Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. (January–March 2018) 88(1):15–26
https://doi.org/10.1007/s40010-016-0278-9

RESEARCH ARTICLE

Arithmetic Operations on Generalized Parabolic Fuzzy Numbers and Its Application

Harish Garg[1] · **Ansha**[1]

Received: 13 April 2014 / Revised: 21 November 2015 / Accepted: 13 June 2016 / Published online: 2 July 2016

Abstract In this paper, we have studied the basic arithmetic operations for two generalized positive parabolic fuzzy numbers by using the concept of the distribution and complementary distribution functions. The major advantage of these operations is that they do not need the computation of α-cut of the fuzzy number and hence it becomes more powerful where the standard method i.e., α-cuts method fails. Based on these operations, some elementary applications on mensuration have been illustrated and compared their results with generalized triangular fuzzy numbers.

Keywords Generalized parabolic fuzzy numbers · Defuzzification · Arithmetic operations · Fuzzy numbers

1 Introduction

Under the growing complexities of the system, problems in the real world quite often turn out to be complex owing to an element of uncertainty either in the parameters which define the problem or in the situations in which the problem occurs. However, it is very difficult to make statistical interference in case of systems where available data is insufficient. As the probability approach has been applied successfully for many real world engineering problems but still there are some limitations to the probabilistic method. For instance, probabilistic methods are based on mass collection of data, which is random in nature, to achieve the requisite confidence level. But in large scale the complicated system has the massive fuzzy uncertainty due to which it is difficult to get the exact probability of the events. Furthermore, the assessment of the systems are usually affected by aleatory and epistemic uncertainty. Aleatory uncertainty arises from heterogeneity or the random character of natural processes while epistemic uncertainty arises from the partial character of our knowledge of the natural world. Epistemic uncertainty can be reduced by further study while aleatory cannot be reduced. Thus, results based on probability theory do not always provide useful information to the practitioners due to the limitation of being able to handle only quantitative information. Due to these limitations, the results based on probability theory do not always provide useful information to the practitioners and hence probabilistic approach is inadequate to account for such built-in uncertainties in the data.

To overcome these difficulties, methodologies based on fuzzy set theory and logic represents a useful tool for dealing with the uncertainties in addition to the probability theory. Fuzzy set theory [1] has been viewed as a useful tool, especially for dealing with the complex systems, in which the interactions of the system's variables may be too complex to be precisely specified. However, we could find that fuzzy logic may obtain different simulated efficiency and performance while adopting various forms of fuzzy arithmetic, and the fuzzy arithmetic operations have necessary condition which operations have to use triangular fuzzy numbers. In the framework of fuzzy arithmetic various operations as, e.g., addition, subtraction, etc., are realized [2]. These operations are made with the use of Zadeh's possibilistic extension principle [3] or its new, improved, and also possibilistic version proposed by Klir [4], which takes into account the so-called requisite

✉ Harish Garg
harishg58iitr@gmail.com;
https://sites.google.com/site/harishg58iitr/

1 School of Mathematics, Thapar University Patiala, Patiala 147004, India

constraints. Arithmetic operations are also performed under the assumption which was introduced by Zadeh [5] that the membership function of a fuzzy set is of a possibilistic character and that each element of the universal set, with a non-zero membership grade, belongs to a fuzzy set. For the past few years, some people have worked on arithmetic operations on fuzzy numbers. Piegat [6] presented a definition of fuzzy which allows for a considerable fuzziness decrease in the number of arithmetic operations. Stefanini and Guerra [7] analyzed decomposition of fuzzy numbers in order to study some properties of fuzzy arithmetic operations and compared the proposed approximation with the results of standard fuzzy mathematics. Gao et al. [8] worked on four methods for solving multiplication operation of two fuzzy numbers. These are non-linear programming method, analytical method, computer method and computer simulation method. Akther and Ahmad [9] presented a way of computing arithmetic operations of fuzzy numbers as well as an analytic form of resultant membership functions. Mahanta et al. [10] gave method that can be utilized in cases where the method of α-cuts fails. Taleshian and Rezvani [11] gave methods for solving multiplication operation of two trapezoidal fuzzy numbers. Chutia et al. [12] developed a method of finding membership function for functions of triangular fuzzy variable from the concept of credibility theory and a method for computation of basic arithmetical operations of fuzzy variables is forwarded. Bansal [13] explored the arithmetic properties of an arbitrary trapezoidal fuzzy number. Oussalah [14] addressed theoretical results about some invariance properties concerning the relationships between the defuzzification outcomes and the arithmetic of fuzzy numbers. Kechagias and Papadopoulos [15] proposed a computational method to evaluate the arithmetic operations on fuzzy numbers with nonlinear membership functions. Deschrijver [16] analyzed the arithmetic operations in both interval and intuitionistic fuzzy set theory. Xue et al. [17] presented an expression for the expected value of a function of fuzzy variable by taking fuzzy variables has a continuous membership function. Banerjee and Roy [18] studied defuzzification method for generalized trapezoidal fuzzy numbers based on the Zadeh's extension principle method, interval method and vertex method. Garg [19] presented an arithmetic operations based on weakest t-norm and compute the various expression of reliability indices corresponding to complex repairable system. Vahidi and Rezvani [20] presented an arithmetic operations on the trapezoidal fuzzy numbers. Garg [21] presented an approach for computing the various arithmetic operations using credibility theory corresponding to different type of intuitionistic fuzzy numbers.

All the above studies have adopted the well-known additions of fuzzy quantities by a well known extension principle and α-cut methods for their computation. Here, a former one is directly considering the membership functions while the latter one is to deal with the α-cut sets without considering the membership functions. Both the methods have their own limitations, such as it is not always possible to compute the α-cut of a fuzzy numbers and hence their approaches are quite restricted. Moreover, it is quite clear that there exists a large amount of uncertainties during the computation when linear membership functions have been taken. Therefore, there is a need of suitable methodology which will handle this problem and compute the arithmetic operations in a fuzzy environment. So, in this study, instead of using these methods, we compute the membership functions by using distribution and complementary distribution functions.

Thus, the objective of the paper is to present an alternative method for obtaining the membership functions of the various arithmetic operations on fuzzy numbers. For this distribution and complementary distribution function has been used for finding the membership function of generalized parabolic fuzzy numbers instead of triangular fuzzy numbers. The major advantages of using their distribution and complementary functions are that they do not need the computation of α-cuts and hence the method is quite useful in those cases where it is difficult to compute the α-cut of the fuzzy numbers. The operations have been validated through some elementary applications and illustrated with their approximated/defuzzified values. Finally results are compared with the α-cut method and shows the supremacy of the result.

2 Preliminaries

Real-world problems are generally associated with different types of uncertainties and imprecision's. In the past, a considerable amount of effort was made to model those uncertainties and imprecision's. Prior to 1965, people used to consider probability theory (which works based on two-valued logic either in or out) as the prime agent for dealing with uncertainties. But the results based on probability theory do not always provide useful information to the practitioners due to the limitation of being able to handle only quantitative information. To overcome this, mathematical modeling of fuzzy concepts (a generalization of crisp or classical set approach) was presented by Zadeh [1] by allowing images of elements to be in the interval [0, 1] rather than being restricted to the two-element set {0,1} and defined the new concept called as fuzzy set.

2.1 Fuzzy Set

Fuzzy sets [1] may be viewed as an extension and generalization of the basic concepts of crisp sets which allows partial membership i.e. between 0 and 1. A fuzzy set $\tilde{A}$ can be defined on the universe of discourse U as

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x) \mid x \in U\} \tag{1}$$

where $\mu_{\tilde{A}}$ is the membership function of the fuzzy set $\tilde{A}$ defined as $\mu_{\tilde{A}} : U \rightarrow [0, 1]$ and $\mu_{\tilde{A}}(x)$ indicates the degree of membership of x in $\tilde{A}$ and its value lies between zero and one. When a set is an ordinary set, its membership function can take on only two values 0 and 1, with $\chi_A(x) = 1$ or 0 according as x does or does not belong to A. $\chi_A(x)$ is referred to as the characteristic function of the set A.

2.2 Convex Fuzzy Set [2, 4]

If the membership function has membership values those are monotonically increasing, or, monotonically decreasing, or they are monotonically increasing and decreasing with increasing values for elements in the universe, those fuzzy set $\tilde{A}$ is called convex fuzzy set. Mathematically, a fuzzy set $\tilde{A}$ in the universe of discourse U is called a convex fuzzy set if and only if [2, 4]

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)] \ \forall\, x_1, x_2 \in U, \lambda \in [0, 1]$$

If above inequality does not hold then it is said to be non-convex fuzzy set.

2.3 Normal Fuzzy Set [2, 4]

A fuzzy set is said to be normal fuzzy set if there exists at least one element $x \in U$ such that $\mu_{\tilde{A}}(x) = 1$. A fuzzy set wherein no membership function has its value equal to 1 is called subnormal fuzzy set.

2.4 Fuzzy Number [2, 4]

A fuzzy number is an extension of a regular number in which the value corresponding to element has its own weight between 0 and 1, called membership functions, instead of one single values. In other words, a fuzzy number is a normal and convex membership function on the real line $\mathbb{R}$ such that

1. there exists at least one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{A}}(x_0) = 1$.
2. $\mu_{\tilde{A}} : \mathbb{R} \longrightarrow [0, 1]$ is piecewise continuous.

and its membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} f_A(x); & \text{if } a_1 \leq x < a_2 \\ 1; & \text{if } x = a_2 \\ g_A(x); & \text{if } a_2 \leq x < a_3 \\ 0; & \text{if otherwise} \end{cases} \tag{2}$$

where $0 \leq \mu_{\tilde{A}}(x) \leq 1$ and $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1 \leq a_2 \leq a_3$ and the two functions $f_A, g_A : \mathbb{R} \rightarrow [0, 1]$ are called the sides of the fuzzy numbers such that f_A and g_A are nondecreasing and nonincreasing continuous functions respectively. Dubois and Prade [3] named $f_A(x)$ as left reference function and $g_A(x)$ as right or complementary reference function of concerned fuzzy number. We denote this fuzzy number as $\tilde{A} = (a_1, a_2, a_3)$ where $\tilde{A}$ represents the fuzzy set of A.

2.5 Parabolic Fuzzy Number

A fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is said to be a parabolic fuzzy number if its membership function is defined as below

$$\mu_{\tilde{A}}(x) = \begin{cases} \left(\dfrac{x - a_1}{a_2 - a_1}\right)^2, & \text{if } a_1 \leq x < a_2 \\ 1 & \text{if } x = a_2 \\ \left(\dfrac{a_3 - x}{a_3 - a_2}\right)^2, & \text{if } a_2 \leq x < a_3 \\ 0, & \text{if otherwise} \end{cases} \tag{3}$$

2.6 Generalized Fuzzy Number

A fuzzy number $\tilde{A} = (a_1, a_2, a_3; \omega)$, defined on the universal set of real numbers $\mathbb{R}$, is said to be generalized fuzzy number it its membership function has the following characteristics

1. $\mu_{\tilde{A}}(x) : \mathbb{R} \rightarrow [0, 1]$ is continuous.
2. $\mu_{\tilde{A}}(x) = 0$, for all $x \in (-\infty, a_1] \cup [a_3, \infty)$.
3. $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a_1, a_2]$ and strictly decreasing on $[a_2, a_3]$.
4. $\mu_{\tilde{A}}(x) = \omega$ for all $x = a_2$ where $0 < \omega \leq 1$.

2.7 Generalized Parabolic Fuzzy Number

A fuzzy number $\tilde{A} = (a_1, a_2, a_3; \omega)$, is called a generalized parabolic fuzzy number if its membership function is defined as

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega\left(\dfrac{x - a_1}{a_2 - a_1}\right)^2, & \text{if } a_1 \leq x < a_2 \\ \omega, & \text{if } x = a_2 \\ \omega\left(\dfrac{a_3 - x}{a_3 - a_2}\right)^2, & \text{if } a_2 \leq x < a_3 \\ 0, & \text{if otherwise} \end{cases} \tag{4}$$

or $\quad \mu_{\tilde{A}}(x) = \max(\min(\omega(\frac{x-a_1}{a_2-a_1})^2, \omega, \omega(\frac{a_3-x}{a_3-a_2})^2), 0)$

A generalized fuzzy number is said to be positive(negative) i.e. $\tilde{A} \geq 0 (\tilde{A} \leq 0)$ if and only if $a_1 \geq 0 (a_3 \leq 0)$.

2.8 Defuzzification

In order to make decisions with respect to maintenance actions it is necessary to convert the fuzzy output into a crisp value. The process of converting the fuzzy output to a crisp value is called defuzzification. Out of the existence of the various defuzzification techniques in the literature, center of area (COA) or center of gravity (COG) method [22] is selected due to its property that it is equivalent to the mean of the data. If the membership function $\mu_{\tilde{A}}(x)$ of the output fuzzy set $\tilde{A}$ is described on the interval $[x_1, x_2]$, then COA defuzzification $\bar{x}$ is defined as

$$\bar{x} = \frac{\int_{x_1}^{x_2} x \cdot \mu_{\tilde{A}}(x)dx}{\int_{x_1}^{x_2} \mu_{\tilde{A}}(x)dx}$$

3 Proposed Membership Function for Function of a Fuzzy Variable

Let Θ be a nonempty set, $\mathfrak{P}(\Theta)$ be the power set of Θ, and *Pos* a possibility measure, then the triplet $(\Theta, \mathfrak{P}(\Theta), \mathfrak{Pos})$ is called possibility space. Based on this space, a fuzzy variable has been defined

Definition 1 A fuzzy variable [23, 24], say ζ, is defined as a function from a possibility space $(\Theta, \mathfrak{P}(\Theta), \mathfrak{Pos})$ to the set of real numbers, then its membership function μ is derived from the possibility measure by

$$\mu(x) = Pos\{\theta \in \Theta \mid \zeta(\theta) = x\}, \quad x \in \Re$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function, and let $\zeta_1, \zeta_2, \ldots, \zeta_n$ be fuzzy variables on the possibility space $(\Theta, \mathfrak{P}(\Theta), \mathfrak{Pos})$. Then $\zeta = f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is a fuzzy variable defined as $\zeta(\theta) = f(\zeta_1(\theta), \zeta_2(\theta), \ldots, \zeta_n(\theta))$ for any $\theta \in \Theta$. If the fuzzy variables are defined on different possibility spaces, then $\zeta = f(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is a fuzzy variable defined on the product possibility space $(\Theta, \mathfrak{P}(\Theta), \mathfrak{Pos})$ as $\zeta(\theta_1, \theta_2, \ldots, \theta_n) = f(\zeta_1(\theta_1), \zeta_2(\theta_2), \ldots, \zeta_n(\theta_n)$ for any $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$.

Let $\zeta = (a_1, a_2, a_3; \omega)$ be the triangular fuzzy variable with height of the variable is ω, then $F(\zeta) = [F(a_1), F(a_2), F(a_3); F(\omega)]$ be the fuzzy variable of the function $F(\zeta)$. Let the membership function of ζ is given as

$$\mu_\zeta(x) = \begin{cases} \omega L_1(x) & \text{if } a_1 \leq x < a_2 \\ \omega & \text{if } x = a_2 \\ \omega R_1(x) & \text{if } a_2 \leq x < a_3 \\ 0 & \text{otherwise} \end{cases}$$

where $L_1(x)$ and $R_1(x)$ are the nondecreasing and nonincreasing functions of x respectively. Let $z = F(x), x \in \zeta$ or $x = \psi(z)$. Hence the density functions for the distribution functions $L_1(x)$ and $R_1(x)$ are obtained as

$$f_1(x) = \frac{d}{dx}(L_1(x)) = \eta_1(z) \text{ at } x = \psi_1(z)$$
$$g_1(x) = \frac{d}{dx}(R_1(x)) = \eta_2(z) \text{ at } x = \psi_2(z)$$

Now, let,

$$\frac{dx}{dz} = \frac{d}{dz}(\psi_1(z)) = m_1(z); \quad \frac{dx}{dz} = \frac{d}{dz}(\psi_2(z)) = m_2(z)$$

Then the distribution function for $F(\zeta)$ would be given as

$$\int_{F(a_1)}^{x} \eta_1(z)m_1(z)dz; \quad F(a_1) \leq x < F(a_2)$$

while their complementary distribution function would be given as

$$\int_{F(a_3)}^{x} \eta_2(z)m_2(z)dz; \quad F(a_2) \leq x < F(a_3)$$

Hence, the membership function for the fuzzy variable function $F(\zeta)$ is given by

$$\mu_{F(\zeta)}(x) = \begin{cases} F(\omega) \int_{F(a_1)}^{x} \eta_1(z)m_1(z)dz & F(a_1) \leq x < F(a_2) \\ F(\omega) & x = F(a_2) \\ F(\omega) \int_{F(a_3)}^{x} \eta_2(z)m_2(z)dz & F(a_2) \leq x < F(a_3) \\ 0 & \text{otherwise} \end{cases}$$

In order to evaluate the fuzzy arithmetic for the parabolic fuzzy numbers, consider the two parabolic fuzzy numbers $X = [a_1, a_2, a_3; \omega_1]$ and $Y = [b_1, b_2, b_3; \omega_2]$ where ω_1, ω_2 represents the degree of their membership functions in crisp environment. Their corresponding membership functions are defined as

$$\mu_X(x) = \begin{cases} \omega_1 L_1(x) & \text{if } a_1 \leq x < a_2 \\ \omega_1 & \text{if } x = a_2 \\ \omega_1 R_1(x) & \text{if } a_2 \leq x < a_3 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and

$$\mu_Y(y) = \begin{cases} \omega_2 L_1(y) & \text{if } b_1 \leq y < b_2 \\ \omega_2 & \text{if } y = b_2 \\ \omega_2 R_1(y) & \text{if } b_2 \leq y < b_3 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where $L_1(x) = (\frac{x - a_1}{a_2 - a_1})^2$, $L_1(y) = (\frac{y - b_1}{b_2 - b_1})^2$ are the left distribution functions and $R_1(x) = (\frac{a_3 - x}{a_3 - a_2})^2$, $R_1(y) =$

$(\frac{b_3 - y}{b_3 - b_2})^2$ are the right distribution functions of X and Y respectively. In order to find the distribution functions of their corresponding arithmetic operations, we start with equating $L_1(x)$ with $L_1(y)$ and $R_1(x)$ with $R_1(y)$ and obtain $y = \phi_1(x)$ and $y = \phi_2(x)$ respectively, where $\phi_1(x) = b_1 \pm (\frac{(x - a_1)(b_2 - b_1)}{(a_2 - a_1)})$, $\phi_2(x) = b_3 \mp (\frac{(a_3 - x)(b_3 - b_2)}{(a_3 - a_2)})$.

Let Z be the resultant of the arithmetic operations of X and Y. Then at $y = \phi_1(x)$ and $y = \phi_2(x)$ we get $x = \psi_1(z)$ and $x = \psi_2(z)$ respectively. Based on these functions, we get the density function corresponding to the distribution and complementary distribution functions as

$$f_1(x) = \frac{d}{dx}(L_1(x)) = \eta_1(z) \text{ at } x = \psi_1(z)$$
$$g_1(x) = \frac{d}{dx}(R_1(x)) = \eta_2(z) \text{ at } x = \psi_2(z)$$

Also,

$$\frac{dx}{dz} = \frac{d}{dz}(\psi_1(z)) = m_1(z); \quad \frac{dx}{dz} = \frac{d}{dz}(\psi_2(z)) = m_2(z)$$

Hence, the distribution function for fuzzy variable $F(z)$ where $F(z) = [z_1, z_2, z_3; \omega]$, $\omega = \min(\omega_1, \omega_2)$ are

$$\mu_{F(\zeta)}(x) = \begin{cases} \omega \int_{z_1}^{x} \eta_1(z) m_1(z)dz & \text{if } z_1 \le x < z_2 \\ \omega & \text{if } x = z_2 \\ \omega \int_{z_3}^{x} \eta_2(z) m_2(z)dz & \text{if } z_2 \le x < z_3 \\ 0 & \text{otherwise} \end{cases}$$

Based on these functions, we obtain the membership functions of functions, such as addition, subtraction, multiplication, inverse etc., as given below

3.1 Addition of Fuzzy Numbers

Theorem 1 *If X and Y be the two parabolic fuzzy number over the universe U whose membership function are defined in Eqs. (5) and (6) respectively then the fuzzy variable $Z = X + Y$ is also a parabolic fuzzy number with membership function*

$$\mu_Z(x) = \begin{cases} \omega\left(\frac{x - (a_1 + b_1)}{a_2 - a_1 + b_2 - b_1}\right)^2 & a_1 + b_1 \le x < a_2 + b_2 \\ \omega & x = a_2 + b_2 \\ \omega\left(\frac{(a_3 + b_3) - x}{a_3 - a_2 + b_3 - b_2}\right)^2 & a_2 + b_2 \le x < a_3 + b_3 \\ 0 & \text{otherwise} \end{cases}$$

Proof Consider two parabolic fuzzy numbers X and Y whose membership functions are defined in Eqs. (5) and (6) respectively. For addition of the fuzzy numbers X and Y, the fuzzy number $Z = X + Y = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$ be the resultant fuzzy number of X and Y. Now let $z = x + y$ we get $z = x + \phi_1(x)$ and $z = x + \phi_2(x)$ which implies that $x = \psi_1(z)$ and $x = \psi_2(z)$ where

$$x = \psi_1(z) = \frac{z - \frac{a_2 b_1}{a_2 - a_1} + \frac{a_1 b_2}{a_2 - a_1}}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}$$

Hence, $\eta_1(z) = \left(\frac{2}{(a_2 - a_1)^2}\right)\left(\frac{z - a_1 - b_1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)$, $m_1(z) = \frac{1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}$

Thus, left sided distribution function for the fuzzy variable $Z = X + Y$ is

$$\begin{aligned} \int_{a_1 + b_1}^{x} \eta_1(z) m_1(z)dz &= \int_{a_1 + b_1}^{x} \left(\frac{2}{(a_2 - a_1)^2}\right)\left(\frac{z - a_1 - b_1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right) \\ &\quad \times \left(\frac{1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)dz \\ &= \left(\frac{2}{(a_2 - a_1)^2}\right)\left(\frac{1}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)^2 \\ &\quad \times \int_{a_1 + b_1}^{x} (z - a_1 - b_1)dx \\ &= \left(\frac{1}{a_2 - a_1}\right)^2\left(\frac{x - (a_1 + b_1)}{1 + \frac{(b_2 - b_1)}{a_2 - a_1}}\right)^2 \\ &= \left(\frac{x - (a_1 + b_1)}{a_2 + b_2 - a_1 - b_1}\right)^2; \\ &\quad a_1 + b_1 \le x < a_2 + b_2 \end{aligned}$$

Similarly, if $y = \phi_2(x)$ then $z = x + y$ becomes $x = \psi_2(z)$ where

$$\psi_2(z) = \frac{z + \frac{-a_2 b_3}{a_3 - a_2} + \frac{a_3 b_2}{a_3 - a_2}}{1 + \frac{b_3 - b_2}{a_3 - a_2}}$$

Here, in this case

$$\eta_2(z) = \left(\frac{-2}{(a_3 - a_2)^2}\right)\left(\frac{a_3 + b_3 - z}{1 + \frac{(b_3 - b_2)}{a_3 - a_2}}\right),$$
$$m_2(z) = \frac{1}{1 + \frac{b_3 - b_2}{a_3 - a_2}}$$

Thus, right sided distribution function for the fuzzy variable $Z = X + Y$ is

$$\int_{a_3+b_3}^{x} \eta_2(z)\, m_2(z)dz = \int_{a_3+b_3}^{x} \left(\frac{-2}{(a_3-a_2)^2}\right)\left(\frac{a_3+b_3-z}{1+\frac{(b_3-b_2)}{a_3-a_2}}\right) \times \left(\frac{1}{1+\frac{(b_3-b_2)}{a_3-a_2}}\right)dz$$
$$= \left(\frac{-2}{(a_3-a_2)^2}\right)\left(\frac{1}{1+\frac{(b_3-b_2)}{a_3-a_2}}\right)^2 \times \int_{a_3+b_3}^{x} (a_3+b_3-z)dz$$
$$= \left(\frac{-2}{(a_3-a_2)^2}\right)\left(\frac{1}{1+\frac{(b_3-b_2)}{a_3-a_2}}\right)^2 \times \frac{(a_3+b_3-x)^2}{-2}$$
$$= \left(\frac{1}{a_3-a_2}\right)^2\left(\frac{(a_3+b_3)-x}{1+\frac{(b_3-b_2)}{a_3-a_2}}\right)^2$$
$$= \left(\frac{(a_3+b_3)-x}{a_3-a_2+b_3-b_2}\right)^2; \quad a_2+b_2 \le x < a_3+b_3$$

Therefore, the membership functions of the fuzzy variable $Z = X + Y$ is given by

$$\mu_Z(x) = \begin{cases} \omega\left(\frac{x-(a_1+b_1)}{a_2-a_1+b_2-b_1}\right)^2 & a_1+b_1 \le x < a_2+b_2 \\ \omega & x = a_2+b_2 \\ \omega\left(\frac{(a_3+b_3)-x}{a_3-a_2+b_3-b_2}\right)^2 & a_2+b_2 \le x < a_3+b_3 \\ 0 & \text{otherwise} \end{cases}$$

where $\omega = \min(\omega_1, \omega_2)$.

3.2 Scalar Multiplication of Fuzzy Variable

Theorem 2 *If X be a parabolic fuzzy number and* $z = kx$ *be the transformation then kX is also a parabolic fuzzy number given by*

$$kX = \begin{cases} (ka_1, ka_2, ka_3; \omega_1) & \text{if k} > 0 \\ (ka_3, ka_2, ka_1; \omega_1) & \text{if } k < 0 \end{cases}$$

Proof Using the transformation $z = kx$, we get $x = z/k$ and hence $\psi(z) = z/k$. Thus $|\frac{d}{dz}x| = \frac{1}{k} = m(z)$. Therefore,

$$\int_{ka_1}^{x} \eta_1(z)m(z)dz = \int_{ka_1}^{x} \left(\frac{2(z-ka_1)}{k(a_2-a_1)^2}\right)\left(\frac{1}{k}\right)dz = \left(\frac{x-ka_1}{ka_2-ka_1}\right)^2$$
$$\int_{ka_3}^{x} \eta_2(z)m(z)dz = \int_{ka_3}^{x} \left(\frac{-2(ka_3-z)}{k(a_3-a_2)^2}\right)\left(\frac{1}{k}\right)dz = \left(\frac{ka_3-x}{ka_3-ka_2}\right)^2$$

Therefore, the membership functions of the fuzzy variable $k\tilde{X}$, $k > 0$ is given by

$$\mu_{kX}(x) = \begin{cases} \omega_1\left(\frac{x-ka_1}{ka_2-ka_1}\right)^2 & ka_1 \le x < ka_2 \\ \omega_1 & x = ka_2 \\ \omega_1\left(\frac{ka_3-x}{ka_3-ka_2}\right)^2 & ka_2 \le x < ka_3 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $k < 0$, the membership functions of the fuzzy variable $k\tilde{X}$, is given by

$$\mu_{kX}(x) = \begin{cases} \omega_1\left(\frac{x-ka_3}{ka_2-ka_3}\right)^2 & ka_3 \le x < ka_2 \\ \omega_1 & x = ka_2 \\ \omega_1\left(\frac{ka_1-x}{ka_1-ka_2}\right)^2 & ka_2 \le x < ka_1 \\ 0 & \text{otherwise} \end{cases}$$

3.3 Subtraction of Fuzzy Variable

Theorem 3 *If X and Y be the two parabolic membership function over the universe U then the fuzzy variable* $Z = X - Y$ *is also a parabolic fuzzy number whose membership function is given by*

$$\mu_Z(x) = \begin{cases} \omega\left(\frac{x-(a_1-b_3)}{a_2-a_1+b_3-b_2}\right)^2 & a_1-b_3 \le x < a_2-b_2 \\ \omega & x = a_2-b_2 \\ \omega\left(\frac{(a_3-b_1)-x}{a_3-a_2+b_2-b_1}\right)^2 & a_2-b_2 \le x < a_3-b_1 \\ 0 & \text{otherwise} \end{cases}$$

Proof The proof is trivial by using the addition and scalar multiplication ($k = -1 < 0$) of two parabolic fuzzy numbers.

3.4 Multiplication of a Fuzzy Variables

Theorem 4 *If X and Y be the two parabolic membership function over the universe U then the fuzzy variable* $Z = X \cdot Y$ *is also a parabolic fuzzy number whose membership function is given by*

$$\mu_{XY}(x) = \begin{cases} \omega\left(\frac{-B_1+\sqrt{B_1^2-4A_1(C_1-x)}}{2A_1}\right)^2 & a_1b_1 \le x < a_2b_2 \\ \omega & x = a_2b_2 \\ \omega\left(\frac{-B_2+\sqrt{B_2^2-4A_2(C_2-x)}}{2A_2}\right)^2 & a_2b_2 \le x < a_3b_3 \\ 0 & \text{otherwise} \end{cases}$$

where $A_1 = (a_2-a_1)(b_2-b_1)$, $B_1 = a_1(b_2-b_1) + b_1(a_2-a_1)$, $C_1 = a_1b_1$, $A_2 = (a_3-a_2)(b_3-b_2)$, $B_2 = -a_3(b_3-b_2) - b_3(a_3-a_2)$ *and* $C_2 = a_3b_3$.

Proof As the parabolic membership functions of X and Y are given in Eqs. (5) and (6) respectively, thus, in order to find the membership functions of the fuzzy variable $Z = XY$

where the distribution functions of X and Y are defined in (5) and (6) respectively. Thus at $y = \phi_1(x)$, $z = xy$ becomes

$$x = \frac{(a_1b_2 - a_2b_1) \pm \sqrt{(a_1b_2 - a_2b_1)^2 + 4(b_2 - b_1)(a_2 - a_1)z}}{2(b_2 - b_1)}$$
$$= \psi_1(z), (say)$$

Take,

$$A_1 = (a_2 - a_1)(b_2 - b_1);$$
$$B_1 = a_1(b_2 - b_1) + b_1(a_2 - a_1); \quad C_1 = a_1b_1$$

Hence,

$$\eta_1(z) = \frac{2}{(a_2 - a_1)^2} \times \left[\frac{-a_1b_2 - a_2b_1 + 2a_1b_1 + \sqrt{B_1^2 - 4A_1(C_1 - z)}}{2(b_2 - b_1)}\right] = \frac{1}{a_2 - a_1}\left[\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - z)}}{A_1}\right]$$

and $\quad m(z) = \mid \frac{dx}{dz} \mid = \frac{a_2 - a_1}{\sqrt{B_1^2 - 4A_1(C_1 - z)}}.$

$$\begin{aligned}
\therefore \int_{a_1b_1}^{x} \eta_1(z)m_1(z)dz
&= \int_{a_1b_1}^{x} \frac{1}{a_2 - a_1}\left[\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - z)}}{A_1}\right] \times \frac{a_2 - a_1}{\sqrt{B_1^2 - 4A_1(C_1 - z)}}dz \\
&= \int_{a_1b_1}^{x} \frac{1}{A_1}\left[\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - z)}}{\sqrt{B_1^2 - 4A_1(C_1 - z)}}\right]dz \\
&= \int_{a_1b_1}^{x} \frac{1}{A_1}\left[\frac{-B_1}{\sqrt{B_1^2 - 4A_1(C_1 - z)}} + 1\right]dz \\
&= \frac{1}{A_1}\left[\frac{(-B_1)^2 - B_1\sqrt{B_1^2 - 4A_1(C_1 - x)} + 2A_1x - 2A_1C_1}{2A_1}\right] \\
&= \left[\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - x)}}{2A_1}\right]^2; \\
& a_1b_1 \le x < a_2b_2
\end{aligned}$$

Similarly, by taking

$$A_2 = (a_3 - a_2)(b_3 - b_2);$$
$$B_2 = -a_3(b_3 - b_2) - b_3(a_3 - a_2); \quad C_2 = a_3b_3$$

we get, the membership function for the complementary distribution functions as

$$\int_{a_3b_3}^{x} \eta_1(z)m_1(z)dz = \left[\frac{-B_2 + \sqrt{B_2^2 - 4A_2(C_2 - x)}}{2A_2}\right]^2; \quad a_2b_2 \le x < a_3b_3$$

Hence, the membership function of the fuzzy variable $Z = XY$ is given by

$$\mu_{XY}(x) = \begin{cases} \omega\left(\frac{-B_1 + \sqrt{B_1^2 - 4A_1(C_1 - x)}}{2A_1}\right)^2 & a_1b_1 \le x < a_2b_2 \\ \omega & x = a_2b_2 \\ \omega\left(\frac{-B_2 + \sqrt{B_2^2 - 4A_2(C_2 - x)}}{2A_2}\right)^2 & a_2b_2 \le x < a_3b_3 \\ 0 & \text{otherwise} \end{cases}$$

3.5 Inverse of a Fuzzy Variable

Theorem 5 *If fuzzy number X represent the parabolic membership function given in Eq. (5) then the inverse of X i.e $X^{-1} = [a_3^{-1}, a_2^{-1}, a_1^{-1}; \omega_1]$ is also a parabolic fuzzy number whose membership function is*

$$\mu_{X^{-1}}(x) = \begin{cases} \omega_1\left(\frac{xa_3 - 1}{x(a_3 - a_2)}\right)^2 & \text{if } a_3^{-1} \le x < a_2^{-1} \\ \omega_1 & \text{if } x = a_2^{-1} \\ \omega_1\left(\frac{1 - a_1x}{x(a_2 - a_1)}\right)^2 & \text{if } a_2^{-1} \le x < a_1^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Proof Consider a fuzzy variable $X = [a_1, a_2, a_3, \omega_1]$ with membership function given in Eq. (5). Let $z = \frac{1}{x}$ so that $\mid \frac{dx}{dz} \mid = \frac{1}{z^2}$. Therefore for X^{-1} we have

$$\int_{x}^{a_1^{-1}} \eta_1(z)m(z)dz = \int_{x}^{a_1^{-1}} \left(\frac{2}{(a_2 - a_1)^2}\left(\frac{1}{z} - a_1\right)\right)\left(\frac{1}{z^2}\right)dz = \left(\frac{1 - a_1x}{x(a_2 - a_1)}\right)^2$$

$$\int_{a_3^{-1}}^{x} \eta_2(z)m(z)dz = \int_{a_3^{-1}}^{x} \left(\frac{2}{(a_3 - a_2)^2}\left(\frac{1}{z} - a_3\right)\right)\left(\frac{1}{z^2}\right)dz = \left(\frac{xa_3 - 1}{x(a_3 - a_2)}\right)^2$$

Thus, based on these distribution functions, fuzzy membership function of X^{-1} are

$$\mu_{X^{-1}}(x) = \begin{cases} \omega_1\left(\frac{xa_3 - 1}{x(a_3 - a_2)}\right)^2 & \text{if } a_3^{-1} \le x < a_2^{-1} \\ \omega_1 & \text{if } x = a_2^{-1} \\ \omega_1\left(\frac{1 - a_1x}{x(a_2 - a_1)}\right)^2 & \text{if } a_2^{-1} \le x < a_1^{-1} \\ 0 & \text{otherwise} \end{cases}$$

3.6 Division of Fuzzy Variables

Theorem 6 *If X and Y be the two parabolic fuzzy numbers over the universe U then, for* $0 \notin Y$, *the fuzzy variable* $Z = \frac{X}{Y} = X \cdot Y^{-1}$ *is also a parabolic fuzzy number.*

Proof By using the Theorems 4 and 5, we get the membership function of $Z = X \cdot Y^{-1}$

4 Illustrative Examples

The above methodology for computing the membership functions of various arithmetic operation has been illustrated through a numerical examples as given below.

Example 1 Addition of two numbers

Let $X = [1, 2, 4; 1]$ and $Y = [3, 5, 6; 1]$ be two parabolic fuzzy numbers with membership functions as

$$\mu_{\tilde{X}} = \begin{cases} (x-1)^2, & \text{if} \quad 1 \le x < 2 \\ \left(\frac{4-x}{2}\right)^2, & \text{if} \quad 2 \le x < 4; \\ 0, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{Y}} = \begin{cases} \left(\frac{y-3}{2}\right)^2, & \text{if} \quad 3 \le y < 5 \\ (6-y)^2, & \text{if} \quad 5 \le y < 6 \\ 0, & \text{otherwise} \end{cases}$$

In order to evaluate the degree of membership of $X + Y$, we start with the equating of the distribution and complementary distribution functions and hence we get $y = 2x + 1 = \phi_1(x)$ and $y = \frac{8-x}{2} = \phi_2(x)$. Now for $Z = X + Y$, we get $x = \psi_1(z) = \frac{z-1}{3}$, $x = \psi_2(z) = \frac{2z-8}{3}$, $\eta_1(z) = 2(\frac{z-4}{3})$, $\eta_2(z) = \frac{10-z}{3}$, $m_1(z) = \frac{1}{3}$ and $m_2(z) = \frac{2}{3}$.

Therefore, the distribution function of the fuzzy variable $Z = X + Y$ would now be given as

$$\int_4^x \eta_1(z)\, m_1(z) dz = \left(\frac{x-4}{3}\right)^2; \quad 4 \le x < 7$$

and the complementary distribution function is

$$\int_{10}^x \eta_2(z)\, m_2(z) dz = \left(\frac{10-x}{3}\right)^2 \quad ; \quad 7 \le x < 10$$

Then the fuzzy membership function of X + Y is

$$\therefore \mu_{X+Y}(x) = \begin{cases} \left(\frac{x-4}{3}\right)^2 & \text{if } 4 \le x < 7 \\ \left(\frac{10-x}{3}\right)^2 & \text{if } 7 \le x < 10 \\ 0 & \text{otherwise} \end{cases}$$

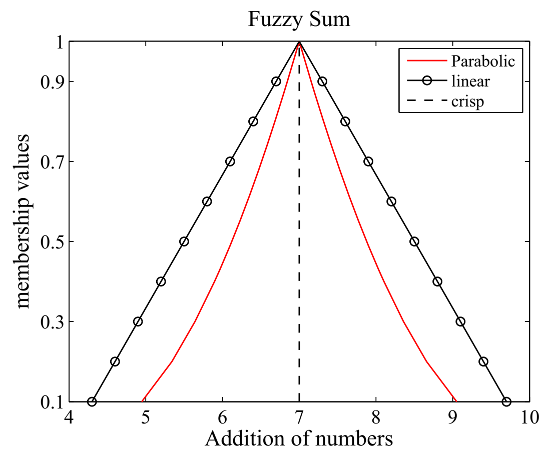

Fig. 1 Membership function of addition of two numbers

The obtained results are depicted graphically in Fig. 1 along with the other existing results, linear and crisp, and are explained as below.

1. The results computed by the crisp or traditional methodology are independent of the uncertainty level and hence it remain constant for all membership values. Therefore, these results are suitable only for a system whose data are precise.
2. The results computed by taking the linear membership functions are shown in Fig. 1 with linear legend. From the figure it is concluded that it contains a wide range of spread in the form of support and hence results are not so much practical as it contains a large amount of uncertainties.
3. On the other hand, the results computed by taking parabolic fuzzy numbers have reduced region and a smaller spread than the other results at any level of satisfaction. This means that uncertainties existing during the analysis are reduced up to the desired degree and hence decision makers/system analyst may use these results for further analysis which leads to a more sound and effective decision for future course of actions in lesser time.

Also, it has been concluded that the value of their resultant number is increasing from 4 to 7 cm at a nonlinear increasing rate $\frac{2}{9}(x-4)$ and then decreases from 7 to 10 cm at a nonlinear decreasing rate $\frac{2}{9}(10-x)$. The corresponding defuzzified value obtained by using COG method is 7 cm.

Example 2 Length of the Rod

Let length of a rod is a parabolic fuzzy number $\tilde{A} = (12, 13.5, 15 \text{ cm}; 0.8)$. If the length $\tilde{B} = (5, 6.5, 8 \text{ cm}; 0.7)$, a parabolic fuzzy number, is cut off from this rod then the remaining length of the rod $\tilde{C}$ is $\tilde{A}(-)\tilde{B}$.

The parabolic membership function corresponding to fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are defined as below

$$\mu_{\tilde{A}}(x) = \begin{cases} 0.8\left(\dfrac{x-12}{1.5}\right)^2 & \text{if } 12 \le x < 13.5 \\ 0.8\left(\dfrac{15-x}{1.5}\right)^2 & \text{if } 13.5 \le x < 15 \\ 0 & \text{otherwise} \end{cases};$$

$$\mu_{\tilde{B}}(x) = \begin{cases} 0.7\left(\dfrac{y-5}{1.5}\right)^2 & \text{if } 5 \le y < 6.5 \\ 0.7\left(\dfrac{8-y}{1.5}\right)^2 & \text{if } 6.5 \le y < 8 \\ 0 & \text{otherwise} \end{cases}$$

Now $\tilde{B} = (-8, -6.5, -5 \text{ cm}; 0.7)$ be the negative of the fuzzy number $\tilde{B}$, then their corresponding membership functions is given as

$$\mu_{-\tilde{B}}(y) = \begin{cases} 0.7\left(\dfrac{y+8}{1.5}\right)^2 & \text{if } -8 \le y < -6.5 \\ 0.7 & \text{if } y = -6.5 \\ 0.7\left(\dfrac{y+5}{1.5}\right)^2 & \text{if } -6.5 \le y < -5 \\ 0 & \text{otherwise} \end{cases}$$

Hence, using the property of the addition of the two parabolic fuzzy numbers, the membership functions of the remaining length of the rod is a parabolic fuzzy number $\tilde{C}$ and is given as:

$$\therefore \mu_{\tilde{C}}(x) = \begin{cases} 0.7\left(\dfrac{x-4}{3}\right)^2 & \text{if } 4 \le x < 7 \\ 0.7 & \text{if } x = 7 \\ 0.7\left(\dfrac{10-x}{3}\right)^2 & \text{if } 7 \le x < 10 \\ 0, & \text{otherwise} \end{cases}$$

From above, we conclude that the remaining length of the rod lies between 4 and 10 cm. Moreover, the value of this length is increased from 4 to 7 cm at a nonlinear increasing rate $\frac{1.4}{9}(x-4)$ and then decreases from 7 to 10 cm at a nonlinear decreasing rate $\frac{1.4}{9}(10-x)$. Also, there are 70 % possibilities that the length takes the value 7 cm. The corresponding membership values are plotted in Fig. 2 at different level of significance and concluded that the proposed one have less range of uncertainties than others. The defuzzified value of the remaining length of the rod is 7 cm.

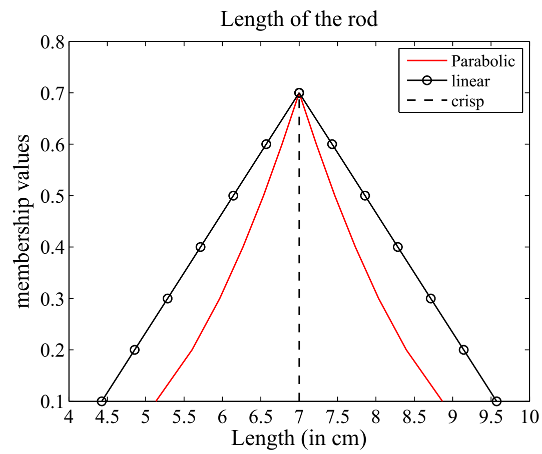

Fig. 2 Membership function of length of the rod

Example 3 Area of the rectangle

Let length and breadth of a rectangle are two parabolic fuzzy numbers given by $\tilde{A} = (1, 2, 4 \text{ cm}; 0.75)$ and $\tilde{B} = (3, 5, 6 \text{ cm}; 0.85)$. Then the area $\tilde{C}$ of the rectangle is $\tilde{A}(\cdot)\tilde{B}$.

In order to evaluate the membership functions of $\tilde{C}$, we equate the distribution and complementary distribution functions respectively of $\tilde{A}$ and $\tilde{B}$ and hence we get $\phi_1(x) = 2x+1$ and $\phi_2(x) = \dfrac{x+8}{2}$. Now for $Z = A.B$ we get $x = \psi_1(z) = \dfrac{-1 \pm \sqrt{1+8z}}{4}$, $x = \psi_2(z) = -4 \pm \sqrt{16+2z}$, $\eta_1(z) = \frac{-5+\sqrt{1+8z}}{2}$ and $\eta_2(z) = \dfrac{4-x}{2} = \dfrac{8-\sqrt{16+2z}}{2}$, $m_1(z) = \frac{1}{\sqrt{1+8z}}$ and $m_2(z) = \frac{1}{\sqrt{16+2z}}$.

Therefore, the distribution function of the fuzzy variable $\tilde{C}$ is given by

$$\int_3^x \eta_1(z)m_1(z)dz = \int_3^x \left(\frac{-5+\sqrt{1+8z}}{2}\right)\left(\frac{1}{\sqrt{1+8z}}\right)dz$$
$$= \frac{1}{2}\int_3^x \left(\frac{-5+\sqrt{1+8z}}{\sqrt{1+8z}}\right)dz$$
$$= \left(\frac{\sqrt{1+8x}-5}{4}\right)^2$$

and the complimentary distribution function is given by

$$\int_{10}^x \eta_2(z)m_2(z)dz = \int_{10}^x \left(\frac{8-\sqrt{16+2z}}{2}\right)\left(\frac{1}{\sqrt{16+2z}}\right)dz$$
$$= \left(\frac{8-\sqrt{16+2x}}{2}\right)^2$$

Hence, the membership functions of the area of the rectangle is given as

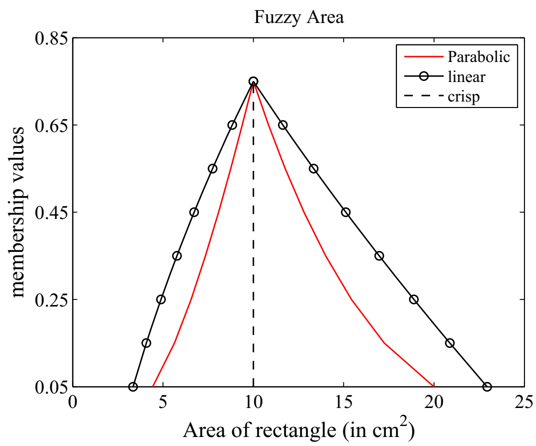

Fig. 3 Membership function of area of the rectangle

$$\therefore\ \mu_{\tilde{C}}(x)=\begin{cases}0.75\left(\dfrac{\sqrt{1+8x}-5}{4}\right)^2 & \text{if } 3\le x<10\\ 0.75 & \text{if } x=10\\ 0.75\left(\dfrac{8-\sqrt{16+2x}}{2}\right)^2 & \text{if } 10\le x<24\\ 0 & \text{otherwise}\end{cases}$$

The variation of their membership functions corresponding to linear and parabolic functions are summarized in Fig. 3 at different value of significance level. From this figure, it is concluded that resultant fuzzy number is a convex-concave type with a nonlinear increasing rate $\frac{\sqrt{1+8x}-5}{2\sqrt{1+8x}}$ from 3 to 10 cm^2 and then decreases from 10 to 24 cm^2 with non-linear decreasing rate $\frac{8-\sqrt{16+2x}}{2\sqrt{16+2x}}$. Also, there is a probability of 75 % that the area of a rectangle is 10 cm^2. Hence, the area of a rectangle lies between 4 and 10 cm^2 i.e. it does not less than 4 cm^2 and does not increase 10 cm^2. The defuzzified values corresponding to linear and parabolic fuzzy numbers are 11.9768 and 11.2806 cm^2 respectively. Thus, there is less variation in their defuzzified values in case of parabolic numbers as compared to linear numbers when compared with their crisp value 10 cm^2.

Example 4 Length of the rectangle

Let area and breadth of the rectangle be given as a parabolic fuzzy numbers $\tilde{A}=(1,2,4\ \text{cm}^2;\ 0.75)$ and $\tilde{B}=(3,5,6\ \text{cm};\ 0.85)$ respectively, then the length of the rectangle is given by $\tilde{A}(\div)\tilde{B}$ or $\tilde{A}(\cdot)\tilde{B}^{-1}$.

Now based on the membership function of $\tilde{B}$ we obtain the membership functions of $\tilde{B}^{-1}=(6^{-1},5^{-1},3^{-1};0.85)$ as

$$\mu_{\tilde{B}^{-1}}(y)=\begin{cases}0.85\left(6-\dfrac{1}{y}\right)^2 & \text{if } 6^{-1}\le y<5^{-1}\\ 0.85 & \text{if } y=5^{-1}\\ 0.85\left(\dfrac{\frac{1}{y}-3}{2}\right)^2 & \text{if } 5^{-1}\le y<3^{-1}\\ 0 & \text{otherwise}\end{cases}$$

Hence the membership function of the length of the rectangle is obtained by multiplying the two fuzzy numbers $\tilde{A}$ and $\tilde{B}^{-1}$ as

$$\mu_{\tilde{A}\cdot\tilde{B}^{-1}}(x)=\begin{cases}0.75\left(\dfrac{6x-1}{x+1}\right)^2 & \text{if } \frac{1}{6}\le x<\frac{2}{5}\\ 0.75 & \text{if } x=\frac{2}{5}\\ 0.75\left(\dfrac{4-3x}{2(x+1)}\right)^2 & \text{if } \frac{2}{5}\le x<\frac{4}{3}\\ 0 & \text{otherwise}\end{cases}$$

From this membership function, it has been concluded that there is a 75 % probability that the length of the rectangle is 0.4 cm and the range of the length of the rectangle is $[\frac{1}{6},\frac{4}{3}]$. The variation of their membership values at different level of membership values are plotted in Fig. 4 which shows that its value is increased from $\frac{1}{6}$ to $\frac{2}{5}$ with a nonlinear increasing rate $(10.5)\frac{6x-1}{(x+1)^3}$ while decreases from $\frac{2}{5}$ to $\frac{4}{3}$ with nonlinear rate $(\frac{10.5}{8})\frac{4-3x}{(x+1)^3}$. Thus membership functions are a concave-convex type instead of linear one as in the case of linear membership functions. The corresponding values of their defuzzified values are 0.52028 and 0.59294 cm for parabolic and linear membership functions while their crisp value is 0.4 cm. Hence there is 23.11 and 32.54 % decrease in the defuzzified values of crisp and linear membership functions when parabolic membership functions have been used.

Example 5 Perimeter of the rectangle

Let the length and breadth of a rectangle are two parabolic fuzzy numbers $\tilde{A}=(12,13.5,14\ \text{cm};\ \ 0.9)$ and $\tilde{B}=(6,7.5,9\ \text{cm};\ 0.8)$, then perimeter $\tilde{C}$ of rectangle is $2[\tilde{A}(+)\tilde{B}]$.

The parabolic membership functions of $\tilde{A}$ and $\tilde{B}$ are given as

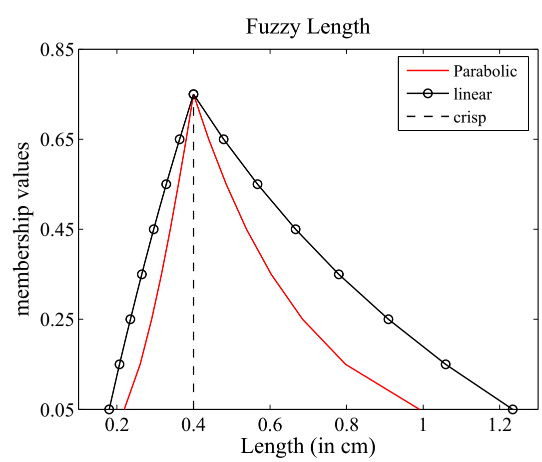

Fig. 4 Membership function of the length of the rectangle

$$\mu_{\tilde{A}}(x) = \begin{cases} 0.9\left(\dfrac{x-12}{1.5}\right)^2 & \text{if } 12 \le x < 13.5 \\ 0.9 & \text{if } x = 13.5 \\ 0.9\left(\dfrac{14-x}{0.5}\right)^2 & \text{if } 13.5 \le x < 14 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{B}}(x) = \begin{cases} 0.8\left(\dfrac{y-6}{1.5}\right)^2 & \text{if } 6 \le y < 7.5 \\ 0.8 & \text{if } y = 7.5 \\ 0.8\left(\dfrac{9-y}{1.5}\right)^2 & \text{if } 7.5 \le y < 9 \\ 0 & \text{otherwise} \end{cases}$$

The corresponding membership function of $2\tilde{A}$ and $2\tilde{B}$ are given as:

$$\mu_{2\tilde{A}}(x) = \begin{cases} 0.9\left(\dfrac{x-24}{3}\right)^2 & \text{if } 24 \le x < 27 \\ 0.9 & \text{if } x = 27 \\ 0.9\left(\dfrac{28-x}{1}\right)^2 & \text{if } 27 \le x < 28 \\ 0 & \text{otherwise} \end{cases};$$

$$\mu_{2\tilde{B}}(x) = \begin{cases} 0.8\left(\dfrac{y-12}{3}\right)^2 & \text{if } 12 \le y < 15 \\ 0.8 & \text{if } y = 15 \\ 0.8\left(\dfrac{18-y}{3}\right)^2 & \text{if } 15 \le y < 18 \\ 0 & \text{otherwise} \end{cases}$$

Now, by property of the addition of the two fuzzy numbers, we get

$$\mu_{\tilde{C}}(x) = \begin{cases} 0.8\left(\dfrac{x-36}{6}\right)^2 & \text{if } 36 \le x < 42 \\ 0.8 & \text{if } x = 42 \\ 0.8\left(\dfrac{46-x}{6}\right)^2 & \text{if } 42 \le x < 46 \\ 0 & \text{otherwise} \end{cases}$$

The membership values corresponding to the perimeter of the rectangle are summarized graphically in Fig. 5 which shows that the level of uncertainties in the form of support are less as compared to the linear membership functions. Thus the results corresponding to parabolic membership functions are beneficial for system analyst for making more sound decision based on these results. Also, it has been concluded from the figure that there is a 80 % probability of getting the perimeter of rectangular 42 cm. On the other hand, there is an increase in their perimeter with a non-linear increasing rate $(\frac{0.8}{18})(x-36)$ when $x \in [36, 42]$ while decreasing with a rate of $(\frac{0.8}{18})(48-x)$ when $x \in [42, 46]$. Their corresponding defuzzified values are 41.3518 and 41.5433 cm respectively, for linear and parabolic membership functions.

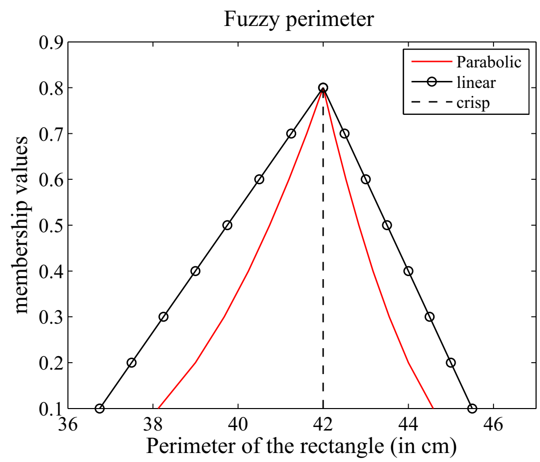

Fig. 5 Membership functions of perimeter of rectangle

5 Conclusion

In this paper, we have worked on the generalized parabolic fuzzy numbers and introduced their corresponding fuzzy arithmetic operations such as addition, subtraction, multiplication, inverse, division etc., based on their distribution and their complementary distribution functions. This method is an alternative and useful for finding the membership functions because the standard method, α-cut, does not always yield results. The variations of the membership functions has been plotted and compared with the linear membership functions and traditional (crisp) methodology. From the analysis it has been concluded that there is less range of uncertainties in the form of support during the analysis and hence proposed one is beneficial for system analyst. The defuzzified values corresponding to linear and parabolic membership functions has been computed by COG method and found that for increasing the performance, the maintenance should be based on the defuzzified values rather than crisp values, as a safe interval is inspected before reaching to the crisp value. The validity of the method has been evaluated by solving some problems of mensuration using generalized parabolic fuzzy numbers and compared their results with the triangular membership functions with the existing method. Further, the proposed approach can be applied to the uncertainty analysis and engineering and mathematical science problems which can be taken for further research.

References

1. Zadeh LA (1965) Fuzzy sets. Inf Control 8:338–353
2. Kaufmann A, Gupta MM (1991) Introduction to fuzzy arithmetic. Van Nostrand, New York
3. Dubois D, Prade H (1988) Possibility theory. Plenum Press, Berlin
4. Klir GJ (1997) Fuzzy arithmetic with requistic constraints. Fuzzy Sets Syst 91:165–175
5. Zadeh LA (1978) Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets Syst 1(28):3–28
6. Piegat A (2005) A new definion of the fuzzy set. Int J Appl Math Comput Sci 15(1):125–140
7. Stefanini L, Guerra ML (2006) On fuzzy arithmetic operations: some properties and distributive approximations. Int J Appl Math 19:171–199
8. Gao S, Zhang Z, Cao C (2009) Multiplication operation on fuzzy numbers. J Softw 4(4):331–338
9. Akther T, Ahmad SU (2009) A computational method for fuzzy arithmetic operations. Daffodil Int Univ J Sci Technol 4(1):18–22
10. Mahanta S, Chutia R, Baruah HK (2010) Fuzzy arithmetic wit-with using the method of α-cuts. Int J Latest Trends Comput 1(2):73–80
11. Taleshian A, Rezvani S (2011) Multiplication operation on trapezoidal fuzzy numbers. J Phys Sci 11:17–26
12. Chutia R, Mahanta S, Datta D (2011) Arithmetic of triangular fuzzy variable from credibility theory. Int J Energy Inf Commun 2(3):9–20
13. Bansal A (2011) Trapezoidal fuzzy numbers (a,b,c,d): arithmetic behavior. Int J Phys Math Sci 2:39–44
14. Oussalah M (2002) On the compatibility between defuzzification and fuzzy arithmetic operations. Fuzzy Sets Syst 128(2):247–260
15. Kechagias P, Papadopoulos BK (2007) Computational method to evaluate fuzzy arithmetic operations. Appl Math Comput 185:169–177
16. Deschrijver G (2007) Arithmetic operators in interval-valued fuzzy set theory. Inf Sci 177:2906–2924
17. Xue F, Tang W, Zhao R (2008) The expected value of a function of a fuzzy variable with a continuous membership function. Comput Math Appl 55:1215–1224
18. Banerjee S, Roy TK (2012) Arithmetic operations on generalized trapzoidal fuzzy number and its applications. Turk J Fuzzy Syst 3(1):16–44
19. Garg H (2014) A novel approach for analyzing the behavior of industrial systems using weakest t-norm and intuitionistic fuzzy set theory. ISA Trans 53:1199–1208
20. Vahidi J, Rezvani S (2013) Arithmetic operations on trapezoidal fuzzy numbers. J Nonlinear Anal Appl, Article ID jnaa–00,111, 8
21. Garg H (2016) A novel approach for analyzing the reliability of series-parallel system using credibility theory and different types of intuitionistic fuzzy numbers. J Braz Soc Mech Sci Eng 38(3):1021–1035
22. Ross TJ (2004) Fuzzy logic with engineering applications, 2nd edn. Wiley, New York
23. Liu B, Liu YK (2002) Expected value of fuzzy variable and fuzzy expected value model. IEEE Trans Fuzzy Syst 10(4):445–450
24. Nahmias S (1978) Fuzzy variables. Fuzzy Sets Syst 1:97–110