Natl. Acad. Sci. Lett. (January–February 2020) 43(1):73–76
https://doi.org/10.1007/s40009-019-00808-y

SHORT COMMUNICATION

q-Double Cesaro Matrices and q-Statistical Convergence of Double Sequences

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Received: 23 May 2017 / Revised: 24 August 2018 / Accepted: 4 March 2019 / Published online: 6 April 2019

Abstract In this study, we introduce and examine the concepts of q-double Cesaro matrices and q-statistical convergence and q-statistical limit point of double sequences. Also, we give some relations connected to these concepts.

Keywords q-Integers · Statistical convergence · Cesaro summability

The idea of statistical convergence was given by Zygmund [1] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [2] and Fast [3] and later reintroduced by Schoenberg [4] independently. Statistical convergence has been discussed with different names in many branches of mathematics in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Many authors have studied statistical convergence in the name of summability theory by Çınar *et al.* [5], Connor [6], Connor and Kline [7], Et *et al.* [8], Fridy ([9, 10]), Colak and Altın [11], Moricz [12], Mursaleen et al. [13] and many others.

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Now we recall some basic definitions about q-integer. For any $n \in \mathbb{N}$, q integer of n denotes $[n]_q$,which is defined as follows

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & q \in R^+ - \{1\} \\ n & q = 1 \end{cases}.$$

In 1910, Jackson [14] was defined and studied q-integral. Later Lupas [15] introduced q-Brenstein polynomials. Bustoz et al. [16] studied q-Fourier theory and defined q-Hausdorff summability. Recently, q-integers and their applications have studied by many mathematicians such as Aktuğlu and Bekar [17].

Let $A = (a_{jk}^{mn})$ be a four-dimensional infinite matrix of real numbers for all $m, n = 0, 1, 2, \ldots$ The sums

$$z_{mn} = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk}^{mn} x_{jk}$$

are called the A-transforms of the double sequence x. We say that a sequence x is A-summable to the limit s if the A-transform of x exists for all $m, n = 0, 1, 2, \ldots$ and convergent in the Pringsheim [18] sense, that is

$$\lim_{p,p\to\infty} \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk}^{mn} x_{jk} = z_{mn} \text{ and } \lim_{m,n\to\infty} z_{mn} = s.$$

The matrix corresponding to the first-order double Cesaro mean defined by

$$\begin{cases} \dfrac{1}{mn} & \text{if } \quad j \le m \quad \text{and} \quad k \le n \\ 0 & \qquad \text{otherwise} \end{cases}. \tag{1}$$

A matrix $A = \left[a_{jk}^{mn}\right]$ is RH-regular if and only if

$$P-\lim_{m,n} a_{jk}^{mn}=0, \quad \text{for each } j,k$$
$$P-\lim_{m,n}\sum_{j}\sum_{k} a_{jk}^{mn}=1,$$
$$P-\lim_{m,n}\sum_{j}\left|a_{jk}^{mn}\right|=0, \quad \text{for each } k$$
$$P-\lim_{m,n}\sum_{k}\left|a_{jk}^{mn}\right|=0, \quad \text{for each } j$$
$$\|A\|=\sup_{m,n}\sum_{j}\sum_{k} a_{jk}^{mn}<\infty.$$

It is obvious that the double Cesaro matrix defined by (1) satisfies the above five conditions.

In the following theorem, we give without proof q-analog of the double Cesaro matrix.

Theorem 1 *$C_{(1,1)}(q)=\left(c_{jkmn}\left(q^{j+k}\right)\right)$ with*

$$c_{jkmn}(1,1)(q)=\begin{cases}\dfrac{q^{j+k}}{[n+1]_q[m+1]_q} \text{ if } j\le m \text{ and } k\le n & n,k=0,1,\ldots\\ 0 \quad \text{otherwise}\end{cases}$$

and $C^2_{(1,1)}(q)=\left(c^2_{jkmn}\left(q^{j+k}\right)\right)$ *with*

$$c^2_{jkmn}(1,1)(q^{j+k})=\begin{cases}\dfrac{[n-k+1]_q[m-j+1]_q q^{2(j+k)}}{\sum_{j=0}^{m}\sum_{k=0}^{n} q^{2(j+k)}[n-k+1]_q[m-j+1]} \text{ if } j\le m \text{ and } k\le n & n,k=0,1,\ldots\\ 0 \quad \text{otherwise}\end{cases}$$

We give the following results without proof.

Lemma 1 *(i) $C_{(1,1)}(q)$ is conservative for each $q\in\mathbb{R}$,*
(ii) $C_{(1,1)}(q)$ is regular for each $q\ge 1$.

Corollary 1 *If $q_1\ne q_2$ then $C_{(1,1)}(q_1)\ne C_{(1,1)}(q_2)$. If $q_1>1$ then $C_{(1,1)}\left(q^{j+k}\right)$ is regular, but $C_{(1,1)}\left(q^{-(j+k)}\right)$ is not regular.*

Theorem 2 *$C_{(1,1)}\left(q_1^{j+k}\right)$ is equivalent to $C_{(1,1)}\left(q_2^{j+k}\right)$ for $1<q_1<q_2$.*

Let A be a nonnegative regular matrix. Freedman and Sember [19] defined a density function by

$$\delta_A(K)=\liminf_{n\to\infty}(A\chi_k)_n \tag{2}$$

where χ_k denotes characteristic function of $K\subseteq N$.

Using C_1 instead of A and ordinary limit of lim inf *in* (2), *we obtain natural density function such as*

$$\delta(K)=\delta_{C_1}(K)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_K(k)$$

provided that limit exists.

Using $C_1(q)$ instead of C_1, Aktuğlu and Bekar [17] defined q-density as follows:

$$\delta_q(K)=\delta_{C_1^q}(K)=\liminf_{n\to\infty}\left(C_1^q\chi_k\right)_n=\liminf\sum_{k\in K}\frac{q^{k-1}}{[n]_q}, q\ge 1.$$

On generalizing the notion of the density of subsets of $\mathbb{N}$, Tripathy [20] introduced the notion of the asymptotic density for subsets of $\mathbb{N}\times\mathbb{N}$ as follows:

A subset E of $\mathbb{N}\times\mathbb{N}$ is said to have density $\delta(E)$ if

$$\delta(E)=\lim_{p,q\to\infty}\sum_{n\le p}\sum_{k\le q}\chi_E(n,k)$$

provided that limit exists.

We can define the q-density function $\delta_q^2(K)$ as follows:

$$\delta_q^2(K)=P-\lim_{m,n}\sum_{(j,k)\in K} C^q_{(1,1)}\chi_K(n,k)$$
$$=P-\lim_{m,n}\sum_{(j,k)\in K}\frac{q^{j+k-2}}{[n]_q[m]_q}, q\ge 1.$$

If K is finite subset of $N\times N$, then obviously $\delta_q^2(K)=0$.

q-density function $\delta_q^2(K)$ is well-defined as seen follows example:

Example *i*) Let $K=\{(2j,2k):j,k\in N\}$, then

$$\delta_q^2(K)=P-\lim_{m,n}\sum_{(j,k)\in K}\frac{q^{j+k-2}}{[n]_q[m]_q}$$
$$=\begin{cases}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{m}{2}}\dfrac{q^{2(j+k)-2}}{[n]_q[m]_q} & \text{if } n,m \text{ is even}\\ \sum_{j=1}^{\frac{n-1}{2}}\sum_{k=1}^{\frac{m}{2}}\dfrac{q^{2(j+k)-2}}{[n]_q[m]_q} & \text{if } n \text{ odd, } m \text{ is even}\\ \sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{m-1}{2}}\dfrac{q^{2(j+k)-2}}{[n]_q[m]_q} & \text{if } n \text{ even, } m \text{ is odd}\\ \sum_{j=1}^{\frac{n-1}{2}}\sum_{k=1}^{\frac{m-1}{2}}\dfrac{q^{2(j+k)-2}}{[n]_q[m]_q} & \text{if } n,\ m \text{ is odd}\end{cases}$$
$$=\begin{cases}\dfrac{q^2}{(1+q)^2}=\dfrac{1}{[2]} & \text{if } n,m \text{ is even}\\ \dfrac{q}{(1+q)^2} & \text{if } n \text{ odd}, m \text{ is even}\\ \dfrac{q}{(1+q)^2} & \text{if } n \text{even}, m \text{ is odd}\\ \dfrac{1}{[2][2]} & \text{if } n,m \text{ is odd}\end{cases}$$

Since $q\ge 1$ we have $\frac{1}{(1+q)^2}\le\frac{q}{(1+q)^2}\le\frac{q^2}{(1+q)^2}$

$$\delta_q^2(K)=P-\liminf_{m,n}\sum_{(j,k)\in K}\frac{q^{j+k-2}}{[n]_q[m]_q}=\frac{1}{[2][2]}.$$

ii) Let $K=\left\{(j^2,k^2):j,k\in N\right\}$. Then we have

$$\delta_q^2(K) = P - \liminf_{m,n} \sum_{(j,k)\in K} \frac{q^{j+k-2}}{[n]_q[m]_q}$$

and

$$t_{(a^2-1)(b^2-1)} = \sum_{j=1}^{a-1}\sum_{k=1}^{b-1} \frac{q^{j^2+k^2-2}}{[a^2-1]_q[b^2-1]_q};$$

$$t_m = \sum_{\substack{(j,k)\in K\\ j\le a,k\le b}} \frac{q^{j+k-2}}{[a]_q[b]_q}.$$

Hence

$$\lim_{a\to\infty}\lim_{b\to\infty} t_{(a^2-1)(b^2-1)} = \lim_{\substack{a\to\infty\\ b\to\infty}} \frac{q^0+q^3+\cdots+q^{(a-1)^2-1}}{[a^2-1]_q}\frac{q^0+q^3+\cdots+q^{(b-1)^2-1}}{[b^2-1]_q}$$
$$\le \lim_{\substack{a\to\infty\\ b\to\infty}} \frac{Mq^{(a-1)^2}}{[a^2-1]_q}\frac{Mq^{(b-1)^2}}{[b^2-1]_q} \to 0$$

So we have

$$\delta_q^2(K) = P - \liminf_{m,n} \sum_{(j,k)\in K} \frac{q^{j+k-2}}{[n]_q[m]_q} = 0.$$

Definition 1 A real double sequence $x = (x_{jk})$ is said to be q-statistically convergent to L if for every $\varepsilon > 0$, $\delta_q^2(K_\varepsilon) = 0$, where $K_\varepsilon = \{(j,k) : j \le n \text{ and } k \le m : |x_{jk} - L| \ge \varepsilon\}$. In this case we write $st_2^q - \lim x = L$.

q-Statistical convergence of double sequences is different from statistical convergence of double sequences. For this consider a double sequence defined by

$$x_{jk} = \begin{cases} 1 & 4^{n-1} \le j+k \le 2.4^{n-1}-1 \\ 0 & \text{otherwise} \end{cases} \quad n = 1,2,\ldots$$

Then $x = (x_{jk})$ is not statistically convergent, but q-statistically convergent to 0, since $st_2^q - \lim x = 0$.

Theorem 3 *If a double sequence $x = (x_{jk})$ for which there is a double sequence $y = (y_{jk})$ that is convergent $x_{jk} = y_{jk}$ for almost all (q) then $x = (x_{jk})$ is q-statistical convergence.*

Proof Omitted. □

Definition 2 Let $C^q_{(1,1)}$ be a double q-Cesaro matrix. A double sequence $x = (x_{jk})$ is said to be strongly $C^q_{(1,1)}$-summable if there is a complex number L such taht

$$P - \lim_{m,n} \sum_{j,k} |x_{jk} - L| a^{mn}_{jk}(q) = 0.$$

Definition 3 Let $x = (x_{jk})$ be a double sequence. A double sequence $x = (x_{jk})$ is said to be $C^q_{(1,1)}$-uniformly integrable if for each $\varepsilon > 0$ there exist $N = N(\varepsilon)$ and $h = h(\varepsilon)$ such that $t > h$

$$\sup_{m,n>N} \sum_{j,k:|x_{jk}|>t} |x_{jk}| \left|a^{mn}_{jk}(q)\right| < \varepsilon.$$

Theorem 4 *Let $x = (x_{jk})$ be a double sequence. Then $x = (x_{jk})$ is strongly $C^q_{(1,1)}$-summable to L if and only if x is $C^q_{(1,1)}$- uniformly integrable to L and $C^q_{(1,1)}$-statistical convergent to L.*

Definition 4 If (x_{j_n,k_n}) a subsequence of $x = (x_{jk})$ and let $K = \{(j_n,k_n) : j_1 < j_2 < \cdots ; k_1 < k_2, \ldots\}$. If $\delta_q^2(K) = 0$, then (x_{j_n,k_n}) is called subsequence of q-density zero or q-thin subsequence. On the other hand (x_{j_n,k_n}) is a q-nonthin subsequence of x if K does not have density zero.

Definition 5 Let $x = (x_{jk})$ be a double sequence. The number L is q-statistical limit point of the double sequence $x = (x_{jk})$ provided that there exist a q-nonthin subsequence of $x = (x_{jk})$ that converges to L. By $\wedge_x^q$ we denote the set of q-statistical limit points of double sequence (x_{jk}).

Definition 6 Let $x = (x_{jk})$ be a double sequence and L is any real number. We say that a real number L is said to be q-statistical cluster point of the double sequence $x = (x_{jk})$ provided that for $\varepsilon > 0$

$$\delta_q^2(\{(j,k) \in N \times N : |x_{jk} - L| < \varepsilon\}) > 0.$$

By Γ_x^q, we denote the set of all q-statistical cluster points of the double sequence $x = (x_{jk})$.

Theorem 5 *If $x = (x_{jk})$ is a double sequence if (x_{jk}) q-statistical convergence L then $\wedge_x^q = \Gamma_x^q = L$.*

Proof Suppose that (x_{jk}) q-statistical convergence L and $x \in \Gamma_x^q$. Suppose that there exist at least $\ell \in \Gamma_x^q$ such that $\ell \ne L$. Thus there exists $\varepsilon > 0$ such that

$$\{(j,k) \in N \times N : |x_{jk} - L| \ge \varepsilon\} \supseteq \{(j,k) \in N \times N : |x_{jk} - \ell| < \varepsilon\}$$

holds. Hence

$$\delta_q^2(\{(j,k) \in N \times N : |x_{jk} - L| \ge \varepsilon\}) \ge \delta_q^2(\{(j,k) \in N \times N : |x_{jk} - \ell| < \varepsilon\}).$$

Since (x_{jk}) q-statistical convergence L we can write

$$\delta_q^2(\{(j,k) \in N \times N : |x_{jk} - \ell| < \varepsilon\}) = 0$$

which is a contradiction $\ell \in \Gamma_x^q$. Therefore $\Gamma_x^q = L$. On the other hand, since (x_{jk}) q-statistical convergence L, we get $\wedge_x^q = \Gamma_x^q = L$. □

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