

Fuzzy Goal Programming Approach to Multi Objective Quadratic Programming Problem

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Abstract This paper presents a fuzzy goal programming (FGP) procedure for multi objective quadratic programming problem. In the FGP model formulation, firstly the objectives are transformed into fuzzy goals (membership functions) by means of assigning an aspiration level to each of them and suitable membership function is defined for each objectives. Then achievement of the highest membership value of each of fuzzy goals is formulated by minimizing the negative deviational variables. The proposed methodology and its efficiency over traditional utility function approach are illustrated by the numerical examples.

Keywords Multi criteria decision analysis · Multiple objective programming · Goal programming · Quadratic programming · Utility function

Introduction

Decision-making problems such as production planning, water resource management etc., involve multiple conflicting objectives with constraints and can be described by multiple objective programming models. Wallenius [21], Zimmermann [25, 26], Yager [22], Hanan [8], Narasimhan [15], Rubin and Narasimhan [18], Ying-Yung [23], Chanas [3], Rommelfanger [19], Gupta and Chakraborty [4, 7] and many researchers used and modified the concept of multi objective decision making problems and also discussed different approaches to tackle the multi objective programming problem. Balbas and Galperin et al. [2] gave a

sensitivity analysis in multi objective optimization. Yan and Wei et al. [24] constructed an efficient solution structure of multi objective linear programming. Jain and Lachhwani [10] considered multi objective programming problem with fuzzy relational equations. Afterwards, Jain and Lachhwani [11] obtained the solution of multi objective linear fractional programming problem by converting it into fuzzy programming problem. Numerous methods for multi objective optimization problems have been suggested in the literature. Each method appears to have some advantages as well as disadvantages. In the context of each application, some of the methods seem more appropriate than others. However, the issue of choosing a proper method in a given context is still a subject of active research. A number of researchers have worked for fuzzy mathematical programming problem using goal programming approach like Pal and Moitra et al. [17] suggested a goal programming procedure for fuzzy multi objective linear fractional programming problem. Chao-Fang et al. [5] proposed a generalized varying domain optimization method using fuzzy goal programming (FGP) for multi objective optimization problem with priorities. Pramanik and Roy [16] gave a procedure for solving multi level programming problems in a large hierarchical decentralized organization through linear FGP approach. Ibrahim [9] presented FGP algorithm for solving decentralised bi-level multi objective (DBL-MOP) problems with a single decision maker at the upper level and multiple decision makers at the lower level. Li and Hu [12] proposed a satisfying optimization method based on goal programming for fuzzy multi objective optimization problem with the aim of achieving the higher desirable satisfying degree. But a very few of the researchers have considered fuzzy quadratic programming problem. This situation inspired us to consider multi-objective quadratic programming (MOQP) problem.

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A quadratic programming problem is a nonlinear programming problem having an objective function that contains both linear and quadratic forms. For a non-linear program, the feasible region may not be convex. However, Wolfe [20] gave a simplex method to solve quadratic programming problem with single objective function with the assumption that the set of feasible solutions is a convex polyhedral with a finite number of extreme points. Mishra and Ghosh [14] proposed an interactive fuzzy programming method for obtaining a satisfactory solution to a bi-level quadratic fractional programming problem. Recently, Ammar [1] considered multi objective quadratic programming problem having fuzzy random coefficients matrix in the objectives and constraints and the decision vector are fuzzy pseudo random variables.

The aim of this paper is to present FGP approach which is introduced by Mohamed [13] to solve MOQP problems. In the FGP model formulation of the problem, firstly the objectives are transformed into fuzzy goals by means of assigning an aspiration level to each of them and suitable membership function is defined for each objective. Then achievement of the highest membership values (unity) to the extent possible of each of the fuzzy goals is considered.

The paper is organized as follows: In “**Problem Formulation**”, we discuss formulation of MOQPP, membership function and related definitions. In next section, we discuss proposed FGP approach to tackle MOQPP and formulate mathematical models related to it. A line diagram of FGP model development is also given in this section. Comparison of proposed methodology with utility function approach is considered in “**Comparison with Utility Function Approach**”. To illustrate the proposed methodology, numerical examples are considered and comparative analysis is performed with utility function approach in “**Numerical Example**”. Concluding remarks are given in the last section.

Problem Formulation

The general format of classical multi objective quadratic programming problem can be stated as:

$$\text{Max.}\{Z_1(X), Z_2(X), \dots, Z_k(X)\} \tag{1}$$

where $Z_i(X) = C_iX + X^T D_iX \quad \forall i = 1, 2, \dots, k$

Subject to, $X \in S$

$$= \left\{ X \in R^n \left| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, b \in R^m \right. \right\} \tag{2}$$

Here C_i and D_i are row vectors with n -components. X and b are column vectors with n and m components,

respectively. $X^T D_i X \quad (\forall i = 1, 2, \dots, k)$ is a strictly concave quadratic function on the convex set S of all its feasible solutions such that a relative maximum of $Z_i(X)$ $Z_i(X) \quad \forall i = 1, 2, \dots, k$ over S is a global maximum which is unique. Let us consider some related definitions as follows:

Definition 1 An ideal solution (ideal point) X_i^* is the finite optimal solution to the single objective programming problem *i.e.* Maximize Z .

Subject to,

$$X \in S = \left\{ X \in R^n \left| AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, b \in R^m \right. \right\}$$

Definition 2 $X^0 \in S$ is an efficient solution to problem (1)–(2) if and only if there exists no other $X \in S$ such that $Z_i \geq Z_i^0$ for all $i = 1, 2, \dots, k$ and $Z_i > Z_i^0$ for at least one i .

For our purpose, we define ideal solution (ideal point) of single objective and compromise efficient solution for multi objective programming problems.

Definition 3 For problem (1)–(2), a compromise optimal solution is an efficient solution selected by the decision maker (DM) as being the best solution where the selection is based on the DM’s explicit or implicit criteria. If an imprecise aspiration level is introduced to each of the objectives, then these objectives are termed as fuzzy goals.

Now, in the field of fuzzy programming, the fuzzy goals are characterized by their associated membership functions. The membership function for the i th fuzzy goal can be defined according to Gupta and Chakraborty [7] as:

$$\mu_i(d_i(X)) = \begin{cases} 0 & \text{if } d_i(X) \geq p \\ \frac{p-d_i(X)}{p} & \text{if } 0 < d_i(X) < p \\ 1 & \text{if } d_i(X) \leq 0 \end{cases} \tag{3}$$

where the distance function d_i with unit weight as: $d_i(X) = |\bar{Z}_i - Z_i(X)|, \quad \forall i = 1, 2, \dots, k$. This distance depends upon X . At $X = \bar{X}$ (ideal point in X -space), $d_i = 0$ and at $X = \underline{X}$ (nadir point in X -space), $Z_i(X) = \underline{Z}_i$, we get the maximum value of $d_i(X)$ as:

$$\bar{d}_i = |\bar{Z}_i - \underline{Z}_i| \tag{4}$$

and

$$p = \sup\{\bar{d}_i\} \tag{5}$$

Now in the fuzzy decision making environment, the achievement of the objective goals to their aspired levels to the extend possible is actually represented by the possible achievement of their respective membership values to the highest degree. Regarding the presently available procedures, a FGP approach seems to be most appropriate for the problem and considered in this paper.

Goal Programming Formulation

In fuzzy programming approach, the highest degree of membership function is 1. So, as given by Mohamed in [13] for the defined membership function in (3), the flexible membership goals with the aspired level 1 can be expressed as:

$$\frac{p - d_i(X)}{p} + D_i^- - D_i^+ = 1$$

i.e.,

$$-\bar{Z}_i + Z_i(X) + pD_i^- - pD_i^+ = 0, \quad \forall i = 1, 2, \dots, k \quad (6)$$

where $D_i^- (\geq 0)$ and $D_i^+ (\geq 0)$ with $D_i^- D_i^+ = 0$ represent the under and over deviational variables, respectively, from the aspired levels. In conventional GP, the under and/or over deviational variables are included in the achievement function for minimizing them and that depends upon the type of the objective functions to be optimized. In this approach, only the under deviational variables D_i^- is required to be minimized to achieve the aspired levels of the fuzzy goals. It may be noted that any over deviation from a fuzzy goal indicate the full achievement of the membership value. Now it can be easily realized that the membership goals in expression (6) are inherently non linear equation and this may reduce computational difficulties in the solution process. The i th membership goal with aspired level 1 can be presented as:

$$C_i X + X^T D_i X - \bar{Z}_i + pD_i^- - pD_i^+ = 0 \quad (7)$$

However, for model simplification the expression (7) can be considered as a general form of goal expression of the above stated membership goals. It may be noted that when a membership goal is fully achieved, $D_i^- = 0$, and when its achievement is zero, $D_i^+ = 1$ are found in the solution. Now, if the most widely used and simplest version of GP (i.e., minsum GP) is introduced to formulate the model of the problem under consideration, then GP model formulation becomes:

Model I Find X so as to minimize $\lambda = \sum_{i=1}^k w_i D_i^-$
 Subject to,

$$C_i X + X^T D_i X - \bar{Z}_i + pD_i^- - pD_i^+ = 0$$

$$X \in S = \left\{ X \in R^n \mid AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, b \in R^m \right\}$$

and

$$D_i^-, D_i^+ \geq 0, \quad \forall i = 1, 2, \dots, k \quad (8)$$

where λ represents the fuzzy achievement function consisting of the weighted under deviational variables where the numerical weights $w_i \geq 0$, ($\forall i = 1, 2, \dots, k$) represent the relative importance of achieving the aspired level of the

respective fuzzy goals subject to the constraint set in the decision making situation. The above model can also be rewritten as:

Model II Find X so as to minimize $\lambda = \sum_{i=1}^k D_i^-$
 Subject to,

$$C_i X + X^T D_i X - \bar{Z}_i + pD_i^- - pD_i^+ \leq 0$$

$$X \in S = \left\{ X \in R^n \mid AX \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b, X \geq 0, b \in R^m \right\}$$

and

$$D_i^-, D_i^+ \geq 0, \quad \forall i = 1, 2, \dots, k \quad (9)$$

However, the above models involve constrains quadratic in nature but the problem (9) can be easily solved using non linear techniques. Following the above discussion, we can construct the proposed FGP algorithm for solving MOQP problems as:

- Step 1: Solve each objective function and calculate maximum and minimum of each objective function under the given constraints.
- Step 2: Calculate the distance $\bar{d}_i = |\bar{Z}_i - \underline{Z}_i|$.
- Step 3: Calculate $p = \sup\{\bar{d}_i\}$.
- Step 4: Construct the membership function (3) for each objective function.
- Step 5: Formulate the FGP model I and II.
- Step 6: Solve the FGP model.
- Step 7: If solution is satisfactory, then stop. Otherwise modify the values of weights of negative deviational variables. The line diagram of proposed FGP model development is shown in Fig. 1 as:

Comparison with Utility Function Approach

Here we consider utility function approach given by Cochrane and Zeleny [6] for a general multi objective programming problem as:

$$U(Z_1, Z_2, \dots, Z_k) = \sum_{i=1}^k W_i Z_i(X) \quad (10)$$

where $W_i \geq 0$ are weights of the i th objective function. Now in order to compare solution procedures (8)–(10), It may be noted that when a membership goal is fully achieved, $D_i^- = 0$, is found in the solution. Then in this situation, the expression (7) can be written as:

$$Z_i(X) = \bar{Z}_i + pD_i^+ \quad \forall i = 1, 2, \dots, k$$

Thus utility function (10) becomes

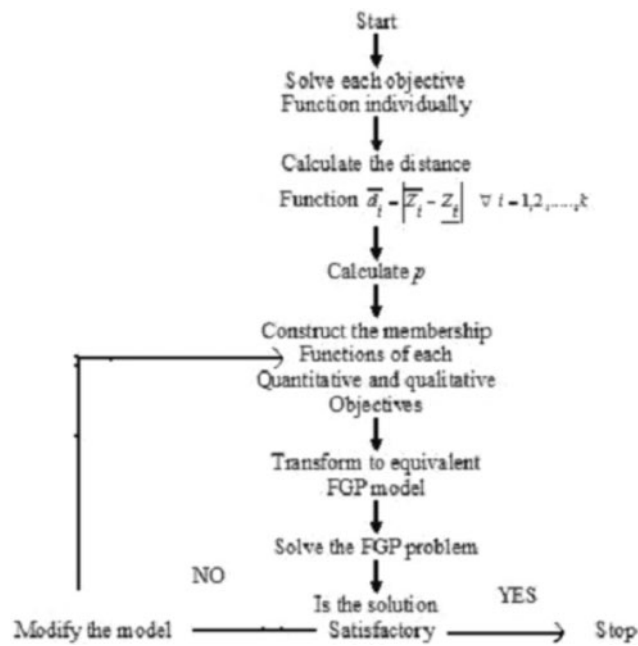


Fig. 1 Line diagram of development of FGP model for MOQP problems

$$U(Z_1, Z_2, \dots, Z_k) = \sum_{i=1}^k W_i \bar{Z}_i + p \sum_{i=1}^k W_i D_i^+$$

Which is utility function with an increase of quantity $p \sum_{i=1}^k W_i D_i^+$. This concludes that the proposed methodology gives increased value of utility function with an increase of quantity $p \sum_{i=1}^k W_i D_i^+$.

Numerical Example

The following examples are considered to illustrate the above approach:

Example 1

Maximize $\{Z_1(X), Z_2(X), Z_3(X)\}$

where

$$Z_1(X) = 4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2$$

$$Z_2(X) = 4x_1 + 6x_2 - x_1^2 - 3x_2^2$$

$$Z_3(X) = 2x_1 + 3x_2 - 2x_2^2$$

subject to,

$$2x_1 + x_2 \leq 6$$

and

$$x_1, x_2 \geq 0$$

To formulate fuzzy membership function (3), we calculate the value of each objective function individually

as: $\bar{Z}_1 = 6.7692, \underline{Z}_1 = -72, \bar{Z}_2 = 7, \underline{Z}_2 = -72, \bar{Z}_3 = 6.5, \underline{Z}_3 = -54,$ and $p = 79$ Thus the equivalent FGP formulation is obtained as: Find $X(x_3, x_4)$ so as to Minimize

$$\lambda = \left(\frac{1}{2} D_1^- + \frac{1}{3} D_2^- + \frac{1}{4} D_3^- \right)$$

And satisfy

$$4x_1 - x_1^2 + 2x_1x_2 - 2x_2^2 - 6.769 + 79D_1^- - 79D_1^+ = 0$$

$$4x_1 + 6x_2 - x_1^2 - 3x_2^2 - 7 + 79D_1^- - 79D_1^+ = 0$$

$$2x_1 + 3x_2 - 2x_2^2 - 6.5 + 79D_1^- - 79D_1^+ = 0$$

subject to,

$$2x_1 + x_2 \leq 6$$

and

$$x_1, x_2 \geq 0$$

Solving the above problem using non linear techniques, the optimal solution obtained as:

$$D_1^- = 0.04034, D_2^- = 0.01009, D_3^- = 0.042476, x_1 = 1.424765, x_2 = 1.394174$$

And the membership values achieved are:

$$\mu_1(d_1(X)) = 0.961837, \mu_2(d_2(X)) = 0.989911, \mu_3(d_3(X)) = 0.95752$$

Example 2

Maximize $\{Z_1(X), Z_2(X), Z_3(X)\}$

where

$$Z_1(X) = 6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2$$

$$Z_2(X) = 4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2$$

$$Z_3(X) = 2x_1 + 3x_2 - 2x_1^2$$

subject to,

$$x_1 + x_2 \leq 2$$

and

$$x_1, x_2 \geq 0$$

To formulate fuzzy membership function (3), we calculate the value of each objective function individually as: $\bar{Z}_1 = 14, \underline{Z}_1 = 0.50, \bar{Z}_2 = 4.6667, \underline{Z}_2 = 0, \bar{Z}_3 = 6, \underline{Z}_3 = -4$ and $p = 13.50$. Thus the equivalent FGP formulation is obtained as:

Find $X(x_3, x_4)$ so as to minimize $\lambda = (D_1^- + D_2^- + D_3^-)$

And satisfy

$$6 - 6x_1 + 2x_1^2 - 2x_1x_2 + 2x_2^2 - 14 + 13.50D_1^- - 13.50D_1^+ = 0$$

$$4x_1 + 6x_2 - 2x_1^2 - 2x_1x_2 - 2x_2^2 - 4.6667 + 13.50D_2^- - 13.50D_2^+ = 0$$

$$2x_1 + 3x_2 - 2x_1^2 - 6 + 13.50D_3^- - 13.50D_3^+ = 0$$

subject to,

$$x_1 + x_2 \leq 2$$

and

$$x_1, x_2 \geq 0$$

Solving the above problem using non linear techniques, the optimal solution obtained as:

$$D_1^- = 0.0000, D_2^- = 0.049385, D_3^- = 0.0000, x_1 = 0.0000, x_2 = 2.0000 \text{ with } Z_1 = 14, Z_2 = 4, Z_3 = 6$$

And the membership values achieved are:

$$\mu_1(d_1(X)) = 1, \mu_2(d_2(X)) = 0.95061, \mu_3(d_3(X)) = 1$$

If we compare the above results of numerical example 2, with the traditional utility function approach given by Cochrane and Zeleny [6] taking utility function as the addition of objective functions, then

$$U(Z_1(X), Z_2(X), Z_3(X)) = Z_1 + Z_2 + Z_3$$

i.e.,

$$U(Z_1, Z_2, Z_3) = 6 + 9x_2 - 2x_1^2 - 4x_1x_2$$

subject to,

$$x_1 + x_2 \leq 2$$

and

$$x_1, x_2 \geq 0$$

Solving the above quadratic programming problem, we obtain

$$U = 0.0000, x_1 = 0.1773, x_2 = 1.7795, \text{ with } Z_1 = 10.7014, Z_2 = 4.3589, Z_3 = 5.6304$$

It is obvious from the results that the proposed FGP approach gives a fair optimal solution than the utility function approach.

Conclusion

An effort has been made to present FGP approach for the solution of multi objective quadratic programming problem. The proposed method is more efficient than the

traditional method which has been illustrated with numerical examples. Also, the present method is simpler than other available methods as it finally converts multi objective quadratic programming problem into non linear programming problem which can be easily solved using non linear techniques or software packages like LINGO, CPLEX, etc. Certainly there are many other points for future research in the area of FGP and should be studied. Some of these points are:

1. An efficient algorithm should be carried out to solve multi objective quadratic fractional programming problems.
2. A solution algorithm is required to treat multi objective integer programming problems with goal programming approach.

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