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Stochastic ordering among inactivity times of coherent systems

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Abstract The concept of "signature" is a useful tool to study the reliability properties of a coherent system. In this paper, we consider a coherent system consisting of n components and assume that the system is not working at time t. Mixture representations of the inactivity times (IT) of the system and IT of the components of the system are obtained under different scenarios on the signatures of the system. Some stochastic comparisons are made on IT of the coherent systems with same type and different type of components and some aging properties of the IT of the system and its components are investigated. It is proved, under some conditions on the vector of signatures of the system, that when the components of the system have decreasing reversed hazard rate, the mean of the IT (MIT) of the system and the MIT of the components of the system are increasing in time. Several examples and illustrative graphs are also provided.

Keywords k-out-of-n systems · Reversed hazard rate · Mean inactivity time · Order statistics · Reliability · Signature · Conditional lifetime

1 Introduction

Samaniego (1985) introduced a useful concept, called "signature" of a coherent system, which enables one to write the lifetime distribution function of a coherent system as a mixture of the distribution function of the ordered lifetimes of its components. Recall that a system of n components is said to be a coherent system if there is no irrelevant component in the system (a component is said to

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be irrelevant if its performance does not effect the performance of the system) and the system is nondecreasing in each vector argument. Consider a coherent system consists of n components with independent identically distributed (i.i.d.) lifetimes $X_1, X_2, ..., X_n$ which are distributed according to a common continuous distribution F. Let $T = T(X_1, X_2, ..., X_n)$ be the system's lifetime. Samaniego (1985) defined the signature of the system to be the probability vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ such that

$$s_i = P\{T = X_{i:n}\}, \quad i = 1, 2, ..., n,$$

where $X_{i:n}$ denotes the ith ordered lifetime of the components. It can be shown that $s_i = \frac{n_i}{n!}$, in which n_i is the number of ways that distinct $X_1, X_2, \ldots, X_n$ can be ordered such that $T(x_1, x_2, ..., x_n) = x_{i:n}$, $i = 1, ..., n$, where $x_{i:n}$ is the value of $X_{i:n}$. As the vector $\mathbf{s}$ is a probability vector and does not depend on the common distribution function of X_i's, the distribution function of T is a mixture of the distribution functions of $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ with weights $s_1, s_2, ..., s_n$. That is

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i P(X_{i:n} > t), \tag{1}$$

where $\bar{F}_T(t) = P(T > t)$ is the survival function of the system. Navarro et al. (2008) have shown that this result is true even when the assumption that the random variables are independent is replaced with the assumption that the random variables are exchangeable. In recent years, researchers have shown intensified interest in the study of the reliability properties of a coherent system based on the properties of its signature. Kochar et al. (1999) gave some applications of the signatures for comparing the coherent systems in the case of i.i.d. components. Navarro et al. (2005, 2007, 2008) used the concept of signature to compare coherent systems when the components are not necessarily independent. Zhang (2010) obtained some stochastic ordering results in coherent systems with exchangeable components.

Among other directions of the research on the properties of coherent systems, investigation on the stochastic properties of residual lifetimes of k-out-of-n systems (as special case of coherent systems) have been of particular interest between the authors. Bairamov et al. (2002) and Asadi and Bairamov (2005) studied the properties of the mean residual life function of parallel systems. Asadi and Bairamov (2006) investigated the mean residual life function of k-out-of-n systems. Most of the results on the stochastic properties of the coherent systems use the concept of signature. Some of the research papers that use the concept of signature are as follows. Navarro and Shaked (2006) studied the limiting behavior of the hazard rate of the coherent systems. Khaledi and Shaked (2007) made some stochastic comparisons between the residual lifetime of the coherent systems under some conditions on the vector of signatures. Asadi and Goliforushani (2008) studied the mean residual life function of the coherent systems, under the condition that some of the components in the system have failed, and explored that when the components have increasing failure rate then the mean residual life of the system is decreasing. Li and Zhang (2008) investigated some stochastic ordering results on residual lifetime of coherent systems. Navarro and Hernandez (2008) investigated the mean residual life functions of finite mixtures and systems. Samaniego et al. (2009) introduced a concept of dynamic signature and used it to compare the reliability of new and used systems.

In a recent paper, Navarro et al. (2008) gave a mixture representation of the residual lifetime of the coherent system. In other words, they have shown that

$$P\left(T-t>x|T>t\right)=\sum_{i=1}^{n}s_i(t)P\left(X_{i:n}-t>x|X_{i:n}>t\right), \qquad (2)$$

where $s_i(t)$,is defined as follows

$$s_i(t)=P\left(T=X_{i:n}|T>t\right).$$

Various properties of Eq. 2 are investigated by Navarro et al. (2008).

In reliability engineering the inactivity times (IT) of a component or system are also considered by researchers. If X denotes the lifetime of an alive organism and we assume that at time t, $X<t$, then the IT of the component is defined to be $t-X|X<t$. The concept of IT is closely related to left censored data. Navarro et al. (1997) have made some stochastic comparisons between lifetime random variables based on IT and Asadi and Berred (2011) investigated some properties of mean of IT (MIT). Some of the n-component systems have the property that when r, ($r\leq n$), components of the system fail, the system still operate. For example, parallel structure is working if at least one component out of n components is alive. Assume that an n-component system with the property described above is put in operation at time $t=0$ and suppose that the system is not monitored continuously. An interesting problem is to get inference about the history of the system, e.g. when the individual components have failed. These kind of problems are related to the problem of analyzing the so-called autopsy data, i.e. information obtained by examining the component states of a failed system. Motivated by this, Asadi (2006) defined and investigated the concept of MIT of a parallel system at the system level. Tavangar and Asadi (2009) have extended the Asadi's (2006) results to k-out-of-n structures. Khaledi and Shaked (2007) and Li and Zhang (2008) obtained some stochastic ordering results on IT of coherent systems. Li and Zhao (2008) studied some stochastic comparison on a general IT of k-out-of-n systems.

This paper is a study on the stochastic and aging properties of IT of an n-component coherent system under different scenarios. We present some results which stochastically compare the IT of coherent systems and components of coherent systems, with the same type or with different types of components. The paper is organized as follows: In Section 2, we use the concept of signature to give a mixture representation of IT of a coherent system in terms of IT of ordered lifetimes of its components. Some stochastic comparisons are made between systems with i.i.d. components. It is shown that, under some conditions, the results can be extended to systems with exchangeable components. Section 3 is an investigation on the MIT of coherent systems. It is known that when a component has a decreasing reversed hazard rate, its MIT is increasing in time. Using an example we show, in this section, that when the components of a coherent system has decreasing reversed hazard rate, the MIT of the system is not necessarily increasing in time. There are some coherent systems with the vector of signature of the form $\mathbf{s} = (s_1, s_2, ..., s_i, 0, ..., 0)$, with $s_i > 0$, $i = 1, 2, \ldots, n$. We prove, for such systems, that when the common reversed hazard rate of the components is decreasing in time, then the MIT of the system is an increasing function of time. In Section 4, we introduce a new concept of IT of the components of the system at the system level. There are many coherent systems in reliability engineering for which the vector of signatures is of the form $\mathbf{s} = (0, \ldots, 0, s_i, s_{i+1}, ..., s_n)$. For instance, it can be shown that a 3-component system with structure $\max(X_1, \min(X_2, X_3))$ has the signature $(0, \frac{2}{3}, \frac{1}{3})$. In this section, we obtain the MIT of the components of the such systems under the assumption that the system has failed at time t. Among other results, it is proved that when the components of the system have decreasing reversed hazard rate, then the MIT of the failed components are increasing functions of time. Several examples and illustrative plots are also provided.

2 Mixture representation of the IT of systems

In this section we study some stochastic properties of the IT of a coherent system. To this end, we obtain a mixture representation of the IT of a coherent system in terms of IT of ordered lifetimes of its components. The representation result is similar to that of Navarro et al. (2008) where they have considered the residual lifetime of the system. Although, based on this similarity, most of the results of Navarro et al. (2008) on residual lifetime of the system can be translated to IT of the system but we only prove some of them here.

Consider a coherent system consists of n components where the lifetimes of the components are denoted by $X_1, X_2, ..., X_n$. Assume that X_i's are independent and identically distributed according to a common continuous distribution F. Let T denote the lifetime of the system. Then from Eq. 1 we have

$$F_T(t) = \sum_{i=1}^{n} s_i P\left(X_{i:n} < t\right),$$

where $F_T(t) = P(T < t)$, $X_{1:n}, X_{2:n}, ..., X_{n:n}$ are representing the order statistics obtained from $X_1, X_2, ..., X_n$ and $s_i = P(T = X_{i:n})$, $i = 1, 2, \ldots, n$, is ith element of the signature vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$.

Now we have for all $x < t$ and $t > 0$,

$$\begin{aligned} P(t - T > x | T < t) &= \frac{P(T < t - x)}{P(T < t)} \\ &= \frac{1}{F_T(t)} \sum_{i=1}^{n} s_i P(X_{i:n} < t - x) \\ &= \sum_{i=1}^{n} \frac{s_i P(X_{i:n} < t)}{F_T(t)} P(t - X_{i:n} > x | X_{i:n} < t) \\ &= \sum_{i=1}^{n} p_i(t) P(t - X_{i:n} > x | X_{i:n} < t), \end{aligned} \tag{3}$$

where $p_i(t) = s_i P(X_{i:n} < t)/F_T(t)$. Representation Eq. 3 shows that the distribution function of IT of the system is a mixture of distribution function of IT of the ordered lifetimes of its components. In the following we show that $p_i(t) = P\{T = X_{i:n} | T < t\}$. That is, $p_i(t)$ may be identified as the probability that $X_{i:n}$ causes the failure of the system given that the lifetime of the system is less than t. We have

$$\begin{aligned} p_i(t) &= \frac{s_i P(X_{i:n} < t)}{F_T(t)} \\ &= \frac{P(T = X_{i:n}) P(X_{i:n} < t | T = X_{i:n})}{F_T(t)} \\ &= \frac{P(T = X_{i:n}) P(T < t | T = X_{i:n})}{F_T(t)} \\ &= P(T = X_{i:n} | T < t), \end{aligned} \tag{4}$$

where the second equality follows from the fact that the events $(T = X_{i:n})$ and $(X_{i:n} < t)$ are independent (see, Navarro et al. 2008). Table 1 depicts the vector $\mathbf{p}(t) = (p_1(t), p_2(t), ..., p_n(t))$ for coherent systems of order $n = 3$ (see also, Navarro et al. 2008).

Note that from Eq. 4 we have, for $i = 1, 2, \ldots, n$,

$$\lim_{t\to\infty} p_i(t) = \lim_{t\to\infty} \frac{s_i P(X_{i:n} < t)}{F_T(t)} = s_i.$$

That is,

$$\lim_{t\to\infty} \mathbf{p}(t) = \mathbf{s}.$$

Table 1 Vectors of coefficients in Eq. 3 with $n = 3$ for coherent systems with 1–3 i.i.d. components

System	$T = T(X_1, X_2, X_3)$	$\mathbf{p}(t)$
1	$X_{1:1} = X_1$	$\left(1 - F(t) + \frac{1}{3}F^2(t), F(t) - \frac{2}{3}F^2(t), \frac{1}{3}F^3(t)\right)$
2	$X_{1:3} = \min(X_1, X_2, X_3)$	$(1, 0, 0)$
3	$X_{2:3}$(2-out-of-3)	$(0, 1, 0)$
4	$X_{3:3} = \max(X_1, X_2, X_3)$	$(0, 0, 1)$
5	$\min(X_1, \max(X_2, X_3))$	$\left(\frac{3 - 3F(t) + F^2(t)}{3 + 3F(t) - 3F^2(t)}, \frac{6F(t) - 4F^2(t)}{3 + 3F(t) - 3F^2(t)}, 0\right)$
6	$\max(X_1, \min(X_2, X_3))$	$\left(0, \frac{6 - 4F(t)}{6 - 3F(t)}, \frac{F(t)}{6 - 3F(t)}\right)$
7	$X_{1:2} = \min(X_1, X_2)$	$\left(\frac{6 - 6F(t) + 2F^2(t)}{6 - 3F(t)}, \frac{3F(t) - 2F^2(t)}{6 - 3F(t)}, 0\right)$
8	$X_{2:2} = \max(X_1, X_2)$	$\left(0, 1 - \frac{2}{3}F(t), \frac{2}{3}F(t)\right)$

Theorem 2.1 *Let T be a coherent system having signature $\mathbf{s} = (0, ..., 0, s_{n-i+1}, s_{n-i+2}, ..., s_n)$, where $s_{n-i+1} > 0$ for an integer $i \in \{1, 2, ..., n\}$, then*

$$\lim_{t\to 0} \mathbf{p}(t) = \left(\underbrace{0, 0..., 0}_{n-i \ times}, 1, \underbrace{0, 0..., 0}_{i-1 \ times} \right).$$

Proof From Eq. 4, we have

$$\lim_{t\to 0} p_k(t) = s_k \lim_{t\to 0} \frac{P(X_{k:n} \le t)}{\sum_{r=n-i+1}^{n} s_r P(X_{r:n} \le t)}.$$

Since

$$P(X_{r:n} \le t) = \sum_{j=r}^{n} (-1)^{j-r} \binom{j-1}{r-1} \binom{n}{j} F^j(t),$$

(see David and Nagaraja 2003) we can write

$$\lim_{t\to 0} \frac{P(X_{r:n} \le t)}{P(X_{k:n} \le t)} = \lim_{t\to 0} \frac{F^r(t)}{F^k(t)} \frac{w_r + w_{r+1}F^1(t) + \cdots + w_n F^{n-r}(t)}{w_k + w_{k+1}F^1(t) + \cdots + w_n F^{n-k}(t)}$$
$$= \lim_{t\to 0} F^{r-k}(t) = \begin{cases} \infty & \text{if } \ r < k \\ 1 & \text{if } \ r = k \\ 0 & \text{if } \ r > k, \end{cases}$$

where $w_r = (-1)^{j-r}\binom{j-1}{r-1}\binom{n}{j}$. Hence

$$\lim_{t\to 0} p_k(t) = s_k \lim_{t\to 0} \frac{P\left(X_{k:n} \le t\right)}{\sum\limits_{r=n-i+1}^{n} s_r P\left(X_{r:n} \le t\right)} = \begin{cases} 0 & \text{if } k \neq n-i+1 \\ 1 & \text{if } k = n-i+1. \end{cases}$$

This completes the proof. □

Before giving next results, we recall that for two non-negative random variables X and Y having distribution functions F and G and probability density functions f and g, respectively:

- X is said to be less than Y in stochastic order, denoted by $X \le_{st} Y$, if $\bar{F}(x) \le \bar{G}(x)$, $x > 0$, where $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$.
- X is said to be less than Y in reversed hazard order, denoted by $X \le_{rh} Y$, if $\frac{F(x)}{G(x)}$ is a decreasing function of x. If the densities of X and Y exist, $X \le_{rh} Y$ is equivalent to say that $r_F(t) \le r_G(t)$, where $r_F(t) = \frac{f(t)}{F(t)}$ and $r_G(t) = \frac{g(t)}{G(t)}$ denote the reversed hazard rates of X and Y, respectively.
- X is said to be less than Y in likelihood ratio order, denoted by $X \le_{lr} Y$, if $\frac{f(x)}{g(x)}$ is a decreasing function of x.

The following theorem gives a comparison between $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ when $t_1 < t_2$.

Theorem 2.2 *Let* $\mathbf{p}(t)$ *be a vector of coefficients in Eq. 3 of a mixed system with* n *i.i.d. components. Then,* $\mathbf{p}(t_1) \le_{st} \mathbf{p}(t_2)$ *for all* $0 \le t_1 \le t_2$ *and* $\mathbf{p}(t) \le_{st} \mathbf{s}$ *for all* $t \ge 0$.

Proof To prove the required result we need show that for all $j \ge 1$

$$\frac{\sum\limits_{k=j}^{n} s_k F_k\left(t_1\right)}{F_T\left(t_1\right)} \le \frac{\sum\limits_{k=j}^{n} s_k F_k\left(t_2\right)}{F_T\left(t_2\right)}$$

or equivalently

$$\sum_{i=1}^{n}\sum_{k=j}^{n} s_i s_k \left(F_i\left(t_2\right) F_k\left(t_1\right) - F_i\left(t_1\right) F_k\left(t_2\right)\right) \le 0.$$

Note that

$$\sum_{i=1}^{n}\sum_{k=j}^{n} s_i s_k \left(F_i\left(t_2\right) F_k\left(t_1\right) - F_i\left(t_1\right) F_k\left(t_2\right)\right)$$

$$= \sum_{i=1}^{j-1}\sum_{k=j}^{n} s_i s_k \left(F_i\left(t_2\right) F_k\left(t_1\right) - F_i\left(t_1\right) F_k\left(t_2\right)\right)$$

$$+ \sum_{i=j}^{n}\sum_{k=j}^{n} s_i s_k \left(F_i\left(t_2\right) F_k\left(t_1\right) - F_i\left(t_1\right) F_k\left(t_2\right)\right) \tag{5}$$

$$= \sum_{i=1}^{j-1}\sum_{k=j}^{n} s_i s_k \left(F_i\left(t_2\right) F_k\left(t_1\right) - F_i\left(t_1\right) F_k\left(t_2\right)\right), \tag{6}$$

where the equality in Eq. 6 follows based on the fact that in the second summation in Eq. 5 the difference in summand is a symmetric function and both summations are from j to n. Using the fact that order statistics from i.i.d. random variables are rh-ordered, that is, $X_{i:n} \leq_{rh} X_{k:n}$ for $i \leq k$, we have for all $t_1 \leq t_2$ and $i \leq k$

$$F_i\left(t_2\right) F_k\left(t_1\right) - F_i\left(t_1\right) F_k\left(t_2\right) \leq 0,$$

which implies that Eq. 6 is non-positive. This completes the proof. □

Now we are ready to prove the following theorem.

Theorem 2.3 *Let $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ be the vectors of coefficients in representation Eq. 3, for a fixed $t > 0$, of two coherent systems of order n, both based on components with i.i.d. lifetimes distributed as the common continuous distribution function F. Denote by T_1 and T_2 the respective lifetimes of the systems.*

(a) *If $\mathbf{p}_1(t) \leq_{st} \mathbf{p}_2(t)$, then $(t - T_1|T_1 < t) \geq_{st} (t - T_2|T_2 < t)$.*
(b) *If $\mathbf{p}_1(t) \leq_{rh} \mathbf{p}_2(t)$, then $(t - T_1|T_1 < t) \geq_{hr} (t - T_2|T_2 < t)$.*
(c) *If $\mathbf{p}_1(t) \leq_{lr} \mathbf{p}_2(t)$, then $(t - T_1|T_1 < t) \geq_{lr} (t - T_2|T_2 < t)$, under the assumption that F is absolutely continuous.*

Proof

Part (a) It can be shown that, for i.i.d. random variables, the order statistics are likelihood ratio ordered . That is $X_{i:n} \leq_{lr} X_{i+1:n}$ for $i = 1, ..., n - 1$. Then it can be shown that

$$(t - X_{i:n}|X_{i:n} < t) \geq_{lr} (t - X_{i+1:n}|X_{i+1:n} < t), \quad \text{for } i = 1, ..., n-1, \tag{7}$$

(see, Shaked and Shanthikumar (2007), Theorems 1.C.8 and 1.C.6). Therefore, based on implications between the *lr, rh* and *st* orders, the IT in Eq. 7 are also *hr* and *st* ordered. Since $\mathbf{p}_1(t) \leq_{st} \mathbf{p}_2(t)$, we have from Theorem 1.A.6 Shaked and Shanthikumar (2007)

$$\begin{aligned} P(t - T_1 > x | T_1 < t) &= \sum_{i=1}^{n} p_{1,i}(t) P(t - X_{i:n} > x | X_{i:n} < t) \\ &\geq \sum_{i=1}^{n} p_{2,i}(t) P(t - X_{i:n} > x | X_{i:n} < t) \\ &= P(t - T_2 > x | T_2 < t). \end{aligned}$$

This completes the proof of Part (a).

Part (b) Now assume that $T_{t,i} = (t - T_i | T_i < t)$, and define $\bar{H}_{t,i}(x) = P(T_{t,i} > x)$, $i = 1, 2$. In order to prove that $T_{t,1} \geq_{hr} T_{t,2}$ we should prove that $\frac{\bar{H}_{t,1}(x)}{\bar{H}_{t,2}(x)}$ is an increasing function of x. This is equivalent to show that for $x_1 < x_2$,

$$\frac{\sum_{i=1}^{n} p_{2,i}(t) P(t - X_{i:n} > x_2 | X_{i:n} < t)}{\sum_{i=1}^{n} p_{2,i}(t) P(t - X_{i:n} > x_1 | X_{i:n} < t)} \leq \frac{\sum_{i=1}^{n} p_{1,i}(t) P(t - X_{i:n} > x_2 | X_{i:n} < t)}{\sum_{i=1}^{n} p_{1,i}(t) P(t - X_{i:n} > x_1 | X_{i:n} < t)}. \tag{8}$$

Now using Theorem 1.B.50 of Shaked and Shanthikumar (2007) one can easily see that, under the assumptions of the theorem, inequality 8 holds. This completes the proof of Part (b).

Part (c) For $j = 1, 2$, denote the density function of the survival function $\bar{H}_{t,i}(x)$ by $h_{t,j}(x)$. That is

$$h_{t,j}(x) = \sum_{i=1}^{n} p_{j,i}(t) f_{t,i}(x),$$

where $f_{t,i}$ denotes the density function of IT of ordered lifetimes. Now, under the assumptions of the theorem, by using Theorem 1.C.17 of Shaked and Shanthikumar (2007) we can show that $\frac{h_{t,1}(x)}{h_{t2}(x)}$ is an increasing function. This completes the proof. □

Example 2.4 It is easy to see that for $t \geq 0$

$$\left(0, 1 - \frac{2}{3}F(t), \frac{2}{3}F(t)\right) \geq_{lr} \left(0, \frac{6 - 4F(t)}{6 - 3F(t)}, \frac{F(t)}{6 - 3F(t)}\right).$$

Hence, from Table 1 and above theorem, if F is absolutely continuous, we have

$$(t - X_{2:2}|X_{2:2} < t) \leq_{lr} (t - T|T > t), \quad \text{for all } t \geq 0,$$

where $T = \max(X_1, \min(X_2, X_3))$.

Example 2.5 Consider the systems $X_{2:2}$ and $T = \max(X_{2:3}, X_4)$ with corresponding signatures of order 4 as $s_{2:2} = (0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ and $s = (0, 0, \frac{3}{4}, \frac{1}{4})$ (for more details on the list of systems with signatures of order 4 see Navarro et al. (2008)). One can easily shown that $X_{2:2}$ and T are not st-ordered. The corresponding vectors of coefficients $\mathbf{p}(t)$ are

$$\mathbf{p}_{2:2}(t) = \left(0, 1 - \frac{8F(t) - 3F^2(t)}{6}, \frac{4F(t) - 3F^2(t)}{3}, \frac{F^2(t)}{2}\right)$$

and

$$\mathbf{p}(t) = \left(0, 0, \frac{12 - 9F(t)}{12 - 8F(t)}, \frac{F(t)}{12 - 8F(t)}\right).$$

It can be shown that $\mathbf{p}_{2:2}(t) \leq_{st} \mathbf{p}(t)$ whenever

$$F(t) \geq \frac{3 - \sqrt{5}}{4}.$$

Hence, if $t_1 = \sup\{t \geq 0 : \ F(t) = (3 - \sqrt{5})/4\}$ then $\mathbf{p}_{1:2}(t) \leq_{st} \mathbf{p}(t)$ for all $t \geq t_1$. Then, from Theorem 2.3 we have $(t - X_{2:2}|X_{2:2} < t) \geq_{st} (t - T|T < t)$ for all $t \geq t_1$. Note that these vectors are not st-ordered for $0 \leq t < t_1$ and that

$$\lim_{t \to 0} \mathbf{p}_{2:2}(t) = (0, 1, 0, 0) \leq_{st} \lim_{t \to 0} \mathbf{p}(t) = (0, 0, 1, 0).$$

Remark 2.6 It can be shown that the mixture representation Eq. 3 is also true for the system having dependent exchangeable components with an absolutely continuous joint distribution. Navarro and Rychlik (2007) showed that if $X = (X_1, ..., X_n)$ is a random vector with an absolutely continuous exchangeable joint distribution and $T = T(X_1, ..., X_n)$ is the lifetime of a coherent system, then Eq. 1 holds. Using this result we can easily see that for $x < t$ and $t > 0$,

$$P(t - T > x|T < t) = \sum_{i=1}^{n} p_i(t) P(t - X_{i:n} > x|X_{i:n} < t), \tag{9}$$

where $X_{i:n}$ denotes the ith ordered lifetime $i = 1, 2, ..., n$ and similar to the independent components, we have $p_i(t) = P(T = X_{i:n}|T < t)$.

The following theorem can be proved for the system having exchangeable components. The proof is similar to Theorem 2.3 and hence is omitted.

Theorem 2.7 *Let $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ be the vectors of coefficients in representation Eq. 9, for a fixed $t > 0$, of two mixed systems of order n, both based on exchangeable components. Let T_1 and T_2 be their respective lifetimes.*

(a) *If $(t - X_{i:n}|X_{i:n} < t) \geq_{st} (t - X_{i+1:n}|X_{i+1:n} < t)$, for $i = 1, ..., n-1$ and $\mathbf{p}_1(t) \leq_{st} \mathbf{p}_2(t)$, then $(t - T_1|T_1 < t) \geq_{st} (t - T_2|T_2 < t)$.*
(b) *If $(t - X_{i:n}|X_{i:n} < t) \geq_{hr} (t - X_{i+1:n}|X_{i+1:n} < t)$, for $i = 1, ..., n-1$ and $\mathbf{p}_1(t) \leq_{rh} \mathbf{p}_2(t)$, then $(t - T_1|T_1 < t) \geq_{hr} (t - T_2|T_2 < t)$.*
(c) *If $X_{i:n} \leq_{lr} X_{i+1:n}$, for $i = 1, ..., n-1$ and $\mathbf{p}_1(t) \leq_{lr} \mathbf{p}_2(t)$, then $(t - T_1|T_1 < t) \geq_{lr} (t - T_2|T_2 < t)$, under the assumption that the underlying distribution is absolutely continuous.*

3 The MIT of the coherent systems

In this section, we obtain some results on the MIT of the coherent systems. We consider a coherent system of order n where the components are assumed to be i.i.d. with lifetimes $X_1, X_2, ..., X_n$. We assume that the lifetimes are distributed according to a common continuous distribution F and denote by T the lifetime of the system. First we assume that at time t the system has failed, i.e. it is known that $T < t$. If $M_T(t)$ denotes the MIT of the system, then

$$M_T(t) = E(t - T|T < t).$$

Based on representation Eq. 3, we can write

$$M_T(t) = \sum_{i=1}^{n} p_i(t) m_{i:n}(t),$$

where

$$m_{i:n}(t) = E\left(t - X_{i:n}|X_{i:n} < t\right), \quad i = 1, 2, ..., n.$$

It is easily seen that $M_T(t)$ is bounded as follows.

$$m_{n:n}(t) \leq M_T(t) \leq m_{1:n}(t).$$

The result follows from the fact that reversed hazard rates of $X_{i:n}$'s, denoted by $r_{i:n}(t)$, $i = 1, 2..., n$, are ordered in terms of i. That is

$$r_{1:n}(t) \leq r_{2:n}(t) \leq \cdots \leq r_{n:n}(t).$$

It is also well known that the reversed hazard rate ordering implies the MIT ordering (see, for example, Finkelstein (2002)). That is,

$$m_{1:n}(t) \geq m_{2:n}(t) \geq \cdots \geq m_{n:n}(t).$$

Hence

$$
\begin{aligned}
M_T(t) &= \sum_{i=1}^{n} p_i(t) m_{i:n}(t) \\
&\leq \sum_{i=1}^{n} p_i(t) m_{1:n}(t) \\
&= m_{1:n}(t).
\end{aligned}
$$

Similarly, it can be shown that $m_{n:n}(t) \leq M_T(t)$.

It is well known that if the reversed hazard rate of a random variable X is decreasing in time, then the MIT of X is an increasing function of time. Now the question is what can we say about the behavior of MIT of a coherent system if the common reversed hazard rate of the components of the system is decreasing. The following example shows that the MIT of the system is not necessarily an increasing function of time in this case.

Example 3.1 Let X_1, X_2, X_3 denote the lifetimes of three independent components which are connected to a system with structure $\min(X_1, \max(X_2, X_3))$. Assume that the components have a common distribution function

$$
F(x) = \frac{e^x - 1}{e^{20} - 1}, \qquad 0 < x < 20.
$$

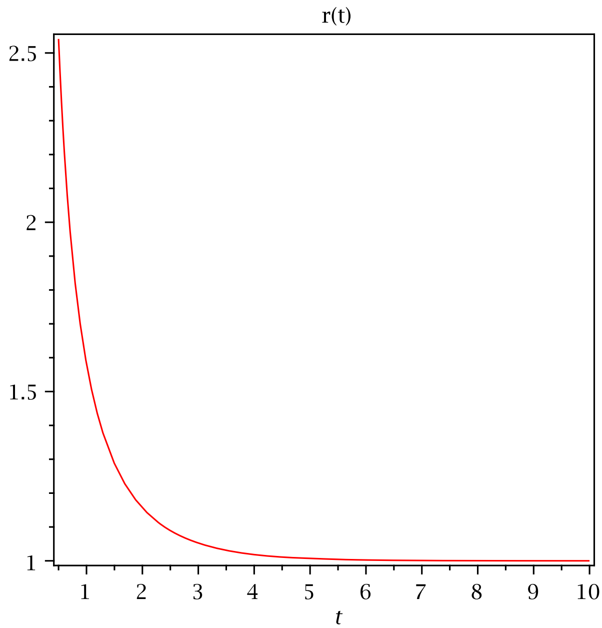

Fig. 1 The plot of reversed hazard rate in Example 3.1

Then, it can be shown that the reversed hazard rate corresponding to F is

$$r(t) = \frac{e^t}{e^t - 1}, \quad 0 < t < 20,$$

which is a decreasing function of time. A plot of $r(t)$ is given in Fig. 1. Figure 2 represents the plots of the MITs $m_1(t)$, $M_T(t)$, $m_2(t)$ and $m_3(t)$ from top, respectively. It is seen from the plots that although $m_1(t)$, $m_2(t)$ and $m_3(t)$ are increasing functions but $M_T(t)$ first increases for a period of time then it decreases for a period of time and eventually increases.

Now consider a coherent system consists of n i.i.d. components having the signature of the form

$$\mathbf{s} = (s_1, ..., s_i, 0, ..., 0). \tag{10}$$

A list of systems with such vectors of signature is given in Table 2.

It can be shown that the survival function of the IT of a coherent system with signature of the form Eq. 10 with i.i.d.components, under the condition that $X_{r:n} < t$, $(r \geq i)$, is (see Khaledi and Shaked 2007)

$$P(t - T > x | X_{r:n} < t) = \sum_{k=1}^{i} s_k P\left(t - X_{k:n} > x | X_{r:n} < t\right).$$

Fig. 2 The plots of the MITs in Example 3.1

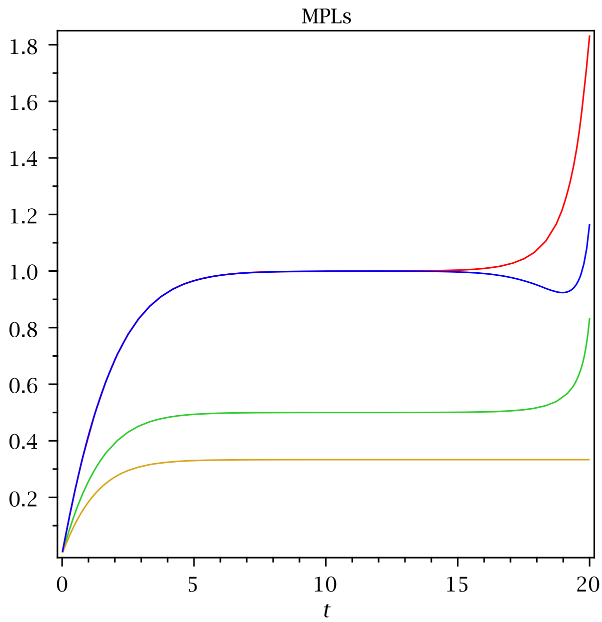

Table 2 Coherent systems of order $n = 4$ with the vector of signatures of form Eq. 10

System	$T = T(X_1, X_2, X_3, X_4)$	Signature
1	$X_{1:4} = \min(X_1, X_2, X_3, X_4)$ (series)	$(1, 0, 0, 0)$
2	$\max(\min(X_1, X_2, X_3), \min(X_2, X_3, X_4))$ (consecutive 3-out-of-4:G)	$\left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$
3	$\min(X_{2:3}, X_4)$	$\left(\frac{1}{4}, \frac{3}{4}, 0, 0\right)$
4	$X_{1:3} = \min(X_1, X_2, X_3)$ (3 − series)	$\left(\frac{3}{4}, \frac{1}{4}, 0, 0\right)$
5	$X_{1:2} = \min(X_1, X_2)$ (2 − series)	$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0\right)$
6	$\min(X_2, \max(X_1, X_3))$ (consecutive 2-out-of-3:G)	$\left(\frac{1}{4}, \frac{5}{12}, \frac{1}{3}, 0\right)$
7	$\min(X_1, \max(X_2, X_3), \max(X_3, X_4))$	$\left(\frac{1}{4}, \frac{7}{12}, \frac{1}{6}, 0\right)$
8	$\min(X_1, \max(X_2, X_3, X_4))$	$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0\right)$

Therefore the MIT of the system, which we denote by M_T^*, under the condition that $X_{r:n} < t$, is

$$M_T^*(t) = E\left(t - T | X_{r:n} < t\right) = \sum_{k=1}^{i} s_k m_n^{k,r}(t),$$

where

$$\begin{aligned} m_n^{k,r}(t) &= E\left(t - X_{k:n} | X_{r:n} \le t\right) \\ &= \int_0^t P\left(t - X_{k:n} > x | X_{r:n} \le t\right) dx. \end{aligned}$$

It can be shown that

$$m_n^{k,r}(t) = \sum_{j=r}^{n} \tau_j(t) H_j^k(t)$$

where

$$H_j^k(t) = E\left(t - X_{k:j} | X_{j:j} \le t\right),$$

is the MIT of a parallel system with j components (see Asadi 2006) and

$$\tau_j(t) = \frac{\binom{n}{j}\phi^j(t)}{\sum_{m=r}^{n} \binom{n}{m}\phi^m(t)},$$

in which $\phi(t) = F(t)/\bar{F}(t)$ (see also Tavangar and Asadi 2009).

Let $r(t)$ denote the reversed hazard rate function of F. Asadi (2006) showed that when $r(t)$ is decreasing in t then $H_j^k(t)$ is an increasing function of t. Tavangar and Asadi (2009) proved that the latter result is also true for $m_n^{k,r}(t)$. That is, they showed that if $r(t)$ is a decreasing function of time then the MIT $m_n^{k,r}(t)$ is an increasing function of time. The following theorem shows, as conclusion of above discussion, the MIT $M_T^*(t)$ is also increasing function of time when $r(t)$ is decreasing in t.

Theorem 3.2 *Let be $X_1, ..., X_n$ the i.i.d. lifetimes of n components which are connected in a coherent system. Suppose that, for some $i \leq n$, the signature of T is $\mathbf{s} = (s_1, ..., s_i, 0, ..., 0)$. If the common reversed hazard rate of the components is decreasing then the MIT $M_T^*(t)$ is increasing in time.*

4 The MIT of the components of the system at the system level

In this section we introduce a new concept of MIT for the components of the system at the system level and study some of aging properties of that. We assume that a coherent system with lifetime T has the property that, with probability 1, it is alive as long as at least $n - i + 1$ components of the system are alive. There are several type of such systems in reliability engineering. Table 3 displays examples of these kind of systems. For the system described above the vector of the signature $\mathbf{s}$ is of the form

$$\mathbf{s} = (0, 0, ..., 0, s_i, s_{i+1}, ..., s_n)\,, . \tag{11}$$

That is,

$$P\,(T = X_{r:n}) = 0, \quad r = 1, .., i - 1,$$

where $X_{1:n}, X_{2:n}, ..., X_{3:n}$ denote the ordered lifetimes of the components. In other words, for a system with the signature of the form Eq. 11, the components with lifetime $X_{r:n}$, $r = 1, ..., i - 1$, never cause the failure of the system.

Now if we assume that, at time t, the system is not working, i.e. $T < t$ then we can make sure that the components with lifetime $X_{r:n}$, $r = 1, ..., i - 1$ had already been failed in the system. As the systems often are not monitored continuously, one might be interested in getting inference about the history of the system, e.g. when the components with lifetime $X_{r:n}$, $r = 1, ..., i - 1$ have failed. We consider the following conditional probability

$$P\,(t - X_{r:n} > x | T < t) = \frac{P\,(t - X_{r:n} > x,\, T < t)}{P(T < t)},$$

where $r \leq i$. This is the probability of the IT of $X_{r:n}$ under the condition that the system has failed at time t. We call this the IT of the components at the system level. To derive the form of this probability let, for $1 \leq k \leq n$, A_k be the set of

Table 3 Coherent systems with four components and signatures of form Eq. 11

System	$T = T(X_1, X_2, X_3, X_4)$	Signature
1	$X_{2:2} = \max(X_1, X_2)$ (2 − parallel)	$\left(0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right)$
2	$\max(X_2, \min(X_1, X_3))$ (consecutive 2-out-of-3:F)	$\left(0, \frac{1}{3}, \frac{5}{12}, \frac{1}{4}\right)$
3	$\max(X_1, \min(X_2, X_3, X_4))$	$\left(0, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$
4	$\max(X_1, \min(X_2, X_3), \min(X_3, X_4))$	$\left(0, \frac{1}{6}, \frac{7}{12}, \frac{1}{4}\right)$
5	$X_{3:3} = \max(X_1, X_2, X_3)$ (3 − parallel)	$\left(0, 0, \frac{1}{4}, \frac{3}{4}\right)$
6	$\max(X_{2:3}, X_4)$	$\left(0, 0, \frac{3}{4}, \frac{1}{4}\right)$
7	$\min(\max(X_1, X_2, X_3), \max(X_2, X_3, X_4))$ (consecutive 3-out-4:F)	$\left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$
8	$X_{4:4} = \max(X_1, X_2, X_3, X_4)$ (parallel)	$(0, 0, 0, 1)$

all the permutation $\pi = (\pi_1, \pi_2, ..., \pi_n)$ of $(1, 2, ..., n)$ such that if $X_{\pi_1} < X_{\pi_2} < ... < X_{\pi_n}$ then $T = X_{\pi_k}$. In other words, A_k is the set of all the permutation $\pi = (\pi_1, \pi_2, ..., \pi_n)$ such that if $X_{\pi_1} < X_{\pi_2} < ... < X_{\pi_n}$ then $T = X_{k:n}$. Hence for $r \le i$, we have

$$
\begin{aligned}
P(t - X_{r:n} > x, T < t) &= \sum_{k=i}^{n} P(t - X_{r:n} > x, T < t, T = X_{k:n}) \\
&= \sum_{k=i}^{n} \sum_{\pi \in A_k} P\left(X_{\pi_r} < t - x, X_{\pi_k} < t, X_{\pi_1} < X_{\pi_2} < ... < X_{\pi_n}\right) \\
&= \sum_{k=i}^{n} \sum_{\pi \in A_k} P(X_r < t - x, X_k < t, X_1 < X_2 < ... < X_n) \\
&= \sum_{k=i}^{n} n! s_k P(X_r < t - x, X_k < t, X_1 < X_2 < ... < X_n) \\
&= \sum_{k=i}^{n} s_k P(X_{r:n} < t - x, X_{k:n} < t) .
\end{aligned}
$$

This implies that

$$
\begin{aligned}
P(t - X_{r:n} > x | T < t) &= \sum_{k=i}^{n} \frac{s_k P(X_{k:n} < t)}{F_T(t)} \frac{P(X_{r:n} < t - x, X_{k:n} < t)}{P(X_{k:n} < t)} \\
&= \sum_{k=i}^{n} p_k(t) P(t - X_{r:n} > x | X_{k:n} < t) ,
\end{aligned}
$$

where $F_T(t)$ is the distribution function T and $p_i(t) = P(T = X_{i:n}|T < t)$. In the sequel, we focus on the MIT of the component $X_{r:n}$, denoted by $M_n^r(t)$, at the system level. That is, we study the properties of

$$M_n^r(t) = E(t - X_{r:n}|T < t) = \sum_{k=i}^{n} p_k(t) M_n^{r,k}(t),$$

where $M_n^{r,k}(t) = E(t - X_{r:n}|X_{k:n} < t)$. This shows that $M_n^r(t)$ is a linear combination of $M_n^{r,k}(t)$ (for some properties of $M_n^{r,k}(t)$ see Tavangar and Asadi (2009)). It is easily seen that $M_n^r(t)$ is decreasing in r, $r = 1, ..., i$. This is true because

$$M_n^r(t) - M_n^{r+1}(t) = E(X_{r+1:n} - X_{r:n}|T < t) \geq 0.$$

Now we prove the following theorems.

Theorem 4.1 *Consider two coherent systems of order n and assume that* $\mathbf{p}_1(t)$ *and* $\mathbf{p}_2(t)$ *are the corresponding vectors of signatures, for a fixed* $t > 0$ *where we assume that the vector of the signatures are of the form Eq. 11. Further assume that the components of the systems are i.i.d. distributed according to the common continuous distribution F and denote by* T_1 *and* T_2 *the lifetimes of the systems, respectively. If*

$$\mathbf{p}_1(t) \leq_{st} \mathbf{p}_2(t),$$

then

$$M_{1n}^r(t) \leq M_{2n}^r(t),$$

where $M_{1n}^r(t)$ *and* $M_{2n}^r(t)$ *are the MITs corresponding to* T_1 *and* T_2*, respectively.*

Proof Let Y_v be a random variable over $1, 2, ..., n$, with probability mass function $\mathbf{p}_v(t) = (0, 0, \ldots, p_{vi}(t), ..., p_{vn}(t))$, $v = 1, 2$. Then we can write

$$M_{1n}^r(t) = E\left(M_n^{r,Y_1}\right) \leq E\left(M_n^{r,Y_2}\right) = M_{1n}^r(t),$$

where the inequality follows from Shaked and Shanthikumar (2007, page 4, relation (1.A.7)) and the fact that $M_n^{r,k}(t)$ is an increasing function of k, $k = 1, ..., n$ (see Tavangar and Asadi 2009). This completes the proof. □

Tavangar and Asadi (2009) have shown that when the components of the system have a decreasing reversed hazard rate then the MIT $M_n^{r,k}(t)$ is an increasing function of time. The following theorem extends their result.

Theorem 4.2 *Consider a coherent system of order n with signature vector of the form Eq. 11 in which* $s_k > 0$, $k = i, i + 1, ..., n$. *Let the components of the system* *have i.i.d. lifetimes with a common absolutely continuous distribution function F. Let also $r(t)$, the reversed hazard rate of the components of the system, be decreasing in t, $t > 0$. Then $M_n^r(t)$ is an increasing function in t.*

Proof Under the assumption that $M_n^r(t)$ is differentiable, we have

$$\frac{d}{dt}M_n^r(t) = \sum_{k=i}^{n}\left(\frac{d}{dt}p_k(t)\right)M_n^{r,k}(t) + \sum_{k=i}^{n}p_k(t)\left(\frac{d}{dt}M_n^{r,k}(t)\right). \tag{12}$$

But the second term in the above equality is nonnegative by Theorem 3.1 of Tavangar and Asadi (2009). Hence we just need to prove that the first term in Eq. 12 is nonnegative. First note that $p_k(t)$ can be written as

$$\begin{aligned} p_k(t) &= \frac{s_k P\left(X_{k:n} < t\right)}{F_T(t)} \\ &= \frac{s_k \sum_{j=k}^{n}\binom{n}{j}\phi^j(t)}{\sum\limits_{m=i}^{n} s_m \sum_{j=m}^{n}\binom{n}{j}\phi^j(t)}, \end{aligned}$$

where $\phi(t) = F(t)/\bar{F}(t)$. Define

$$W_m(t) = \sum_{j=m}^{n}\binom{n}{j}(\phi(t))^j, \quad m = 1, .., n.$$

Then

$$p_k(t) = \frac{s_k W_k(t)}{\sum\limits_{m=i}^{n} s_m W_m(t)}.$$

Therefore we have

$$\sum_{k=i}^{n}\left(\frac{d}{dt}p_k(t)\right)M_n^{r,k}(t) = \frac{\sum\limits_{k=i}^{n}\left(s_k W_k'(t)\sum\limits_{m=i}^{n} s_m W_m(t) - s_k W_k(t)\sum\limits_{m=i}^{n} s_m W_m'(t)\right)M_n^{r,k}(t)}{\left(\sum\limits_{m=i}^{n} s_m W_m(t)\right)^2}.$$

The numerator of the above expression can be written as

$$
\begin{aligned}
&\sum_{k=i}^{n}\sum_{m=i}^{n} s_k s_m W_k^{'}(t) W_m(t) M_n^{r,k}(t) - \sum_{k=i}^{n}\sum_{m=i}^{n} s_k s_m W_k(t) W_m^{'}(t) M_n^{r,k}(t) \\
&\quad = \sum_{k=i}^{n}\sum_{m=i}^{n} s_k s_m W_k^{'}(t) W_m(t) \left(M_n^{r,k}(t) - M_n^{r,m}(t)\right) \\
&\quad = \sum_{k=i}^{n}\sum_{m=i}^{k} s_k s_m W_k^{'}(t) W_m(t) \left(M_n^{r,k}(t) - M_n^{r,m}(t)\right) \\
&\qquad + \sum_{m=i}^{n}\sum_{k=i}^{m} s_k s_m W_k^{'}(t) W_m(t) \left(M_n^{r,k}(t) - M_n^{r,m}(t)\right) \\
&\quad = \sum_{k=i}^{n}\sum_{m=i}^{k} s_k s_m W_k^{'}(t) W_m(t) \left(M_n^{r,k}(t) - M_n^{r,m}(t)\right) \\
&\qquad + \sum_{k=i}^{n}\sum_{m=i}^{k} s_k s_m W_k(t) W_m^{'}(t) \left(M_n^{r,m}(t) - M_n^{r,k}(t)\right) \\
&\quad = \sum_{k=i}^{n}\sum_{m=i}^{k} s_k s_m \left(W_k^{'}(t) W_m(t) - W_k(t) W_m^{'}(t)\right) \left(M_n^{r,k}(t) - M_n^{r,m}(t)\right).
\end{aligned}
$$

But $M_n^{r,k}(t) - M_n^{r,m}(t) \geq 0$ since $m \leq k$ and $M_n^{r,k}(t)$ is increasing in k, $k = r, ..., n$. Now for $m \leq k$ we have

$$
\begin{aligned}
W_k^{'}(t) W_m(t) - W_k(t) W_m^{'}(t) &= \phi^{'}(t) \sum_{j=k}^{n}\sum_{l=m}^{n} \binom{n}{j}\binom{n}{l} j\phi^{j+l-1}(t) - \phi^{'}(t) \\
&\quad \times \sum_{j=k}^{n}\sum_{l=m}^{n} \binom{n}{j}\binom{n}{l} l\phi^{j+l-1}(t) \\
&= \phi^{'}(t) \sum_{j=k}^{n}\sum_{l=m}^{k-1} \binom{n}{j}\binom{n}{l} j\phi^{j+l-1}(t) + \phi^{'}(t) \\
&\quad \times \sum_{j=k}^{n}\sum_{l=k}^{n} \binom{n}{j}\binom{n}{l} j\phi^{j+l-1}(t) \\
&\quad - \phi^{'}(t) \sum_{j=k}^{n}\sum_{l=m}^{k-1} \binom{n}{j}\binom{n}{l} l\phi^{j+l-1}(t) - \phi^{'}(t)
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{j=k}^{n} \sum_{l=k}^{n} \binom{n}{j} \binom{n}{l} l \phi^{j+l-1}(t) \\
&= \phi^{'}(t) \sum_{j=k}^{n} \sum_{l=m}^{k-1} \binom{n}{j} \binom{n}{l} (j-l) \phi^{j+l-1}(t) \\
&\geq 0.
\end{aligned}
$$

This shows that the first part of Eq. 12 is also nonnegative and hence the proof is complete. □

In the following theorem we compare the MITs of two coherent systems in terms of their reversed hazard rates.

Theorem 4.3 *Let $\mathcal{S}_1$ and $\mathcal{S}_1$ be two coherent systems with n independent and identical components that are distributed as F and G, respectively. If $r_F(t) \geq r_G(t)$, $t > 0$ and $\mathbf{p}_{\mathcal{S}_1}(t) \leq_{st} \mathbf{p}_{\mathcal{S}_2}(t)$, then*

$$M^r_{n,\mathcal{S}_1}(t) \leq M^r_{n,\mathcal{S}_2}(t).$$

Proof Following Khaledi and Shaked (2007) (see also, Tavangar and Asadi 2009) when $r_F(t) \geq r_G(t)$, $t > 0$, then

$$M^{r,k}_{n,\mathcal{S}_1}(t) \leq M^{r,k}_{n,\mathcal{S}_2}(t), \quad r = 1, ..., k. \tag{13}$$

This implies that

$$
\begin{aligned}
M^r_{n,\mathcal{S}_1}(t) &= \sum_{k=1}^{n} p_{\mathcal{S}_1,k}(t) M^{r,k}_{n,\mathcal{S}_1}(t) \\
&\leq \sum_{k=1}^{n} p_{\mathcal{S}_1,k}(t) M^{r,k}_{n,\mathcal{S}_2}(t) \\
&\leq \sum_{k=i}^{n} p_{\mathcal{S}_2,k}(t) M^{r,k}_{n,\mathcal{S}_2}(t) \\
&= M^r_{n,\mathcal{S}_2}(t),
\end{aligned}
$$

where the first inequality follows from Eq. 13 and the second inequality follows from the result that $M^{r,k}_{n,\mathcal{S}_j}(t)$ is increasing in k, $j = 1, 2$ and $\mathbf{p}_{\mathcal{S}_1}(t) \leq_{st} \mathbf{p}_{\mathcal{S}_2}(t)$. Hence the proof is complete. □

Example 4.4 Let we have a coherent system with n components such that they have lifetimes $X_1, ..., X_n$ that are i.i.d exponential random variables with mean 1 and the signature vector of the system is

$$\mathbf{s} = (0, 0, ..., 0, s_i, s_{i+1}, ..., s_n).$$

Then for $r \le i \le n$ we have

$$M_n^r(t) = \sum_{k=i}^{n} p_k(t) M_n^{r,k}(t),$$

where

$$M_n^{r,k}(t) = \frac{\sum_{m=k}^{n} \binom{n}{m} (e^t - 1)^m S_m^r(t)}{\sum_{m=k}^{n} \binom{n}{m} (e^t - 1)^m}$$

and

$$S_m^r(t) = \sum_{j=r}^{m} \binom{m}{j} \sum_{l=0}^{m-j} (-1)^l \binom{m-j}{l} \frac{1 + \sum_{u=1}^{j+l} (-1)^u \binom{j+l}{u} \frac{1}{u}(1 - e^{-ut})}{(1 - e^{-t})^{j+l}}.$$

Remark 4.5 We have to point out here that the results of Theorems 4.1, 4.2 and 4.3 can be strengthen. In other words, in Theorem 4.1 using the same steps, it can be proved that if $\mathbf{p}_1(t) \le_{st} \mathbf{p}_2(t)$ then $(t - X_{r:n}|T_1 < t) \le_{st} (t - X_{r:n}|T_2 < t)$ which in turn implies that $M_{1n}^r(t) \le M_{2n}^r(t)$. In Theorem 4.2 under the assumption that $r(t)$ is decreasing then, using the same argument as used to prove the theorem, it can be verified that when $0 < t_1 < t_2$ then $(t_1 - X_{r:n}|T < t_1) \le_{st} (t_2 - X_{r:n}|T < t_2)$ which implies that $M_n^r(t)$ is increasing in t. Also in Theorem 4.3 we can show under the assumption that $r_F(t) \ge r_G(t)$, $(t - X_{r:n}|T_1 < t) \le_{st} (t - X_{r:n}|T_2 < t)$.

References

Asadi, M. 2006. On the mean past lifetime of components of a parallel system. *Journal of Statistical Planning and Inference* 136:1197–1206.

Asadi, M., and I. Bairamov. 2005. A note on the mean residual life function of a parallel system. *Communications in Statistics. Theory and Methods* 34:475–484.

Asadi, M., and I. Bairamov. 2006. The mean residual life function of a k-out-of-n structure at the system level. *IEEE Transactions on Reliability* 55:314–318.

Asadi, M., and A. Berred. 2011. Properties and estimation of mean past lifetime. *Statistics*. doi:10.1080/02331888.2010.540666.

Asadi, M., and S. Goliforushani. 2008. On the mean residual life function of coherent systems. *IEEE Transactions on Reliability* 57:574–580.

Bairamov, I., M. Ahsanullah, and I. Akhundov. 2002. A residual life function of a system having parallel or series structures. *Journal of Statistical Theory and Methods* 1:119–132.

David, H., and H.N. Nagaraja. 2003. Order statistics. 3th edn. John Wiley and Sons.

Finkelstein, M.S. 2002. On the reversed hazard rate. *Reliability Engineering & Systems Safety* 78:71–75.

Khaledi, B.E., and M. Shaked. 2007. Ordering conditional lifetimes of coherent systems. *Journal of Statistical Planning and Inference* 137:1173–1184.

Kochar, S., H. Mukerjee, and F.J. Samaniego. 1999. The “signature” of a coherent system and its application to comparison among systems. *Naval Research Logistics* 46:507–523.

Li, X., and Z. Zhang. 2008. Some stochastic comparisons of conditional coherent systems. *Applied Stochastic Models in Business and Industry* 24:541–549.
Li, X., and P. Zhao. 2008. Stochastic comparisons on general inactivity times and general residual life of k-out-of-n systems. *Communications in Statistics. Simulation and Computation* 37:1005–1019.
Navarro, J., and P.J. Hernandez. 2008. Mean residual life functions of finite mixtures and systems. *Metrika* 67:277–298.
Navarro, J., and T. Rychlik. 2007. Reliability and expectation bounds for coherent systems with exchangeable components. *Journal of Multivariate Analysis* 98:102–113.
Navarro, J., and M. Shaked. 2006. Hazard rate ordering of order statistics and systems. *Journal of Applied Probability* 43:391–408.
Navarro, J., F. Belzunce, and J.M. Ruiz. 1997. New stochatic orders based on double truncation. *Probability in the Engineering and Informational Sciences* 11:395–402.
Navarro, J., J.M. Ruiz, and C.J. Sandoval. 2005. A note on comparisons among coherent systems with dependent components using signatures. *Statistics & Probability Letters* 72:179–185.
Navarro, J., J.M. Ruiz, and C.J. Sandoval. 2007. Properties of coherent systems with dependent components. *Communications in Statistics. Theory and Methods* 36:175–191.
Navarro, J., N. Balakrishnan, and F.J. Samaniego. 2008. Mixture reptesentations of residual lifetimes of used systems. *Journal of Applied Probability* 45:1097–1112.
Navarro, J., F.J. Samaniego, N. Balakrishnan, and D. Bhattacharya. 2008. On the application and extension of system signatures in engineering reliability. *Naval Research Logistics* 55:313–327.
Samaniego, F.J. 1985. On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability* 34:69–72.
Samaniego, F.J., N. Balakrishnan, and J. Navarro. 2009. Dynamic signatures and their use in comparing the reliability of new and used systems. *Naval Research Logistics* 56:577–591.
Shaked, M., and J.G. Shanthikumar. 2007. Stochastic orders. Springer, New York.
Tavangar, M., and M. Asadi. 2009. A study on the mean past lifetime of the components of $(n - k + 1)$-out-of-n system at the system level. *Metrika* 72:59–73.
Zhang, Z. 2010. Ordering conditional general coherent systems with exchangeable components. *Journal of Statistical Planning and Inference* 140:454–460.