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Mathematical modeling on investments in burning and recycling of dumped waste by plastic industry

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Abstract

Every year tones of plastic are produced and people become the victim of many types of waste including domestic, agronomic and industrialized waste. In this paper, we have constructed the mathematical model to examine the rising pollution level due to industrial plastic waste and the process of destroying or renovating this plastic waste using several toxic chemicals. The dynamical model suggests investing the influential part of the budget into civilization health policy, like reducing the plastic burning and support the installation of the eco-friendly plastic recycling machine to reuse dumped industrial plastic. Both of these policies are strides to the health as well as the economic progress of the country. In the model, the local stability of the system of nonlinear differential equations is qualitatively analyzed with appropriate conditions. Graph theory results help to distinguish the global stability behavior of the model. Techniques to reduce plastic pollution are outlined by optimizing both the policies which are simulated numerically with given validated data.

Introduction

Million tons of plastic are consumed each year. Estimation said that out of this only one-quarter part is recycled. Plastic is prepared from toxic materials like vinyl hydrochloride, benzene, etc. These chemicals cause the precarious health issues and also contaminate our air, soil and water. The huge bulk of industrial plastic is non-biodegradable and if it is degradable then it only starts degrading in about 700 years which means that all the plastic that has ever been produced has not degraded yet. Therefore, these dumped industrial plastics are either burned or recycled. One should recycle all the things of plastic rather than new production and burning. Keep it away from environmental sources such as

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landfill and ocean. Because when the industrial waste plastic composed in dumping sites, it keeps rotting, spreading odor which cause the air pollution. Hence, rivers and landfill become the target of this various types of pollution which are produced by industrial waste plastic.

About three decades ago, as people's requirement was not too big so that easy to control the garbage causes. But today's world has none of the activities which do not generate plastic debris. Instead of this, all stated processes like producing, dumping, burning of plastics, one needs to think beyond this. This paper includes two occurrences to invest particular total budget into health policy and installing recycling machine policy rather than the budget used in new production, burning or dumping of plastic.

In statistical way, Wilcox et al. (2015) has developed a model on how sea birds do effective waste management which help to reduce pollution threat. Song et al. (1999) has tuned a model to determine the optimal conditions and identify environmentally favorable recycle routes by analyzing the sensitivity and optimization methods. Dubey (2010) used optimal investment policy for clean environment and human health policy by considering a nonlinear dynamical model of resource biomass. Pathinathan et al. (2014) took an algorithmic approach and using induced fuzzy cognitive maps which analyze the threats of plastic pollution. Nganda

(2007) prepared two models to manage the municipal waste by using integer and mixed integer programming problem. Matar et al. (2014) established a paper for production, recycling and reuses of inventory plastic beverages bottles like EOQ model. Shah et al. (2019) formed a model to optimize two investment policies to control environmental pollution. Donovan et al. (1975) prepared a mathematical model using the concept of two-shot modeling for recycling of plastic. Cairns et al. (2005) has framed a paper on improving options to reuse, reduce and recycle of e-waste to protect consumers, public health and an environment.

In epidemics model proposed by Swan (1990), Branicky et al. (1998), De Pillis and Radunskaya (2001), Gaff and Schaefer (2009), Okosun and Makinde (2014), Rodrigues et al. (2014), Rachah and Torres (2015), Ding et al. (2016) and many more control theory have been applied.

A brief survey tells the information about the plastics and way to optimize the environmental pollution using different approaches of mathematics. This paper has obtained the results using mathematical dynamical model. "Modeling" section deals with development of mathematical model of the problem under consideration. "Stability" section analyzes the local and global stability of the proposed model. Two cost policies for burning and installing recycling machine is optimized in "Optimal investment in health and recycled machine policy" section. All the results are numerically interpreted in "Numerical simulation" section. "Conclusion" section includes the remarkable conclusions from the model.

Material and methods

Modeling

This paper supports for reducing eminence of pollution by moderating plastic consumption and promoting recycling of industrial plastic. This industrial plastic ($P_L(t)$)(in tons) is either sends for burning ($B_n(t)$)(in tons), dumping ($D(t)$) (in tons) or recycling ($R_C(t)$)(in tons) which sources the pollution ($P(t)$)(in ppm) at any time $t \geq 0$. Here, total budget (C) (in \$) is divided into two useful optimum policy, health policy for the society (C_h) (in \$) and installing recycling machine policy ($C_m = C - C_h$) (in \$).

The model given in Fig. 1 is formulated by using all parameters mentioned in Table 1 are the rates which connects the state variables known as compartments. The parametric values have been assumed hypothetically.

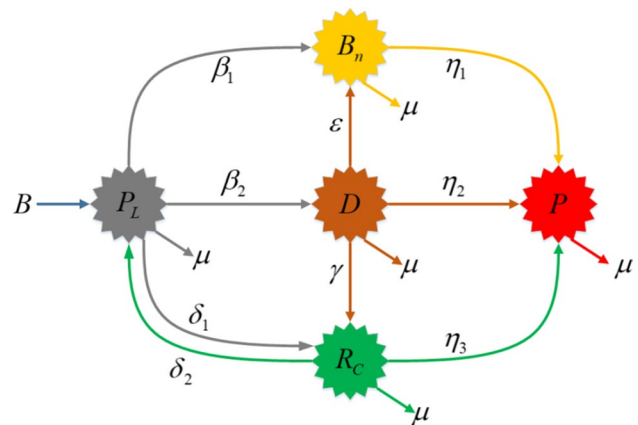Hence, the system of differential equation is constructed as,

Fig. 1 Diagram of model

$$\frac{dP_L}{dt} = B - \beta_1 P_L B_n - \beta_2 P_L D - \delta_1 P_L R_C + \delta_2 R_C - \mu P_L$$

$$\frac{dB_n}{dt} = \beta_1 P_L B_n + \varepsilon D - \eta_1 B_n - \mu B_n - \theta_1 C_h B_n$$

$$\frac{dD}{dt} = \beta_2 P_L D + \varepsilon D - \eta_2 D - \gamma D - \mu D + \theta_2 C_m D \qquad (1)$$

$$\frac{dR_C}{dt} = \delta_1 P_L R_C - \delta_2 R_C + \gamma D - \eta_3 R_C - \mu R_C$$

$$\frac{dP}{dt} = \eta_1 B_n + \eta_2 D + \eta_3 R_C - \mu P$$

with $P_L > 0$ and $D, B_n, R_C, P \geq 0$.

Summing the above differential equations of system (1), we have

$$\frac{d}{dt}\left(P_L + B_n + D + R_C + P\right)$$
$$= B - \mu P_L - \mu B_n - \mu D - \mu R_C - \mu P - \theta_1 C_h B_n + \theta_2 C_m D$$
$$= B - \mu(P_L + B_n + D + R_C + P) \geq 0$$

$$\Rightarrow \lim_{t \to \infty} \sup(P_L + B_n + D + R_C + P) \leq \frac{B}{\mu}.$$

Thus, the feasible region of the model is $\Lambda = \left\{ (P_L, B_n, D, R_C, P) \in R^5 : P_L + B_n + D + R_C + P \leq \frac{B}{\mu} \right\}$. where $R_+^5 = \left\{ (P_L, B_n, D, R_C, P) \in R^5 : P_L > 0, B_n \geq 0, D \geq 0, R_C \geq 0, P \geq 0 \right\}$.

By solving system (1), the solutions are finding out called equilibrium points:

(i) $E_0\left(\frac{B}{\mu}, 0, 0, 0, 0\right)$

(ii) $E_1\left(\frac{C_h\theta_1+\eta_1+\mu}{\beta_1}, \frac{-C_h\mu\theta_1+B\beta_1-\eta_1\mu-\mu^2}{\beta_1(C_h\theta_1+\eta_1+\mu)}, 0, 0, \frac{\eta_1(-C_h\mu\theta_1+B\beta_1-\eta_1\mu-\mu^2)}{\beta_1\mu(C_h\theta_1+\eta_1+\mu)}\right)$

(iii) $E_2\left(\frac{\delta_2+\eta_3+\mu}{\delta_1}, 0, 0, \frac{B\delta_1-\delta_2\mu-\eta_3\mu-\mu^2}{\delta_1(\eta_3+\mu)}, \frac{\eta_3(B\delta_1-\delta_2\mu-\eta_3\mu-\mu^2)}{\mu\delta_1(\eta_3+\mu)}\right)$

(iv) $E^*\left(P_L^*, B_n^*, D^*, R_C^*, P^*\right)$

Table 1 Notation, parameter description and parametric values

Notation	Parameter description	Parametric values
B	Growth rate of industrial plastic usage (in tons)	0.05
β_1	The rate of burning of plastic	0.35
β_2	The rate of dumping of plastic	0.50
δ_1	The rate of recycling of plastic	0.40
δ_2	The rate at which recycled industrial plastic is being used	0.2
ε	The rate at which dumped plastic is burned	0.5
γ	The rate at which dumped plastic is recycled	0.30
$\eta_1/\eta_2/\eta_3$	The rate at which pollution is created due to burning/dumping/recycling	0.40/0.35/0.30
μ	Wastage rate	0.30
θ_1	Depletion rate of burned industrial plastic used due to investment in health policy	0.5
θ_2	Growth rate of dumped recycled plastic used due to investment in recycling policy	0.3
C	Total budget	5 (in 000′ \$/ton)
C_h	Investment to be made for health policy	2 (in 000′ \$/ton)
C_m	Investment to be made for installing recycling machine	2.5 (in 000′ \$/ton)

where

$$P_L^* = \frac{\varepsilon + \eta_2 + \gamma + \mu - C_m\theta_2}{\mu}$$

$$
\begin{aligned}
B_n^* = &\ \varepsilon((-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)(-\delta_1\mu(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu) + \beta_2 B\delta_1) + (\delta_2 + \eta_3 + \mu)(-B\beta_2^2 \\
&+ \beta_2\mu(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)))/-\beta_2^2\delta_2(-C_m\theta_2 + \varepsilon + \eta_2 + \mu)(C_h\theta_1 + \eta_1 + \mu) + \beta_1\delta_1(-C_m\theta_2 \\
&+ \varepsilon + \eta_2 + \mu)^2(-C_m\theta_2 - \eta_2 - \mu) + (-C_m\theta_2 + \varepsilon + \eta_2 + \mu)((C_h\theta_1 + \eta_1 + \mu)(\beta_2\delta_1(-C_m\theta_2 + \varepsilon + \eta_2 \\
&+ \mu) - \beta_2^2(\eta_3 + \mu)) + \beta_1\beta_2\delta_2(\eta_3 + \mu) + \beta_1\beta_2\eta_3(\eta_2 + \gamma + \mu))
\end{aligned}
$$

$$
\begin{aligned}
D^* = &\ [\delta_1(-C_m\theta_2 + \varepsilon + \eta_2 + \mu) - \beta_2(\delta_2 + \eta_3 + \mu)][(C_h\theta_1 + \eta_1 + \mu)(\beta_2\mu(-C_m\theta_2 + \varepsilon + \eta_2 + \mu) \\
&- B\beta_2^2) + (-C_m\theta_2 + \varepsilon + \eta_2 + \mu)(-\beta_1\mu(-C_m\theta_2 + \varepsilon + \eta_2 + \mu) + B\beta_1\beta_2)]/\beta_2\{\beta_2^2\delta_2(C_h\theta_1 + \eta_1 \\
&+ \mu)(-C_m\theta_2 - \varepsilon - \mu) + (-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)[\beta_1\delta_1(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)(C_m\theta_2 - \eta_2) \\
&+ \beta_2\delta_1(-C_m\theta_2 + \varepsilon + \eta_2 + \mu)(\theta_1 + \eta_1 + \mu) - \beta_2^2(C_h\theta_1 + \eta_1 + \mu)(\eta_3 + \mu) + \beta_1\beta_2\delta_2(-C_m\theta_2 + \eta_2 \\
&+ \mu) + \beta_1\beta_2\eta_3(-C_m\theta_2 + \eta_2 + \gamma + \mu) + \beta_1\beta_2\mu(-C_m\theta_2 + \gamma + \mu) - \beta_1\delta_1\mu(-C_m\theta_2 + \varepsilon + \gamma + \mu) \\
&+ \beta_1\delta_1\mu(C_m\gamma\theta_2 - \eta_2(\varepsilon + \eta_2 + \gamma + \mu))]\}
\end{aligned}
$$

$$
\begin{aligned}
R_C^* = &\ \gamma(\beta_2(C_h\theta_1 + \eta_1 + \mu)(-B\beta_2 + \mu(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)) + \beta_1(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma \\
&+ \mu)(B\beta_2 - \mu(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu))/\beta_2(C_h\theta_1 + \eta_1 + \mu)(-C_m\delta_1\theta_2(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu) \\
&- \beta_2\delta_2(-C_m\theta_2 + \varepsilon + \eta_2 + \mu)) + \beta_1\delta_1\mu(C_m\varepsilon\theta_2 - \eta_2\gamma) + \beta_2\delta_1(\eta_2\mu(-C_m\theta_2 + \eta_1) + \eta_1(\mu(\gamma + \mu) \\
&+ \varepsilon\eta_2)) + (-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)(\beta_1\delta_1(-C_m\theta_2 + \varepsilon + \eta_2 + \gamma + \mu)(C_m\theta_2 - \mu) + \beta_2(C_h\theta_1 + \eta_1 \\
&+ \mu)(-\beta_2(\eta_3 + \mu) + \delta_1(\varepsilon + \eta_2 - \mu) - \beta_1\beta_2(C_m\theta_2 - \mu)(\delta_2 + \eta_3 + \mu) + \beta_1\beta_2(\eta_2(\delta_2 + \eta_3 + \mu) \\
&+ \gamma(\eta_3 + \mu)) - \beta_1\delta_1\eta_2(\varepsilon + \eta_2 + \gamma + \mu))
\end{aligned}
$$

$$P^* = \eta_2\mu\{C_h\theta_1\beta_2[\gamma(\beta_2\delta_2 - \gamma\delta_1) + \beta_2(-\mu(C_m\theta_2 + \delta_2 + \gamma) + \eta_3(1 + \eta_2 + 2\gamma) - \eta_2\delta_1(\eta_2 + 2\mu))$$
$$- \delta_1(\mu^2 + 2\gamma(\eta_2 + \mu))] + C_m\theta_2[C_m\theta_2(-\beta_1\beta_2(\delta_2 + \eta_3) + \beta_1(3\delta_1\gamma - \beta_2\eta_1)) + \beta_1\beta_2(2\delta_2(\eta_2 + \mu)$$
$$+ 2\eta_3(\eta_2 + \mu)) + \beta_1\gamma(2\mu - 3\delta_1(2\eta_2 - 1 - 2\mu) - \beta_2\eta_1(\delta_2 + \eta_3 - 2\mu(\delta_1 + \delta_2) - 2\eta_2\delta_1)] + \beta_2[2\varepsilon\gamma($$
$$- \beta_1(\delta_2 + 2\eta_3) - 2\delta_1\eta_1) + \gamma(\beta_1(-2\delta_2(\eta_2 + \mu) - \eta_3(3\eta_2 + 3\gamma + 4\gamma\mu)) + 2\eta_1(\beta_2\eta_3 - \delta_1(\eta_2 + \beta_2\mu))$$
$$+ \beta_2\mu^2(C_h\theta_1(\eta_2 + \mu) - \mu C_m\theta_2) - \beta_1((\delta_2 + \eta_3)(\eta_2 + \mu)^2 + \gamma\mu(2\eta_2 + \gamma + \mu)) - \eta_1(\beta_2(\delta_2 + \eta_3)(\eta_2$$
$$+ \mu) + \mu\gamma(1 + 2\eta_3)) + \beta_2\mu^2(\eta_2 + \mu) - C_m\theta_2(C_h\theta_1(2\delta_1\gamma - \beta_2(\delta_2 + \eta_3)) + 2\beta_1\gamma(\delta_2 + 2\eta_3)$$
$$+ 2\delta_1\eta_1\gamma)] - 3\beta_1\delta_1\gamma(3\eta_2(\eta_2 + \gamma) + \mu(2\eta_2 + \gamma + \mu))\} + C_m\theta_2\{\beta_1[\beta_2\eta_3\gamma\mu(2\mu - C_m\theta_2 + 2\gamma - \beta_2\eta_1)]$$
$$+ \mu[\varepsilon(\delta_1 C_m\theta_2(3\beta_1\eta_2 - \beta_2\eta_1) + \beta_1\eta_2(2\mu\beta_2 - 3\delta_1(\varepsilon + 2\eta_2 + 2\mu)) - \beta_2(\beta_2\eta_1(\delta_2 + \eta_3) - 2\delta_1(\eta_1(\varepsilon$$
$$+ \mu) + \mu\eta_2))) + \beta_2(-\mu\beta_2(\eta_3(\gamma + \eta_2) + \eta_2(\delta_2 + \eta_1)) + 2\eta_2\mu\delta_1(\eta_2 + \mu)) + C_m\theta_2\eta_2(-\beta_2\delta_1(C_m\theta_2$$
$$+ \mu) + \beta_1(-\beta_2\mu + 3\delta_1(\eta_2 + \mu))) + B\beta_2(\delta_1\eta_2(C_m\theta_2\beta_1 - C_h\theta_1\beta_2) + 2\beta_1\eta_2\delta_1(\varepsilon + \gamma + \mu))$$
$$- C_h\theta_1\beta_2^2\eta_3\gamma]\} + B\beta_2\{\beta_2[C_h\theta_1(\delta_1\eta_2(\varepsilon + \mu) + \beta_2(\eta_2(\delta_2 + \mu) + \eta_3(\eta_2 + \gamma))) + C_m\theta_2 - \beta_1(\eta_2(\delta_2$$
$$+ \eta_3 + \mu) - \eta_3\gamma + \delta_2\eta_2(\varepsilon + \gamma + \mu) + \varepsilon(\eta_2(\eta_2 + \mu) + \gamma\eta_3) + \eta_3\eta_2(\gamma + \mu) + \gamma\mu(\eta_3 + \eta_2)) + \delta_1(-\eta_1(\varepsilon$$
$$+ \eta_2) - \eta_2\mu - \eta_2^2 + \varepsilon\eta_1(2\eta_2 + \gamma + \mu) + \beta_2\eta_1(\varepsilon^2 + \eta_2^2) + \beta_2\eta_2\mu(\eta_2 + \mu) + \delta_1\mu\eta_2(\mu(\varepsilon + \eta_1) + \gamma(\eta_2$$
$$+ \mu)) - \beta_2^2(\delta_2(\eta_1(\varepsilon + \eta_2) + \eta_2\mu) - \eta_1(\eta_3 + \mu)(\varepsilon + \eta_2))] + \beta_1[\delta_1\eta_2(2(C_m\eta_2\theta_2 - \varepsilon(\gamma + \mu) - \gamma\mu)(\varepsilon$$
$$+ \mu^2 + \gamma^2 + 2\eta_2(\eta_2 + \gamma + \mu))) + \eta_2\beta_2(\delta_2 + \eta_3 + \mu(\eta_2 + \mu)) + \beta_2\eta_3\gamma^2 - \delta_1\eta_2^3)]\} + \gamma\mu[\gamma\beta_2(-\beta_1\eta_3(\gamma$$
$$+ 2\mu) + \mu\eta_2(\beta_2\delta_2 - \delta_1) + \beta_2\eta_1\eta_3) + \gamma^2\beta_1\delta_1\eta_2 + \beta_2\mu\eta_3(\mu\beta_1 + \beta_2\eta_1)] + \beta_2\mu\{\beta_2\mu[2\eta_3\gamma(\gamma + \mu)$$
$$- 2B\beta_2\eta_2 + 2\varepsilon(\varepsilon\eta_1 + \mu\eta_2) - \beta_1\eta_2((\eta_2 + \mu)^2 + \eta_2^3 + \mu^3 + 3\eta_2\mu(\eta_2 + \mu)) + \beta_2\eta_2\mu^2\delta_2(\eta_2 + 1) + \eta_1\eta_2$$
$$+ \eta_3\eta_2(\eta_2 + \mu)] + \beta_2\delta_1[-\mu\eta_2(\mu^2 + \eta_2^2 + \mu\eta_2) + \gamma^2\eta_1(\varepsilon - \eta_2)]\} + \varepsilon\mu\{C_h\theta_1[\beta_2\eta_2(\beta_2(\delta_2 + \eta_3)$$
$$- \delta_1(\varepsilon + 2\eta_2 + 2\mu)) + \beta_2^2\eta_3\gamma + 2\beta_2\delta_1\eta_2(C_m\theta_2 - \gamma)] + C_m\theta_2[\beta_1\beta_2(\eta_2(\delta_2 + \eta_3) + \gamma\eta_3) + 2\delta_1(\eta_2($$
$$- 3\beta_1\gamma + 2\beta_2\eta_1) + \beta_2\eta_1\gamma)] - \delta_1[\eta_1(\varepsilon^2 + \mu^2) + \varepsilon\mu(2\eta_1 + \eta_2) + 2\mu\eta_2(\eta_2 + \mu)] + \beta_1[\beta_2(\eta_2(-(\varepsilon + 2\eta_2$$
$$+ 2\mu)(\delta_2 + \eta_3) - 2\gamma\mu) - \varepsilon\eta_3\gamma - 2\gamma\eta_3(\gamma + \mu)) + 3\delta_1\eta_2\gamma(\varepsilon + 2\eta_2 + \gamma + 2\mu) + 3\delta_1\eta_2(\mu(\varepsilon + \eta_2) + \mu^2$$
$$+ \eta_2^2) - \beta_2\eta_2\mu(\varepsilon + \eta_2 + 2\mu) + \varepsilon\delta_1\eta_2(\varepsilon + \eta_2)] + \beta_2[-\delta_1(2\eta_1\gamma(\varepsilon + \mu) + \eta_1\eta_2(3\eta_2 + 4\mu) - 2\mu\eta_2\gamma)$$
$$+ \beta_2\eta_1(2\eta_2(\delta_2 + \eta_3) + 4\gamma\eta_3 + \eta_3(\varepsilon + \mu) + \mu(2\eta_2 + \gamma)) + \beta_2\delta_2(\eta_1(\varepsilon + \mu) + \eta_2\mu) + \eta_3(\eta_2 + \gamma)]\}$$
$$/\beta_2\mu\{\beta_2^2\delta_2(C_m\theta_2 - \varepsilon - \eta_2 - \gamma - \mu)(C_h\theta_1 + \eta_1 + \mu) + (C_m\theta_2 - \varepsilon - \eta_2 - \gamma - \mu)[\beta_2\delta_1(C_m\theta_2 - \varepsilon - \eta_2$$
$$- \mu)(C_h\theta_1 + \eta_1 + \mu) + \beta_2^2(C_h\theta_1 + \eta_1 + \mu)(\eta_3 + \mu) + \beta_1\delta_1(C_m\theta_2 - \varepsilon - \eta_2 - \gamma - \mu)(C_m\theta_2 - \eta_2$$
$$- \mu) + \beta_1\beta_2((C_m\theta_2 - \eta_2 - \mu)(\delta_2 + \eta_3 + \mu) + \gamma(\eta_3 + \mu))]\}$$

First is pollution-free equilibrium point. Second point is recycling free, third is dumping free and fourth is optimum issue point.

Stability

To scrutinize the stability analysis of model for equilibrium points E_0, E_1, E_2 and E^* is very useful. Stepwise stability for each equilibrium point is carried out as follows locally and globally.

Local stability

Jacobian matrix using system (1) is stated which will be used to derive local stability for the model.

$$J = \begin{bmatrix} J_{11} & -\beta_1 P_L & -\beta_2 P_L & -\delta_1 P_L + \delta_2 & 0 \\ \beta_1 B_n & J_{22} & \varepsilon & 0 & 0 \\ \beta_2 D & 0 & J_{33} & 0 & 0 \\ R_C\delta_1 & 0 & \gamma & J_{44} & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & -\mu \end{bmatrix}$$

where $J_{11} = -\beta_1 B_n - \beta_2 D - R_C\delta_1 - \mu$, $J_{22} = -C_h\theta_1 + \beta_1 P_L - \eta_1 - \mu$, $J_{33} = C_m\theta_2 + \beta_2 P_L - \varepsilon - \eta_2 - \gamma - \mu$ and $J_{44} = -\delta_1 P_L - \delta_2 - \eta_3 - \mu$.

Theorem 3.1.1 *The pollution-free equilibrium point E_0 of model is locally asymptotically stable with conditions.*

Proof Jacobian matrix at E_0 is as follows:

$$J_0 = \begin{bmatrix} -\mu & \frac{-\beta_1 B}{\mu} & \frac{-\beta_2 B}{\mu} & \frac{-\delta_1 B}{\mu} + \delta_2 & 0 \\ 0 & -C_h\theta_1 + \frac{\beta_1 B}{\mu} - \eta_1 - \mu & \varepsilon & 0 & 0 \\ 0 & 0 & C_m\theta_2 + \frac{\beta_2 B}{\mu} - \varepsilon - \eta_2 - \gamma - \mu & 0 & 0 \\ 0 & 0 & \gamma & \frac{\delta_1 B}{\mu} - \delta_2 - \eta_3 - \mu & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & -\mu \end{bmatrix}$$

has eigenvalues,

$$\omega_1 = -\mu, \quad \omega_2 = -\mu, \quad \omega_3 = \frac{\delta_1 B}{\mu} - (\delta_2 + \eta_3 + \mu),$$
$$\omega_4 = -C_h\theta_1 + \frac{\beta_1 B}{\mu} - \eta_1 - \mu \quad \text{and}$$
$$\omega_5 = C_m\theta_2 + \frac{\beta_2 B}{\mu} - \varepsilon - \eta_2 - \gamma - \mu.$$

Equilibrium point is stable if all eigenvalues related to Jacobian matrix are negative. Here, only first two eigenvalues are negative. Therefore, the conditions for the equilibrium point $E_0\left(\frac{B}{\mu}, 0, 0, 0, 0\right)$ to be asymptotically stable is,

(i)   (i) $\delta_2 + \eta_3 + \mu > \frac{\delta_1 B}{\mu}$,
(ii)   (ii) $C_h\theta_1 + \eta_1 + \mu > \frac{\beta_1 B}{\mu}$ and
(iii) (iii) $\varepsilon + \eta_2 + \gamma + \mu > C_m\theta_2 + \frac{\beta_2 B}{\mu}$.

$\square$

Theorem 3.1.2 *The recycling free equilibrium point E_1 of model is locally asymptotically stable with conditions.*

Proof Let $x_1 = \frac{C_h\theta_1 + \eta_1 + \mu}{\beta_1}$ and the Jacobian matrix at E_1 is,

$$J_1 = \begin{bmatrix} J_{111} & -\beta_1 x_1 & -\beta_2 x_1 & -\delta_1 x_1 + \delta_2 & 0 \\ \beta_1\left(\frac{-\mu}{\beta_1} + \frac{B}{\beta_1 x_1}\right) & J_{122} & \varepsilon & 0 & 0 \\ 0 & 0 & J_{133} & 0 & 0 \\ 0 & 0 & \gamma & J_{144} & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & -\mu \end{bmatrix}$$

where $J_{111} = -\beta_1\left(\frac{-\mu}{\beta_1} + \frac{B}{\beta_1 x_1}\right) - \mu$, $J_{122} = -C_h\theta_1 + \beta_1 x_1 - \eta_1 - \mu$,
$J_{133} = C_m\theta_2 + \beta_2 x_1 - \varepsilon - \eta_2 - \gamma - \mu$ and $J_{144} = \delta_1 x_1 - \delta_2 - \eta_3 - \mu$ has eigenvalues,

$$\omega_1 = -\mu, \omega_2 = C_m\theta_2 + \beta_2 x_1 - \varepsilon - \eta_2 - \gamma - \mu, \omega_3 = \delta_1 x_1 - \delta_2 - \eta_3 - \mu \text{ and}$$

$$\omega_4 = -\frac{1}{2x_1}\left(\begin{array}{l}(\theta_1 C_h x_1 - x_1^2\beta_1 + \eta_1 x_1 + \mu x_1 + B) \pm (C_h^2\theta_1^2 x_1^2 \\ -2\theta_1 C_h x_1^3\beta_1 + x_1^4\beta_1^2 + 2\theta_1 C_h \eta_1 x_1^2 + 2\theta_1 C_h \mu x_1^2 \\ -2x_1^3\beta_1\eta_1 + 2x_1^3\beta_1\mu - 2B\theta_1 C_h - 2Bx_1^2\beta_1 \\ +\eta_1^2 x_1^2 + 2\eta_1\mu x_1^2 + \mu^2 x_1^2 - 2B\eta_1 x_1 - 2B\mu x_1 + B^2)^{\frac{1}{2}}\end{array}\right).$$

Recycling free equilibrium point E_1 is asymptotically stable if following conditions hold:

(i) $\delta_2 + \eta_3 + \mu > \delta_1 x_1$,
(ii) $\varepsilon + \eta_2 + \gamma + \mu > C_m\theta_2 + \beta_2 x_1$, and
(iii) $B\left(C_h\theta_1 + \eta_1 + x_1\right) > \beta_1\mu x_1^2$.

$\square$

Theorem 3.1.3 *The dumping free equilibrium point E_2 of model is locally asymptotically stable with conditions.*

Proof Let $x_2 = \frac{\delta_2 + \eta_3 + \mu}{\delta_1}$ and $x_3 = \frac{B - \mu x_2}{\eta_3 + \mu}$ and the Jacobian matrix at E_2 is

$$J_2 = \begin{bmatrix} -\delta_1 x_3 - \mu & -\beta_1 x_2 & -\beta_2 x_2 & -\delta_1 x_2 + \delta_2 & 0 \\ 0 & J_{222} & \varepsilon & 0 & 0 \\ 0 & 0 & J_{233} & 0 & 0 \\ x_3\delta_1 & 0 & \gamma & \delta_1 x_2 - \delta_2 - \eta_3 - \mu & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & -\mu \end{bmatrix}$$

where $J_{222} = -C_h\theta_1 + \beta_1 x_2 - \eta_1 - \mu$, $J_{233} = C_m\theta_2 + \beta_2 x_2 - \varepsilon - \eta_2 - \gamma - \mu$ has eigenvalues, $\omega_1 = -\mu, \omega_2 = -C_h\theta_1 + \beta_1 x_2 - \eta_1 - \mu$, $\omega_3 = C_m\theta_2 + \beta_2 x_2 - \varepsilon - \eta_2 - \gamma - \mu$ and $\omega_4 = -\frac{1}{2}$

$$\left(\begin{array}{l}(-\delta_1 x_2 + x_3\delta_1 + \delta_2 + \eta_3 + 2\mu) \pm (\delta_1^2 x_2^2 - \delta_1^2 x_2 x_3 + \delta_1^2 x_3^2 - 2\delta_1\delta_2 x_3 + \eta_3^2 \\ +2\delta_1\delta_2 x_3 - 2\delta_1\eta_3 x_2 + \delta_2^2 + 2\delta_2\eta_3)\end{array}\right)^{\frac{1}{2}}.$$

Dumping free equilibrium point E_2 is asymptotically stable if following conditions hold:

(i) $C_h\theta_1 + \eta_1 + \mu > \beta_1\left(\frac{\delta_2 + \eta_3 + \mu}{\delta_1}\right)$,
(ii) $\varepsilon + \eta_2 + \gamma + \mu > C_m\theta_2 + \beta_2\left(\frac{\delta_2 + \eta_3 + \mu}{\delta_1}\right)$ and
(iii) $\left(\eta_3 + \mu\right)(x_3\delta_1 + \mu) + \mu\delta_2 > \mu\left(\delta_2 + \eta_3 + \mu\right)$.

$\square$

Theorem 3.1.4 *The endemic equilibrium point $E^*\left(P_L^*, B_n^*, D^*, R_C^*, P^*\right)$ of model is locally asymptotically stable with conditions.*

Proof Let, $x_{11} = \beta_1 B_n^* + \beta_2 D^* + R_C^*\delta_1 + \mu$, $x_{22} = C_h\theta_1 - \beta_1 P_L^* + \eta_1 + \mu$,
$x_{33} = \varepsilon + \eta_2 + \gamma + \mu - C_m\theta_2 - \beta_2 P_L^*$ and $x_{44} = \delta_2 + \eta_3 + \mu - \delta_1 P_L^*$.
Jacobian matrix at E^* is,

$$J^* = \begin{bmatrix} -x_{11} & -\beta_1 P_L^* & -\beta_2 P_L^* & -\delta_1 P_L^* + \delta_2 & 0 \\ \beta_1 B_n^* & -x_{22} & \varepsilon & 0 & 0 \\ \beta_2 D^* & 0 & -x_{33} & 0 & 0 \\ R_C^*\delta_1 & 0 & \gamma & -x_{44} & 0 \\ 0 & \eta_1 & \eta_2 & \eta_3 & -\mu \end{bmatrix}$$ and has char-

acteristic equation, $\lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0$, where

$$a_1 = \mu + x_{44} + x_{33} + x_{22} + x_{11}$$

$$\begin{aligned}a_2 =& R_C^*\delta_1\left(P_L^*\delta_1 - \delta_2\right) + B_n^*P_L^*\beta_1^2 + D^*P_L^*\beta_2^2 \\ &+ \mu\left(x_{11} + x_{22} + x_{33} + x_{44}\right) + \left(x_{11} + x_{44}\right) \\ &\left(x_{22} + x_{33}\right) + x_{11}x_{44} + x_{22}x_{33}\end{aligned}$$

$$\begin{aligned}a_3 =& \left(P_L^*\delta_1 - \delta_2\right)\left(D^*\beta_2\gamma + R_C^*\delta_1\left(x_{22} + x_{33} + \mu\right)\right) \\ &+ \left(x_{11}x_{22} + B_n^*P_L^*\beta_1^2\right)\left(x_{33} + x_{44}\right) + x_{33}x_{44}(x_{11} + x_{22}) \\ &+ \mu\left(x_{11}x_{44} + x_{22}x_{33} + \left(x_{22} + x_{33}\right)\left(x_{11} + x_{44}\right)\right) \\ &+ D^*P_L^*\beta_2^2(x_{22} + x_{44} + \mu) + P_L^*\beta_1(D^*\beta_2\varepsilon + B_n^*\beta_1\mu)\end{aligned}$$

$$\begin{aligned}a_4 =& \mu\left(x_{33} + x_{44}\right)\left(B_n^*P_L^*\beta_1^2 + x_{11}x_{22}\right) + \mu D^*P_L^*\beta_1\beta_2\varepsilon \\ &+ \mu D^*\beta_2^2 P_L^*\left(x_{22} + x_{44}\right) + \left(P_L^*\delta_1 - \delta_2\right) \\ &\left[D^*\beta_2\gamma\left(\mu + x_{22}\right) + R_C^*\delta_1\left(\mu\left(x_{22} + x_{33}\right) + x_{22}x_{33}\right)\right] \\ &+ x_{33}x_{44}(\mu\left(x_{11} + x_{22}\right) + B_n^*P_L^*\beta_1^2 + x_{11}x_{22}) \\ &+ D^*P_L^*\beta_2 x_{44}\left(\beta_1\varepsilon + \beta_2 x_{22}\right)\end{aligned}$$

$$\begin{aligned}a_5 =& \mu(x_{22}(P_L^*\delta_1 - \delta_2)(D^*\beta_2\gamma + R_C^*\delta_1 x_{33}) \\ &+ x_{33}x_{44}(B_n^*P_L^*\beta_1^2 + x_{11}x_{22}) + D^*P_L^*\beta_2 x_{44}(\beta_1\varepsilon + \beta_2))\end{aligned}$$

A criterion satisfies if all coefficients a_i's are positive. This gives condition $P_L^*\delta_1 > \delta_2$. □

Global stability

A graph is basically a pair of sets $G = (V, E)$, where V is the set of vertices and E is the set of edges, formed by pairs (i, j) of vertices. (i, j) stands for the initial vertex and the terminal vertex, respectively. If these vertices are same that is $i = j$ known as a self-loop, it connects a vertex to itself. A directed graph is made up of a set of vertices which are connected by edges, where these edges have a direction associated with their vertices. A directed graph has path in which the edges are in the same direction is known as directed path. Out-degree ($d^+(i)$) of refers to the number of edges incident from initial vertex i and the number of edges away from the vertex i, is denoted by in-degree ($d^-(i)$). A weighted graph directed graph (G, A) where $A = (a_{ij})_{n\times n}$, $a_{ij} > 0$ is weight of each edge with numerical positive weight if it exists otherwise zero. Product of weight on all edges gives the weight $w(K)$ of sub-graph. The root vertex has degree one is called root tree. The sub-graph contains all vertices with minimum number of edges are called a spanning tree. The Laplacian matrix $L = (l_{ij})$ of G is defined as:

$$l_{ij} = \begin{cases} -a_{ij}, & i \neq j, \\ \sum_{i \neq k} a_{ik}, & i = k. \end{cases} \quad (2)$$

Theorem 3.2.1 (Kirchhoff's matrix tree theorem) *For $n \geq 2$, assume that c_i is the cofactor of l_{ii} in L. Then $c_i = \sum_{T \in T_i} \mathcal{W}(T)$, $i = 1, 2, ..., n$, where T_i is the set of all spanning tree T of graph (G, A) that are rooted at vertex i. Moreover, if (G, A) is strongly connected, then $c_i > 0$ for $1 \leq i \leq n$.*

Lemma 3.2.1 *Let c_i be as given in Kirchhoff's matrix tree theorem. If $a_{ij} > 0$ and $d^+(j) = 1$ for some $1 \leq i, j \leq n$, then $c_i a_{ij} = \sum_{k=1}^{n} c_j a_{jk}$.*

Lemma 3.2.2 *Let c_i be as given in the Kirchhoff matrix tree theorem. If $a_{ij} > 0$ and $d^-(i) = 1$ for some $1 \leq i, j \leq n$, then $c_i a_{ij} = \sum_{k=1}^{n} c_k a_{ki}$.*

Theorem 3.2.2 *Suppose that the following assumptions are satisfied*:

1. There exist functions $V_i : U \to \mathbb{R}$, $G_{ij} : U \to \mathbb{R}$ and constants $a_{ij} > 0$ such that for every $1 \leq i \leq n$, $V_i' \leq \sum_{j=1}^{n} G_{ij}(x)$ for $x \in U$.
2. For $M = [a_{ij}]$, each directed cycle C of (G, M) has $\sum_{(s, r) \in \varepsilon(C)} G_{rs}(x) \leq 0$ for $x \in U$, where $\varepsilon(C)$ denotes the arc set of the directed cycle C.

Then, the function $V(x) = \sum_{i=1}^{n} c_i V_i(x)$, with constants $c_i \geq 0$ as given in the theorem 3.2.1,

satisfies $V' \leq 0$, hence, V is a Lyapunov function for the system.

Theorem 3.2.3 *The endemic equilibrium point $E^*\left(P_L^*, B_n^*, R_C^*, D^*, P^*\right)$ is globally asymptotically stable.*

Proof Using graph theoretical results for global stability, we generate Lyapunov function $V(t)$ and each V_i is defined as,

$$V_1 = P_L - P_L^* - P_L^* ln\frac{P_L}{P_L^*}, \qquad V_2 = B_n - B_n^* - B_n^* ln\frac{B_n}{B_n^*},$$

$$V_3 = D - D^* - D^* ln\frac{D}{D^*}, V_4 = R_C - R_C^* - R_C^* ln\frac{R_C}{R_C^*} \quad \text{and}$$

$$V_5 = P - P^* - P^* ln\frac{P}{P^*}.$$

Now, each V_i, for $1 \leq i \leq 5$ differentiating with respect to t we have,

$$\begin{aligned}V_1' =& \left(1 - \frac{P_L^*}{P_L}\right)P_L' = \left(1 - \frac{P_L^*}{P_L}\right)(B - \beta_1 P_L B_n \\ &- \beta_2 P_L D - \delta_1 P_L R_C + \delta_2 R_C - \mu P_L) \\ =& \beta_1 P_L^* B_n^*\left(1 - \frac{P_L^*}{P_L}\right)\left(1 - \frac{P_L B_n}{P_L^* B_n^*}\right) + \beta_2 P_L^* D^*\left(1 - \frac{P_L^*}{P_L}\right)\left(1 - \frac{P_L D}{P_L^* D^*}\right) \\ &+ \delta_1 P_L^* R_C^*\left(1 - \frac{P_L^*}{P_L}\right)\left(1 - \frac{P_L R_C}{P_L^* R_C^*}\right) - \delta_2 R_C^*\left(1 - \frac{P_L^*}{P_L}\right)\left(1 - \frac{R_C}{R_C^*}\right) \\ &+ \mu P_L^*\left(1 - \frac{P_L^*}{P_L}\right)\left(1 - \frac{P_L}{P_L^*}\right) \leq a_{12}G_{12} + a_{13}G_{13} + a_{14}G_{14}\end{aligned}$$

$$V_2' = \left(1 - \frac{B_n^*}{B_n}\right)\left(\beta_1 P_L B_n + \varepsilon D - \eta_1 B_n - \mu B_n - \theta_1 C_h B_n\right)$$

$$\leq \beta P_L^* B_n^*\left(1 - \frac{B_n^*}{B_n}\right)\left(1 - \frac{P_L B_n}{P_L^* B_n^*}\right) + \varepsilon D^*\left(1 - \frac{B_n^*}{B_n}\right)\left(1 - \frac{D}{D^*}\right)$$

$$= a_{21} G_{21} + a_{23} G_{23}$$

$$V_3' = \left(1 - \frac{D^*}{D}\right)\left(\beta_2 P_l D - \varepsilon D - \eta_2 D - \gamma D - \mu D + \theta_2 C_m D\right)$$

$$\leq \beta_2 P_L^* D^*\left(1 - \frac{D^*}{D}\right)\left(1 - \frac{P_L D}{P_L^* D^*}\right) + \theta_2 C_m D^*\left(1 - \frac{D^*}{D}\right)\left(1 - \frac{D}{D^*}\right)$$

$$= a_{31} G_{31} + a_{33} G_{33}$$

$$V_4' = \left(1 - \frac{R_C^*}{R_C}\right)\left(\delta_1 P_L R_C - \delta_2 R_C + \gamma R_C - \eta_3 R_C - \mu R_C\right)$$

$$\leq \delta_1 P_L^* R_C^*\left(1 - \frac{R_C^*}{R_C}\right)\left(1 - \frac{P_L R_C}{P_L^* R_C^*}\right) + \gamma D^*\left(1 - \frac{R_C^*}{R_C}\right)\left(1 - \frac{D}{D^*}\right)$$

$$= a_{41} G_{41} + a_{43} G_{43}$$

and

$$V_5' = \left(1 - \frac{P^*}{P}\right)\left(\eta_1 B_n + \eta_2 D + \eta_3 R_C - \mu P\right)$$

$$\leq \eta_1 B_n^*\left(1 - \frac{P^*}{P}\right)\left(1 - \frac{B_n}{B_n^*}\right)$$

$$+ \eta_2 D^*\left(1 - \frac{P^*}{P}\right)\left(1 - \frac{D}{D^*}\right) + \eta_3 R_C^*\left(1 - \frac{P^*}{P}\right)\left(1 - \frac{R_C}{R_C^*}\right)$$

$$= a_{52} G_{52} + a_{53} G_{53} + a_{54} G_{54}$$

where $\quad a_{12} = a_{21} = \beta_1 P_L^* B_n^*$, $\quad a_{13} = a_{31} = \beta_2 P_L^* D^*$, $a_{14} = a_{41} = \delta_1 P_L^* R_C^*$, $a_{23} = \varepsilon D^*$, $a_{33} = \theta_2 C_m D^*$, $a_{43} = \gamma D^*$, $a_{52} = \eta_1 B_n^*$, $a_{53} = \eta_2 D^*$ and $a_{54} = \eta_3 R_C^*$. $\qquad \square$

This above calculation leads to a weighted graph containing each compartment as below:

This weighted graph (G, A) includes four vertices presenting the three cycles $G_{12} + G_{21} = 0$, $G_{13} + G_{31} = 0$ and $G_{14} + G_{41} = 0$ (Fig. 2). Theorem 3.2.2 gives the constants c_i's, for $1 \leq i \leq 5$ to construct a Lyapunov function $V(x) = \sum_{i=1}^{n} c_i V_i(x)$. Lemma 3.2.2 helps to relate the constants c_i's which turns out to,

$$d^-(3) = 1 \quad \Rightarrow c_3 a_{31} = c_1 a_{13} + c_2 a_{23} + c_4 a_{43} + c_5 a_{53}$$

Let us take $c_1 = k_1, c_2 = k_2, c_4 = k_3, c_5 = k_4$

$$\Rightarrow c_3 = \frac{k_1 \beta_2 P_L^* + k_2 \varepsilon + k_3 \gamma + k_4 \eta_2}{\beta_2 P_L^*}$$

Since, $V(t) = \sum_{i=1}^{5} c_i V_i(t) = c_1 V_1 + c_2 V_2 + c_3 V_3 + c_4 V_4 + c_5 V_5$

$$= k_1 V_1 + k_2 V_2 + \left(\frac{k_1 \beta_2 P_L^* + k_2 \varepsilon + k_3 \gamma + k_4 \eta_2}{\beta_2 P_L^*}\right) V_3 + k_3 V_4 + k_4 V_5$$
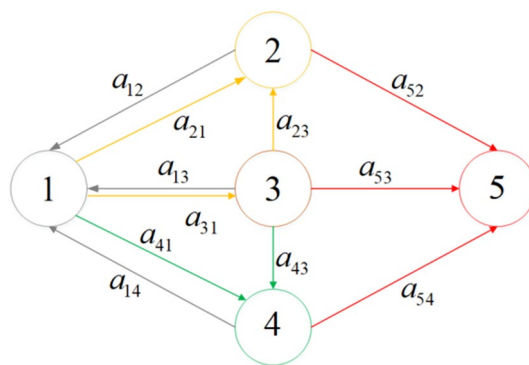

Fig. 2 The weighted graph

where k_i's for $1 \leq i \leq 4$ are arbitrary constants. $\{E^*\}$.

Here, $V' \leq 0$ at equilibrium point *i.e.*, $P_L = P_L^*, B_n = B_n^*$, $R_C = R_C^*, D = D^*$ and $P = P^*$ which suggest that $\{E^*\}$ is the only invariant set in int (Λ) indicates that E^* is globally asymptotically stable (Harary 1969; West 1996).

Optimal investment in health and recycled machine policy

Optimal control theory deals with state equations and control variables for finding control by optimizing objective function. The purpose of this paper is to optimizing pollution. Therefore, the objective function is,

$$J(c_i, \Lambda) = \int_0^T (A_1 P_L^2 + A_2 B_n^2 + A_3 D^2$$

$$+ A_4 R_C^2 + A_5 P^2 + w_1 C_h^2 + w_2 C_m^2)\, dt \qquad (3)$$

where Λ denotes set of all compartmental variables, A_1, A_2, A_3, A_4, A_5 denote non-negative weight constants for compartments P_L, B_n, D, R_C, P, respectively. w_1 and w_2 are the weight constants for investments on health policy (C_h) and installing recycled machine policy (C_m), respectively.

As the weight parameter w_1 and w_2 are constants of the cost applied on the density of burned and recycled plastic for both the policies, respectively, from which the optimal investment condition is normalized. C_h and C_m are the investments for optimizing the burning and recycling, respectively. Now, we will calculate both the values of investment policy from $t = 0$ to $t = T$ such that,

$$J(C_h(t), C_m(t)) = min\{J(C_h^*, \Lambda), J(C_m^*, \Lambda)/(C_h, C_m) \in \phi\}$$

where ϕ is a smooth function on the interval [0, 1]. The optimal investment policy denoted by C_h and C_m are found by accumulating all the integrands of Eq. (3) using the lower

bounds and upper bounds, respectively, with the results of Fleming and Rishel (2012).

For minimizing the investment function in (3), using the Pontrygin's principle from Pontryagin et al. (1986) by constructing Lagrangian function consisting of state equations and adjoint variables $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ as follows:

$$L(\Lambda, A_i) = A_1 P_L^2 + A_2 B_n^2 + A_3 D^2 + A_4 R_C^2 + A_5 P^2 + w_1 C_h^2 + w_2 C_m^2$$
$$+ \lambda_1 (B - \beta_1 P_L B_n - \beta_2 P_L D - \delta_1 P_L R_C + \delta_2 R_C - \mu P_L)$$
$$+ \lambda_2 (\beta_1 P_L B_n + \varepsilon D - \eta_1 B_n - \mu B_n - \theta_1 C_h B_n)$$
$$+ \lambda_3 (\beta_2 P_L D - \varepsilon D - \eta_2 D - \gamma D - \mu D + \theta_2 C_m D)$$
$$+ \lambda_4 (\delta_1 P_L R_C - \delta_2 R_C + \gamma D - \eta_3 R_C - \mu R_C)$$
$$+ \lambda_5 (\eta_1 B_n + \eta_2 D + \eta_3 R_C - \mu P)$$

The partial derivatives of the Lagrangian function with respect to each variable of the compartment gives the adjoint equation variables $\lambda_i = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ corresponding to the system which are:

$$\dot{\lambda}_1 = -\frac{\partial L}{\partial P_L} = -2A_1 P_L + (\lambda_1 - \lambda_2)\beta_1 B_n + (\lambda_1 - \lambda_3)\beta_2 D$$
$$+ (\lambda_1 - \lambda_4)\delta_1 R_C + \lambda_1 \mu$$

$$\dot{\lambda}_2 = -\frac{\partial L}{\partial B_n} = -2A_2 B_n + (\lambda_1 - \lambda_2)\beta_1 P_L + \eta_1(\lambda_2 - \lambda_5)$$
$$+ \lambda_2 \mu + \lambda_2 \theta_1 C_h$$

$$\dot{\lambda}_3 = -\frac{\partial L}{\partial D} = -2A_3 D + (\lambda_1 - \lambda_3)\beta_2 P_L + (\lambda_3 - \lambda_2)\varepsilon$$
$$+ (\lambda_3 - \lambda_5)\eta_2 + (\lambda_3 - \lambda_4)\gamma + \lambda_3 \mu - \lambda_3 \theta_2 C_m$$

$$\dot{\lambda}_4 = -\frac{\partial L}{\partial R_C} = -2A_4 R_C + (\lambda_1 - \lambda_4)(\delta_1 P_L - \delta_2) + (\lambda_4 - \lambda_5)\eta_3 + \mu \lambda_4$$

$$\dot{\lambda}_5 = -\frac{\partial L}{\partial P} = -2A_5 P + \lambda_5 \mu$$

The necessary conditions for Lagrangian function L to be optimal are,

$\dot{C}_h = -\frac{\partial L}{\partial C_h} = -2w_1 C_h + \lambda_2 \theta_1 B_n = 0$ and $\dot{C}_m = -\frac{\partial L}{\partial C_m} = -2w_2 C_m - \lambda_3 \theta_2 D = 0$.

Hence, we get $C_h = \frac{\lambda_2 \theta_1 B_n}{2w_1}$ and $C_m = \frac{-\lambda_3 \theta_2 D}{2w_2}$.

The whole computation gives the optimal investment conditions resulting as,

$C_h^* = max\left(a_1, min\left(b_1, \frac{\lambda_2 \theta_1 B_n}{2w_1}\right)\right)$ and $C_m^* = max\left(a_2, min\left(b_2, \frac{-\lambda_3 \theta_2 D}{2w_2}\right)\right)$.

In this manner, analytical outcomes have been determined for optimized investment. Next section contains the numerical interpretation.

Result and discussion

Numerical simulation

This section illustrates the simulation results which are used to verify the efficacy of this existing model by using the parametric values given in Table 1.

Figure 3 consists of five activities, plastic; it's burning, dumping, recycling and causes the pollution, which can be depicted from a graph. After approximately 24 days, industrial plastic is sent for dumping then after one month it is recycled; during that period, it is also burned and this results pollution in two months.

Figure 4 gives the idea that how total budget is functioning. Initially, for almost two years, it is increasing and then gradually decreases. For simulating the model, here we have considered the total budget as 1 (in 000′ \$/ton). Figure 5 advocates that out of total budget, the investment in health policy (C_h) is 30% by decreasing the burning of plastic and
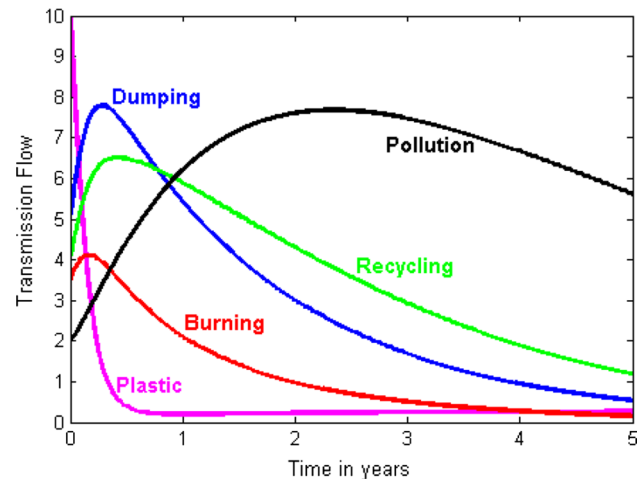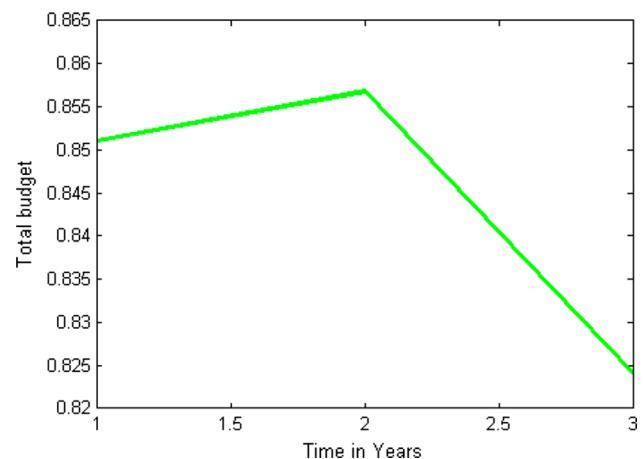

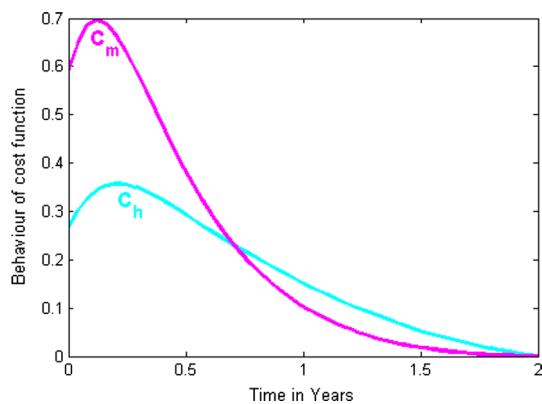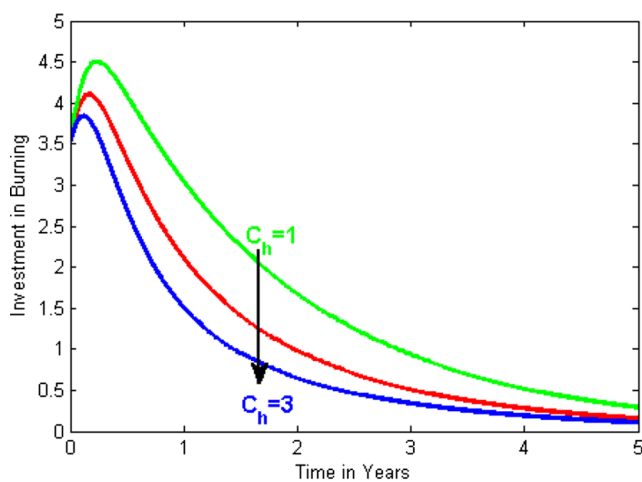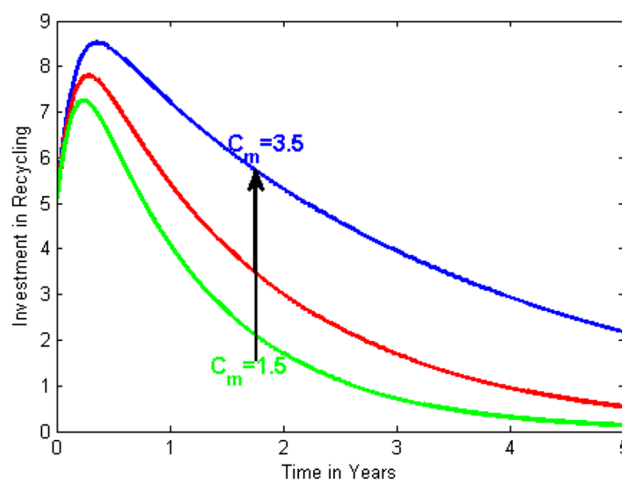**Fig. 3** Transmission of pollution

Fig. 4 Total budget

Fig. 5 Cost functions

another in installing recycling machine policy (C_m) is 60%. Both cost functions are increasing in same manner then with

passage of time it decreases in accordance with the total budget.

Figure 6a and b gives the effect of investment as cost functions on decreasing burning and installing recycling machine, respectively. Investments benefit that burning decreases by 14.88% which can be used for health policy for the society. In the same manner to promote plastic recycling, the cost is invested in installing recycling machine which results in recycling increases by 14.90%.

These trajectory graphs in Fig. 7 are plotted with growth rate value of $B = 60\%$. It recommends that the direction of both the process to pollution is toward to the solution of system (1) that is equilibrium point. As time increases or decreases for an identified period; trajectory remains approaches to critical point.

Figure 8 analyzes the global dynamical behavior of pollution due to dumped plastic. One can observe from this
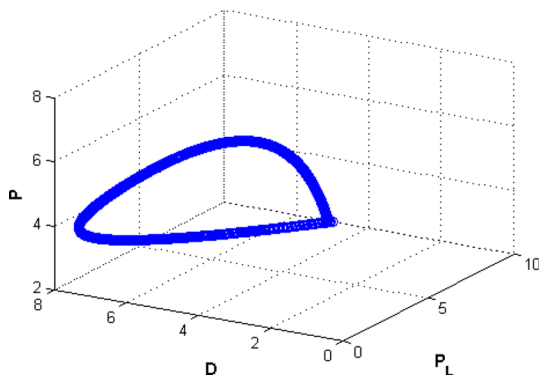

(a) Effect of C_h on burning

(b) Effect of C_m on recycling

Fig. 6 **a** Effect of C_h on burning **b** Effect of C_m on recycling
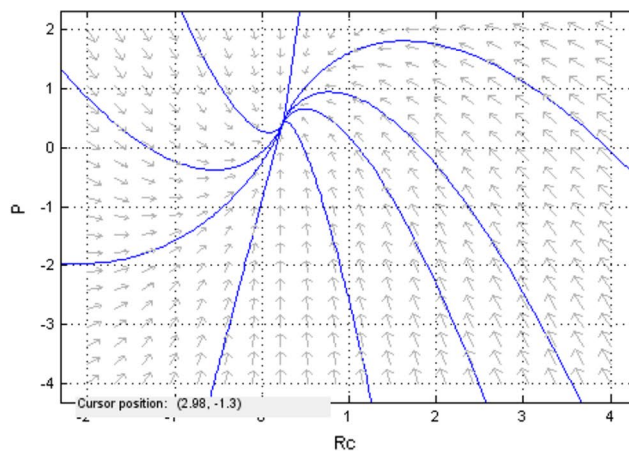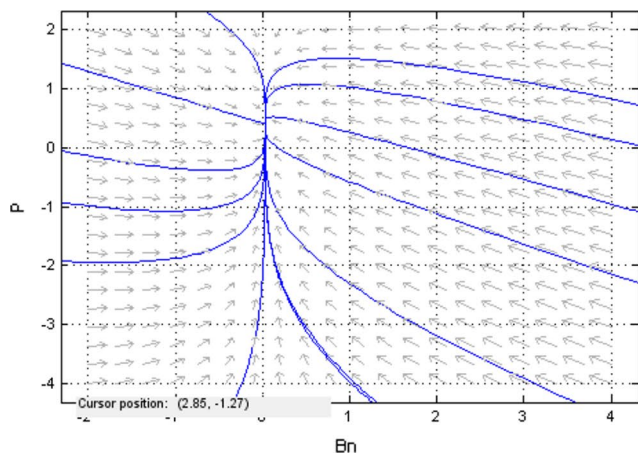
Fig. 7 Trajectory curve of burning and recycling to pollution

Fig. 8 Pollution due to dumping

scatter diagram that dumped plastic garbage from industries emits the pollution. It is periodic in nature.

Conclusion

Industrial dumped plastics are sent for either burning or recycling. With our increasing awareness, this paper is useful in how to invest in health policy by reducing the burning of industrial plastic and in installing recycling machine policy for dumped plastic. Local stability for nonlinear dynamic with its differential equations gives sufficient conditions. By using graph theory results, global stability implemented advocates that the model is globally asymptotically stable. This model has been extended by optimizing the cost function. Detailed computations along with its numerical simulation seek to show that the cost function is very effective for the model as it is concerned with the society by controlling pollution and invest in health policy. Supporting recycling is the one good approach to prevent dumped plastic pollution. The time has come and should know that only we people are the solution by either reducing or recycling the plastic usage by re-thinking our life style.

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