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RESEARCH ARTICLE

Picture Fuzzy Subring of a Crisp Ring

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Abstract In this paper, the concept of picture fuzzy subring of a crisp ring is introduced and some related basic results are studied. Also, some properties of picture fuzzy subring under classical ring homomorphism are investigated.

Keywords Picture fuzzy subring · Ring homomorphism of picture fuzzy subring · Picture fuzzy set

Mathematics Subject Classification 08A72

1 Introduction

Fuzzy set was introduced by Zadeh [1] as an extension of the concept of classical set theory to deal with uncertainty in human life. Later on several researchers applied fuzzy set theory in different fields. Fuzzy group was introduced by Rosenfeld [2], and fuzzy invariant subgroups and fuzzy ideals were studied by Liu [3]. Fuzzy ideals and quotient fuzzy rings were investigated by Ren [4]. As a generalization of fuzzy set theory, intuitionistic fuzzy set theory was propounded by Atanassov [5]. Based on this idea of intuitionistic fuzzy set proposed by Atanassov, intuitionistic fuzzy subgroup was introduced by Biswas [6]. Notion of intuitionistic fuzzy ring was propounded by Hur et al. [7]. Further works on intuitionistic fuzzy subring and intuitionistic fuzzy ideals were done by Banerjee and Basnet [8]. Including more possible types of uncertainty, picture fuzzy set was introduced by Cuong [9] which is a generalization of intuitionistic fuzzy set. It is necessary to mention that in intuitionistic fuzzy set, each element of the set of universe has two components namely measure of membership and measure of non-membership, whereas in picture fuzzy set, each element of the set of universe has three components namely measure of positive membership, measure of neutral membership and measure of negative membership. As the time goes, several works were done by several researchers using picture fuzzy set in different directions [10–13].

In this paper, we introduce the concept of picture fuzzy subring of a crisp ring and study some basic results related to it. Also, we investigate some properties of picture fuzzy subring under classical ring homomorphism.

Significance statement Subring of a ring is an important algebraic structure in classical sense. Fuzzy set is the generalization of classical set. So, to study about fuzzy algebraic structure is the study of generalized version of classical algebraic structure. Picture fuzzy set is the extension of fuzzy set. Here, we study subring of a ring in picture fuzzy environment which can be thought as the study of an important type of advanced fuzzy algebraic structure.

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2 Preliminaries

In the current section, we recapitulate some basic concepts of intuitionistic fuzzy set (IFS), intuitionistic fuzzy subring (IFSR), picture fuzzy set (PFS) and some operations on picture fuzzy sets (PFSs).

Definition 1 [5] Let A be the set of universe. Then an IFS P over A is defined as $P = \{(a, \mu_P, \nu_P) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of membership and $\nu_P(a) \in [0, 1]$ is the measure of non-membership of a in P satisfying the condition $0 \leqslant \mu_P(a) + \nu_P(a) \leqslant 1$ for all $a \in A$.

Definition 2 [8] Let $(R, +, \cdot)$ be a crisp ring. Then an IFS $P = \{(a, \mu_P(a), \nu_P(a)) : a \in R\}$ in R is said to be IFSR of R if the below stated conditions are fulfilled.

(i) $\mu_P(a - b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\nu_P(a - b) \leqslant \nu_P(a) \vee \nu_P(b)$
(ii) $\mu_P(a \cdot b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\nu_P(a \cdot b) \leqslant \nu_P(a) \vee \nu_P(b)$ for all $a, b \in R$.

Definition 3 [9] Let A be the set of universe. Then a PFS P over A is defined as $P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}$, where $\mu_P(a) \in [0, 1]$ is the measure of positive membership, $\eta_P(a) \in [0, 1]$ is the measure of neutral membership and $\nu_P(a) \in [0, 1]$ is the measure of negative membership of a in P satisfying the condition $0 \leqslant \mu_P(a) + \eta_P(a) + \nu_P(a) \leqslant 1$ for all $a \in A$.

Definition 4 [9] Let $P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}$ and $Q = \{(a, \mu_Q(a), \eta_Q(a), \nu_Q(a)) : a \in A\}$ be two PFSs over the universe A. Then

(i) $P \subseteq Q$ iff $\mu_P(a) \leqslant \mu_Q(a)$, $\eta_P(a) \leqslant \eta_Q(a)$ and $\nu_P(a) \geqslant \nu_Q(a)$ for all $a \in A$.
(ii) $P = Q$ iff $\mu_P(a) = \mu_Q(a)$, $\eta_P(a) = \eta_Q(a)$ and $\nu_P(a) = \nu_Q(a)$ for all $a \in A$.
(iii) $P \cup Q = \{(a, \max(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \min(\nu_P(a), \nu_Q(a))) : a \in A\}$.
(iv) $P \cap Q = \{(a, \min(\mu_P(a), \mu_Q(a)), \min(\eta_P(a), \eta_Q(a)), \max(\nu_P(a), \nu_Q(a))) : a \in A\}$.

Definition 5 Let $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), \nu_P(a_1)) : a_1 \in A_1\}$ and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), \nu_Q(a_2)) : a_2 \in A_2\}$ be two PFSs in A_1 and A_2, respectively. Then Cartesian product of P and Q is the PFS $P \times Q = \{((a, b), \mu_{P\times Q}((a, b)), \eta_{P\times Q}((a, b)), \nu_{P\times Q}((a, b)))\}$, where $\mu_{P\times Q}((a, b)) = \mu_P(a) \wedge \mu_Q(b)$, $\eta_{P\times Q}((a, b)) = \eta_P(a) \wedge \eta_Q(b)$ and $\nu_{P\times Q}((a, b)) = \nu_P(a) \vee \nu_Q(b)$ for all $(a, b) \in A_1 \times A_2$.

Definition 6 Let $h : A_1 \to A_2$ be a surjective mapping and $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), \nu_P(a_1)) : a_1 \in A_1\}$ be a PFS in A_1. Then image of P under the map h is the PFS $h(P) = \{(a_2, \mu_{h(P)}(a_2), \eta_{h(P)}(a_2), \nu_{h(P)}(a_2)) : a_2 \in A_2\}$, where $\mu_{h(P)}(a_2) = \bigvee_{a_1 \in h^{-1}(a_2)} \mu_P(a_1)$, $\eta_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \eta_P(a_1)$ and $\nu_{h(P)}(a_2) = \bigwedge_{a_1 \in h^{-1}(a_2)} \nu_P(a_1)$ for all $a_2 \in A_2$.

Definition 7 Let $h : A_1 \to A_2$ be a mapping and $Q = \{(a_2, \mu_Q(a_2), \eta_Q(a_2), \nu_Q(a_2)) : a_2 \in A_2\}$ be a PFS in A_2. Then inverse image of Q under the map h is the PFS $h^{-1}(Q) = \{(a_1, \mu_{h^{-1}(Q)}(a_1), \eta_{h^{-1}(Q)}(a_1), \nu_{h^{-1}(Q)}(a_1)) : a_1 \in A_1\}$, where $\mu_{h^{-1}(Q)}(a_1) = \mu_Q(h(a_1))$, $\eta_{h^{-1}(Q)}(a_1) = \eta_Q(h(a_1))$ and $\nu_{h^{-1}(Q)}(a_1) = \nu_Q(h(a_1))$ for all $a_1 \in A_1$.

Definition 8 Let $P = \{(a, \mu_P, \eta_P, \nu_P) : a \in A\}$ be a PFS over the universe A. Then (θ, ϕ, ψ)-cut of P is the crisp set in A denoted by $C_{\theta,\phi,\psi}(P)$ and is defined as $C_{\theta,\phi,\psi}(P) = \{a \in A : \mu_P(a) \geqslant \theta, \eta_P(a) \geqslant \phi, \nu_P(a) \leqslant \psi\}$, where $\theta, \phi, \psi \in [0, 1]$ with the condition $0 \leqslant \theta + \phi + \psi \leqslant 1$.

Proposition 1 [13] *Let $P = \{(a_1, \mu_P(a_1), \eta_P(a_1), \nu(a_1)) : a_1 \in A_1\}$ and $Q = \{(a_2, \mu_P(a_2), \eta_P(a_2), \nu_P(a_2)) : a_2 \in A_2\}$ be two PFSs over the sets of universe A_1 and A_2, respectively. Also, let $h : A_1 \to A_2$ be a mapping. Then the followings hold.*

(i) $C_{\theta,\phi,\psi}(P) \subseteq C_{\theta,\phi,\psi}(Q)$ *whenever* $P \subseteq Q$.
(ii) $C_{\theta,\phi,\psi}(P \cap Q) = C_{\theta,\phi,\psi}(P) \cap C_{\theta,\phi,\psi}(Q)$
(iii) $C_{\theta,\phi,\psi}(P \cup Q) \supseteq C_{\theta,\phi,\psi}(P) \cup C_{\theta,\phi,\psi}(Q)$
(iv) $C_{\theta,\phi,\psi}(P \times Q) = C_{\theta,\phi,\psi}(P) \times C_{\theta,\phi,\psi}(Q)$
(v) $h^{-1}((C_{\theta,\phi,\psi}(Q)) = C_{\theta,\phi,\psi}(h^{-1}(Q))$

Throughout the paper, we write PFS $P = \{(a, \mu_P(a), \eta_P(a), \nu_P(a)) : a \in A\}$ as $P = (\mu_P, \eta_P, \nu_P)$.

3 Picture Fuzzy Subring

Let us define picture fuzzy subring (PFSR) generalizing the concept of IFSR.

Definition 9 Let $(R, +, \cdot)$ be a crisp ring and $P = (\mu_P, \eta_P, \nu_P)$ be a PFS in R. Then P is said to be PFSR of R if the below stated conditions are meet.

(i) $\mu_P(a - b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a - b) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $\nu_P(a - b) \leqslant \nu_P(a) \vee \nu_P(b)$,
(ii) $\mu_P(a \cdot b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a \cdot b) \geqslant \eta_P(a) \wedge \eta_P(b)$ and $\nu_P(a \cdot b) \leqslant \nu_P(a) \vee \nu_P(b)$ for all $a, b \in R$.

Example 1 Let us consider the crisp ring $R = (Z, +, \cdot)$ and a PFS $P = (\mu_P, \eta_P, \nu_P)$ in R defined by

$$\mu_P(a) = \begin{cases} 0.4, & \text{when } a = 0 \\ 0.2, & \text{when } a \neq 0 \end{cases}$$

$$\eta_P(a) = \begin{cases} 0.4, & \text{when } a = 0 \\ 0.15, & \text{when } a \neq 0 \end{cases}$$

and

$$\nu_P(a) = \begin{cases} 0.2, & \text{when } a = 0 \\ 0.3, & \text{when } a \neq 0. \end{cases}$$

It can be shown that P is a PFSR of R.

Proposition 2 *Let $(R, +, \cdot)$ be a ring and $P = (\mu_P, \eta_P, \nu_P)$ be a PFSR of R. Then*

(i) *$\mu_P(0) \geqslant \mu_P(a)$, $\eta_P(0) \geqslant \eta_P(a)$ and $\nu_P(0) \leqslant \nu_P(a)$*
(ii) *$\mu_P(-a) = \mu_P(a)$, $\eta_P(-a) = \eta_P(a)$ and $\nu_P(-a) = \nu_P(a)$ for all $a \in R$, where 0 is the additive identity in R and $-a$ is the additive inverse of a.*

Proof

(i) Since P is a PFSR of R therefore

$$\mu_P(0) = \mu_P(a-a) \geqslant \mu_P(a) \wedge \mu_P(a) = \mu_P(a),$$
$$\eta_P(0) = \eta_P(a-a) \geqslant \eta_P(a) \wedge \eta_P(a) = \eta_P(a)$$
$$\text{and } \nu_P(0) = \nu_P(a-a) \leqslant \nu_P(a) \vee \nu_P(a) \text{ for all } a \in R.$$

Thus, $\mu_P(0) \geqslant \mu_P(a)$, $\eta_P(0) \geqslant \eta_P(a)$ and $\nu_P(0) \leqslant \nu_P(a)$ for all $a \in R$.

(ii) For all $a \in R$, we have,

$$\mu_P(-a) = \mu_P(0-a) \geqslant \mu_P(0) \wedge \mu_P(a) = \mu_P(a) \text{ [by (i)]}$$
$$\eta_P(-a) = \eta_P(0-a) \geqslant \eta_P(0) \wedge \eta_P(a) = \eta_P(a) \text{ [by (i)]}$$
$$\text{and } \nu_P(-a) = \nu_P(0-a) \leqslant \nu_P(0) \vee \nu_P(a) = \nu_P(a) \text{ [by (i)].}$$

Thus, $\mu_P(-a) \geqslant \mu_P(a)$, $\eta_P(-a) \geqslant \eta_P(a)$ and $\nu_P(-a) \leqslant \nu_P(a)$ for $a \in R$.

Now, replacing a by $-a$ we get, $\mu_P(a) \geqslant \mu_P(-a)$, $\eta_P(a) \geqslant \eta_P(-a)$ and $\nu_P(a) \leqslant \nu_P(-a)$ for all $a \in R$. Consequently, $\mu_P(-a) = \mu_P(a)$, $\eta_P(-a) = \eta_P(a)$ and $\nu_P(-a) = \nu_P(a)$ for all $a \in R$. □

Proposition 3 *Let $(R, +, \cdot)$ be a crisp ring and $P = (\mu_P, \eta_P, \nu_P)$ be a PFSR of R. Then $C_{\theta,\phi,\psi}(P)$ is a crisp subring of R, provided that $\mu_P(0) \geqslant \theta$, $\eta_P(0) \geqslant \phi$ and $\nu_P(0) \leqslant \psi$, where 0 is the additive identity in the ring R.*

Proof Clearly, $C_{\theta,\phi,\psi}(P)$ is non-empty.

Let $a, b \in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$, $\nu_P(a) \leqslant \psi$ and $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$, $\nu_P(b) \leqslant \psi$. Since P is a PFSR of R therefore

$$\mu_P(a-b) \geqslant \mu_P(a) \wedge \mu_P(b) \geqslant \theta \wedge \theta = \theta,$$
$$\eta_P(a-b) \geqslant \eta_P(a) \wedge \eta_P(b) \geqslant \phi \wedge \phi = \phi$$
$$\text{and } \nu_P(a-b) \leqslant \nu_P(a) \vee \nu_P(b) \leqslant \psi \vee \psi = \psi.$$

Thus, $\mu_P(a-b) \geqslant \theta$, $\eta_P(a-b) \geqslant \phi$ and $\nu_P(a-b) \leqslant \psi$.

It follows that $a - b \in C_{\theta,\phi,\psi}(P)$.

Also, since P is a PFSR of R therefore

$$\mu_P(a \cdot b) \geqslant \mu_P(a) \wedge \mu_P(b) \geqslant \theta \wedge \theta = \theta,$$
$$\eta_P(a \cdot b) \geqslant \eta_P(a) \wedge \eta_P(b) \geqslant \phi \wedge \phi = \phi$$
$$\text{and } \nu_P(a \cdot b) \leqslant \nu_P(a) \vee \nu_P(b) \leqslant \psi \vee \psi = \psi.$$

Thus, $\mu_P(a \cdot b) \geqslant \theta$, $\eta_P(a \cdot b) \geqslant \phi$ and $\nu_P(a \cdot b) \leqslant \psi$.

It follows that $a \cdot b \in C_{\theta,\phi,\psi}(P)$. Consequently, $C_{\theta,\phi,\psi}(P)$ is a crisp subring of R. □

Proposition 4 *Let $(R, +, \cdot)$ be a crisp ring and $P = (\mu_P, \eta_P, \nu_P)$ be a PFS in R. Then P is a PFSR of R if all (θ, ϕ, ψ)-cuts of P are crisp subrings of R.*

Proof Let $a, b \in R$. Also, let $\theta = \mu_P(a) \wedge \mu_P(b)$, $\phi = \eta_P(a) \wedge \eta_P(b)$ and $\psi = \nu_P(a) \vee \nu_P(b)$. Clearly, $\theta \in [0,1]$, $\phi \in [0,1]$ and $\psi \in [0,1]$ with $0 \leqslant \theta + \phi + \psi \leqslant 1$.

It is observed that

$$\mu_P(a) \geqslant \mu_P(a) \wedge \mu_P(b) = \theta,$$
$$\eta_P(a) \geqslant \eta_P(a) \wedge \eta_P(b) = \phi$$
$$\text{and } \nu_P(a) \leqslant \nu_P(a) \vee \nu_P(b) = \psi$$

Thus, $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $\nu_P(a) \leqslant \psi$.

Also, we have

$$\mu_P(b) \geqslant \mu_P(a) \wedge \mu_P(b) = \theta,$$
$$\eta_P(b) \geqslant \eta_P(a) \wedge \eta_P(b) = \phi$$
$$\text{and } \nu_P(b) \leqslant \nu_P(a) \vee \nu_P(b) = \psi$$

Thus, $\mu_P(b) \geqslant \theta$, $\eta_P(b) \geqslant \phi$ and $\nu_P(b) \leqslant \psi$.

It follows that $a, b \in C_{\theta,\phi,\psi}(P)$. Since $C_{\theta,\phi,\psi}(P)$ is a crisp subring of R therefore $a - b$ and $a \cdot b \in C_{\theta,\phi,\psi}(P)$.

This yields

$$\mu_P(a-b) \geqslant \theta = \mu_P(a) \wedge \mu_P(b),$$
$$\eta_P(a-b) \geqslant \phi = \eta_P(a) \wedge \eta_P(b)$$
$$\nu_P(a-b) \leqslant \psi = \nu_P(a) \vee \nu_P(b)$$
$$\text{and } \mu_P(a \cdot b) \geqslant \theta = \mu_P(a) \wedge \mu_P(b),$$
$$\eta_P(a \cdot b) \geqslant \phi = \eta_P(a) \wedge \eta_P(b)$$
$$\nu_P(a \cdot b) \leqslant \psi = \nu_P(a) \vee \nu_P(b).$$

Since a, b are arbitrary elements of R therefore $\mu_P(a-b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a-b) \geqslant \eta_P(a) \wedge \eta_P(b)$, $\nu_P(a-b) \leqslant \nu_P(a) \vee \nu_P(b)$ and $\mu_P(a \cdot b) \geqslant \mu_P(a) \wedge \mu_P(b)$, $\eta_P(a \cdot b) \geqslant \eta_P(a) \wedge \eta_P(b)$, $\nu_P(a \cdot b) \leqslant \nu_P(a) \vee \nu_P(b)$ for all $a, b \in R$.

Consequently, P is a PFSR of R. □

Proposition 5 *Let $(R, +, \cdot)$ be a crisp ring and $L = (\mu_L, \eta_L, \nu_L)$ be a PFSR of R. Then $\mu_L(ra) \geqslant \mu_P(a)$, $\eta_L(ra) \geqslant \eta_L(a)$ and $\nu_L(ra) \leqslant \nu_L(a)$ for all $a \in R$ and for all integers r.*

Proof Case 1: Let r be a positive integer. Let $P(r): \mu_L(ra) \geqslant \mu_L(a)$, $\eta_L(ra) \geqslant \eta_L(a)$ and $\nu_L(ra) \leqslant \nu_L(a)$ for all $a \in R$. Here, $P(1)$ is trivially true. Since L is a PFSR of R therefore $\mu_L(a^2) = \mu_L(a \cdot a) \geqslant \mu_L(a) \wedge \mu_L(a) = \mu_L(a)$, $\eta_L(a^2) = \eta_L(a \cdot a) \geqslant \eta_L(a) \wedge \eta_L(a) = \eta_L(a)$ and $\nu_L(a^2) = \nu_L(a \cdot a) \leqslant \nu_L(a) \vee \nu_L(a) = \nu_L(a)$. Therefore, $P(2)$ is true. Let us assume that $P(r)$ is true for $r = m$, i.e. $P(m)$ is true. Then $\mu_L(ma) \geqslant \mu_L(a)$, $\eta_L(ma) \geqslant \eta_L(a)$ and $\nu_L(ma) \leqslant$

$\nu_L(a)$ for all $a \in R$. Since L is a PFSR of R therefore for all $a \in R$,

$$\begin{aligned}\mu_L((m+1)a) &= \mu_L(ma+a)\\ &\geqslant \mu_L(ma) \wedge \mu_L(a)\\ &\geqslant \mu_L(a) \wedge \mu_L(a) = \mu_L(a),\\ \eta_L((m+1)a) &= \eta_L(ma+a)\\ &\geqslant \eta_L(ma) \wedge \eta_L(a)\\ &\geqslant \eta_L(a) \wedge \eta_L(a) = \eta_L(a)\\ \text{and } \nu_L((m+1)a) &= \nu_L(ma+a)\\ &\leqslant \nu_L(ma) \vee \nu_L(a)\\ &\leqslant \nu_L(a) \vee \nu_L(a) = \nu_L(a).\end{aligned}$$

Therefore $P(r)$ is true for all positive integers r.

Case 2: Let r is a negative integer. Also, let $s = -r$. Since r is a negative integer therefore $r \leqslant -1$ which implies that $s \geqslant 1$, i.e. s is a positive integer. Now, $\mu_L(ra) = \mu_L(-sa) = \mu_L(sa) \geqslant \mu_L(a)$, $\eta_L(ra) = \eta_L(-sa) = \eta_L(sa) \geqslant \eta_L(a)$ and $\nu_L(ra) = \nu_L(-sa) = \nu_L(sa) \leqslant \nu_L(a)$ [by Proposition 2 and Case 1]

Case 3: When $r = 0$ then we see that $P(r)$ is trivially true because it is known from Proposition 2 that $\mu_L(0) \geqslant \mu_L(a)$, $\eta_L(0) \geqslant \eta_L(a)$ and $\nu_L(0) \leqslant \nu_L(a)$ for all $a \in R$. □

Proposition 6 *Let $(R, +, \cdot)$ be a crisp ring and $P = (\mu_P, \eta_P, \nu_P)$ be a PFSR of R. If a be the additive generator of R with $a \in C_{\theta,\phi,\psi}(P)$ then $C_{\theta,\phi,\psi}(P) = R$.*

Proof We know that $C_{\theta,\phi,\psi}(P) \subseteq R$. Let a be the additive generator of R with $a \in C_{\theta,\phi,\psi}(P)$. Then $\mu_P(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $\nu_P(a) \leqslant \psi$. Also, let $t \in R$. Since R is an additive cyclic group therefore $t = pa$ for some integer p. Now, we have,

$$\begin{aligned}\mu_P(t) &= \mu_P(pa) \geqslant \mu_P(a) \text{ [by Proposition 5]}\\ &\geqslant \theta,\\ \eta_P(t) &= \eta_P(pa) \geqslant \eta_P(a) \text{ [by Proposition 5]}\\ &\geqslant \phi\\ \text{and } \nu_P(t) &= \nu_P(pa) \leqslant \nu_P(a) \text{ [by Proposition 5]}\\ &\leqslant \psi.\end{aligned}$$

Thus, we get $\mu_P(t) \geqslant \theta$, $\eta_P(t) \geqslant \phi$ and $\nu_P(t) \leqslant \psi$. Therefore, $t \in R \Rightarrow t \in C_{\theta,\phi,\psi}(P)$ which yields $R \subseteq C_{\theta,\phi,\psi}(P)$. Consequently, $C_{\theta,\phi,\psi}(P) = R$. □

Proposition 7 *Let $P = (\mu_P, \eta_P, \nu_P)$ and $Q = (\mu_Q, \eta_Q, \nu_Q)$ be two PFSRs of a ring $(R, +, \cdot)$. Then $P \cap Q$ is a PFSR of R.*

Proof It is known from Proposition 1 that $C_{\theta,\phi,\psi}(P \cap Q) = C_{\theta,\phi,\psi}(P) \cap C_{\theta,\phi,\psi}(Q)$. Since P and Q are PFSRs therefore by Proposition 3, $C_{\theta,\phi,\psi}(P)$ and $C_{\theta,\phi,\psi}(Q)$ are crisp subrings of R. Also, it is known that the intersection of any two crisp subrings is a crisp subring. As a result, $C_{\theta,\phi,\psi}(P \cap Q)$ is a crisp subring of R. Consequently, by Proposition 4, $P \cap Q$ is a PFSR of R. □

Proposition 8 *Let P and Q be two PFSRs of a ring $(R, +,)$. Then $P \cup Q$ is a PFSR of R if either $P \subseteq Q$ or $Q \subseteq P$.*

Proof Case 1: Let $P \subseteq Q$. Then $\mu_P(a) \leqslant \mu_Q(a)$, $n_P(a) \leqslant n_Q(a)$ and $\nu_P(a) \geqslant \nu_Q(a)$ for all $a \in R$. Therefore, $\mu_{P\cup Q}(a) = \mu_P(a) \vee \mu_Q(a) = \mu_Q(a)$, $\eta_{P\cup Q}(a) = \eta_P(a) \wedge \eta_Q(a) = \eta_P(a)$ and $\nu_{P\cup Q}(a) = \nu_P(a) \wedge \nu_Q(a) = \nu_Q(a)$ for all $a \in R$. It is observed that $Q \subseteq P \cup Q \Rightarrow C_{\theta,\phi,\psi}(Q) \leqslant C_{\theta,\phi,\psi}(P \cup Q)$ [by Proposition 1].

Let $a \in C_{\theta,\phi,\psi}(P \cup Q)$. Then $\mu_{P\cup Q}(a) \geqslant \theta$, $\eta_{P\cup Q}(a) \geqslant \phi$ and $\nu_{P\cup Q}(a) \leqslant \psi$, i.e. $\mu_Q(a) \geqslant \theta$, $\eta_P(a) \geqslant \phi$ and $\nu_Q(a) \leqslant \psi$, i.e. $\mu_Q(a) \geqslant \theta$, $\eta_Q(a) \geqslant \phi$ and $\nu_Q(a) \leqslant \psi$. Thus, $a \in C_{\theta,\phi,\psi}(Q)$. Therefore, $C_{\theta,\phi,\psi}(P \cup Q) \subseteq C_{\theta,\phi,\psi}(Q)$. Thus, finally, $C_{\theta,\phi,\psi}(P \cup Q) = C_{\theta,\phi,\psi}(Q)$. Since P and Q are PFSRs of R therefore by Proposition 3, $C_{\theta,\phi,\psi}(P)$ and $C_{\theta,\phi,\psi}(Q)$ are crisp subrings of R. As a result, $C_{\theta,\phi,\psi}(P \cup Q)$ is a crisp subring of R. Consequently, by Proposition 4, $P \cup Q$ is a PFSR of R.

Case 2: When $Q \subseteq P$, it can be proved in the similar way that $P \cup Q$ is a PFSR of R.

The converse of the above proposition does not necessarily hold which is clear from the following example, i.e. if P and Q are two PFSRs of a crisp ring R then $P \cup Q$ is a PFSR of R does not necessarily imply that either $P \subseteq Q$ or $Q \subseteq P$. □

Example 2 Let us consider the ring $(R, +, \cdot)$ and the PFSR $P = (\mu_P, \eta_P, \nu_P)$ of R given in Example 1. Also, a PFSR $Q = (\mu_Q, \eta_Q, \nu_Q)$ of R is defined as follows.

$$\mu_Q(a) = \begin{cases} 0.35, & \text{when } a = 0\\ 0.3, & \text{when } a \neq 0\end{cases}$$

$$\eta_Q(a) = \begin{cases} 0.35, & \text{when } a = 0\\ 0.2, & \text{when } a \neq 0\end{cases}$$

and

$$\nu_Q(a) = \begin{cases} 0.1, & \text{when } a = 0\\ 0.4, & \text{when } a \neq 0.\end{cases}$$

Thus, $P \cup Q$ is given by

$$\mu_{P\cup Q}(a) = \begin{cases} 0.4, & \text{when } a = 0\\ 0.3, & \text{when } a \neq 0\end{cases}$$

$$\eta_{P\cup Q}(a) = \begin{cases} 0.35, & \text{when } a = 0\\ 0.15, & \text{when } a \neq 0\end{cases}$$

and

$$\nu_{P\cup Q}(a) = \begin{cases} 0.1, & \text{when } a = 0\\ 0.3, & \text{when } a \neq 0.\end{cases}$$

Here, $P \cup Q$ is a PFSR of R but neither $P \subseteq Q$ nor $Q \subseteq P$.

Proposition 9 *Let $P = (\mu_P, \eta_P, \nu_P)$ and $Q = (\mu_Q, \eta_Q, \nu_Q)$ be two PFSRs of a ring $(R, +, \cdot)$. Then $P \times Q$ is a PFSR of $R \times R$.*

Proof It is known from Proposition 1 that $C_{\theta,\phi,\psi}(P \times Q) = C_{\theta,\phi,\psi}(P) \times C_{\theta,\phi,\psi}(Q)$. Since P and Q are PFSRs of R therefore by Proposition 3, $C_{\theta,\phi,\psi}(P)$ and $C_{\theta,\phi,\psi}(Q)$ are crisp subrings of R. Also, it is known that the Cartesian product of two crisp subrings is a crisp subring. As a result, $C_{\theta,\phi,\psi}(P \times Q)$ is a crisp subring of $R \times R$. Consequently, by Proposition 4, $P \times Q$ is a PFSR of $R \times R$. □

4 Ring Homomorphism of Picture Fuzzy Subring

In the current section, we study some important properties of PFSR under classical ring homomorphism.

Proposition 10 *Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ be two crisp rings and $Q = (\mu_Q, \eta_Q, \nu_Q)$ be a PFSR of R_2. Then for a ring homomorphism, $h : R_1 \to R_2$, $h^{-1}(Q)$ is a PFSR of R_1.*

Proof Let $h^{-1}(Q) = (\mu_{h^{-1}(Q)}, \eta_{h^{-1}(Q)}, \nu_{h^{-1}(Q)})$.
Also, let $a, b \in C_{\theta,\phi,\psi}(h^{-1}(Q))$. Then

$$\mu_{h^{-1}(Q)}(a) \geqslant \theta, \eta_{h^{-1}(Q)}(a) \geqslant \phi, \nu_{h^{-1}(Q)}(a) \leqslant \psi$$

and $\mu_{h^{-1}(Q)}(b) \geqslant \theta, \eta_{h^{-1}(Q)}(b) \geqslant \phi, \nu_{h^{-1}(Q)}(b) \leqslant \psi$.
This implies,

$$\mu_Q(h(a)) \geqslant \theta, \eta_Q(h(a)) \geqslant \phi, \nu_Q(h(a)) \leqslant \psi$$

and $\mu_Q(h(b)) \geqslant \theta, \eta_Q(h(b)) \geqslant \phi, \nu_Q(h(b)) \leqslant \psi$.

$$\begin{aligned}
\text{Therefore, } \mu_Q(h(a-b)) &= \mu_Q(h(a) - h(b)) \\
&\geqslant \mu_Q(h(a)) \wedge \mu_Q(h(b)) \\
&\geqslant \theta \wedge \theta = \theta, \\
\eta_Q(h(a-b)) &= \eta_Q(h(a) - h(b)) \\
&\geqslant \eta_Q(h(a)) \wedge \eta_Q(h(b)) \\
&\geqslant \phi \wedge \phi = \phi, \\
\nu_Q(h(a-b)) &= \nu_Q(h(a) - h(b)) \\
&\leqslant \nu_Q(h(a)) \vee \nu_Q(h(b)) \\
&\leqslant \psi \vee \psi = \psi \\
\text{and } \mu_Q(h(a \cdot b)) &= \mu_Q(h(a) \cdot h(b)) \\
&\geqslant \mu_Q(h(a)) \wedge \mu_Q(h(b)) \\
&\geqslant \theta \wedge \theta = \theta, \\
\eta_Q(h(a \cdot b)) &= \eta_Q(h(a) \cdot h(b)) \\
&\geqslant \eta_Q(h(a)) \wedge \eta_Q(h(b)) \\
&\geqslant \phi \wedge \phi = \phi, \\
\nu_Q(h(a \cdot b)) &= \nu_Q(h(a) \cdot h(b)) \\
&\leqslant \nu_Q(h(a)) \vee \nu_Q(h(b)) \\
&\leqslant \psi \vee \psi = \psi \\
&\quad [\text{as } h \text{ is a ring homomorphism}].
\end{aligned}$$

Thus, $h(a-b)$ and $h(a \cdot b) \in C_{\theta,\phi,\psi}(Q)$.

This implies, $a - b$ and $a \cdot b \in h^{-1}\ (C_{\theta,\phi,\psi}(Q)) = C_{\theta,\phi,\psi}(h^{-1}(Q))$.

Thus, $C_{\theta,\phi,\psi}(h^{-1}(Q))$ is a crisp subring of R_1. Therefore, by Proposition 4, $h^{-1}(Q)$ is PFSR of R_1. □

Proposition 11 *Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ be two crisp rings and $P = (\mu_P, \eta_P, \nu_P)$ be a PFSR of R_1. Then for a bijective ring homomorphism $h : R_1 \to R_2$, $h(P)$ is a PFSR of R_2.*

Proof Let $h(P) = (\mu_{h(P)}, \eta_{h(P)}, \nu_{h(P)})$.

$$\begin{aligned}
\text{Then } \quad \mu_{h(P)}(b_1) &= \bigvee_{a_1 \in h^{-1}(b_1)} \mu_P(a_1), \\
\eta_{h(P)}(b_1) &= \bigwedge_{a_1 \in h^{-1}(b_1)} \eta_P(a_1) \\
\text{and } \nu_{h(P)}(b_1) &= \bigwedge_{a_1 \in h^{-1}(b_1)} \nu_P(a_1).
\end{aligned}$$

Since h is bijective therefore $h^{-1}(b_1)$ must be a singleton set. So, for each $b_1 \in R_2$, there exists an unique $a_1 \in R_1$ such that $a_1 = h^{-1}(b_1)$, i.e. $h(a_1) = b_1$. Thus, in this case, $\mu_{h(P)}(b_1) = \mu_{h(P)}(h(a_1)) = \mu_P(a_1)$, $\eta_{h(P)}(b_1) = \eta_{h(P)}(h(a_1)) = \eta_P(a_1)$ and $\nu_{h(P)}(b_1) = \nu_{h(P)}(h(a_1)) = \nu_P(a_1)$.

Let $b \in C_{\theta,\phi,\psi}(h(P))$. Then

$$\mu_{h(P)}(b) \geqslant \theta, \eta_{h(P)}(b) \geqslant \phi \text{ and } \nu_{h(P)}(b) \leqslant \psi.$$

That is,
$\mu_{h(P)}(h(a)) \geqslant \theta, \eta_{h(P)}(h(a)) \geqslant \phi$ and $\nu_{h(P)}(h(a)) \leqslant \psi$ [where $b = h(a)$ for unique $a \in R_1$].

That is, $\mu_P(a) \geqslant \theta, \eta_P(a) \geqslant \phi$ and $\nu_P(a) \leqslant \psi$.
This gives, $a \in C_{\theta,\phi,\psi}(P)$.
This implies, $h(a) \in h(C_{\theta,\phi,\psi}(P))$.
That is, $b \in h(C_{\theta,\phi,\psi}(P))$.
Therefore, $C_{\theta,\phi,\psi}(h(P)) \subseteq h(C_{\theta,\phi,\psi}(P))$.

Now, let $d \in h(C_{\theta,\phi,\psi}(P))$. Then there exists an unique $c \in C_{\theta,\phi,\psi}(P)$ such that $d = h(c)$. Therefore,

$$\mu_P(c) \geqslant \theta, \eta_P(c) \geqslant \phi \text{ and } \nu_P(c) \leqslant \psi.$$

That is,

$$\mu_{h(P)}(h(c)) \geqslant \theta, \eta_{h(P)}(h(c)) \geqslant \phi \text{ and } \nu_{h(P)}(h(c)) \leqslant \psi.$$

That is,

$$\mu_{h(P)}(d) \geqslant \theta, \eta_{h(P)}(d) \geqslant \phi \text{ and } \nu_{h(P)}(d) \leqslant \psi.$$

This gives, $d \in C_{\theta,\phi,\psi}(h(P))$.

Therefore, $h(C_{\theta,\phi,\psi}(P)) \subseteq C_{\theta,\phi,\psi}(h(P))$.

Thus, finally, it is obtained that $C_{\theta,\phi,\psi}(h(P)) = h(C_{\theta,\phi,\psi}(P))$.

Let us suppose that $b_1, b_2 \in C_{\theta,\phi,\psi}(h(P))$. Then

$$\mu_{h(P)}(b_1) \geqslant \theta, \eta_{h(P)}(b_1) \geqslant \phi, \nu_{h(P)}(b_1) \leqslant \psi$$

and $\mu_{h(P)}(b_2) \geqslant \theta, \eta_{h(P)}(b_2) \geqslant \phi, \nu_{h(P)}(b_2) \leqslant \psi$.
That is,

$$\mu_{h(P)}(h(a_1)) \geqslant \theta, \eta_{h(P)}(h(a_1)) \geqslant \phi, \nu_{h(P)}(h(a_1)) \leqslant \psi$$

and $\mu_{h(P)}(h(a_2)) \geqslant \theta, \eta_{h(P)}(h(a_2)) \geqslant \phi, \nu_{h(P)}(h(a_2)) \leqslant \psi$ [where $b_1 = h(a_1)$ and $b_2 = h(a_2)$ for unique $a_1, a_2 \in R_1$].

That is,

$\mu_P(a_1) \geqslant \theta, \eta_P(a_1) \geqslant \phi, \nu_P(a_1) \leqslant \psi$ and $\mu_P(a_2) \geqslant \theta, \eta_P(a_2) \geqslant \phi, \nu_P(a_2) \leqslant \psi$.

This gives, $a_1 \in C_{\theta,\phi,\psi}(P)$ and $a_2 \in C_{\theta,\phi,\psi}(P)$.

This implies, $a_1 - a_2 \in C_{\theta,\phi,\psi}(P)$ and $a_1 \cdot a_2 \in C_{\theta,\phi,\psi}(P)$ [as $C_{\theta,\phi,\psi}(P)$ is a crisp subring of R_1].

This implies, $h(a_1 - a_2) \in h(C_{\theta,\phi,\psi}(P)) = C_{\theta,\phi,\psi}(h(P))$ and $h(a_1 \cdot a_2) \in h(C_{\theta,\phi,\psi}(P)) = C_{\theta,\phi,\psi}(h(P))$.

This implies, $h(a_1) - h(a_2) \in C_{\theta,\phi,\psi}(h(P))$ and $h(a_1) \cdot h(a_2) \in C_{\theta,\phi,\psi}(h(P))$ [as h is a ring homomorphism].

This gives, $b_1 - b_2$ and $b_1 \cdot b_2 \in C_{\theta,\phi,\psi}(h(P))$.

Thus, $C_{\theta,\phi,\psi}(h(P))$ is a crisp subring of R_2. Consequently, by Proposition 4, $h(P)$ is a PFSR of R_2. □

5 Conclusion

We notice that exploration of the theory of subring in context of PFS plays a vital role in the field of algebra. In this paper, we established the notion of PFSR of a crisp ring and investigated some basic results related to it. We studied some basic properties of PFSR in the environment of classical ring homomorphism. It is our hope that these works will help the researchers to develop the theory of subring in context of some other types of sets.

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