Natl. Acad. Sci. Lett. (November–December 2018) 41(6):391–393
https://doi.org/10.1007/s40009-018-0689-z

SHORT COMMUNICATION

Influence of the Edge Removal, Edge Addition and Edge Subdivision on the Double Vertex–Edge Domination Number of a Graph

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Received: 7 January 2016 / Revised: 29 January 2016 / Accepted: 20 June 2018 / Published online: 24 September 2018

Abstract A vertex v of a graph $G=(V,E)$ is said to *ve*-dominate every edge incident to v, as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a vertex–edge dominating set (double vertex–edge dominating set, respectively) if every edge of E is *ve*-dominated by at least one vertex (at least two vertices) of S. The minimum cardinality of a vertex–edge dominating set (double vertex–edge dominating set, respectively) of G is the vertex–edge domination number $\gamma_{ve}(G)$ (the double vertex–edge domination number $\gamma_{dve}(G)$, respectively). The influence of edge removal, edge addition and edge subdivision on the double vertex–edge domination number of a graph are investigated in this paper.

Keywords Vertex–edge dominating set · Double vertex–edge dominating set · Edge removal · Edge addition · Edge subdivision

Mathematics Subject Classification 05C69

Let $G=(V,E)$ be a simple graph. The *open neighborhood* of a vertex $v \in V$ is $N(v)=\{u \in V \mid uv \in E\}$ and the *closed neighborhood* is $N[v]=N(v)\cup\{v\}$. The *degree* of a vertex v is the cardinality of its open neighborhood, denoted $d_G(v)=|N(v)|$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*.

A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G)\backslash D$ has a neighbor in D. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. For a comprehensive survey of domination in graphs, refer [1, 2].

In a graph G, a vertex dominates itself and its neighbors. A subset $S \subset V(G)$ is a double dominating set of G, abbreviated as DDS, if every vertex in $V-S$ has at least two neighbors in S and every vertex of S has a neighbor in S. The double domination number $\gamma_2(G)$ is the minimum cardinality of a double dominating set of G. A double domination set of G with minimum cardinality is called a $\gamma_2(G)$-set. Double domination was introduced by Harary and Haynes [3] and was further studied for a class of graphs by Chellali [4].

A vertex v *ve-dominates* every edge uv incident to v, as well as every edge adjacent to these incident edges, that is, a vertex v *ve*-dominates every edge incident to a vertex in $N[v]$. A set $S \subseteq V$ is a *vertex–edge dominating set* (or a *ve-dominating set*) if for every edge $e \in E$, there exists a vertex $v \in S$ such that v *ve*-dominates e. The minimum cardinality of a *ve*-dominating set of G is called the *vertex–edge domination number* $\gamma_{ve}(G)$. The concept of vertex–edge domination was introduced by Peters [5] in 1986 and studied further in [6–8].

A set $D \subseteq V$ is a *double vertex–edge dominating set* (or simply, a *double ve-dominating set*) of G, abbreviated as DVEDS, if every edge of E is *ve*-dominated by at least two vertices of D. The *double vertex–edge domination number* of G, denoted by $\gamma_{dve}(G)$, is the minimum cardinality of a double *ve*-dominating set of G. A double *ve*-dominating set of G of minimum cardinality is called a $\gamma_{dve}(G)$-set.

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It is well known that the removal of an edge does not decrease the domination number γ and increases it by at most one. That is, for any edge e of G, $\gamma(G) \leq \gamma(G-e) \leq \gamma(G)+1$. Here we present similar inequalities for a double vertex–edge domination number γ_{dve} on edge removal, edge addition and edge subdivision.

Theorem 1.1 *For every graph G and any edge e of G we have $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G-e) \leq \gamma_{dve}(G)+2$.*

Proof We first prove that $\gamma_{dve}(G-e) \leq \gamma_{dve}(G)+2$ for any edge $e=uv$. Let D be a minimum DVEDS of G. We consider three cases.

Case 1 Assume that $u, v \notin D$. Then D is a DVEDS of the graph $G-e$ and $\gamma_{dve}(G-e) \leq |D| \leq \gamma_{dve}(G) < \gamma_{dve}(G)+2$.

Case 2 Now let $|\{u,v\} \cap D| = 1$. Without loss of generality assume that $u \in D$. Thus $D \cup \{v\}$ is a DVEDS of the graph $G-e$. We have $\gamma_{dve}(G-e) \leq |D|+1 \leq \gamma_{dve}(G)+1 < \gamma_{dve}(G)+2$.

Case 3 Now let $u, v \in D$. The vertex u dominates the edges incident at v and the vertex v dominates the edges incident at u. In the graph $G-e$, the set $D \cup \{a,b\}$ where $a \in N(u) \backslash \{v\}$ and $b \in N(v) \backslash \{u\}$ is a DVEDS. Thus $\gamma_{dve}(G-e) \leq |D|+2 \leq \gamma_{dve}(G)+2$. Moreover the set $N(u) \backslash \{v\}$ and $N(v) \backslash \{u\}$ are not simultaneously empty. For in that case $G = K_2$ and $G-e = 2K_1$. Without loss of generality assume $N(u) \backslash \{v\}$ is non empty. Then $(D \cup \{a\}) \backslash \{v\}$ is a DVEDS of the graph $G-e$. We have $\gamma_{dve}(G-e) \leq |D|-1+1 = |D| \leq \gamma_{dve}(G) < \gamma_{dve}(G)+2$.

We now prove that $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G-e)$. Let D' be a minimum DVEDS of the graph $G-e$, where $e=uv$ is an edge of G. Let us first assume that both $u, v \in D'$. Then D' is a DVEDS of the graph G. Thus $\gamma_{dve}(G)-1 \leq |D'|-1 \leq \gamma_{dve}(G-e)-1 < \gamma_{dve}(G-e)$.

Now assume that $|\{u,v\} \cap D'| = 1$. Without loss of generality assume $u \in D'$. To dominate the edges incident with v in G, the vertex $v \in D'$. Thus $D' \cup \{v\}$ is a DVEDS of the graph G. Thus $\gamma_{dve}(G) \leq |D'|+1 \leq \gamma_{dve}(G-e)+1$. We have $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G-e)$.

Assume that both $u, v \notin D'$. If u is a pendant vertex in $G-e$, then the support vertex adjacent to $u \in D'$. Thus $D' \cup \{u\}$ is a DVEDS of the graph G. We have $\gamma_{dve}(G) \leq |D'|+1 \leq \gamma_{dve}(G-e)+1$. This gives $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G-e)-1+1 \leq \gamma_{dve}(G-e)$. Now assume that u is not a pendant vertex in $G-e$. To dominate the edges incident with u and v twice, a neighbor of u and v, say a, b should be in D'. These two vertices a and b dominate the edge e in G. Hence D' is a DVEDS of the graph G. We get $\gamma_{dve}(G) \leq |D'| \leq \gamma_{dve}(G-e)$. This gives $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G-e)-1 < \gamma_{dve}(G-e)$. Thus $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G-e) \leq \gamma_{dve}(G)+2$. $\square$

Let $e \in E(\overline{G})$. Then $G+e$ denotes the graph obtained by adding the edge e.

Theorem 1.2 *For every graph G and any edge $e \in E(\overline{G})$ we have $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G+e) \leq \gamma_{dve}(G)$.*

Proof We first prove that $\gamma_{dve}(G+e) \leq \gamma_{dve}(G)$ for any edge $e=uv \in E(\overline{G})$. Let D be a minimum DVEDS of G. We consider the following cases.

Case 1 Assume that $u, v \in D$. Then D is a DVEDS of the graph $G+e$ as the vertices u and v together dominates the edge e. Thus $\gamma_{dve}(G+e) \leq |D| \leq \gamma_{dve}(G)$.

Case 2 Now let $|\{u,v\} \cap D| = 1$. Without loss of generality assume that $u \in D$. Then the vertex u dominates the edges incident to it. Suppose that to dominate the edges incident with v, a neighbor of v, say a is in D. Thus D is also a DVEDS of the graph $G+e$ as the vertex a dominates the edge uv in $G+e$. We get $\gamma_{dve}(G+e) \leq |D| \leq \gamma_{dve}(G)$.

Case 3 Now assume that both $u, v \notin D$. If the neighbor of u and v, say a, b are in D, then D is a DVEDS of the graph $G+e$. Thus $\gamma_{dve}(G+e) \leq |D| \leq \gamma_{dve}(G)$.

We now prove that $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G+e)$. Let D' be a minimum DVEDS of the graph $G+e$, where $e=uv \in E(\overline{G})$. Let us first assume that both $u, v \in D'$. Then D' is a DVEDS of the graph G. Thus $\gamma_{dve}(G)-1 \leq |D'|-1 \leq \gamma_{dve}(G+e)-1 < \gamma_{dve}(G+e)$.

Now assume that $|\{u,v\} \cap D'| = 1$. Without loss of generality assume $u \in D'$. To dominate the edge uv, a neighbor of v, say $a \in D'$. Thus $D' \cup \{a\}$ is a DVEDS of the graph G. Thus $\gamma_{dve}(G) \leq |D'|+1 \leq \gamma_{dve}(G+e)+1$. We have $\gamma_{dve}(G)-1 \leq \gamma_{dve}(G+e)-1+1 \leq \gamma_{dve}(G+e)$.

Now when both $u, v \notin D'$ then it is obvious that D' is itself a DVEDS of the graph G. We get $\gamma_{dve}(G)-1 \leq |D'|-1 \leq \gamma_{dve}(G+e)-1 < \gamma_{dve}(G+e)$. $\square$

The subdivision of some edge $e=uv$ in a graph G yields a graph containing one new vertex w, and with an edge set replacing e by two new edges with endpoints uw and wv.

Let us denote by $G_{uv} \oplus w$ the graph obtained from a graph G by subdivision of an edge uv in a graph G.

Theorem 1.3 *For every graph G and any edge $e=uv$ of G we have $\gamma_{dve}(G) \leq \gamma_{dve}(G_{uv} \oplus w) \leq \gamma_{dve}(G)+1$.*

Proof We first prove that $\gamma_{dve}(G_{uv} \oplus w) \leq \gamma_{dve}(G)+1$ for any edge $e=uv$ which is subdivided by a new vertex w with the edge set $\{uw, wv\}$ replacing e. Let D be a minimum DVEDS of G. We consider three cases.

Case 1 Assume that $u, v \in D$. Then D is a DVEDS of the graph $G_{uv} \oplus w$, since u and v dominates the edges incident to them and also the edges incident to a vertex adjacent to them. Thus the edges uw and wv of $G_{uv} \oplus w$ are dominated by the vertices u, v. We get $\gamma_{dve}(G_{uv} \oplus w) \leq |D| \leq \gamma_{dve}(G) < \gamma_{dve}(G) + 1$.

Case 2 Now let $|\{u, v\} \cap D| = 1$. Without loss of generality assume that $u \in D$. Then u dominates the edges uw, wv in $G_{uv} \oplus w$. To dominate the edges incident with v, the neighbor of v other than u, say a is in D. It is clear that $D \cup \{v\}$ is a DVEDS of the graph $G_{uv} \oplus w$. We get $\gamma_{dve}(G_{uv} \oplus w) \leq |D| + 1 \leq \gamma_{dve}(G) + 1$.

Case 3 Now let both $u, v \notin D$. The edges incident with u are dominated by a neighbor of u, say a other than v and the edges incident with v are dominated by a neighbor of v, say b other than u. It is clear that $D \cup \{u\}$ is a DVEDS of the graph $G_{uv} \oplus w$. Thus $\gamma_{dve}(G_{uv} \oplus w) \leq |D| + 1 \leq \gamma_{dve}(G) + 1$.

We now prove that $\gamma_{dve}(G) \leq \gamma_{dve}(G_{uv} \oplus w)$. Let D' be a minimum DVEDS of the graph $G_{uv} \oplus w$. Let us first assume that both $u, v \in D'$. Then D' is a DVEDS of the graph G as u, v dominates the edge uv of G. We have $\gamma_{dve}(G) \leq |D'| \leq \gamma_{dve}(G_{uv} \oplus w)$.

Now assume that $|\{u, v\} \cap D'| = 1$. Without loss of generality assume $u \in D'$. To dominate the edges incident with v other than wv, a neighbor of v, say a, other than w is in D'. It is clear that D' is a DVEDS of the graph G. We have $\gamma_{dve}(G) \leq |D'| \leq \gamma_{dve}(G_{uv} \oplus w)$.

Now assume that both $u, v \notin D'$. To dominate the edges incident with u, a neighbor of u, say a, other than w is in D' and to dominate the edges incident with v, a neighbor of v, say b, other than w is in D'. To dominate the edges incident with u and v twice, the vertex $w \in D'$. It is clear that $D' \backslash \{w\}$ is a DVEDS of the graph G. We get $\gamma_{dve}(G) \leq |D'| - 1 \leq \gamma_{dve}(G_{uv} \oplus w) - 1 < \gamma_{dve}(G_{uv} \oplus w)$.

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