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SHORT COMMUNICATION

The Differential Transform Method as a New Computational Tool for Laplace Transforms

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Abstract In this short communication, the recent differential transform method is proposed to compute Laplace transforms in an innovative manner. Unlike the common method of finding Laplace transforms, the method is free of integration and hence is of computational interest. A number of illustrative examples are given to show the efficiency and simplicity of the new technique.

Keywords Differential transform method · Laplace transforms · Differential equations · Engineering mathematics · Engineering applications

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Since its inception, the differential transform method (DTM) has established a good reputation for providing convenient solutions to differential equations, both linear and nonlinear. The DTM converges to analytical solutions in form of polynomial series rapidly and does not require any discretization, linearization or perturbation [1, 2]. The mathematical literature abounds with applications of the DTM in various fields of science and engineering; e.g. see [3–7]. In a recent effort, Babolian et al. have ingeniously exploited the Adomian decomposition method to compute Laplace transforms [8]. In a similar theme, Abbasbandy has adopted He's homopoty perturbation method to derive Laplace transforms [9]. Our objective in this short note is to propose an alternative way to compute Laplace transforms by means of the DTM. This scheme eliminates the inherent need for integration, which sometimes is not analytically tractable, to obtain Laplace transforms.

The differential transform of a given univariate function $u(x)$ is given by the following formula:

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=0}, \tag{1}$$

where $U(k)$ is the transformed function.

Also, the relevant differential inverse transform for function $U(k)$ is defined as.

$$u(x) = \sum_{k=0}^{\infty}\left\{U(k)x^k\right\}. \tag{2}$$

For brevity, authors do not discuss the n-dimensional differential transform and refer the interested reader to [3]. Some basic operations of the one-dimensional differential transform are listed in Table 1. Proofs of the given operations in Table 1 can be extracted from [2, 3].

According to the proposed method, in order to have the DTM provide Laplace transform of a desired function, say Q, it is necessary to devise a differential equation whose solution is related to $L\{Q\}$. Let us consider the very fundamental first-ordered ODE:

Dedicated to Professor Esmaiel Babolian, Iran's Father of Modern Mathematics.

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Table 1 A list of operations for the one-dimensional differential transform

Original function	Transformed function
$u(x) \pm v(x)$	$U(K) \pm V(k)$
$\alpha u\ (x)$	$\alpha U\ (k)$
$\frac{d^m u(x)}{dx^m}$	$\frac{(k+m)!}{k!} U(k+m)$
x^m	$\delta(k-m) = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases}$
e^{ax}	$\frac{a^k}{k!}$
$\sin\ (ax)$	$\frac{a^k}{k!}\sin\left(\frac{k\pi}{2}\right) = \begin{cases} 0, & k \text{ is even} \\ \frac{a^k(-1)^{\frac{k-1}{2}}}{k!}, & k \text{ is odd} \end{cases}$
$\cos\ (ax)$	$\frac{a^k}{k!}\cos\left(\frac{k\pi}{2}\right) = \begin{cases} \frac{a^k(-1)^{\frac{k}{2}}}{k!}, & k \text{ is even} \\ 0, & k \text{ is odd} \end{cases}$
$f(x) = \int_{x_0}^{x} u(t)dt$	$F(k) = \frac{U(k-1)}{k}, \quad k \geq 1; \quad F(0) = 0$

m is a non-negative integer and δ represents the Kronecker delta function

$$\begin{cases} u'(x) = su(x) + Q(x), \\ u(0) = 0, \end{cases} \tag{3}$$

where s is a positive constant. Through the use of the integrating factor, the analytical solution to (3) is given by

$$ue^{-sx} = \int Qe^{-sx}dx. \tag{4}$$

The left hand-side of Eq. (4) equates Laplace transform of function Q, if the integration is considered from zero to infinity. That is

$$L\{Q\} = [e^{-sx}u]|_{x=0}^{x=+\infty}. \tag{5}$$

Thus, in order to obtain $L\{Q\}$, we first employ the DTM to solve Eq. (3) and simply substitute the solution into Eq. (5).

In what follows, we give some examples, which illustrate how our approach yields Laplace transform of a desired function. For better reference, we will name Eq. (3) as the "*generator equation*", henceforth.

Example 1 Let us determine $L\{e^{ax}\}$.

As explained above, we build the generator equation as

$$\begin{cases} u' = su + e^{ax}, \\ u(0) = 0, \end{cases} \tag{6}$$

and its transformed pair yields

$$\begin{cases} U(k+1) = \frac{s}{k+1} U(k) + \frac{a^k}{(k+1)!}, \\ U(0) = 0. \end{cases} \tag{7}$$

Through Eq. (7), we calculate

$$\begin{cases} U(1) = 1,\ U(2) = \frac{s}{2} + \frac{a}{2!},\ U(3) = \frac{s^2}{3!} + \frac{sa}{3!} + \frac{a^2}{3!}, \\ U(4) = \frac{s^3}{4!} + \frac{s^2a}{4!} + \frac{sa^2}{4!} + \frac{a^3}{4!}, \\ U(5) = \frac{s^4}{5!} + \frac{s^3a}{5!} + \frac{s^2a^2}{5!} + \frac{sa^3}{5!} + \frac{a^4}{5!}, \ \ldots. \end{cases} \tag{8}$$

So, we easily revert $U(k)$ to

$$\begin{aligned} u = {} & \frac{1}{s}\left(sx + \frac{(sx)^2}{2!} + \frac{(sx)^3}{3!} + \cdots\right) \\ & + \frac{a}{s^2}\left(\frac{(sx)^2}{2!} + \frac{(sx)^3}{3!} + \frac{(sx)^4}{4!} + \cdots\right) \\ & + \frac{a^2}{s^3}\left(\frac{(sx)^3}{3!} + \frac{(sx)^4}{4!} + \frac{(sx)^5}{5!} + \cdots\right) \\ & + \frac{a^3}{s^4}\left(\frac{(sx)^4}{4!} + \frac{(sx)^5}{5!} + \frac{(sx)^6}{6!} + \cdots\right) + \cdots, \end{aligned} \tag{9}$$

or equivalently,

$$\begin{aligned} u = {} & \frac{1}{s}(e^{sx} - 1) + \frac{a}{s^2}(e^{sx} - 1 - sx) \\ & + \frac{a^2}{s^3}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!}\right) \\ & + \frac{a^3}{s^4}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!} - \frac{(sx)^3}{3!}\right) + \cdots. \end{aligned} \tag{10}$$

Hence,

$$L\{e^{ax}\} = [e^{-sx}u]|_{x=0}^{x=+\infty} = \frac{1}{s} + \frac{a}{s^2} + \frac{a^2}{s^3} + \frac{a^3}{s^4} + \cdots. \tag{11}$$

One can readily identify that the sequence in Eq. (11) is a geometric progression with $1/s$ and a as its common ratio and scale factor, respectively. Thus, by increasing the number of components toward infinity, we can calculate the result of the infinite series in (11) as

$$L\{e^{ax}\} = \lim_{n \to +\infty} \frac{1}{s} \frac{\left(1 - \left(\frac{a}{s}\right)^{n+1}\right)}{1 - \frac{a}{s}} = \frac{\frac{1}{s}}{1 - \frac{a}{s}} = \frac{1}{s-a}, \quad s > |a|. \quad \square \tag{12}$$

Example 2 Calculate $L\left\{\frac{\sin(x)}{x}\right\}$.

The transformed equivalent of the generator equation for this problem is

$$\begin{cases} U(k+1) = \frac{sU(k)}{k+1} + \begin{cases} \frac{(-1)^{\frac{k}{2}}}{(k+1)(k+1)!}, & k \text{ is even} \\ 0, & k \text{ is odd} \end{cases} \\ U(0) = 0. \end{cases} \tag{13}$$

Recursively, it follows from Eq. (13) that

$$\begin{cases} U(1) = 1,\ U(2) = \dfrac{s}{2!},\ U(3) = \dfrac{s^2}{3!} - \dfrac{1}{3 \times 3!}, \\ U(4) = \dfrac{s^3}{4!} - \dfrac{s}{3 \times 4!}, U(5) = \dfrac{s^4}{5!} - \dfrac{s^2}{3 \times 5!} + \dfrac{1}{5 \times 5!}, \\ U(6) = \dfrac{s^5}{6!} - \dfrac{s^3}{3 \times 6!} + \dfrac{s}{5 \times 6!}, \\ U(7) = \dfrac{s^6}{7!} - \dfrac{s^4}{3 \times 7!} + \dfrac{s^2}{5 \times 7!} - \dfrac{1}{7 \times 7!}, \\ U(8) = \dfrac{s^7}{8!} - \dfrac{s^5}{3 \times 8!} + \dfrac{s^3}{5 \times 8!} - \dfrac{s}{7 \times 8!}, \ldots. \end{cases} \tag{14}$$

Therefore,

$$\begin{aligned} u = {} & \frac{1}{s}(e^{sx} - 1) - \frac{1}{3s^3}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!}\right) \\ & + \frac{1}{5s^5}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!} - \frac{(sx)^3}{3!} - \frac{(sx)^4}{4!}\right) \\ & - \frac{1}{7s^7}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!} - \frac{(sx)^3}{3!} - \frac{(sx)^4}{4!} - \frac{(sx)^5}{5!} - \frac{(sx)^6}{6!}\right) \\ & + \cdots. \end{aligned} \tag{15}$$

Thus,

$$L\left\{\frac{\sin(x)}{x}\right\} = [e^{-sx}u]\big|_{x=0}^{x=+\infty} = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots. \tag{16}$$

Knowing that

$$\begin{aligned} \arctan(x) &= \int_0^x \frac{d\xi}{1+\xi^2} \\ &= \int_0^x d\xi - \int_0^x \xi^2 d\xi + \int_0^x \xi^4 d\xi - \int_0^x \xi^6 d\xi \\ &\quad + \cdots,\ 0 \le x < 1, \end{aligned} \tag{17}$$

it follows that

$$\arctan\left(\frac{1}{s}\right) = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - \frac{1}{7s^7} + \cdots,\ 0 \le \frac{1}{s} < 1. \tag{18}$$

In view of Eqs. (16) and (18), we conclude

$$L\left\{\frac{\sin(x)}{x}\right\} = \arctan\left(\frac{1}{s}\right),\ 0 \le \frac{1}{s} < 1.\ \square \tag{19}$$

Example 3 Calculate $L\{J_0(x)\}$, where J_0 denotes the Bessel function of first kind of zero order.

Similar to the previous examples, we calculate the following differential transform series components by the correct choice of the generator equation:

$$\begin{cases} U(0) = 0,\ U(1) = 1,\ U(2) = \dfrac{s}{2},\ U(3) = \dfrac{s^2}{6} - \dfrac{1}{12}, \\ U(4) = \dfrac{s^3}{24} - \dfrac{s}{48},\ U(5) = \dfrac{s^4}{120} - \dfrac{s^2}{240} + \dfrac{1}{320}, \\ U(6) = \dfrac{s^5}{720} - \dfrac{s^3}{1440} + \dfrac{s}{1920}, \\ U(7) = \dfrac{s^6}{5040} - \dfrac{s^4}{10080} + \dfrac{s^2}{13440} - \dfrac{1}{16128}, \ldots. \end{cases} \tag{20}$$

Now, from definition (2), it follows that

$$\begin{aligned} u = {} & \frac{1}{s}(e^{sx} - 1) - \frac{1}{2s^3}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!}\right) \\ & + \frac{3}{8s^5}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!} - \frac{(sx)^3}{3!} - \frac{(sx)^4}{4!}\right) \\ & - \frac{5}{16s^7}\left(e^{sx} - 1 - sx - \frac{(sx)^2}{2!} - \frac{(sx)^3}{3!} - \frac{(sx)^4}{4!}\right. \\ & \left. - \frac{(sx)^5}{5!} + \frac{(sx)^6}{6!}\right) + \cdots. \end{aligned} \tag{21}$$

Therefore,

$$L\{J_0(x)\} = [e^{-sx}u]\big|_{x=0}^{x=+\infty} = \frac{1}{s} - \frac{1}{2s^3} + \frac{3}{8s^5} - \frac{5}{16s^7} + \cdots.\ \square \tag{22}$$

By virtue of the MacLaurin expansion series, we find the closed form for the previous sequence as $1\big/\sqrt{(s^2+1)}$.

To conclude, unlike the classic routine that imposes integration, our scheme only requires simple algebraic operations plus differentiation for the derivation of Laplace transforms.

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