Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. (July–September 2020) 90(3):421–428
https://doi.org/10.1007/s40010-018-0591-6

RESEARCH ARTICLE

Thermal Instability of a Micropolar Fluid Layer with Temperature-Dependent Viscosity

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Received: 12 August 2015 / Revised: 12 September 2018 / Accepted: 22 December 2018 / Published online: 9 January 2019
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Abstract In this paper, the effect of temperature-dependent viscosity on the onset of thermal convection in a micropolar fluid layer heated from below for each combination of rigid (the surfaces with non-slip condition) and dynamically free (the surfaces with stress-free condition) boundaries is investigated. It is shown here analytically that the principle of exchange of stabilities is valid for the problem, which means that instability sets in as stationary convection. The expressions for Rayleigh numbers for each combination of rigid and dynamically free boundary conditions are derived using Galerkin method. The effects of micropolar parameters and viscosity variation parameter on critical wave numbers and consequently on the critical Rayleigh numbers are computed numerically.

Keywords Thermal convection · Temperature-dependent viscosity · Principle of exchange of stabilities · Galerkin method · Rayleigh number · Microrotation

1 Introduction

Microfluids exhibiting certain microscopic effects arising from the local structure and micromotions of the fluid elements were introduced and developed by Eringen [1]. These fluids support stress moments and body moments and are influenced by the spin inertia. Eringen's theory has provided a good model to study a number of complicated fluids, including the flow of low-concentration suspensions, liquid crystal, blood and turbulent shear flows. However, Eringen [2] introduced a subclass of microfluids named micropolar fluid, which exhibits microrotational inertia. Physically, micropolar fluids may represent fluids consisting of rigid, randomly oriented particles suspended in a viscous medium, where the deformation of the particles is ignored. In this theory, the local fluid elements have the usual translatory degrees of freedom reckoned by the velocity vector and have in addition, degrees of freedom enabling the intrinsic rotatory motions described by the gyration vector. This constitutes a substantial generalization of the Navier–Stokes model since a new vector field, namely the angular velocity field or rotation of particles, is introduced. With the introduction of this new vector, one more vector equation is added in Navier–Stokes model which represents the conservation of angular momentum. Furthermore, four new viscosities are also introduced in the system of equations. If one of these viscosities, namely microrotation viscosity, becomes zero, the equation of conservation of the linear momentum becomes independent of the microstructure. Thus, the size of the microrotation viscosity coefficient allows us to measure the deviation of flows of micropolar fluids from that of the Navier–Stokes model.

Thermal effects in micropolar fluid flow problems have been extensively investigated due to a large number of applications in engineering problems which include engineering structure such as infinite fiber composites, sandwich structures, grid structures and honeycombs. Examples of successful application of micropolar theory to these materials are calculation of moduli of crystals exhibiting polar phenomenon by Akshar [3] and numerical analysis of steel concrete grid structures in civil engineering performed by Bazant and Christtensen [4]. Micropolar fluid also acts as lubrications to human joints. The other diverse

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areas to which the micropolar theory has been applied, include blood flow in porous biomaterials, cross-diffusion dialysis flows, sediment transport in rivers, application in power generators, refrigeration coils, transmission lines, electric transformers, aeronautics and submarine navigation, MHD accelerators, refrigeration coils, transmission lines, purification of crude oil, polymeric fluids, continuous casting of glass fiber, porous media, cooling of electronic equipments and metal extraction. Extensive reviews on micropolar theory and its applications are given by Ariman et al. [5, 6], Eringen [7], Lukaszewicz [8] and Bég et al. [9].

Walzer [10] analyzed the problem of convective instability of a micropolar fluid layer confined between two rigid boundaries and pointed out that the analysis of the instability finds applications in the area of geophysics. One of the applications is understanding the phenomena of rising of volcanic liquid with bubbles and convective processes inside the earth's mantle. Ahmadi [11] studied the stability of a layer of micropolar fluid heated from below using linear theory as well as energy method. He proved the validity of principle of exchange of stabilities (PES) for the problem. He further showed that the micropolar fluid layer is stable than the classical (Bénard) fluid layer. Datta and Sastry [12] discussed the stationary convection of the micropolar fluid layer heated from below and derived an exact solution for the problem. Dhiman et al. [13] have studied the convective instability of micropolar fluid layer for general nature of boundaries using variational technique and derived the values of critical Rayleigh number for the onset of stationary convection.

Review of the literature reveals that the most studies related to the problems of heat transfer are based on the constant physical properties of the ambient fluid. However, to accurately predict the flow and heat transfer rates, it is necessary to take into account variation of viscosity. In all of the earlier mentioned studies by various authors, the viscosity of the fluid was assumed to be constant, but in recent years, many authors have investigated the influence of temperature-dependent viscosity in the problems of Rayleigh–Bénard convection. Palm [14] and Stengel et al. [15] have studied the convection in fluids with temperature-dependent viscosity. Booker and Stengel [16] have shown that there is a decrease in convective heat transport due to the increase in critical Rayleigh number with variable viscosity. Jenkins [17] considered the general dependence of viscosity on temperature and studied both linear and exponential dependence of viscosity and concluded that the linear dependence is realistic for fluids with small values of viscosity, while exponential dependence is more realistic for fluids with high viscosity. Many other authors including Selak and Lebon [18], Nield [19] and Straughan [20] have also investigated the onset of convection for the ordinary fluids with strongly temperature-dependent viscosity under various assumptions of hydrodynamics. Dhiman and Kumar [21] investigated the Rayleigh–Bénard convection with temperature-dependent viscosity for all combinations of rigid and dynamically free boundaries. Dhiman and Sharma [22] have also studied the effect of temperature-dependent viscosity on the thermal convection of nanofluid layer heated from below for general cases of boundaries.

Since in recent years, multi-physical micropolar flows have particularly emerged as a robust area of interest. Such flows involve magnetic effects, porous media and multi-mode mass heat transfer. Also, it is well known that in many industrial processes and applications, heat transfer is an integral part of the flow. Thus, micropolar fluid provides a mathematical model for accurately simulating the flow and heat transfer characteristics of polymeric additives, gel propellants, colloidal suspensions, liquid crystals, lubricants, bubbly liquids, paints, physiological fluids (blood), smoke-laden air and geological flows containing suspended sediments, in different geometries of fluid flow problem with different physical conditions to various situations of practical interest (*cf.* Eringen [7], Lukaszewicz [8]).

The above discussion provides us a motivation for the present investigations. Since the micropolar fluid model is both a significant and simple generalization of classical Navier–Stokes model and is adequate for exocitic lubricants, certain biological fluids and colloidal or suspensions solutions, convection is a dominant and important mode of heat transport in fluid flow problem and the incorporation of the variation in viscosity due to variation in temperature may have certain significant effect on the onset of convection. Our aim here is to study the effect of temperature-dependent viscosity on the onset of thermal convection in micropolar fluid layer heated from below. The validity of the principle of exchange of stabilities (PES) is investigated for this more general problem by Pellew and Southwell [23] method of conjugate eigenfunctions. A single-term Galerkin method is used to find general expressions for Rayleigh numbers for each combination of rigid and dynamically free boundaries. The values of critical Rayleigh numbers for each case of boundary combinations are computed numerically, for the case of stationary convection. The effects of microrotation parameters and the viscosity variation parameter on critical Rayleigh numbers are computed numerically.

2 Physical Configuration and the Eigenvalue Problem

Consider a viscous, incompressible micropolar liquid layer of infinite horizontal extension and finite vertical depth statically confined between two horizontal plane boundaries $z = 0$ and $z = d$, which are, respectively, maintained at uniform temperatures T_0 and $T_1 (T_0 > T_1)$ in the force field of gravity $(0, 0, -g)$. Since the liquid layer is heated from below, this maintains an adverse temperature gradient in the vertical direction. Our objective here is to investigate the stability of the above physical configuration by taking into account the variation in viscosity $(\mu = \mu_0 f(z))$ due to the temperature variation for all combinations of bounding surfaces.

Following the usual steps of linear stability theory (*cf.* Datta and Sastry [12] and Chandrasekhar [24]), the non-dimensional linearized perturbation equations and boundary conditions governing the problem, with time dependence of the perturbations w, θ and G, respectively, represent the perturbations in the vertical velocity, the temperature and the microrotation of the form; $(\emptyset(z))e^{[i(k_x x + k_y y) + nt]}$ can be easily obtained from the basic equations governing the problem of the thermal instability of micropolar fluid which are given by

$$\begin{aligned}(f + K)\left(D^2 - a^2\right)^2 w - \frac{p}{\sigma}\left(D^2 - a^2\right)w \\ + 2(Df)D\left(D^2 - a^2\right)w + D^2 f\left(D^2 + a^2\right)w \\ = -K\left(D^2 - a^2\right)G + Ra^2\theta\end{aligned} \tag{1}$$

$$\left(D^2 - a^2 - p\right)\theta = -w \tag{2}$$

$$\left(\left(D^2 - a^2\right) - 2A - \frac{n_1 Ap}{K\sigma}\right)G = A\left(D^2 - a^2\right)w \tag{3}$$

subject to the following cases of boundary conditions;

Case 1 Both boundaries rigid

$$w = 0 = \theta = G = Dw \text{ at } z = 0 \text{ and } z = 1 \tag{4}$$

Case 2 Both boundaries dynamically free

$$w = 0 = \theta = G = D^2 w \text{ at } z = 0 \text{ and } z = 1 \tag{5}$$

Case 3 Lower rigid and upper boundary free

$$\left.\begin{aligned} w = 0 = \theta = G = Dw \quad \text{at } z = 0 \\ w = 0 = \theta = G = D^2 w \quad \text{at } z = 1 \end{aligned}\right\} \tag{6}$$

Case 4 Lower free and upper boundary Rigid

$$\left.\begin{aligned} w = 0 = \theta = G = D^2 w \quad \text{at } z = 0 \\ w = 0 = \theta = G = Dw \quad \text{at } z = 1 \end{aligned}\right\} \tag{7}$$

In the foregoing equations, p is the dimensionless growth rate; a is the dimensionless wave number; $\sigma = \frac{\upsilon}{\kappa_0}$ is the Prandtl number; $K = \frac{\kappa}{\mu_0}$ is micropolar viscous parameter; $n_1 = \frac{J}{d^2}$; $A = \frac{\kappa d^2}{\gamma}$ is micropolar couple stress; $R = \frac{g\alpha\beta d^4}{\kappa_0 \upsilon}$ is the Rayleigh number; μ_0 is the Newtonian viscosity of the fluid; κ is the microrotation viscosity; ρ_o is the density of the fluid; g is the gravitational force; α is the coefficient of thermal expansion; $\beta = \left(\frac{T_0 - T_1}{d}\right)$ is the maintained uniform gradient; $f(z)$ is the non-dimensional temperature-dependent viscosity variation function; v is the kinematic viscosity; and κ_0 is coefficient of thermometric diffusivity.

The system of Eqs. (1)–(3) along with either of the boundary conditions (4)–(7) constitutes an eigenvalue problem (or characteristic value problem) for Rayleigh number R for the given values of the other parameters, namely a^2, p, σ, A, K and n_1. A given state of system is stable, neutral or unstable with p_r (real part of p) being negative, zero or positive, respectively. Further, if $p_r = 0$ implies $p_i = 0$ for every wave number a, then the principle of exchange of stability (PES) is valid, which means that stability sets in as stationary convection; otherwise, we shall have overstability at least when instability sets in as certain modes.

3 Principle of Exchange of Stabilities

Multiplying both sides of Eq. (1) by w^* (the complex conjugate of w), integrating the resulting equation over the vertical range of z and integrating the first term of the right-hand side of the resulting equation twice, we have

$$\begin{aligned}\int_0^1 w^*\Big[(f + K)\left(D^2 - a^2\right)^2 w - \frac{p}{\sigma}\left(D^2 - a^2\right)w \\ + 2(Df)D\left(D^2 - a^2\right)w + D^2 f\left(D^2 + a^2\right)w\Big]\mathrm{d}z \\ = -K\int_0^1 G\left(D^2 - a^2\right)w^*\mathrm{d}z + Ra^2\int_0^1 w^*\theta\mathrm{d}z\end{aligned} \tag{8}$$

Now, substituting the values of $(D^2 - a^2)w^*$ and w^* obtained, respectively, from Eqs. (2) and (3) in the first and second terms on right-hand side of the above equation, we get

$$\begin{aligned}\int_0^1 w^*\Big[(f + K)\left(D^2 - a^2\right)^2 w - \frac{p}{\sigma}\left(D^2 - a^2\right)w \\ + 2(Df)D\left(D^2 - a^2\right)w + D^2 f\left(D^2 + a^2\right)w\Big]\mathrm{d}z \\ = -K\int_0^1 G\left(\left(D^2 - a^2\right) - 2A - \frac{n_1 Ap^*}{K\sigma}\right)G^* \\ - Ra^2\int_0^1 \left(D^2 - a^2 - p^*\right)\theta^*\end{aligned} \tag{9}$$

Integrating the various integrals in Eq. (9) by parts a suitable number of times over the vertical range of z and using either of the boundary conditions (4)–(7), we have

$$\int_0^1 (f+K)\int_0^1\left[\left|D^2w\right|^2+2a^2|Dw|^2+a^4|w|^2\right]dz + a^2\int_0^1 \left(D^2f\right)|w|^2dz+\frac{p}{\sigma}\int_0^1\left[|Dw|^2+a^2|w|^2\right]dz = Ra^2\left[\int_0^1 |D\theta|^2+a^2|\theta|^2+p^*|\theta|^2dz\right] + \frac{K}{A}\left[\int_0^1 |DG|^2+a^2|G|^2dz+\int_0^1\left(2A+\frac{n_1Ap^*}{K\sigma}|G|^2\right)dz\right] \tag{10}$$

Equating the imaginary parts of both sides of Eq. (10), we get

$$p_i\left(\frac{1}{\sigma}\int_0^1 |Dw|^2+a^2|w|^2+n_1|G|^2dz+\int_0^1 Ra^2|\theta|^2dz\right)=0 \tag{11}$$

Since n_1, σ and R are positive constants, therefore from Eq. (11), we have $p_i = 0$.

This establishes that the PES is valid for the present problem. It is remarkable to note that the validity of PES leads to a notable mathematical simplification since the transition from stability to instability occurs via a marginal stationary state characterized by $p = 0$. Mathematically, this means that the marginally stable modes with $p_r = 0$ also have $p_i = 0$.

4 Numerical Analysis

As has been proved above that the onset of convection is through stationary modes, therefore taking $p = 0$ in Eqs. (1)–(3), we have the following reduced forms of the equations

$$(f+K)\left(D^2-a^2\right)^2w+2(Df)D\left(D^2-a^2\right)w + D^2f\left(D^2+a^2\right)w = -K\left(D^2-a^2\right)G+Ra^2\theta \tag{12}$$

$$\left(D^2-a^2\right)\theta=-w \tag{13}$$

$$\left[\left(D^2-a^2\right)-2A\right]G=A\left(D^2-a^2\right)w \tag{14}$$

Following the analysis of Finlayson [25], we apply the Galerkin method to find the value of Rayleigh number by taking single term in the expansions of the functions: w, θ and G as

$$\left.\begin{aligned} w &= lw_1(z)\\ \theta &= m\theta_1(z)\\ G &= nG_1(z)\end{aligned}\right\} \tag{15}$$

where w_1, θ_1 and G_1 are suitably chosen trial functions which satisfy the respective boundary conditions (4)–(7) and l, m and n are constants.

Now, multiplying equations obtained after substituting the above trial functions in Eqs. (12)–(14) by w_1, θ_1 and G_1, respectively, integrating each of the resulting equation by parts using relevant boundary conditions and eliminating l, m and n from the resulting equations, we obtain the following expression for Rayleigh number as

$$R=\left(\int_0^1 (f+K)\left[\left(D^2w_1\right)^2+2a^2(Dw_1)^2+a^4(w_1)^2\right]dz + a^2\int_0^1\left(D^2f\right)(w_1)^2dz - KA\frac{\left(\int_0^1 G_1(D^2-a^2)w_1dz\right)^2}{\int_0^1\left[(DG_1)^2+a^2(G_1)^2+2A(G_1)^2\right]dz}\right) \times\frac{\int_0^1\left[(D\theta_1)^2+a^2(\theta_1)^2\right]dz}{a^2\left(\int_0^1\theta_1w_1dz\right)^2} \tag{16}$$

It is to note that the expression (16) is valid for all cases of boundary conditions. We shall now obtain the values of the Rayleigh numbers for each of the case of the boundary conditions, separately.

4.1 The Values of Critical Rayleigh Numbers

We now derive the expressions for Rayleigh numbers for each case of boundary conditions (4)–(7) and consequently the values of the critical Rayleigh numbers R_c for each of these cases, for both the exponential and linear cases of viscosity variations.

Let us consider the following non-dimensional exponential and linear viscosity variation laws (*cf.* Dhiman and Kumar [21])

$$f=e^{\delta z}\quad\text{and}\quad f=(1+\delta z) \tag{17}$$

where δ is the viscosity variation parameter and is given by $\gamma\beta d=\delta$. Here, γ measures the viscosity variation with temperature, β is the temperature gradient across the layer, and d is the vertical depth of the fluid layer. It is clear from the above expressions that the variation in the viscosity is a function of the magnitude of β.

In the following, we shall discuss the case when both the boundaries are rigid in detail and obtain the values of Rayleigh numbers for each linear and exponential variation of viscosity numerically. It is remarkable to note that this case of combination of boundaries is realistic one and for which the exact solution in closed form is not obtainable.

4.2 Both Boundaries are Rigid

For this case of boundary conditions, the chosen trial functions are

$$w = z^4 - 2z^3 + z^2,\ \theta = z(z-1) \text{ and } G = z(z-1) \tag{18}$$

With these choice of trial functions and the viscosity variation laws given in Eq. (17), we obtain the following expressions for Rayleigh numbers, respectively, for linear and exponential variation of viscosity

$$\begin{aligned}(R_{rr}^{\exp}) = \frac{1960(10+a^2)}{3a^2}\Bigg[&\frac{-AK(28+3a^2)^2}{5880(10+a^2+2A)} + \frac{K}{630}(504+24a^2+a^4)\\ &+\frac{4}{\delta^9}\{6a^4(-1680-840\delta-180\delta^2-20\delta^3-\delta^4\\ &+e^{\delta}(1680+840\delta+180\delta^2-20\delta^3+\delta^4))\\ &+\delta^4(-864-432\delta-96\delta^2-12\delta^3-\delta^4\\ &+e^{\delta}(864-432\delta+96\delta^2-12\delta^3+\delta^4))\\ &+2a^2\delta^2(-7920-3960\delta-852\delta^2-96\delta^3-5\delta^4\\ &+e^{\delta}(7920-3960\delta+852\delta^2-96\delta^3+5\delta^4))\}\Bigg]\end{aligned} \tag{19}$$

and

$$\begin{aligned}(R_{rr}^{\text{Lin}}) = \frac{1960(10+a^2)}{3a^2}\Bigg(&\frac{-AK(28+3a^2)^2}{5880(10+a^2+2A)}\\ &+\frac{K}{630}(504+24a^2+a^4)\Bigg)\\ &+\frac{14(2520(4+2\delta)+a^2(1488+744\delta)+a^4(68+34\delta)+a^6(2+\delta))}{27a^2}\end{aligned} \tag{20}$$

The expression representing the value of critical wave number a_c at which the minimum value of $(R_{rr}^{\exp})$ given by expression (19) exists is obtained by performing some numerical calculations. The expression so obtained is too lengthy and hence is omitted here for the sake of compactness. However, we have calculated the values of critical wave numbers a_c from the said expression and consequently the critical Rayleigh numbers from expression (19) for particular values of δ. The values so obtained are presented in Table 1 for different values of A and in Table 2 for different values of K.

Further, for the case of linear variation of viscosity, (R_{rr}^{Lin}) given in Eq. (20) attains its minimum value $\left(R_{rr}^{\text{Lin}}\right)_c$ for different values of a_c, given by positive root of equation

$$\begin{aligned}&A_2(A,K,\delta)a^{10} + B_2(A,K,\delta)a^8 + C_2(A,K,\delta)a^6\\ &+D_2(A,K,\delta)a^4 + E_2(A,K,\delta)a^2 + F_2 = 0\end{aligned} \tag{21}$$

where $A_2(A,K,\delta) = [14(2+\delta+2K)]$

$$B_2(A,K,\delta) = [(2+\delta)\{518+112A\} + 1036K + 197AK]$$

$$C_2(A,K,\delta) = [\{14(2+\delta)(100+8A^2)+112A\} + 2800K] + 17\{(2+\delta)(280+112A) + 560K\} + 116A^2K + 5508AK]$$

$$D_2(A,K,\delta) = [-35280(2+\delta+2K) + 17\{(2+\delta)\{14(100+8A^2) + 1120A\} + 2800K\} + +2260A^2 + 37732AK]$$

$$E_2(A,K,\delta) = [-2520\{(2+\delta)(280+112A) + 560K\} - 517440AK]$$

$$F_2(A,K,\delta) = [-35280\{(2+\delta)(100+112A^2 + 1120A) + 200K\} - 518520A^2K - 5409600AK]$$

In the absence of micropolar parameters (i.e., $A = 0 = K$), Eq. (21) yields

$$(2+\delta)(10+a^2)^2(a^6+17a^4-2520) = 0 \tag{22}$$

Here, $a_c = 3.11$ is the only positive root of Eq. (22), and (R_{rr}^{Lin}) attains its minimum value $\left(R_{rr}^{\text{Lin}}\right)_c = 874.988(2+\delta)$ at $a_c = 3.11$.

This is the same value of Rayleigh number as obtained by Dhiman and Kumar [21] for the Rayleigh–Bénard convection with temperature-dependent viscosity. We also observe that in the absence of micropolar parameters and for the case of constant viscosity (i.e., when $A = 0 = \delta = K$), the value of critical Rayleigh number $(R_{rr})_c = 1750$, obtained using Galerkin method, is very close to the well-known exact value 1707.7 at $a_c = 3.11$, as obtained by Chandrasekhar [24] for the Rayleigh–Bénard convection with constant viscosity.

It is clear from Eq. (21) that all the coefficients of fifth-degree equation in a^2 involve A and K implicitly, and hence, it is difficult to obtain the values of critical wave numbers a_c given by any positive root of Eq. (21) analytically. Therefore, for particular values of δ, we have calculated the values of critical wave numbers a_c from expression (21) and consequently the critical Rayleigh numbers $(R_{rr}^{\text{Lin}})_c$ from expression (20). The results so obtained are also presented in Table 1 for different values

Table 1 Variation of $(R_{rr})_c$ with respect to δ and A for different values of a_c^2 and fixed value of $K = 0.2$, for Case 2 of boundary conditions

Linear variation				Exponential variation		
A	δ	a_c^2	$(R_{rr})_c$	δ	a_c^2	$(R_{rr})_c$
10	0.0	9.843	2532.1	0.0	9.843	2532.1
	0.2	9.835	2707.5	0.2	9.835	2721.8
	0.5	9.824	2969.6	0.5	9.837	3065.4
	0.7	9.818	3144.6	0.7	9.845	3344.2
	0.9	9.812	3319.6	0.9	9.860	3671.2
100	0.0	10.153	2455.6	0.0	10.153	2455.6
	0.2	10.123	2630.7	0.2	10.122	2645.0
	0.5	10.084	2839.4	0.5	10.091	2989.1
	0.7	10.063	3068.5	0.7	10.078	3268.0
	0.9	10.043	3243.5	0.9	10.073	3594.9
	0.0	10.230	2422.0	0.0	10.230	2422.0
500	0.2	10.194	2617.2	0.2	10.193	2631.4
	0.5	10.149	2879.9	0.5	10.15	2975.6
	0.7	10.123	3055.0	0.7	10.135	3257.5
	0.9	10.100	3230.1	0.9	10.125	3581.5

Table 2 Variation of $(R_{rr})_c$ with respect to δ and K for different values of a_c^2 and fixed value of $A = 200$, for Case 2 of boundary conditions

Linear variation				Exponential variation		
K	δ	a_c^2	$(R_{rr})_c$	δ	a_c^2	$(R_{rr})_c$
0.2	0.0	9.942	2029.4	0.0	9.942	2029.4
	0.2	9.924	2204.6	0.2	9.924	2218.9
	0.5	9.901	2466.9	0.5	9.915	2562.7
	0.7	9.888	2641.9	0.7	9.917	2841.5
	0.9	9.877	2817.0	0.9	9.927	3168.5
0.5	0.0	10.199	2447.1	0.0	10.199	2447.1
	0.2	10.165	2622.5	0.2	10.164	2636.7
	0.5	10.123	2885.2	0.5	10.128	2980.9
	0.7	10.099	3060.3	0.7	10.112	3259.8
	0.9	10.077	3235.4	0.9	10.104	3586.7
	0.0	10.487	3142.5	0.0	10.487	3142.5
1	0.2	10.444	3321.5	0.2	10.441	3335.1
	0.5	10.387	3580.8	0.5	10.385	3676.5
	0.7	10.354	3756.1	0.7	10.35	3955.5
	0.9	10.324	3931.3	0.9	10.328	4282.6

of A and Table 2 for different values of K for the case of linear and exponential variation of viscosity. Further, Fig. 1a–b depicts the variation of Rayleigh number $(R_{rr}^{\exp})$ with respect to δ for fixed values of $A$and K.

Now, proceeding exactly as in Case 1 of boundary conditions above, we obtain the analogous expressions for Rayleigh numbers for other cases of boundary combinations (Case 2, Case 3 and Case 4) using the suitably chosen trial functions (given below) satisfying the relevant boundary conditions, for both linear and exponential cases of viscosity variation as

$$w = z^4 - 2z^3 + z, \quad \theta = z(z-1)$$

and $G = z(z-1)$

(*Both boundaries are dynamically free*)

$$w = 2z^4 - 5z^3 + 3z^2, \ \theta = z(z-1)$$

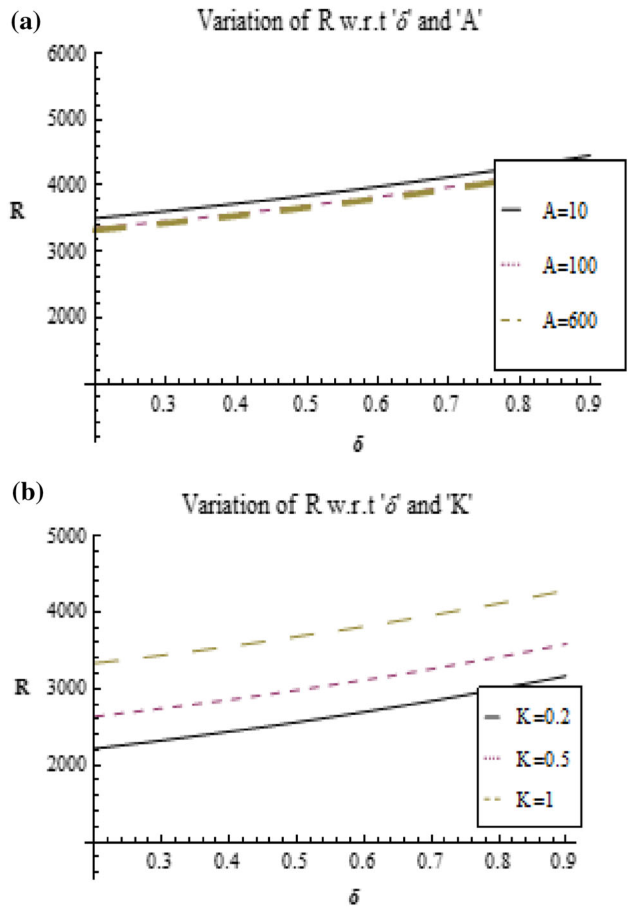

Fig. 1 **a** Variation of R with respect to δ for fixed value of $A = 500$ and different values of $K = 0.2, 0.5, 1$, for the case of both rigid boundaries. **b** Variation of R with respect to δ for fixed value of $K = 1$ and different values of $A = 10, 100, 600$, for the case of both rigid boundaries

and $G = z(z - 1)$

(*Lower rigid and upper free boundaries*)

$$w = 2z^4 - 3z^3 + z, \ \theta = z(z - 1)$$

and $G = z(z - 1)$

(*Lower free and upper rigid boundaries*)

The expressions for Rayleigh numbers so obtained in each of these cases of boundary conditions represent the analogous behavior with respect to a^2 for different values of δ and for fixed values of $A$and K for linear and exponential cases of variation of viscosity.

5 Discussion and Conclusions

A linear stability of micropolar liquid layer heated from below with temperature-dependent viscosity is investigated in the present analysis. The PES is proved to be valid for the problem which yields that the onset of convection in this general problem is through stationary mode. The expressions for the Rayleigh numbers (for both linear and exponential cases of viscosity variation for stationary mode) for each case of boundary conditions are obtained using a single-term Galerkin method.

From Fig. 1a and b, we conclude that for large values of temperature-dependent viscosity parameter δ (which also includes the temperature gradient β), for increasing values of micropolar viscous parameter K and for marginally decreasing value of coupling parameter A, the value of Rayleigh number increases and hence viscosity parameter δ has stabilizing effect on the onset of stationary convection of micropolar fluid layer in each case of boundary conditions. This may be due to the fact that for large values of δ, i.e., for more viscous fluid or for large temperature gradient β, the flow rate decreases and consequently degrades the heat transfer performances. Thus, the value of critical Rayleigh number (R_c) increases. Hence, the viscosity parameter has stabilizing effect on the system.

From the variation of Rayleigh numbers presented in Table 1, we observe that increasing values of micropolar coupling parameter (A), decrease the couple stress of the fluid which causes a decrease in microrotation and hence makes the system more unstable as a result the value of Rayleigh number decreases and hence R_c also decreases. This implies that A has destabilizing effect on the onset of stationary convection in micropolar fluid layer in Case 1 of boundary conditions.

Similarly, from Table 2, we observe that the values of Rayleigh number increase with increasing values of micropolar viscous parameter K. Since an increase in K indicates the increase in the concentration of microelements, these microelements consume the greater part of the energy of the system in developing the gyrational velocities of the fluid and as a result the onset of convection is delayed. This implies that K has a stabilizing effect on the onset of stationary convection.

It is to point out here that the Rayleigh numbers show the analogous variation with respect to a^2 for different values of δ and for fixed values of A and K for linear and exponential cases of variation of viscosity in other cases of boundary conditions also. Hence, the tables of values of variations are omitted here for the sake of compactness and repetition.

We also observe from the values presented in tables that a_c depends upon δ for both exponential and linear variation of viscosity for each case of boundary condition. Further, we found that for fixed value of δ, the values of critical Rayleigh numbers (R_c) are higher in the case of exponential viscosity variation than the linear viscosity variation in each case of boundary condition, which means that the exponential varying viscosity is more stabilizing than the linear varying viscosity.

From the above analysis, we also observe that in the absence of micropolar parameters and variable viscosity (i.e., $A = 0 = \delta = K$, the classical Bénard problem case), the values of critical Rayleigh numbers for each case of boundary conditions are very close to the values obtained by Chandrasekhar [24] for the Rayleigh–Bénard convection problem with constant viscosity for ordinary fluid and are also in close agreement with the values obtained by Dhiman and Kumar [21] for Rayleigh–Bénard convection problem with temperature-dependent viscosity (i.e., taking $A = 0 = K$). Thus, the above remarks clearly establish the generality of the results derived herein and thus are of wider applicability.

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