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Exact solutions to Ernst-like equation in (2+2) Hamiltonian reduction

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Abstract

We apply the method of Hamiltonian reduction without isometry as a way to find exact solutions to Einstein's equations. To find exact solutions, we introduce two spatial Killing vector fields to the Einstein's equations obtained through the Hamiltonian reduction, and derive the Ernst-like equation in the privileged coordinates. By solving the Ernst-like equation, we found a four-parameter family of exact solutions, one of which is interpreted as a deformation of the general Kasner spacetime. We extend our method to spacetimes where two independent gravitational degrees of freedom co-exist and interact with each other, and obtain a set of two partial differential equations satisfied by them. If we substitute a pre-fixed diagonal mode into these equations, and then the equations reduce to a single non-linear partial differential equation, which is interpreted as the equation of non-diagonal mode of gravitational waves propagating on the "background" spacetime determined by the diagonal mode. We choose three simplest "background" spacetimes, and discuss the corresponding non-diagonal modes in each case.

Keywords Hamiltonian reduction · 2+2 Formalism · Exact solutions · Ernst equation

1 Introduction

The idea of Hamiltonian reduction is to describe general relativity by true gravitational degrees of freedom only, after solving the constraints associated with the spacetime diffeomorphisms. It was first suggested by ADM using the canonical (3+1) decomposition decades ago, and they showed that Hamiltonian reduction can be done successfully in asymptotic region of asymptotically flat spacetimes by isolating the true gravitational degrees of freedom propagating in asymtotically flat zone[1]. Beyond asymptotically flat spacetimes; however, the Hamiltonian reduction was only partially successful, namely, one must introduce extra Killing symmetries to isolate the true gravitational degrees of freedom, free from spacetime diffeomorphisms [2–8].

Recently, one of the authors has shown that, using the (2+2) formalism based on the null hypersurface decomposition of spacetimes, Hamiltonian reduction can be done without assuming any isometry [9, 10]. In this method, a set of privileged spacetime coordinates must be introduced, which are chosen as functions defined on the phase space of Einstein's theory. In these coordinates, the spacetime constraints are solved, in the sense that they turn out to be the local conservation equations such as energy and momentum conservation equations in ordinary field theories.

In this paper, we will introduce a new method of solving Einstein's equations using the (2+2) Hamiltonian reduction. We will first present the complete set of Einstein's equations obtained after Hamiltonian reduction in privileged coordinates, and then impose two spacetime Killing symmetries to put the Einstein's equations in the Ernst form [11]. We will solve this Ernst-like equation, which turn out different from the usual Ernst equation, and show that it generates a four-parameter family of exact solutions. We show that some of them correspond to general Kasner solution and its deformation, after suitable coordinate transformations from the privileged coordinates back to the usual spacetime coordinates.

We also study more general case where two gravitational polarizations co-exist and interact with each other. Although we were not able to find explicit solutions in this case, we were able to write down the non-linear partial differential equations for one polarization interacting with the pre-determined

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another polarization that defines the "background" solution spacetime.

2 Einstein's equations in the (2+2) Hamiltonian reduction

In the theory of the (2+2) Hamiltonian reduction, it is known that the most general form of the spacetime metric in the privileged coordinates (τ, R, Y^a) is given by [9, 10]

$$ds^2 = -4h dR d\tau - 2h dR^2 + \tau\rho_{ab}(dY^a + A_R{}^a dR)(dY^b + A_R{}^b dR). \tag{2.1}$$

If we assume the "zero-twist" condition, namely, $A_R{}^a = 0$, then the Einstein's equations can be written as the following set of equations (i), (ii), (iii), and (iv):

(i) The four constraint equations define the local Hamiltonian density $-\pi_\tau$ and momentum densities π_R and $\tau^{-1}\pi_a$ given by

$$-\pi_\tau = \mathcal{H} - 2\partial_R \ln(-h), \tag{2.2}$$

$$\pi_R = -\pi^{ab}\partial_R\rho_{ab}, \tag{2.3}$$

$$\begin{aligned}\tau^{-1}\pi_a = &-\pi^{bc}\frac{\partial}{\partial Y^a}\rho_{bc} + 2\frac{\partial}{\partial Y^b}(\pi^{bc}\rho_{ac})\\ &-\frac{\partial}{\partial Y^a}\{\tau(\mathcal{H}+\pi_R)\},\end{aligned} \tag{2.4}$$

where $\mathcal{H}$ is given by

$$\begin{aligned}\mathcal{H} = &\tau^{-1}\rho_{ab}\rho_{cd}\pi^{ac}\pi^{bd} + \frac{1}{4}\tau\rho^{ab}\rho^{cd}(\partial_R\rho_{ac})(\partial_R\rho_{bd})\\ &+\pi^{ac}\partial_R\rho_{ac} + \frac{1}{2\tau},\end{aligned} \tag{2.5}$$

and π^{ab} is the conjugate momentum of the metric ρ_{ab} of the transverse two-surface N_2 with a unit determinant ($\det \rho_{ab} = 1$).

(ii) The four equations that relate the superpotential $\ln(-h)$ to $\mathcal{H} - \tau^{-1}$, π_R, and $\tau^{-1}\pi_a$

$$\partial_\tau \ln(-h) = \mathcal{H} - \tau^{-1}, \tag{2.6}$$

$$\partial_R \ln(-h) = -\pi_R, \tag{2.7}$$

$$\partial_a \ln(-h) = -\tau^{-1}\pi_a. \tag{2.8}$$

(iii) The evolution equations of ρ_{ab} and π^{ab} are given by

$$\frac{\partial}{\partial\tau}\rho_{ab} = 2\tau^{-1}\rho_{ac}\rho_{bd}\pi^{cd} + \partial_R\rho_{ab}, \tag{2.9}$$

$$\begin{aligned}\frac{\partial}{\partial\tau}\pi^{ab} = &-2\tau^{-1}\rho_{cd}\pi^{ac}\pi^{bd} + \partial_R\pi^{ab} + \frac{\tau}{2}\rho^{ac}\rho^{bd}(\partial_R^2\rho_{cd})\\ &-\frac{\tau}{2}\rho^{ai}\rho^{bj}\rho^{ck}(\partial_R\rho_{ic})(\partial_R\rho_{jk})\\ &+2h\rho^{ac}\rho^{bd}\{\mathrm{R}^{(2)}_{cd} - \frac{1}{2}\tau^{-2}\pi_c\pi_d\\ &+\nabla^{(2)}_c(\tau^{-1}\pi_d)\}.\end{aligned} \tag{2.10}$$

(iv) The *topological* constraint equation [12–16]

$$\tau R^{(2)} - \frac{1}{2}\tau^{-2}\rho^{ab}\pi_a\pi_b + \nabla^{(2)}_a(\tau^{-1}\rho^{ab}\pi_b) = 0, \tag{2.11}$$

where $R^{(2)}$ is the Ricci scalar of N_2, and $\nabla^{(2)}_a$ is the covariant derivative on N_2.

3 Dynamics of two gravitational degrees of freedom with two Killing vectors

In general, the conformal two-metric ρ_{ab} with a unit determinant has two polarizations, and therefore, it is a functional of two independent functions V and W of (τ, R, Y^a). The most general form of the conformal two-metric with two polarizations can be written as [17]

$$\rho_{ab} = \begin{pmatrix} e^V \cosh W & \sinh W \\ \sinh W & e^{-V}\cosh W \end{pmatrix}. \tag{3.1}$$

From the defining Eq. (2.9) of the conjugate momentum, π^{ab} is found to be

$$\begin{aligned}\pi^{ab} = &\frac{\pi_V}{2\cosh W}\begin{pmatrix} e^{-V} & 0 \\ 0 & -e^V \end{pmatrix}\\ &+\frac{\pi_W}{2}\begin{pmatrix} -e^{-V}\sinh W & \cosh W \\ \cosh W & -e^V\sinh W \end{pmatrix},\end{aligned} \tag{3.2}$$

where π_V and π_W are conjugate momentum of V and W, respectively, which satisfy the relation

$$\pi^{ab}\partial_\tau\rho_{ab} = \pi_V\partial_\tau V + \pi_W\partial_\tau W, \tag{3.3}$$

and π^{ab} is traceless

$$\pi^{ab}\rho_{ab} = 0. \tag{3.4}$$

From now on, we will assume that $\partial/\partial Y^a$ $(a = 1, 2)$ are two Killing vectors, and write down the Einstein's equations in terms of V, W, π_V and π_W, which are functions of τ and R only. Substitution of (3.1) into the evolution Eqs. (2.9) and (2.10) yields the following four equations:

$$\pi_V = \tau\cosh^2 W(\partial_\tau V - \partial_R V), \tag{3.5}$$

$$\pi_W = \tau(\partial_\tau W - \partial_R W), \tag{3.6}$$

$$\partial_\tau \pi_V - \partial_R \pi_V = \tau \cosh^2 W (\partial_R^2 V) + 2\tau \cosh W \sinh W (\partial_R V)(\partial_R W), \tag{3.7}$$

$$\partial_\tau \pi_W - \partial_R \pi_W = \tau \partial_R^2 W + \tau \cosh W \sinh W \times \left\{ (\partial_\tau V - \partial_R V)^2 - (\partial_R V)^2 \right\}. \tag{3.8}$$

Equations (2.4) and (2.8) are trivial due to the Killing condition, and Eqs. (2.6) and (2.7) become

$$\partial_\tau \ln(-h) = \frac{1}{2\tau \cosh^2 W} \left\{ \pi_V + \tau \cosh^2 W (\partial_R V) \right\}^2 + \frac{1}{2\tau} \left(\pi_W + \tau \partial_R W \right)^2 - \frac{1}{2\tau}, \tag{3.9}$$

$$\partial_R \ln(-h) = \pi_V (\partial_R V) + \pi_W (\partial_R W). \tag{3.10}$$

By Eqs. (3.5) and (3.6), Eqs. (3.7), (3.8), (3.9), and (3.10) become

$$(\partial_\tau - \partial_R)^2 V - \partial_R^2 V + \frac{1}{\tau}(\partial_\tau V - \partial_R V) = -2 \tanh W \left\{ (\partial_\tau V - \partial_R V)(\partial_\tau W - \partial_R W) - (\partial_R V)(\partial_R W) \right\}, \tag{3.11}$$

$$(\partial_\tau - \partial_R)^2 W - \partial_R^2 W + \frac{1}{\tau}(\partial_\tau W - \partial_R W) = \cosh W \sinh W \left\{ (\partial_\tau V - \partial_R V)^2 - (\partial_R V)^2 \right\}, \tag{3.12}$$

$$\partial_\tau \ln(-h) = \frac{\tau}{2} \cosh^2 W (\partial_\tau V)^2 + \frac{\tau}{2} (\partial_\tau W)^2 - \frac{1}{2\tau}, \tag{3.13}$$

$$\partial_R \ln(-h) = \tau \cosh^2 W (\partial_\tau V - \partial_R V)(\partial_R V) + \tau (\partial_\tau W - \partial_R W)(\partial_R W), \tag{3.14}$$

respectively. Equations (3.11) and (3.12) are second-order partial differential equations for V and W. The function h is determined by integrating the r.h.s. of the Eqs. (3.13) and (3.14), after solving Eqs. (3.11) and (3.12) for V and W. The local Hamiltonian $-\pi_\tau$ and momentum densities π_R are also determined by V and W through Eqs. (2.2) and (2.3)

$$-\pi_\tau = \frac{1}{2\tau \cosh^2 W} \left\{ \pi_V - \tau \cosh^2 W (\partial_R V) \right\}^2 + \frac{1}{2\tau} \times \left(\pi_W - \tau \partial_R W \right)^2 + \frac{1}{2\tau}, \tag{3.15}$$

$$\pi_R = -\pi_V (\partial_R V) - \pi_W (\partial_R W), \tag{3.16}$$

respectively. The remaining Einstein's equation [Eq. (2.11)] is trivial by the Killing condition. Thus, the spacetime metric is completely determined by V and W that satisfies Eqs. (3.11) and (3.12).

4 Derivation of Ernst-like equation in privileged coordinates

The line element in the privileged coordinate (τ, R, Y^a) is given by

$$ds^2 = -2h(2d\tau dR + dR^2) + \tau \cosh W \{ e^V (dY^1)^2 + e^{-V} (dY^2)^2 \} + 2\tau \sinh W dY^1 dY^2. \tag{4.1}$$

In order to derive the Ernst-like equation, it is useful to introduce the double null coordinates (u, v) defined by

$$u = \tau + R/2, \quad v = R/2. \tag{4.2}$$

Then, the metric (4.1) becomes

$$ds^2 = -8h\, du dv + (u - v) \cosh W \{ e^V (dY^1)^2 + e^{-V} (dY^2)^2 \} + 2(u - v) \sinh W dY^1 dY^2, \tag{4.3}$$

where $u \geq v$. In these coordinates, Eqs. (3.11) and (3.12) become

$$2\partial_u \partial_v V = \frac{1}{u - v}(\partial_u V - \partial_v V) + 2 \tanh W \{ (\partial_u V)(\partial_v W) + (\partial_v V)(\partial_u W) \}, \tag{4.4}$$

$$2\partial_u \partial_v W = \frac{1}{u - v}(\partial_u W - \partial_v W) - 2 \cosh W \sinh W (\partial_u V)(\partial_v V), \tag{4.5}$$

respectively. Let us introduce a complex function Z defined as[11]

$$Z = e^{-V} (\operatorname{sech} W + i \tanh W). \tag{4.6}$$

Then, we find that the two Eqs. (4.4) and (4.5) can be written as a single complex equation

$$(Z + \bar{Z}) \left\{ 2\partial_u \partial_v Z - \frac{1}{u - v}(\partial_u Z - \partial_v Z) \right\} = 4(\partial_u Z)(\partial_v Z), \tag{4.7}$$

which can be compactly written as

$$(Z + \bar{Z}) \nabla^2 Z = 2(\nabla Z)^2. \tag{4.8}$$

Here, ∇ is the covariant derivative associated with the metric (4.3), and $\nabla^2 Z$ is given by

$$\nabla^2 Z = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu Z). \tag{4.9}$$

Equation (4.7) or (4.8) is the sought-for Ernst-like equation in the Hamiltonian reduction.

5 Solutions to the Ernst-like equation

In this section, we find some solution to the Ernst-like Eq. (4.8) in the following two cases, where either V or W polarization is present.

5.1 *V* polarization solutions

In this case, we assume $W = 0$, and consider the V polarization only. Then, Eq. (4.5) becomes trivial, and (4.4) becomes [18]

$$2\partial_u\partial_v V - \frac{1}{u-v}(\partial_u V - \partial_v V) = 0. \tag{5.1}$$

We found that a whole class of solutions for V

$$V = -\ln b_0 - n\ln(u-v) - a_0(u+v), \tag{5.2}$$

and Z is given by

$$Z = e^{-V} = b_0(u-v)^n e^{a_0(u+v)}, \tag{5.3}$$

where a_0, b_0, and n are constant. In the original (τ, R) coordinates, V becomes

$$V = -\ln b_0 - n\ln\tau - a_0(\tau+R). \tag{5.4}$$

The superpotential Eqs. (3.13) and (3.14) become

$$\partial_\tau \ln(-h) = \frac{\tau}{2}(\partial_\tau V)^2 - \frac{1}{2\tau}, \tag{5.5}$$

$$\partial_R \ln(-h) = \tau(\partial_\tau V - \partial_R V)(\partial_R V), \tag{5.6}$$

which reduce to

$$\partial_\tau \ln(-h) = \frac{{a_0}^2}{2}\tau + a_0 n + \frac{n^2-1}{2\tau}, \tag{5.7}$$

$$\partial_R \ln(-h) = a_0 n, \tag{5.8}$$

respectively. By integrating these equations, we find that $-2h$ becomes

$$-2h = c_0\tau^{(n^2-1)/2}e^{\{{a_0}^2\tau^2/4 + a_0 n(\tau+R)\}}, \tag{5.9}$$

where c_0 is an arbitrary constant. Thus, we found a four parameter family of exact solutions to Einstein's equations, parametrized by four constants (a_0, b_0, c_0, n)

$$\begin{aligned} ds^2 =& c_0\tau^{(n^2-1)/2}e^{\{{a_0}^2\tau^2/4 + a_0 n(\tau+R)\}}(2dRd\tau + dR^2) \\ &+ {b_0}^{-1}\tau^{-n+1}e^{-a_0(\tau+R)}dX^2 + b_0\tau^{n+1}e^{a_0(\tau+R)}dY^2, \end{aligned} \tag{5.10}$$

where we introduced the coordinates X and Y defined as $X = Y^1$ and $Y = Y^2$, respectively. Thus, we found a four-parameter family of solutions to the Ernst-like Eq. (4.7) for a diagonal V polarization.

Let us examine this metric in the following special cases. When the constants (a_0, b_0, c_0) are chosen as $a_0 = 0,\ b_0 = c_0 = 1$, but n is left arbitrary, the metric (5.10) becomes

$$ds^2 = \tau^{(n^2-1)/2}(2d\tau dR + dR^2) + \tau^{-n+1}dX^2 + \tau^{n+1}dY^2, \tag{5.11}$$

which turns out to be a general Kasner solution. To show this, let us make the following coordinate transformations:

$$T = \tau, \quad Z = \tau + R. \tag{5.12}$$

Then the metric (5.11) becomes

$$ds^2 = T^{(n^2-1)/2}(-dT^2 + dZ^2) + T^{-n+1}dX^2 + T^{n+1}dY^2. \tag{5.13}$$

If we introduce a new coordinate t defined as

$$t = \frac{4}{n^2+3}T^{(n^2+3)/4}, \tag{5.14}$$

then the metric (5.13) can be written in a standard Kasner form [17]

$$ds^2 = -dt^2 + (\alpha t)^{2p_1}dX^2 + (\alpha t)^{2p_2}dY^2 + (\alpha t)^{2p_3}dZ^2, \tag{5.15}$$

where α, p_1, p_2, and p_3 are constants defined as

$$\begin{aligned} \alpha =& \frac{n^2+3}{4}, \quad p_1 = \frac{2(-n+1)}{n^2+3}, \quad p_2 = \frac{2(n+1)}{n^2+3}, \\ p_3 =& \frac{n^2-1}{n^2+3}, \end{aligned} \tag{5.16}$$

respectively, and p_1, p_2, and p_3 satisfy the relations

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1. \tag{5.17}$$

More generally, when both n and a_0 are arbitrary with $b_0 = c_0 = 1$, the metric (5.10) becomes

$$\begin{aligned} ds^2 =& \tau^{(n^2-1)/2}e^{\{{a_0}^2\tau^2/4 + a_0 n(\tau+R)\}}(2dRd\tau + dR^2) \\ &+ \tau^{-n+1}e^{-a_0(\tau+R)}dX^2 + \tau^{n+1}e^{a_0(\tau+R)}dY^2. \end{aligned} \tag{5.18}$$

This metric contains a non-trivial extra term that depends on arbitrary constant a_0, and therefore, it should be regarded as a *deformation* of the general Kasner solution by a free parameter a_0.

5.2 *W* polarization solutions

In this subsection, we shall solve the Ernst-like equation in the opposite case, namely, by assuming $V = 0$. The Ernst-like potential Z then becomes

$$Z = \operatorname{sech} W + i \tanh W. \tag{5.19}$$

Let us define a new function Θ by

$$\sin\Theta = \tanh W. \tag{5.20}$$

Then the Eq. (5.19) becomes

$$Z = e^{i\Theta} = \cos\Theta + i\sin\Theta. \tag{5.21}$$

By substituting this equation into (4.7), we found two independent solutions Θ_1 and Θ_2, which are given by

$$\sin\Theta_1 = \tanh W_1 = \frac{b_0 e^{a_0(u+v)} - 1}{b_0 e^{a_0(u+v)} + 1}, \tag{5.22}$$

$$\sin\Theta_2 = \tanh W_2 = \frac{\tilde{b}_0(u-v)^n - 1}{\tilde{b}_0(u-v)^n + 1}, \tag{5.23}$$

respectively, and where n, a_0, b_0, and $\tilde{b}_0$ are arbitrary constants, and the polarization W_1 and W_2 are given by

$$W_1 = \ln b_0 + a_0(u+v), \tag{5.24}$$

$$W_2 = \ln \tilde{b}_0 + n\ln(u-v), \tag{5.25}$$

respectively. Let us notice that, when $V=0$, Eq. (4.5) reduces to

$$2\partial_u\partial_v W - \frac{1}{u-v}(\partial_u W - \partial_v W) = 0, \tag{5.26}$$

which is a linear differential equation for W. By superposing the two solutions W_1 and W_2 given by (5.24) and (5.25), we obtain a more general solution of the type

$$\begin{aligned} W &= \ln b_0 + n\ln(u-v) + a_0(u+v) \\ &= \ln b_0 + n\ln\tau + a_0(\tau+R). \end{aligned} \tag{5.27}$$

One can determine the superpotential $-h$ by solving the following equations:

$$\partial_\tau \ln(-h) = \frac{\tau}{2}(\partial_\tau W)^2 - \frac{1}{2\tau}, \tag{5.28}$$

$$\partial_R \ln(-h) = \tau(\partial_\tau W - \partial_R W)(\partial_R W), \tag{5.29}$$

which reduce to

$$\partial_\tau \ln(-h) = \frac{{a_0}^2}{2}\tau + a_0 n + \frac{n^2-1}{2\tau}, \tag{5.30}$$

$$\partial_R \ln(-h) = a_0 n, \tag{5.31}$$

respectively. By integrating these equations, we find that $-2h$ becomes

$$-2h = c_0 \tau^{(n^2-1)/2} e^{\{{a_0}^2\tau^2/4 + a_0 n(\tau+R)\}}, \tag{5.32}$$

where c_0 is an arbitrary constant. Therefore, the metric is given by

$$\begin{aligned} \mathrm{d}s^2 =& c_0\tau^{(n^2-1)/2} e^{\{a_0^2\tau^2/4 + a_0 n(\tau+R)\}}(2\mathrm{d}R\mathrm{d}\tau + \mathrm{d}R^2) \\ &+ \frac{b_0}{2}\tau^{n+1}e^{a_0(\tau+R)}(\mathrm{d}X + \mathrm{d}Y)^2 \\ &+ \frac{1}{2b_0}\tau^{-n+1}e^{-a_0(\tau+R)}(\mathrm{d}X - \mathrm{d}Y)^2, \end{aligned} \tag{5.33}$$

where $X = Y^1$ and $Y = Y^2$. This is another four-parameter family of solutions to the Ernst-like equation with a non-diagonal W polarization only. However, by making the following coordinate transformations:

$$\tilde{X} = \frac{X - Y}{\sqrt{2}}, \quad \tilde{Y} = \frac{X + Y}{\sqrt{2}}, \tag{5.34}$$

then one can show that the metric (5.33) becomes

$$\begin{aligned} \mathrm{d}s^2 =& c_0\tau^{(n^2-1)/2} e^{\{a_0^2\tau^2/4 + a_0 n(\tau+R)\}}(2\mathrm{d}R\mathrm{d}\tau + \mathrm{d}R^2) \\ &+ b_0^{-1}\tau^{-n+1}e^{-a_0(\tau+R)}\mathrm{d}\tilde{X}^2 + b_0\tau^{n+1}e^{a_0(\tau+R)}\mathrm{d}\tilde{Y}^2, \end{aligned} \tag{5.35}$$

which is exactly the same as the metric (5.10).

5.3 Solutions with two polarizations *V* and *W*

In previous subsections, we found solutions with a single polarization, which are given by (5.4) and (5.27), which correspond to V polarization solutions with $W=0$ and W polarization solutions with $V=0$, respectively. In this subsection, we will find solutions that contain two polarizations simultaneously. For this purpose, we will study the equations of the W excitations propagating on the background spacetime determined by V polarization, which are given by Eq. (5.4)

$$V = -\ln b_0 - n\ln\tau - a_0(\tau+R), \tag{5.36}$$

where a_0, b_0, and n are arbitrary constants. We will consider the following 3 cases separately.

(i) $n = a_0 = 0$

In this case, the solution (5.36) becomes

$$V = -\ln b_0 = \text{constant}, \tag{5.37}$$

and Eq. (3.11) is trivially satisfied, and the equation (3.12) becomes

$$(\partial_\tau - \partial_R)^2 W - \partial_R^2 W + \frac{1}{\tau}(\partial_\tau W - \partial_R W) = 0. \tag{5.38}$$

A particular solution of this equation given by

$$W = \ln\tilde{b}_0 + \tilde{n}\ln\tau + \tilde{a}_0(\tau+R), \tag{5.39}$$

where $\tilde{a}_0$, $\tilde{b}_0$, and $\tilde{n}$ are constants. This solution is identical to the solution (5.27) that we found in Sect. 5.2, which was shown to reproduce a class of spacetimes interpreted as a *deformation* of the general Kasner solution after the prescribed coordinate transformations.

(ii) $a_0 = b_0 = 0$

In this case, the solution (5.36) becomes

$$V = -n \ln \tau \quad (n = \text{constant}), \tag{5.40}$$

and Eqs. (3.11) and (3.12) become

$$\partial_\tau W - \partial_R W = 0, \tag{5.41}$$

$$\begin{aligned} &(\partial_\tau - \partial_R)^2 W - \partial_R^2 W + \frac{1}{\tau}(\partial_\tau W - \partial_R W) \\ &\quad - \frac{n^2}{\tau^2} \cosh W \sinh W = 0, \end{aligned} \tag{5.42}$$

respectively. Equation (5.41) states that W is a function of $\tau + R$ only, and Eq. (5.42) becomes

$$\partial_R^2 W + \frac{n^2}{\tau^2} \cosh W \sinh W = 0, \tag{5.43}$$

where $W = W(\tau + R)$. Unfortunately, we were not able to find any solution of this equation, except the trivial one $W = 0$.

(iii) $n = b_0 = 0$

In this case, the solution (5.36) becomes

$$V = -a_0(\tau + R) \quad (a_0 = \text{constant}), \tag{5.44}$$

and Eqs. (3.11) and (3.12) become

$$\partial_R W = 0, \tag{5.45}$$

$$\begin{aligned} &(\partial_\tau - \partial_R)^2 W - \partial_R^2 W + \frac{1}{\tau}(\partial_\tau W - \partial_R W) \\ &\quad - a_0^2 \cosh W \sinh W = 0, \end{aligned} \tag{5.46}$$

respectively. By Eq. (5.45), W is a function of τ only, so that Eq. (5.46) becomes

$$\partial_\tau^2 W + \frac{1}{\tau}\partial_\tau W - a_0^2 \cosh W \sinh W = 0. \tag{5.47}$$

This is an ordinary differential equation of τ only, which admits a trivial solution $W = 0$. However, we were not be able to find any non-trivial solution to this equation.

6 Discussion

In this paper, we presented the Einstein's equations obtained by Hamiltonian reduction in the privileged coordinates, and then, derived the Ernst-like equation assuming two Killing symmetries. By solving the Ernst-like equation, we were able to find a four parameter family of exact solutions when a single polarization is present, which we interpret as a deformation of the general Kasner spacetime. We believe that it is a new solution, but detailed studies of the deformation solution of the general Kasner spacetime are necessary and would be interesting in its own right.

We also studied more general case where two gravitational polarizations co-exist and interact with each other. Although we were not able to find explicit solutions in this case, we were able to write down the non-linear differential Eqs. (5.43) and (5.47) for W polarization interacting with the pre-determined V polarization that defines the "background" solution spacetimes. Problems of finding non-trivial solutions to Eqs. (5.43) and (5.47) and interpreting them physically are left for a future project.

It would be interesting to find cosmological solutions whose spatial topologies are, for example, $S^1 \times S^2$ or S^3. Finding solutions with spatial $S^1 \times S^2$ topology is a doable work with the present zero-twisting condition, but to find solution with S^3 topology, one needs to drop this condition, which is beyond the scope of the present paper. The authors thank the referee for suggesting this problem, which is left for a future work.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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