



# Experimental Investigations on Crack Detection Using Modal Analysis and Prediction of Properties for Multiple Cracks by Neural Network

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**Abstract** In the present study, a method is proposed for detection and prediction of properties of multiple transverse cracks on simply supported stepped rotor shaft. Two cases of cracks are considered. Initially, both cracks are perpendicular to axis. Later, both cracks are inclined to vertical plane and also inclined with each other. Modal analysis is performed to extract natural frequency and mode shapes. Finite element method (FEM) is treated as basis for numerical analysis. For validation, experimentation is performed using fast Fourier transform analyzer. Based on natural frequency, cracks are detected. The results of FEM and experimentation are found in agreement. Crack properties are predicted in forward technique using artificial neural networks (ANN). The database of natural frequencies is used to train the network of ANN to predict the crack properties. Applicability of the method is verified by comparing the predictions of ANN with FEM and experimentation. The predictions of ANN and results given by FEM and experimentation are found in agreement. It envisages that the method is competent, suitable and would be alternate to the existing methods. It enhances the performance of structural integrity assessment and online conditioning and monitoring.

**Keywords** Bayesian statistics · Crack configuration · Modal analysis · Natural frequency · Neural network

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## List of symbols

$a$	Crack depth
$a_1$	Depth of first crack
$a_2$	Depth of second crack
$D$	Diameter of beam/shaft
$L$	Length of beam
$\omega$	Natural frequency
$M$	Mass matrix
$C$	Damping matrix
$K$	Stiffness matrix
$\ddot{x}$	Acceleration
$\dot{x}$	Velocity
$x$	Displacement
$\theta_1$	Angle made by plane of first crack
$\theta_2$	Angle made by plane of second crack
$\gamma$	Angle between the two planes of cracks
$L_{c1}$	Location of first crack from support
$L_{c2}$	Location of second crack from support
$A$	Cross-sectional area of the beam/shaft

## Introduction

Integrity assessment of structures and monitoring the health of machines comprising of rotating elements has always been very crucial. They are prone to catastrophic failure due to cracks. Early detection reduces machine down time and further consequences. It becomes necessary to predict the cracks by observing single parameter. A method is proposed which is competent to detect as well as predict the cracks with convenience and ease. It utilizes the natural frequency which is global parameter and susceptible to minor changes. Presently ultrasonic testing, magnetic particle testing, X-ray techniques, radiographic testing techniques are widely used. Beam forming of lamb waves

is promising option used for structural health monitoring [1, 2]. Wavelet transform analysis is also used for the assessment of health of shafts which can be further extended to beams [3]. Natural frequency measurement can be utilized. The presence of crack in structure changes its dynamic characteristics like modal frequencies and mode shapes associated with each modal frequency due to reduction in stiffness and increase in flexibility [4]. The structural parameters like mass, damping matrix, stiffness are altered due to crack of fault [5]. These parameters are helpful for assessment of behaviour in vibrations [6]. Fault causes non linearity [7] which can be incorporated by modelling the breathing crack with truncated Fourier series [8]. Linearity of system is accommodated by replacing the crack with an equivalent linear spring for longitudinal vibration and the torsion spring for transverse vibrations [9]. Crack strain energy function is utilized for calculating equivalent stiffness [10]. The expression for the spring stiffness representing a crack depth ratio is essential for crack detection [11]. This method can be implemented for assessment of multiple cracks also [12]. The change in local flexibility can be interpreted by  $6 \times 6$  matrix which reduces to  $3 \times 3$  matrix for bending. Low level noise is indication of the crack in the shaft [13]. The changes in eigen frequencies due to cracks are appreciable for slender shafts/beams with low slenderness ratio [14]. The closed cracks in shafts can be converted to open by modifications in stiffness [15]. The crack can be modelled as bending and shear compliance of equivalent incremental strain energy using J-integral approach, strain energy release rate. Additional crack in the component can be analyzed using two higher natural frequencies [16, 17]. The curvature mode shapes are also reported be useful for crack detection [18]. The alternate method for prediction of crack properties is reported based on constitutive relation updating method [19]. Finite Element Analysis is powerful tool which performs near exact analysis [20]. The existing methods are based on observation in changes of stiffness and efficient for prediction of single crack with precision. Genetic Algorithm can be used to solve forward problem to detect the cracks based on natural frequency changes. These are designed to search large, non-linear, discrete and poorly understood search space where expert knowledge is difficult to model and traditional optimization techniques may not give accurate results [21]. Likewise, artificial neural networks (ANN) in a wide sense belongs to the class of evolutionary computing algorithms that try to simulate natural evolution of information handling [22].

In the present work, a method is proposed for detection of multiple cracks in shaft on the basis of natural frequency reduction due to increasing depth of crack. Inverse technique is utilized wherein the cracks of known

location and known depth are generated. Shaft of uniform and varying cross sectional area with simply supported and cantilever boundary condition are considered. Initially both cracks are perpendicular to axis. To simulate the random orientation of cracks, in later case both cracks are inclined to axis as well as inclined with each other. Effect of orientation of cracks on natural frequency is studied. Natural frequencies are extracted by performing modal analysis. The experimental analysis is conducted using Fast Fourier Transform (FFT) analyzer whereas ANSYS<sup>TM</sup> is used to perform finite element method (FEM). Frequencies of bending mode are identified from the results of finite element analysis. A method is proposed for prediction of crack properties which utilizes forward problem solving technique wherein the network of Artificial Neural Network (ANN) is trained using the database of natural frequencies. Frequency values for intermediate crack properties are obtained from the trend lines presented in graphical interpretation. These frequencies are input to network which gives the prediction. Predictions given by ANN are verified by simulating the conditions and performing analysis using FEM and performing experimental analysis.

#### Geometry of Shaft and Crack Configuration

Three stepped simply supported rotor shaft as shown in detail in Fig. 1 is considered. Two transverse cracks are considered having two different configurations. To simulate the cracks with random orientation and investigate its effect on natural frequency, initially both cracks are perpendicular to the axis. In later case, both cracks have different orientation. The plane of first crack is inclined at  $30^\circ$  with reference to vertical designated as  $\theta_1$ . The plane of second crack is inclined at  $60^\circ$  in clockwise direction with reference to vertical plane designated as  $\theta_2$ . First and second crack has inclination with each other designated as  $\gamma = \theta_1 - \theta_2 = 30^\circ$ . The inclination between the two cracks  $\gamma$  is kept constant. The range of variation of first crack is  $0^\circ$ – $150^\circ$  while for second crack  $30^\circ$ – $180^\circ$ . Figure 1 shows the orientation of two cracks for both configurations discussed above. Location of first crack is interpreted as ratio  $\beta_1 = (L_{c1}/L)$  and second crack is  $\beta_2 = (L_{c2}/L)$ . The depth of first and second crack is represented as ratio  $(a_1/D)$  and  $(a_2/D)$ .

#### Detection of Crack Properties

The procedure of the proposed fault identification method relies on changes in natural frequency. The increase in the severity reports larger reduction in natural frequency. Specific correlation between the reduction in natural

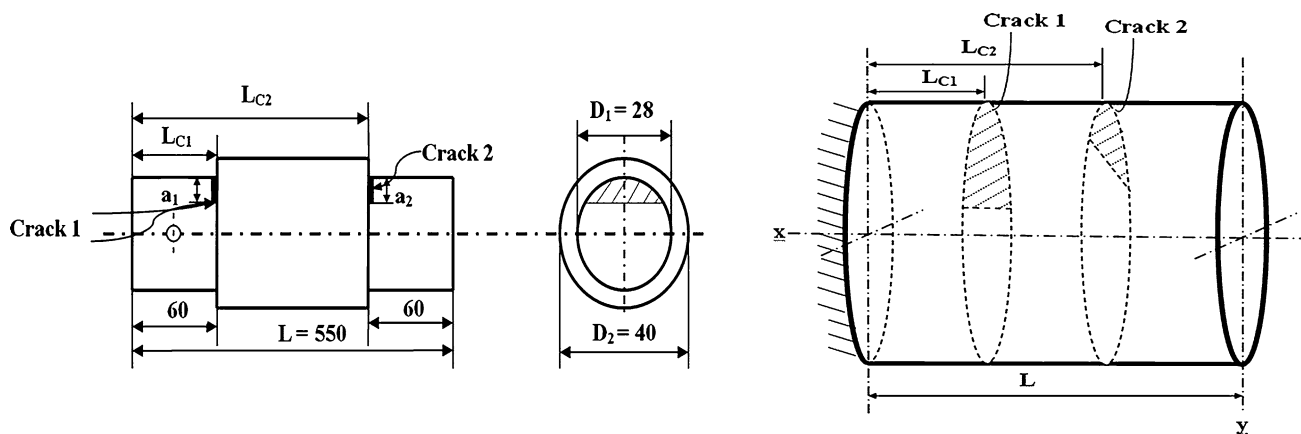


Fig. 1 Schematic diagram showing shaft geometry and crack configuration

frequency and crack depth could not be found as reduction and depth does not have any trend. Also, with change in location of crack, natural frequency changes randomly so that the location cannot be predicted by observing the natural frequency values. The rate of change of natural frequency of different modes is also in random fashion. The multiple regression analysis is suitable for such data leading towards eradicating the randomness in the function. The analysis is done using *STATISTICA*<sup>TM</sup> software with the option of advanced multiple regression analysis.

Equation (2a–e) can be written as,

$$[a_{11}, a_{12}, a_{13}, a_{14}, a_{15}; a_{21}, a_{22}, a_{23}, a_{24}, a_{25}; a_{31}, a_{32}, a_{33}, a_{34}, a_{35}; a_{41}, a_{42}, a_{43}, a_{44}, a_{45}; a_{51}, a_{52}, a_{53}, a_{54}, a_{55}] [A, B, C, E, F] = [\omega_1, \omega_2, \omega_3, \omega_4, \omega_5]. \quad (3)$$

where A is constant. The depth of the either is varied keeping the depth of another crack constant. The parameter associated with the constant crack vanishes and the Eq. (3) reduces to

$$\begin{bmatrix} a_{11}, a_{12}, 0, a_{14}, a_{15}; a_{21}, a_{22}, 0, a_{24}, a_{25}; a_{31}, a_{32}, 0, a_{34}, a_{35}; a_{41}, a_{42}, 0, a_{44}, a_{45}; a_{51}, a_{52}, 0, a_{54}, a_{55} \end{bmatrix} [A, B, C, E, F] = [\omega_1, \omega_2, \omega_3, \omega_4, \omega_5]. \quad (4)$$

$$\begin{aligned} \omega_1 &= A (B^{q1})(C^{r1})(E^{s1})(F^{t1}); \\ \omega_2 &= A (B^{q2})(C^{r2})(E^{s2})(F^{t2}); \\ \omega_3 &= A (B^{q3})(C^{r3})(E^{s3})(F^{t3}); \\ \omega_4 &= A (B^{q4})(C^{r4})(E^{s4})(F^{t4}); \\ \omega_5 &= A (B^{q5})(C^{r5})(E^{s5})(F^{t5}). \end{aligned} \quad (1a–e)$$

where A is constant,  $B = (L_{c1}/L)$ ,  $C = (a_1/D)$ ,  $E = (L_{c2}/L)$ ,  $F = (a_2/D)$ . Minimum three mode shapes are required to be extracted. By taking log on both sides, Eq. (1a–e) can be deduced to,

$$\begin{aligned} \ln \omega_1 &= \ln A + q_1 \ln(B) + r_1 \ln(C) + s_1 \ln(E) + t_1 \ln(F); \\ \ln \omega_2 &= \ln A + q_2 \ln(B) + r_2 \ln(C) + s_2 \ln(E) + t_2 \ln(F); \\ \ln \omega_3 &= \ln A + q_3 \ln(B) + r_3 \ln(C) + s_3 \ln(E) + t_3 \ln(F); \\ \ln \omega_4 &= \ln A + q_4 \ln(B) + r_4 \ln(C) + s_4 \ln(E) + t_4 \ln(F); \\ \ln \omega_5 &= \ln A + q_5 \ln(B) + r_5 \ln(C) + s_5 \ln(E) + t_5 \ln(F). \end{aligned} \quad (2a–e)$$

Eliminating 0 terms in Eq. (4) the matrix reduces to  $4 \times 4$ . Taking antilog, the values of B, C, E, F are calculated. The same system can be generated for three, four as well as five modes of natural frequencies. The proposed study takes in to account two cracks having different orientation as proposed in second configuration in Sect. 2 making angle with the vertical plane. In such situation, transverse vibration in both vertical and horizontal direction along with torsional vibration about X axis is essential for consideration. In the present analysis, the model of shaft with two cracks of different orientation is generated and modal analysis is performed using *ANSYS*<sup>TM</sup> to find natural frequencies. Figure 1 shows the shaft geometry and crack configurations.

### Modal Analysis Using Finite Element Analysis

A modal analysis is the procedure to calculate the natural frequency and find out mode shapes for a given system.

Natural frequency is dependent of the stiffness and mass of the system. The knowledge of natural frequency enables to design the system so that frequency of excitation will not match with natural frequency of component to cause resonance which in turn leads towards catastrophic failure.

$M(\ddot{x}) + C(\dot{x}) + K(x) = F$ , Neglecting damping  $M(\ddot{x}) + K(x) = 0$ ,  $\ddot{x} + (K/M)x = 0$  which can be written as  $\ddot{x} + \omega_n^2 x = 0$ , where  $\omega_n^2 = \sqrt{K/M}$ .

Finite Element Analysis is performed using ANSYS<sup>TM</sup>. The model of the entity to be analyzed is generated and used to perform modal analysis. The modal analysis is used to determine the natural frequencies and mode shapes of a structure. The element used is SOLID 87. It is higher order 3-D, 10 node element. It is quadratic element having three degrees of freedom at each node. It has capabilities to incorporate plasticity, hyperelasticity, creep, stress stiffening, large deflection and large strain. It is competent under the circumstances of either simulating deformations of nearly incompressible elastoplastic materials or totally incompressible hyperelastic materials. The Lanczos method is used to perform modal analysis. It finds eigen values and eigen vectors of a square matrix by decomposing large matrices. The material used for shaft is mild steel having properties Young's modulus (E) 1.96 e11 N/m<sup>2</sup>, density ( $\rho$ ) 7,860 kg/m<sup>3</sup> and Poisson's ratio ( $\nu$ ) 0.3. The geometry of shaft is described in detail in Sect. 2. The crack location and crack depth of first crack with reference to left support is fixed and the other crack is varied [12]. Table 1 shows the natural frequencies of shaft for different inclination configurations extracted using ANSYS.

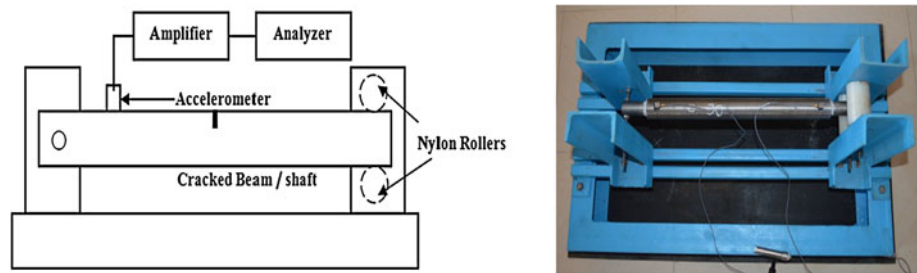
## Experimental Analysis

The experimentation is carried on simply supported shaft to perform the modal analysis which extracts the natural frequencies. In first case, cracks have first configuration wherein both cracks are perpendicular to axis. In the second case, cracks are inclined to vertical plane having second configuration. The cracks are introduced at steps where cross sectional area changes. The properties of shaft material are Young's modulus (E) 1.96 e11 N/m<sup>2</sup>, density ( $\rho$ ) 7,860 kg/m<sup>3</sup> and Poisson's ratio ( $\nu$ ) 0.3. The experimentation is performed using fast Fourier transform (FFT) analyzer, accelerometer, impact hammer and accessories. The FFT analyzer is 4 channel Bruel and Kjaer make with measuring range 10–200 dB, amplitude stability  $\pm 0.1$  dB, impedance 10 G  $\Omega$ , frequency limit 1 Hz–20 kHz. RT-PRO<sup>TM</sup> software, compatible with the FFT analyzer is used. Neoprene rubber pad of 5 mm thickness is inserted between the fixture and ground to minimize the structure prone vibrations. The piezoelectric, miniature type unidirectional accelerometer is mounted near the crack using mounting clips to capture the frequency response functions. The impact hammer used to excite the structure has the range of excitation 1–5,000 Hz. Excitation is applied in the plane parallel to the neutral plane. For every test, the location of excitation of structure by impact hammer is kept constant [23]. Initially, the frequencies of intact shaft are extracted. A fine crack is generated using cutting saw and the crack severity is increased. Table 1 shows the natural frequencies of shaft for different inclination

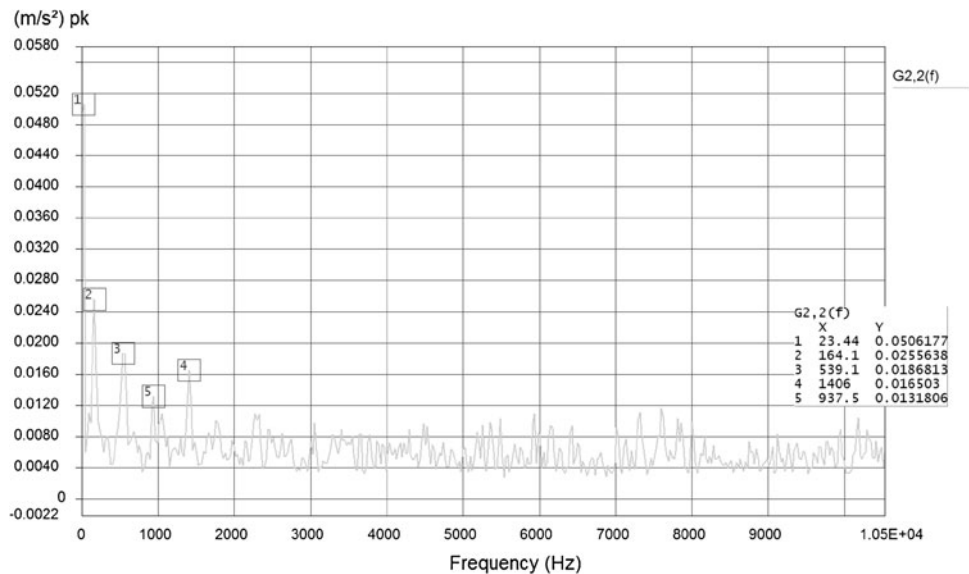
**Table 1** Natural frequency of shaft with cracks of different inclination

Crack depth (mm)	Inclination of first and second crack				
	0–0	0–30	30–60	60–90	90–120
Natural frequency of bending mode (Hz)					
2.8–2.8					
FEM $\omega_5$	1,943.2	1,938.4	1,932.2	1,936.9	1,936.1
Expt. $\omega_5$	2,173.42	2,170.33	2,176.58	2,180.18	2,179.02
FEM $\omega_6$	2,231.2	2,237.8	2,236	2,235.9	2,234.5
Expt. $\omega_6$	2,282.26	2,274.36	2,280.83	2,279.95	2,280.12
5.6–5.6					
FEM $\omega_5$	1,871.1	1,875.4	1,875.5	1,873.2	1,879.7
Expt. $\omega_5$	1,921.77	1,930.13	1,927.57	1,937.27	1,909.02
FEM $\omega_6$	2,194.2	2,202	2,200.7	2,198.2	2,179.9
Expt. $\omega_6$	2,251.67	2,217.79	2,633.89	2,241.95	2,205.77
11.2–11.2					
FEM $\omega_5$	1,737.7	1,735.1	1,739.9	1,731	1,733.8
Expt. $\omega_5$	1,785.2	1,839.29	1,751.49	1,960.4	1,864.11
FEM $\omega_6$	1,965	1,985.4	2,004	2,009.2	2,000.4
Expt. $\omega_6$	2,012.78	2,097.36	2,169.17	2,149.95	2,131.42

**Fig. 2** Line diagram and snap shot of experimental set-up for



**Fig. 3** Output of FFT analyzer



configurations extracted using experimental technique. Figure 2 shows line diagram of the experimental set up for simply supported boundary conditions and snap shot of experimental set up. The histograms captured by FFT analyzer are shown in Fig. 3.

### Prediction of Crack Properties Using (Ann)

Tools of Artificial Intelligence like Genetic algorithm, Fuzzy Logic, ANN are useful to convert inverse problem solving into forward technique [24]. Genetic algorithms are stochastic search algorithms which are based on the mechanics of nature selection and natural genetics. They have been frequently accepted as optimization methods in various fields and have also been proved to be excellent in solving complicated optimization problem [24]. These techniques minimize the errors in frequencies determined by numerical simulation and experimental investigation as they are designed to search large, discrete, non-linear and poorly understood search space. In the genetic algorithm, this error is used to evaluate the fitness of each individual in the population. The present paper checks the applicability of this tool to predict the crack location and depth in

the component depending upon the input. The network is trained for unsupervised learning with the principles of Bayesian statistics. The input to the ANN is the natural frequency of three or more number of modes and output is crack location and crack depth. In case of simply supported stepped shaft with two cracks, the output will be prediction of two parameters i.e. crack depth of both crack as the location of crack is assumed to be at the step. Back propagation method is used. Amongst the available data, 90 % data is used to train the network in ANN whereas 10 % of the data is used for validation [25]. The network is trained using the data obtained by FEM and experimental analysis. Feed forward network is selected and weights are balanced. The network decides the predominant input parameter on its own. The iterations are conducted till the average training error and average validating error is minimized [26]. In the present analysis, accuracy of predictions given by ANN is compared for single, double as well as three hidden layers. Three hidden layers give minimum error for average training and average validating error. The network can be utilized to predict crack location and depth for any intermediate input values of natural frequencies. During the routine assessment of the health of component or online



conditioning and monitoring, if decrease in natural frequency is observed, these frequency values can be given as input in the form of new query to the network.

Results and Discussion

The effect of multiple (two) cracks; crack orientation and its effect on the natural frequency are studied. Table 1 shows the natural frequencies of bending mode for cracks with different inclination. No significant variation is observed. The focus is on detection of multiple cracks and prediction of crack properties. Figures 4, 5 and 6 show the graphical comparison of natural frequencies extracted by experimental method and FEM for bending mode. The

abscissa denotes non dimensional crack depth in terms of ratio is considered while ordinate denotes non dimensional frequency ratio. Non dimensional frequency ratio is ratio of natural frequency of cracked shaft to natural frequency of uncracked shaft. With the increase in the severity of crack, natural frequency decreases. No significant variation is observed when either the first crack or the second is varied. This database is used to detect the crack properties using Eq. (4). Table 2 show the results of detection of crack properties for the cracks of different inclination. It is in the error of 4–17 %. The predictions given by ANN are verified with FEM and experimentation. Table 3 shows that the variation in the predicted results with results obtained by FEM and experimentation is in the range of 2–16 %.

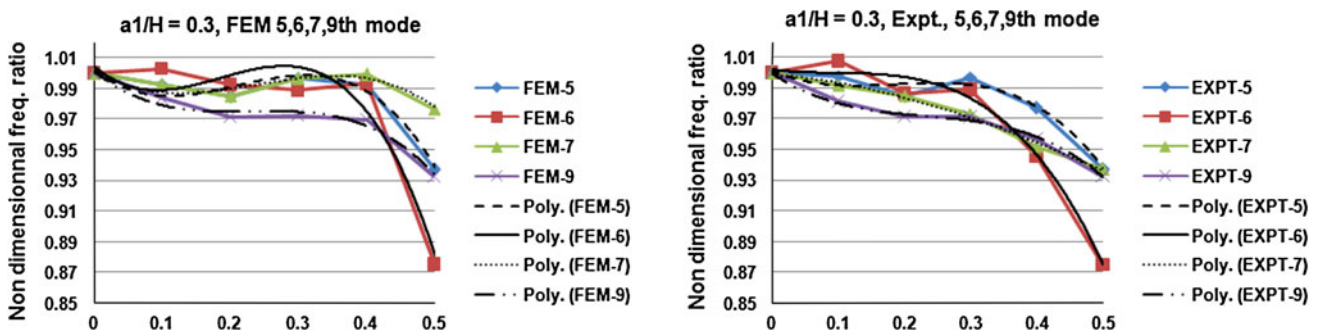


Fig. 4 Graph of natural frequency for varying depth of second crack (both cracks perpendicular to axis,  $\gamma = 0^\circ$ )

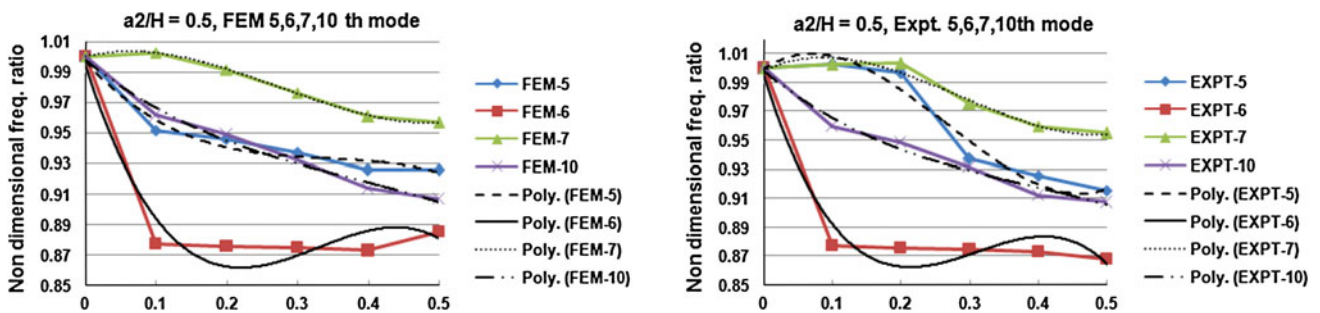


Fig. 5 Graph of natural frequency for varying depth of first crack (both cracks perpendicular to axis  $\gamma = 0^\circ$ )

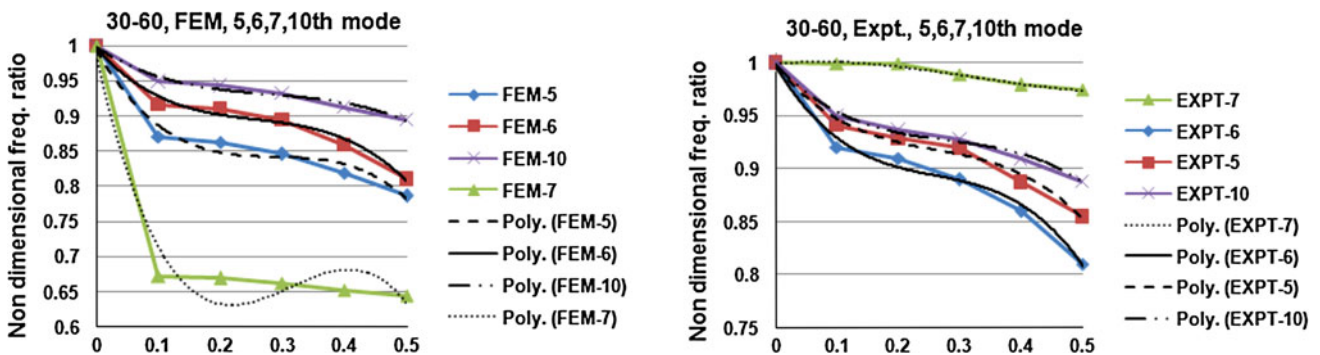


Fig. 6 Comparison of natural frequency for varying crack depth ( $\theta_1 = 60^\circ$ ,  $\theta_2 = 90^\circ$ ,  $\gamma = 30^\circ$ )

**Table 2** Comparison of results of detection of cracks by the proposed method

First crack orientation 0°			Second crack orientation 0°		
First crack (a <sub>1</sub> /D) mm			Second crack (a <sub>2</sub> /D) mm		
Actual	By proposed	% error	Actual	By proposed	% error
0.1	0.107	7	0.3	0.3196	6.53
0.2	0.235	17.5	0.4	0.4283	7.07
0.3	0.284	-5.33	0.5	0.4629	-7.42
0.4	0.3813	-4.67	0.5	0.523	4.6
First crack orientation 0°			Second crack orientation 30°		
0.1	0.11	10	0.3	0.3296	9.86
0.2	0.231	15.5	0.4	0.373	-6.75
0.3	0.314	4.66	0.5	0.4129	-17.42
0.4	0.3803	-4.92	0.5	0.523	4.6
0.5	0.427	-14.6	0.5	0.5343	6.86
First crack orientation 90°			Second crack orientation 120°		
0.1	0.0875	-12.5	0.3	0.285	-5
0.2	0.226	13	0.4	0.354	-11.5
0.3	0.329	9.66	0.5	0.562	12.4
0.4	0.373	-6.75	0.5	0.548	9.6
0.5	0.538	7.6	0.5	0.447	-10.6

**Table 3** Verification of predictions by ANN with FEM and experimentation

Predictions by ANN		Verification by FEM		Verification by Expt.		Comparison of % error			
						ANN and FEM		ANN and Expt.	
Crack orientation (first crack 0°–second crack 0°)									
a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D
0.473	0.465	0.4523	0.471	0.5	0.45	-4.37	1.29	5.70	-3.22
Crack orientation (first crack 0°–second crack 30°)									
a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D
0.358	0.54	0.317	0.478	0.35	0.45	-11.45	-11.48	-2.23	-16.66
0.451	0.457	0.443	0.516	0.45	0.45	-1.77	12.91	-3.22	-1.53
0.334	0.24	0.303	0.274	0.3	0.3	-9.28	14.16	-10.17	25
Crack orientation (first crack 90°–second crack 120°)									
a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D	a <sub>1</sub> /D	a <sub>2</sub> /D
0.361	0.553	0.331	0.488	0.4	0.55	-8.31	-11.75	10.80	-0.54
0.481	0.472	0.453	0.515	0.5	0.5	-5.82	9.11	3.95	5.93

Intermediate values of natural frequency with corresponding crack properties can be sought from the trend line drawn in graphs which is cubic polynomial.

**Conclusions**

This work attempts to establish a systematic method of crack detection and prediction of multiple crack properties. For the detection of cracks, technique is proposed. In forward technique, ANN is used to predict crack properties.

The ANN network gives the prediction of crack properties which is verified by FEM and experimental analysis and found in agreement. Based on numerical and experimental analysis, following conclusions can be drawn

- i. The inclination of crack does not affect the natural frequency.
- ii. The results of crack detection are in agreement with error in the range of 4–17 %. Maximum error of 17 % is observed due to variation in natural frequencies extracted during experimental analysis.

- iii. The trend lines in graphical interpretation can be used to facilitate the consideration of general crack.
- iv. The predictions of crack location by ANN are verified with the results of FEM and experimental analysis. The results are in agreement with error of 2–16 %. Maximum of 16 % error is observed because of variation in natural frequencies extracted in modal analysis which are used to train the network.
- v. The proposed method envisages fault diagnosis in beams, shafts or rotating machine element. It may be useful to enhance the performance of fault diagnosis methods.

## References

1. Y. Jun, U. Lee, *J Mech. Sci. Tech.* **23**, 2964–2972 (2012)
2. G.M. Owolabi, A.S.J. Swamidass, R. Seshadri, *J Sound Vib.* **265**, 1–22 (2003)
3. A.K. Darpe, *J Sound Vib.* **305**, 151–171 (2007)
4. M. Radzienski, M. Krawczuk, M. Palacz, *Mech. Sys. Sig. Proc.* **25**, 2169–2190 (2011)
5. A. Papadopoulos, *J Sound Vib.* **278**, 1205–1211 (2004)
6. M. Karthikeyan, R. Tiwari, S. Talukdar, *J Vib. Acoust.* **129**, 395–401 (2007)
7. Y. Seo, C. Lee, K.C. Park, *J. Vib. Acoust.* **131**, 011–012 (2009)
8. J.-J. Sinou, A. Lees, *J. Sound Vib.* **282**, 1015–1037 (2005)
9. S.M.A. Said, *J. Sound Vib.* **300**, 863–876 (2007)
10. R.Y. Liang, F.K. Choy, *J. Frankl. Inst.* **328**(4), 505–518 (1991)
11. S.K. Maiti, D.P. Patil, *J. Adv. Vib. Engg. Universities Press (India) Pvt. Ltd.* **4**, 348–369 (2001)
12. D.P. Patil, S.K. Maiti, *J. Eng. Frac. Mech.* **70**, 1553–1572 (2003)
13. G.M. Dong, J. Chen, *J. Mech. Sci. Tech.* **23**, 2964–2972 (2009)
14. A.S. Sekhar, *J. Sound Vib.* **223**(4), 497–512 (1999)
15. J. Wauer, *J. Solids Struct.* **26**(8), 901–914 (1990)
16. S.S. Naik, S. Maiti, *J. Sound Vib.* **324**, 1067–1085 (2009)
17. M. Kisa, M.A. Guriel, *J. Eng. Sci.* **45**, 364–380 (2007)
18. S.S. Naik, *J. Inst. Eng. India Series C* (2012) doi:[10.1007/s40032-012-0027-z](https://doi.org/10.1007/s40032-012-0027-z)
19. J.-J. Faverjon, Sinou. *Eur. J. Mech. Solids* **28**, 75–85 (2009)
20. A. El-kafrawy, *J. Mech. Mat. Des.* (2011) doi:[10.1007/s10999-011-9164-4](https://doi.org/10.1007/s10999-011-9164-4)
21. J. Xiang, Y. Zhong et al., *J. Solids Struct.* **45**, 4782–4795 (2008)
22. K. Worden et al., *Mech. Sys. Sig. Proc.* **25**, 4–111 (2011)
23. H. Nahvi, M. Jabbari, *J. Mech. Sci.* **47**, 1477–1497 (2005)
24. S. Rajendraboopathy, T. Sasikumar, K.M. Usha, *J. Adv. Manu. Tech.* (2009) doi:[10.1007/s001700081874-x](https://doi.org/10.1007/s001700081874-x)
25. S. Dzung et al., *Mech. Sys. Sig. Proc.* **39**, 181–194 (2013)
26. G.F. Bin et al., *Mech. Sys. Sig. Proc.* **27**, 696–711 (2012)