookrightarrow \mathbb{R}[\mathfrak{t}] \xrightarrow{\text{rest}} \mathbb{R}[\mathfrak{s}].$$

In particular, elements of H_K^4 correspond to W-invariant quadratic forms on $\mathfrak{t}$ and $H_K^4 \longrightarrow H_S^4$ is surjective if any such form does not vanish on $\mathfrak{s}$. But the Killing form B of K is a $(\mathrm{Ad}\,K)$-invariant bilinear form on $\mathfrak{k}$, negative definite since K is semisimple ([11], Prop. V.(5.13), p. 214), so precomposing the diagonal inclusion $\mathfrak{t} \rightarrowtail \mathfrak{t}^2 \hookrightarrow \mathfrak{k}^2$ yields a W-invariant quadratic form on $\mathfrak{t}$ restricting nontrivially to any one-dimensional subspace $\mathfrak{s}$. □

Remark A.5 The author's original proof of this lemma proceeded laboriously by cases through the simple groups. He is indebted to Mathew Wolak for pointing out the Killing form is invariant and definite.

B. Partial reductions

Some fragments of the results we are interested in persist even in the case G is merely assumed to be a pro-Lie group, not necessarily connected, but as the surviving results are not so powerful as one might like, they have been deferred to this appendix. We can nevertheless prove the expected result when the isotropy group remains a circle.

B.1. Connected groups

Let G be a topological group and K a closed subgroup. We would like to reduce the question of when (G, K) is isotropy-formal to the same for connected components (G_0, K_0) of the identity in each, but that is too much to hope. There is at least the following diagram:

$$\begin{array}{ccccc}
G_0/K_0 & \longrightarrow & G/K_0 & \xrightarrow{\delta} & G/K \\
\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle k} \\
{}_{K_0}G_0/K_0 & \longrightarrow & {}_{K_0}G/K_0 & \xrightarrow{\epsilon} & {}_{K_0}G/K \\
 & & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \theta} \\
 & & {}_K G/K_0 & \xrightarrow{\zeta} & {}_K G/K.
\end{array}$$

As K_0 lies in G_0, the map j can be understand as the disjoint union of $\pi_0 G$ parallel copies of i, so the one surjects in cohomology just if the other does. Less can be said about the other maps.

Proposition B.1 *Assume $\pi_0 K$ is finite. If $j^*\colon H^*_{K_0}(G/K_0) \longrightarrow H^*(G/K_0)$ is surjective, then so is $k^*\colon H^*_{K_0}(G/K) \longrightarrow H^*(G/K)$, and $(\theta \circ k)^*$ is surjective as well if and only if additionally the left action of K on G/K induces a trivial action of $\pi_0 K$ on $H^*(G/K)$. Suppose additionally K lies in G_0 and $H^*_{K_0}$ is free over H^*_K. Then if k^* is surjective, so also are $(\theta \circ k)^*$ and $(\eta \circ j)^*$ and j^*.*

Proof The maps δ, ϵ, and ζ in the diagram are coverings induced by a right $\pi_0 K$-action in such a way that j and η and hence j^* and $(\eta \circ j)^*$ are $\pi_0 K$-equivariant. Since we assume $\pi_0 K$ is finite, a standard lemma ([30], Prop. 3G.1) identifies δ^*, ϵ^*, and ζ^* with inclusions of invariants so that k^* becomes the restriction $H^*_{K_0}(G/K_0)^{\pi_0 K} \longrightarrow H^*(G/K_0)^{\pi_0 K}$ and $(\theta \circ k)^*$ the restriction $H^*_K(G/K_0)^{\pi_0 K} \longrightarrow H^*(G/K_0)^{\pi_0 K}$. If j^* is surjective, then k^* must be as well, and if $(\eta \circ j)^*$ is, then so is $(\theta \circ k)^*$, in both cases by averaging. Now, the map $(\theta \circ k)^*\colon H^*_K(G/K) \longrightarrow H^*(G/K)$ is surjective if and only if the Serre spectral sequence associated to the Borel fibration $G/K \to {}_K G/K \to BK$ collapses at E_2 and the action of $\pi_1 BK$ on the cohomology of the fiber G/K is trivial ([5], Prop. 4.1, p. 129) so triviality of the action is necessary. On the other hand, if k^* is surjective and the action is trivial, then the map of Serre spectral sequences induced by the map ${}_{K_0}G/K \twoheadrightarrow {}_K G/K$ is represented on the E_2 page by an injection $H^*_K \otimes H^*(G/K) \rightarrowtail H^*_{K_0} \otimes H^*(G/K)$, and since all differentials vanish on $E_2^{0,\bullet} \cong \mathbb{Q} \otimes H^*(G/K)$ in the larger sequence, the same holds in the smaller, so it also collapses and $(\theta \circ k)^*$ is surjective.

In general, in a Borel fibration $X \to X_K \to BK$, the action of $\pi_1 BK = \pi_0 K$ on the fiber X descends from the action of K on X, so if we assume K lies in G_0, then by path-connectedness of the latter, $\pi_0 K$ acts trivially on the right on the fibers G, ${}_{K_0}G$, and ${}_K G$ of the Borel fibrations over BK, the cohomology of whose total spaces is in question. If we assume additionally that $H^*_{K_0}$ is free over H^*_K, then Corollary B.3 applies to identify j^* with $\mathrm{id}_{H^*_{K_0}} \otimes_{H^*_K} k^*$ and $(\eta \circ j)^*$ with $\mathrm{id}_{H^*_{K_0}} \otimes_{H^*_K} (\theta \circ k)^*$, meaning in either pair of maps, the latter is surjective if and only if the former is. If k^* is surjective, then, by the argument of the previous paragraph, so also is $(\theta \circ k)^*$, and then by Corollary B.3 so also are $(\eta \circ j)^*$ and j^*. □

As limiting as the hypotheses seem, they are necessary. We will discuss their disappointing asymmetry in Remark B.4.

Lemma B.2 *Let a map of fibrations with homotopy fiber F be given as in (3.1) such that $\pi_1 B_0$ acts trivially on H^*F and H^*B is a flat module over H^*B_0. Then there is an H^*E_0-algebra isomorphism*

$$\psi\colon H^*B \underset{H^*B_0}{\otimes} H^*E_0 \xrightarrow{\sim} H^*E$$

natural in ξ.

Proof The map induces a map (ψ_r^0) of Serre spectral sequences $(E_r^0, d_r^0) \longrightarrow (E_r, d_r)$. As each E_r^0 is an H^*B_0-algebra and each E_r an H^*B-algebra, we obtain a collection of maps

$$\psi_r\colon E'_r := H^*B \underset{H^*B_0}{\otimes} E_r^0 \longrightarrow H^*B \underset{H^*B}{\otimes} E_r \xrightarrow{\sim} E_r.$$

If we assign E'_r the differential $d'_r := \mathrm{id} \otimes d_r^0$, then (E'_r, d'_r) is a spectral sequence by flatness:

$$H^*E'_r = H^*B \underset{H^*B_0}{\otimes} H^*E_r^0 = H^*B \underset{H^*B_0}{\otimes} E_{r+1}^0 = E'_{r+1}.$$

Since (ψ_r^0) was a spectral sequence map, so also is (ψ_r). As we assume simple coefficients, ψ_2 is the canonical isomorphism. Inductively, since each cochain map ψ_r is an isomorphism, so also is the map ψ_{r+1} it induces in cohomology. Thus ψ_∞ is an isomorphism. As ψ_∞ is the map of associated graded algebras induced from ψ, it follows ψ is an isomorphism as well. □

Corollary B.3 *Let a Lie group K act on X in such a way that the action of $\pi_1 BK$ on H^*X induced by the Borel fibration $X \to X_K \to BK$ is trivial. Suppose H is a subgroup of K such that H_H^* is free as an H_K^*-module. Then there is an isomorphism $H_H^* \otimes_{H_K^*} H_K^*X \xrightarrow{\sim} H_H^*X$ natural in X.*

Proof Apply Lemma B.2 to the map $X_H \longrightarrow X_K$. □

Remark B.4 In case H_K^* is not free over H_G^*, Corollary B.3 can fail. To see this, consider the block-diagonal inclusion of $H = \mathrm{SU}(3)^2$ in $K = \mathrm{SU}(6)$ and let each act on the right of $X = \mathrm{U}(6)$ by multiplication. We want to determine whether the map

$$H^*_{\mathrm{SU}(6)} \underset{H^*_{\mathrm{SU}(3)^2}}{\otimes} H^*\big(\mathrm{U}(6)/\mathrm{SU}(6)\big) \longrightarrow H^*\big(\mathrm{U}(6)/\mathrm{SU}(3)^2\big)$$

is an isomorphism. But $\mathrm{U}(6)/\mathrm{SU}(6) \cong S^1$ has cohomology ring Λz_1 concentrated in odd degree, so the map $H^*_{\mathrm{SU}(6)} \longrightarrow H^*\big(\mathrm{U}(6)/\mathrm{SU}(6)\big)$ is trivial and the domain is isomorphic to

$$\big(H_{\mathrm{SU}(3)^2} \underset{H^*_{\mathrm{SU}(6)}}{\otimes} \mathbb{Q}\big) \otimes \Lambda z_1.$$

On the other hand, it is easy to see from the Cartan algebra of Appendix A that $H^*\big(\mathrm{U}(6)/\mathrm{SU}(3)^2\big) \cong H^*\big(\mathrm{SU}(6)/\mathrm{SU}(3)^2\big) \otimes \Lambda z_1$, so the map in question is an isomorphism only if $H_{\mathrm{SU}(3)^2} \otimes_{H^*_{\mathrm{SU}(6)}} \mathbb{Q} \longrightarrow H^*\big(\mathrm{SU}(6)/\mathrm{SU}(3)^2\big)$ is. This is, however, untrue ([29], pp. 486–488): the target is the ring $\mathrm{Tor}^*_{H^*_{\mathrm{SU}(6)}}(\mathbb{Q}, H^*_{\mathrm{SU}(3)^2})$ and the sources the proper subring $\mathrm{Tor}^0_{H^*_{\mathrm{SU}(6)}}(\mathbb{Q}, H^*_{\mathrm{SU}(3)^2})$.

The condition that K lie within G_0 is severe as well, but without it, the right action of K on G/K_0 already induces a nontrivial action of $\pi_1 BK = \pi_0 K$ on $H^0(G/K_0)$.

B.2. Lie groups

To make as complete as possible the attempted reduction of the problem of isotropy-formality to the case of a torus in a semisimple group, we include the case of compact groups. We get surprisingly far, as there are relatively few algebraic obstacles, but we only achieve a complete reduction if the isotropy group is Lie. In case the isotropy group is a circle, we do get back a version of Theorem 1.5, namely Corollary B.15.

Every compact Hausdorff group G can be realized as an inverse limit of Lie group homomorphisms ([32], Ex. 3.4, p. 137), which is to say the limit in the category of topological groups of a directed system

$$(G_\alpha,\ \phi_{\alpha\to\beta}\colon G_\alpha \longrightarrow G_\beta)_{\alpha\geqslant\beta}$$

of Lie groups, the maps $\phi_{\alpha\to\beta}$ between which may be taken surjective ([31], Prop. 1.33, p. 21). Such a realization comes equipped with unique surjections $\phi_\alpha\colon G \longrightarrow G_\alpha$ for each G_α such that $\phi_\beta = \phi_{\alpha\to\beta} \circ \phi_\alpha$ whenever $\alpha \geqslant \beta$. If K is a closed subgroup of G, let $K_\alpha := \phi_\alpha K \leqslant G_\alpha$; then the restrictions $\phi_{\alpha\to\beta} \restriction K_\alpha$ realize K as $\varprojlim K_\alpha$. The inclusion map of inverse systems $(K_\alpha, \phi_{\alpha\to\beta} \restriction K_\alpha) \longrightarrow (G_\alpha, \phi_{\alpha\to\beta})$ induces a quotient system $(G_\alpha/K_\alpha, \bar\phi_{\alpha\to\beta})$ of continuous surjections of homogeneous spaces and the left action of $(K_\alpha, \phi_{\alpha\to\beta} \restriction K_\alpha)$ induces a system $({}_{K_\alpha}G_\alpha/K_\alpha, \bar\phi'_{\alpha\to\beta})$ of homotopy quotients. The canonical map $G/K \longrightarrow \lim G_\alpha/K_\alpha$ is a continuous bijection of compact Hausdorff spaces, hence a homeomorphism (in fact, this is still the case if G and K are non-compact pro-Lie groups ([43]), Lem. 1). We take as our realization of $E(-) \longrightarrow B(-)$ the Milnor construction [42]. The functorially induced map $EG \longrightarrow \lim EG_\alpha$ is actually a G-equivariant homeomorphism, inducing a homeomorphism $BG \longrightarrow \lim BG_\alpha$. Thus the map $EK \times G/K \longrightarrow \lim(EK_\alpha \times G_\alpha/K_\alpha)$ is a K-equivariant homeomorphism as well, so finally we can write ${}_K G/K$ as $\lim {}_{K_\alpha}G_\alpha/K_\alpha$. Then the fiber inclusion $i\colon G/K \longrightarrow {}_K G/K$ is identified with $\lim(i_\alpha\colon G_\alpha/K_\alpha \longrightarrow {}_{K_\alpha}G_\alpha/K_\alpha)$.

Čech cohomology (with coefficients in the constant sheaf $\underline{\mathbb{Q}}$, henceforth) converts inverse limits to direct limits ([50], pp. 318–9); the essential point is that an inverse limit can be viewed as an intersection.

Example B.5 The solenoid Ξ which is the inverse limit of the sequence $\cdots \to S^1 \xrightarrow{2} S^1 \xrightarrow{2} S^1$, though connected, has continuum-many path components, so particularly $H^0\Xi$ is large. Nevertheless, applying $\check{H}^0$ to the sequence yields isomorphisms $\cdots \leftarrow \mathbb{Q} \xleftarrow{\mathrm{id}} \mathbb{Q} \xleftarrow{\mathrm{id}} \mathbb{Q}$ and $\check{H}^1$ isomorphisms $\cdots \leftarrow \mathbb{Q} \xleftarrow{2} \mathbb{Q} \xleftarrow{2} \mathbb{Q}$, so

$\check{H}^0\Xi \cong \mathbb{Q} \cong \check{H}^1\Xi$. If we identify the map $\check{H}^1 S^1 \xrightarrow{\sim} \check{H}^1\Xi$ induced by projection to the last circle with $\mathrm{id}_{\mathbb{Q}}$, then projection to the n^{th}-from-last induces multiplication by $1/2^n$.

Thus we can identify the restriction $\check{H}^*(G/K \hookrightarrow {}_K G/K)$ with

$$\varinjlim \big(H^*_{K_\alpha}(G_\alpha/K_\alpha) \xrightarrow{i^*_\alpha} H^*(G_\alpha/K_\alpha)\big). \tag{B.1}$$

The following is then clear.

Proposition B.6 *If there is a cofinal subset of indices α such that the associated i^*_α are surjective, then so is i^*.*

But i^* can be surjective though no individual map $H^*_{K_\alpha}(G_\alpha/K_\alpha) \longrightarrow H^*(G_\alpha/K_\alpha)$ be.

Example B.7 Set $H_1 = \mathrm{SU}(6)$ and for each $k \geqslant 2$ set $H_k = \mathrm{S}\big(\mathrm{U}(3)\times\mathrm{U}(6)\big)$. Let G be the product $\prod_{k\geqslant 1} H_k$ and K the subgroup $\big\{(A_1 \oplus B_1)^\frown(B_{k-1}, A_k \oplus B_k)_{k\geqslant 2} \in H_1 \times \prod_{k\geqslant 2} H_k : A_k, B_k \in \mathrm{SU}(3)\big\}$, where $A_k \oplus B_k \in \mathrm{SU}(6)$ denotes the 6×6 block-diagonal matrix with nonzero 3×3 blocks A_k and B_k. Then (G, K) is isotropy-formal, and is the limit of the quotients $G_n = \prod_{k\leqslant n} H_k$, with the expected projections $\phi_n\colon G \longrightarrow G_n$ and $K_n = \phi_n K$, but none of the pairs (G_n, K_n) is isotropy-formal.

There is an evident artifice to this example. The groups H_k for $k \geqslant 2$ contain subgroups $H'_k = \mathrm{SU}(3)\times\mathrm{SU}(6)$ and also admit $\widetilde{H}_k = \mathrm{SU}(3)\times\mathrm{SU}(6)\times S^1$ as six-fold covers, and these are decomposable. Replacing G with $\widetilde{G} = H_1 \times \prod_{k\geqslant 2}\widetilde{H}_k$, with $\widetilde{G}_n = \prod_{k\leqslant n}\widetilde{H}_k$, and K with the isomorphic subgroup $\widetilde{K}$ of G' with entries 1 in all S^1 factors and A_k, B_k in special unitary factors as before, or replacing G with $G' = H_1 \times \prod_{k\geqslant 2} H'_k$ and maintaining the old K, the cohomological behavior of $\big(\widetilde{G}_n, \mathrm{im}(\widetilde{K} \to \widetilde{G}_n)\big)$ is the same as before, each $\widetilde{G}_n \longrightarrow G_n$ being a 6^{n-1}-fold central cover, and the behavior of $\big(G'_n, \mathrm{im}(K \to G'_n)\big)$ is similar except that all the H^*S^1 tensor factors are lost. But $\widetilde{G}$ is also the limit of the groups $\widetilde{G}'_n = \big(\mathrm{SU}(6)\times\mathrm{SU}(3)\times S^1\big)^n$, and G' of the groups $G''_n = \big(\mathrm{SU}(6)\times\mathrm{SU}(3)\times\{1\}\big)^n$, and the images $\widetilde{K}'_n$ of $\widetilde{K} \to \widetilde{G}'_n$ and K'_n of $K \to G'_n$ are both isomorphic to $\big\{(A_k \oplus B_k, B_k, 1)_{k\leqslant n} : A_k, B_k \in \mathrm{SU}(3)\big\}$, so the pairs $(\widetilde{G}'_n, \widetilde{K}'_n)$ and (G'_n, K'_n) *are* all isotropy-formal. Thus in a sense we only obtained this counterexample by perversely choosing a bad inverse system when better— up to finite coverings—were plainly available. The author still does not know if more meaningful counterexamples exist.

In the event G and K are connected, the pure Sullivan models of Cartan and Kapovitch ([16], Thm. 5, p. 216) ([5], Thm. 25.2) ([37], Prop. 1) ([23], Thm. 3.50) express each i^*_α from (B.1) as the map induced in cohomology by CDGA maps

$$\big(H^*_{K_\alpha} \otimes H^*_{K_\alpha} \otimes H^*G_\alpha, \tilde{d}_\alpha\big) \longrightarrow \big(H^*_{K_\alpha} \otimes H^*G_\alpha, d_\alpha\big). \tag{B.2}$$

With some care, we can realize $i^* = \varinjlim i^*_\alpha$ as the cohomology of a colimit of these models.

Proposition B.8 *Let* (G, K) *be a pair of compact, connected Hausdorff groups. Then the cohomology of the fiber inclusion* $G/K \longrightarrow {}_KG_K$ *is induced by a map*

$$(\check{H}^*_K \otimes \check{H}^*_K \otimes \check{H}^*G, \tilde{d}) \longrightarrow (\check{H}^*_K \otimes \check{H}^*G, d). \tag{B.3}$$

of pure Sullivan algebras given as follows.[5] *The differential* d *is the unique derivation vanishing on* H^*_K *and extending the composition*

$$P\check{H}^*G \xrightarrow[\sim]{\tau} Q\check{H}^*_G =: \check{H}^{\geqslant 1}_G \,/\, \check{H}^{\geqslant 1}_G \check{H}^{\geqslant 1}_G \xrightarrow{s} \check{H}^*_G \xrightarrow{\rho^*} \check{H}^*_K,$$

where τ *is the transgression in the Serre spectral sequence of* $G \to EG \to BG$*, the map* s *is a certain graded linear lifting of the indecomposables of* $\check{H}^*_G$ *to generators, and* $\rho = B(K \hookrightarrow G)$ *is the canonical map. The differential* $\tilde{d}$ *is the unique derivation vanishing on* $\check{H}^*_K \otimes \check{H}^*_K$ *and taking* $z \in P\check{H}^*G$ *to* $1 \otimes dz - dz \otimes 1 \in \check{H}^*_K \otimes \check{H}^*_K$*. The map of differential graded algebras is that quotienting out the ideal* $\check{H}^{\geqslant 1}_K \otimes \check{H}^*_K \otimes \check{H}^*G$*.*

Proof First we show this is a map of pure Sullivan algebras, then that it computes the map in cohomology claimed. For the former, we need only see the commutative graded algebras underlying the proposed models are free, which is to say $\check{H}^*_K$ is a polynomial ring and $\check{H}^*G$ an exterior algebra. This results from the rather restricted nature of surjective homomorphisms $G \twoheadrightarrow G'$ between compact, connected Lie groups: such a map induces a surjection $\mathfrak{g} \twoheadrightarrow \mathfrak{g}'$ of reductive Lie algebras, which is a factor projection. The group map is thus finitely covered by a factor projection:

$$\begin{array}{ccc} G'' \times \widetilde{G}' & \longrightarrow\!\!\!\!\to & \widetilde{G}' \\ \downarrow & & \downarrow \\ G & \longrightarrow\!\!\!\!\to & G'. \end{array}$$

If K is a subgroup of G and K' its image in G', then $K \twoheadrightarrow K'$ is likewise finitely covered by a factor projection $K'' \times \widetilde{K}' \twoheadrightarrow \widetilde{K}'$.[6] Since we take rational coefficients, by Proposition 3.12 the maps in cohomology are then tensor factor inclusions of the form $H^*G' \overset{\sim}{\to} H^*\widetilde{G}' \longrightarrow H^*G' \otimes H^*G'' \overset{\sim}{\to} H^*G$ and $H^*BK' \longrightarrow H^*BK' \otimes H^*BK'' \overset{\sim}{\to} H^*BK$. Thus each of the maps between the models of ${}_{K_\alpha}G_\alpha/K_\alpha$ and G_α/K_α may be replaced with a tensor factor inclusion. But the direct limit of such a system is a tensor product by definition, and a tensor product of free commutative algebras is again free. As Čech cohomology converts inverse limits to direct limits, we can substitute $\check{H}^*_K$ for $\varinjlim H^*_{K_\alpha}$ and $\check{H}^*G$ for $\varinjlim H^*G_\alpha$.

[5] It is tempting to call these algebras Sullivan models, but to do so would require CGA quasi-isomorphisms from our algebras to the algebras of polynomial differential forms $A_{\mathrm{PL}}({}_KG/K)$ and $A_{\mathrm{PL}}(G/K)$. We can construct such maps at each level of the inverse system, but as A_{PL} computes *singular* cohomology, it is unclear we will still have quasi-isomorphisms when we are done.

[6] But not necessarily in such a way that K'' is contained in G'' and $\widetilde{K}'$ in $\widetilde{G}'$. For example, let $K = \Delta K'$ be a diagonally embedded copy of $K < G$ in $G \times G$ and consider the factor projection $G \times G \twoheadrightarrow G$.

To see the cohomology of the map is as claimed, we must construct the differentials to be the colimit of the Cartan and Kapovitch differentials for the Lie pairs (G_α, K_α). Note that these models are not quite functorial, in the sense that the differentials d_α and $\tilde{d}_\alpha$—given as in the statement of the theorem if $(G, K) = (G_\alpha, K_\alpha)$—each depend on an arbitarily chosen section $QH^*_{G_\alpha} \longrightarrow H^*_{G_\alpha}$ of the reduction $H^*_{G_\alpha} \longrightarrow QH^*_{G_\alpha}$. For there to be a colimit at all, the sections must be chosen coherent in the sense the obvious squares

$$\begin{array}{ccc} QH^*_{G_\alpha} & \xleftarrow{Q(B\phi_{\alpha\to\beta})^*} & QH^*_{G_\beta} \\ \downarrow s_\alpha & & \downarrow s_\beta \\ H_{G_\alpha} & \xleftarrow[(B\phi_{\alpha\to\beta})^*]{} & H^*_{G_\beta} \end{array} \tag{B.4}$$

commute for all $\alpha \geqslant \beta$. One might hope to achieve this by defining s first and then restricting, but then it is not necessarily the case that the image of the composition $QH^*_{G_\alpha} \rightarrowtail Q\check{H}^*_G \xrightarrow{s} \check{H}^*_G$ lies in the image of $H_{G_\alpha} \rightarrowtail \check{H}^*_G$. Instead, note that the limit G will not change if we extend the diagram to include all quotients of all G_α, so we do. Next, since a finite covering induces an isomorphism in rational cohomology, we may, by picking one ring in each isomorphism class, replace the diagram of graded rings $H^*_{G_\alpha}$ by a skeleton in which no nonidentity arrow is an isomorphism.[7] Since the G_α are Lie groups, the indexing partial order is discrete and has minimal elements, which are now of the form $H^*_{S^1}$ or $H^*_{G_\alpha}$ for G_α simple. Now an induction is possible. For the base case, $QH^*_{G_\alpha}$ is one-dimensional and s_α is uniquely determined. For the induction step, because we have included all quotient groups in the diagram, each $H^*_{G_\alpha}$ is $\lim_{\beta<\alpha} H^*_{G_\beta}$. As the s_β have been chosen to make the squares (B.4) commute, the limit $s_{\leqslant\alpha} := \lim_{\beta<\alpha} s_\beta$ makes sense and we may take s_α to be the composition

$$QH^*_{G_\alpha} \xrightarrow{\sim} \lim_{\beta<\alpha} QH^*_{G_\beta} \xrightarrow{s_{\leqslant\alpha}} \lim_{\beta<\alpha} H^*_{G_\beta} \xrightarrow{\sim} H^*_{G_\alpha}.$$

This constructs s_α for all α; now we may take $s = \lim s_\alpha$.

It is now clear that (B.3) is the colimit of the maps (B.2), so as colimit is an exact functor, we may commute the colimit in (B.1) with cohomology to arrive at an identification of $\check{H}^*(G/K \hookrightarrow {}_K G/K)$ with the cohomology of the model (B.3) as claimed. □

From the existence of these pure Sullivan algebras we can with little effort extract generalizations of results known if G is a Lie group. The common thread in the proofs is that the assumption from the Lie case that the space of generators is finite-dimensional

[7] To appreciate how drastic this reduction is, note that if the solenoid Ξ of Example B.5 is a quotient of G, say $G = H \times \Xi$, then the corresponding parts of the diagram of $\mathbb{Q}$-algebras comprise solely factors $H^*_{H_\alpha}$ and $H^*_{H_\alpha} \otimes H^*_{S^1}$. Of course, this staggering swindle is only possible because we have already passed to a diagram of graded vector spaces; nothing like this can be hoped to hold in the original diagram of Lie groups.

is actually irrelevant. We say a CDGA is *formal* if it can be connected through a zig-zag of CDGA quasi-isomorphisms to its own cohomology, viewed as a CDGA with differential zero. A space is formal if its algebra $A_{\mathrm{PL}}(X)$ of polynomial differential forms is.

Proposition B.9 (Cf. ([29, pp. 83, 152], ([44, Thm. 12.6.2, p. 211], [13, Thm. 7.4.7,8]) *Let (G, K) be a pair of compact, connected Hausdorff groups. The model $(\check{H}^*_K \otimes \check{H}^*G, d)$ is formal if and only if the ideal of $\check{H}^*_K$ generated by the image of $\rho^*\colon \check{H}^{\geqslant 1}_G \longrightarrow \check{H}^{\geqslant 1}_K$ is also generated by a regular sequence contained in this image. (For any finitely-generated pure Sullivan algebra these conditions are equivalent to the equality* $\dim \hat{P} = \dim P - \dim Q$.*)*

Proposition B.10 (Cf. ([15, Thms. A, 3.4]) *If a pair (G, K) of compact, connected Hausdorff groups is isotropy-formal for Čech cohomology with rational coefficients, then the models of G/K and ${}_K G/K$ considered above are formal, and the cohomology of the latter is isomorphic to*

$$\check{H}^*_K \underset{\check{H}^*_G}{\otimes} \check{H}^*_K \otimes \operatorname{im}(\check{H}^*_K G_K \longrightarrow \check{H}^*G)$$

as an $(\check{H}^*_K \otimes \check{H}^*_K)$-algebra.

Remark B.11 If G is a Lie group, these propositions mean G/K and ${}_K G/K$ are formal in the sense of rational homotopy theory, but we do not recover this statement in general because Čech and singular cohomology $H^*X = H^*\big(A_{\mathrm{PL}}(X)\big)$ will differ.

Proposition B.12 (Cf. ([16, p. 218], [29, Thm. 2.15.V, p. 73], [44, §8.4]) *Let (G, K) be a pair of compact, connected Hausdorff groups such that the model of G/K considered above is formal and let $\check{Q} < \check{H}^*_G$ be a graded vector subspace sent bijectively by $\rho^*\colon \check{H}^*_G \longrightarrow \check{H}^*_K$ to the space spanned by a regular sequence generating $(\rho^* \check{H}^{\geqslant 1}_G) \lhd \check{H}^*_K$. Suppose there is a graded subspace $\hat{Q} < \check{H}^*_G$, meeting $\check{Q}$ trivially, such that $\check{H}^*_G$ is the symmetric algebra on $\check{Q} \oplus \hat{Q}$ and $\rho^*\hat{Q} \leqslant (\rho^*\check{H}^{\geqslant 1}_G) \cdot (\rho^*\check{H}^{\geqslant 1}_G)$. Then (G, K) is isotropy-formal for Čech cohomology with rational coefficients.*

Proof We have the liberty to choose the section $Q\check{H}^*_G \longrightarrow \check{H}^*_G$ to take $\ker \rho^* + \check{H}^1_G \cdot \check{H}^1_G$ into $\ker \rho^*$ itself. By assumption for each x in a homogeneous basis of $\hat{Q}$ we can find $a_j, b_j \in \check{H}^1_G$ with $x' = x - \sum a_j b_j \in \ker \rho^*$. Replacing each x with x', we obtain from $\hat{Q}$ a different set $\hat{Q}'$ such that $\rho^*\hat{Q}' = 0$ but $\check{Q} + \hat{Q}'$ still irredundantly generates $\check{H}^*_K$. The suspension maps $\check{H}^*_{G_\alpha} \twoheadrightarrow \check{H}^{\geqslant 1}_{G_\alpha} / \check{H}^{\geqslant 1}_{G_\alpha} \cdot \check{H}^{\geqslant 1}_{G_\alpha} \xrightarrow{\sim} P\check{H}^*G_\alpha \hookrightarrow \check{H}^*G_\alpha$ colimit to a map $\sigma\colon \check{H}^*_G \longrightarrow \check{H}^*G$ taking $\check{Q} + \hat{Q}$ bijectively onto a space of exterior generators $P < \check{H}^*G$. If we write $\check{H}^*G = \Lambda P$ and $\sigma\check{Q} = \check{P}$ and $\sigma\hat{Q}' = \hat{P}$ then the model $(\check{H}^*_K \otimes \check{H}^*_K \otimes \check{H}^*G, \tilde{d})$ factors as $(\check{H}^*_K \otimes \check{H}^*_K \otimes \Lambda\check{P}, \tilde{d}) \otimes (\Lambda\hat{P}, 0)$ and the resulting map

$$(\check{H}^*_K \otimes \check{H}^*_K \otimes \check{H}^*G, \tilde{d}) \longrightarrow \Big(\check{H}^*_K \otimes \check{H}^*_K \big/ (\tilde{d}\check{P}) \otimes \Lambda\hat{P}, 0\Big)$$

is a quasi-isomorphism. Likewise, formality of the model of G/K implies $(\check{H}^*_K \otimes \check{H}^*G, d) \longrightarrow \big(\check{H}^*_K/(d\check{P}) \otimes \Lambda\hat{P}, 0\big)$ is a quasi-isomorphism, so the cohomology of the restriction map (B.3) modeling the fiber inclusion $G/K \hookrightarrow {}_K G/K$ can be identified with the surjection $(\check{H}^*_K \otimes \check{H}^*_K)/(\tilde{d}\check{P}) \otimes \Lambda\hat{P} \twoheadrightarrow \check{H}^*_K/(d\check{P}) \otimes \Lambda\hat{P}$, and (G, K) is isotropy-formal for Čech cohomology. □

These models give us the desired converse of Proposition B.6 if K is a Lie group. In this case $K_\alpha \cong K$ far enough up in the partial order. We can loosen this obvious sufficient condition a bit by asking only that the images of the differentials $\tilde{d}_\alpha$ stabilize in a suitable sense.

Proposition B.13 *Let (G, K) be a pair of compact, connected Hausdorff groups, presented as a projective limit of compact, connected Lie groups (G_α, K_α). Endow the Cartan and Kapovitch models with differentials such that the obvious ring maps are* DGA *homomorphisms, as in the proof of Proposition B.8, and suppose there is some index ω such that for all $\alpha \geqslant \omega$ the ideal $(\tilde{d}_\alpha PH^*G_\alpha)$ of $H^*_{K_\alpha} \otimes H^*_{K_\alpha}$ is generated by the image of $\tilde{d}_\omega PH^*G_\omega$ under $(B\phi'_{\alpha\to\omega})^* \otimes (B\phi'_{\alpha\to\omega})^*$. Then (G, K) is isotropy-formal if and only if (G_ω, K_ω) is.*

Proof This heavy-handed hypothesis ensures that for all $\alpha \geqslant \omega$ the primitive elements $\hat{P}^\perp_\alpha$ of H^*G_α not in the image of $H^*G_\omega \longrightarrow H^*G_\alpha$ lie in the Samelson subspace for both the Kapovitch and the Cartan algebras, so that we can coherently tensor-factor the exterior algebra on $\hat{P}^\perp_\alpha := PH^*G_\alpha/PH^*G_\omega$, equipped with trivial differential, out of these models. Moreover the induced differentials of the nontrivial factors $(H^*_{K_\alpha})^{\otimes 2} \otimes H^*G_\omega$ are determined by the compositions

$$PH^*G_\omega \longrightarrow (H^*_{G_\omega})^{\otimes 2} \rightarrowtail (H^*_{K_\omega})^{\otimes 2} \rightarrowtail (H^*_{K_\alpha})^{\otimes 2} \rightarrowtail (H^*_{K_\beta})^{\otimes 2},$$

for $\beta \geqslant \alpha \geqslant \omega$, where the last two maps represent each target ring as a free module over the source so the Kapovitch model $\big((H^*_{K_\alpha})^{\otimes 2} \otimes H^*G_\alpha, \tilde{d}_\alpha\big)$ of ${}_{K_\alpha}G_\alpha/K_\alpha$ factors as

$$\big((H^*_{K_\alpha})^{\otimes 2}, 0\big) \underset{\big((H^*_{K_\omega})^{\otimes 2}, 0\big)}{\otimes} \big((H^*_{K_\omega})^{\otimes 2} \otimes H^*G_\omega, \tilde{d}_\omega\big) \otimes (\Lambda\hat{P}^\perp_\alpha, 0)$$

for $\alpha \geqslant \omega$ and likewise the Cartan model $(H^*_{K_\alpha} \otimes H^*G_\alpha, d_\alpha)$ of G_α/K_α factors as

$$(H^*_{K_\alpha}, 0) \underset{(H^*_{K_\omega}, 0)}{\otimes} (H^*_{K_\omega} \otimes H^*G_\omega, d_\omega) \otimes (\Lambda\hat{P}^\perp_\alpha, 0).$$

In the colimit this describes a decomposition of the model of $G/K \hookrightarrow {}_K G/K$ inducing the map

$$(\check{H}_K^*)^{\otimes 2} \underset{(H_{K_\omega}^*)^{\otimes 2}}{\otimes} H^*\big((H_{K_\omega}^*)^{\otimes 2} \otimes H^*G_\omega, d_\omega\big) \otimes \Lambda \hat{P}^\perp$$
$$\longrightarrow \check{H}_K^* \underset{H_{K_\omega}^*}{\otimes} H^*(H_{K_\omega}^* \otimes H^*G_\omega, d_\omega) \otimes \Lambda \hat{P}^\perp. \tag{B.5}$$

in cohomology.

If the map $H^*\big((H_{K_\omega}^*)^{\otimes 2} \otimes H^*G_\omega, d_\omega\big) \longrightarrow H^*(H_{K_\omega}^* \otimes H^*G_\omega, d_\omega)$ of Proposition B.8 arising from $G_\omega/K_\omega \hookrightarrow {}_{K_\omega}G_\omega/K_\omega$ is surjective, clearly (B.5) is too. On the other hand, as $\check{H}_K^*$ is a free module $A \otimes H_{K_\omega}^*$ over $H_{K_\omega}^*$, reduction modulo the augmentation ideal of A makes $H_{K_\omega}^*$ a $\check{H}_K^*$-module, and because $\check{H}_K^*$ acts on the ring on the right-hand side of (B.5), so also does $(\check{H}_K^*)^{\otimes 2}$ by the reduction $\check{H}_K^* \otimes \check{H}_K^* \twoheadrightarrow \mathbb{Q} \otimes \check{H}_K^*$. This action makes (B.5) a map of modules over $(\check{H}_K^*)^{\otimes 2} \otimes \Lambda \hat{P}^\perp$. If this map is surjective, then the map obtained by applying $\big((H_{K_\omega}^*)^{\otimes 2} \otimes \mathbb{Q}\big) \otimes_{(\check{H}_K^*)^{\otimes 2} \otimes \Lambda \hat{P}^\perp} -$ is also surjective; but this is just the cohomology of the model of $G_\omega/K_\omega \hookrightarrow {}_{K_\omega}G_\omega/K_\omega$ from Proposition B.8. □

If the K_α themselves stabilize to K, then the ideals $(\tilde{d}_\alpha PH^*G_\alpha)$ in Proposition B.13 must stabilize as well simply since $H_K^* \otimes H_K^*$ is Noetherian.

Corollary B.14 *Let (G, K) be a pair of compact, connected Hausdorff groups. If K is a Lie group, then (G, K) is isotropy-formal if and only if G admits some Lie quotient $\phi\colon G \twoheadrightarrow \overline{G}$ such that $K \cap \ker\phi = 1$ and $(\overline{G}, \phi K)$ is isotropy-formal.*

This gives us back a version of our circle result.

Corollary B.15 *Let G be a compact, connected Hausdorff group and S a circle subgroup. Then (G, S) is isotropy-formal if and only if S is not contained in the commutator subgroup of G or otherwise there is some Lie quotient $\overline{G}$ of G in which the image of S is a reflected circle, as described in Theorem 1.5.*

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