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Continuous-time MISO fractional system identification using higher-order-statistics

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Abstract

In this paper, the problem of identifying Multiple-Input-Single-Output (MISO) systems with fractional models from noisy input-output available data is studied. The proposed idea is to use Higher-Order-Statistics (HOS), like fourth-order cumulants (*foc*), instead of noisy measurements. Thus, a fractional fourth-order cumulants based-simplified and refined instrumental variable algorithm (*frac-foc-sriv*) is first developed. Assuming that all differentiation orders are known a priori, it consists in estimating the linear coefficients of all Single-Input-Single-Output (SISO) sub-models composing the MISO model. Then, the *frac-foc-sriv* algorithm is combined with a nonlinear optimization technique to estimate all the parameters: coefficients and orders. The performances of the developed algorithms are analyzed using numerical examples. Thanks to fourth-order cumulants, which are insensitive to Gaussian noise, and the iterative strategy of the instrumental variable algorithm, the parameters estimation is consistent.

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1 Introduction

Continuous-time (CT) system identification involves using measured input-output data to build mathematical models that characterize the behavior of a system. Several methods have therefore been developed in the literature to identify CT systems using rational models [9–11, 20, 28, 35]. In many real applications such as heat transfer, electrochemical and lithium-ion batteries, system behavior can be described more accurately with fractional models than with rational models [1, 8, 15–17, 21, 26, 34, 36, 37]. This greater accuracy is due to the global characterization of fractional differentiation. Consequently, various approches have been developed in the literature to deal with system identification with fractional models (see [7, 22, 29, 31, 32, 38–40, 42] and references therein for an overview). Most of the developed approaches have focused on the problem of identifying SISO (Single-Input-Single-Output) and MIMO (Multi-Input-Multi-Output) systems. All approches have been developed when only the output measurements are noisy. In the case where both input and output measurements are affected by additive noise, the system is called with Errors-In-Variables (EIV). However, the estimation cannot be consistent when using the classical approaches. This problem has been solved in [41] by using Second-Order-Statistics (SOS) and in [4, 5] by using Higher-Order-Statistics (HOS), such us third-order and fourth-order cumulants.

One of the main difficulties in CT system identification is the computing of time derivatives. Since the goal is to obtain an estimate of a CT system, the knowledge of input and output derivatives is required. However, these derivatives are firstly not exactly computable when using sampled input-output data and secondly may amplify the noise or the error of measurements effect. To overcome this problem, the simplified and refined based-instrumental variable (*sriv*) algorithm has been proposed. Firstly, it has been developed for rational and fractional SISO systems identification in the case where only the output signal is noisy [12, 13, 18, 19, 27, 32, 43]. Then, it has been extended for system identification with fractional models in the EIV context. The developed estimator is based on fourth-order cumulants [2]. The obtained results were satisfactory, the estimator gave unbiased estimates with minimal variances in presence of important noise corrupting the input and output signals. In [33], it has been developed for fractional MISO systems identification in the case where only the output is noisy. In [25], the *sriv* estimator based on third-order cumulants has been developed for MISO system identification with fractional models in the EIV framework. But the assumption of non-symmetry of the input signals on which the method is relying is restrictive. To solve this problem and in early work developed in [3], the ordinary least squares method based on fourth-order cumulants has been proposed. It gave consistent results in presence of noise corrupting both input and output measurements. However, the variance of the estimates becomes larger when the noise level becomes significant.

Hence, motivated by the studied works mentioned above, the main contribution of this paper is to develop a new *sriv* estimator based on fourth-order cumulants. Firstly, the differentiation orders are supposed to be known and only the linear coefficients are estimated. The developed algorithm is called fractional fourth-order cumulants based-simplified and refined instrumental variable (*frac-foc-sriv*). Secondly, all parameters of
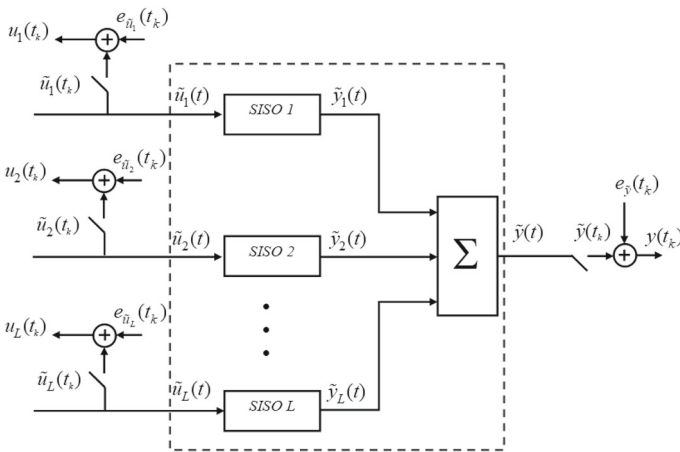
Fig. 1 Fractional LTI MISO system with Errors-In-Variables

MISO fractional model are estimated by assuming that the structures of all subsystems composing the MISO fractional system are the same and have a prior knowledge. Indeed, three cases are established: in the first case a global commensurate order is estimated. In the second case, the local commensurate orders for all SISO subsystems are estimated. In the third case, all differentiation orders are estimated. The proposed idea consists in combining the *frac-foc-sriv* with a nonlinear optimization technique. The developed algorithm is called *frac-foc-sriv* combined with optimization orders (*frac-foc-oosriv*).

The remainder of this paper is organized as follows: the MISO fractional systems with Errors-In-Variables are presented is Section 2. Section 3 describes the problem statement of CT MISO system identification with fractional models in the EIV framework. In Sections 4 and 5 the developed algorithms are detailed. Section 6 concludes the paper.

2 MISO fractional systems with errors-in-variables

As illustrated in Figure 1, a fractional MISO Linear-Time-Invariant (LTI) system with noise-affected input and output signals is represented by the following model:

$$(\mathcal{H}) \begin{cases} \left(1 + \sum_{n=1}^{N_\ell} a_{n,\ell} p^{\alpha_{n,\ell}}\right) \tilde{y}_\ell(t) = \left(\sum_{m=0}^{M_\ell} b_{m,\ell} p^{\beta_{m,\ell}}\right) \tilde{u}_\ell(t), \ \ell = 1, ..., L \\ \tilde{y}(t) = \sum_{\ell=1}^{L} \tilde{y}_\ell(t) \\ u_\ell(t_k) = \tilde{u}_\ell(t_k) + e_{\tilde{u}_\ell}(t_k), \ \ell = 1, ..., L \\ y(t_k) = \tilde{y}(t_k) + e_{\tilde{y}}(t_k). \end{cases} \tag{2.1}$$

L is the number of SISO subsystems.

$\{\tilde{u}_1(t), ..., \tilde{u}_\ell(t), ..., \tilde{u}_L(t)\}$ and $\{\tilde{y}_1(t), ..., \tilde{y}_\ell(t), ..., \tilde{y}_L(t)\}$ denote, respectively, the noise-free input and output signals. The discrete-time available input and output measurements are $\{u_1(t_k), ..., u_l(t_k), ..., u_L(t_k)\}$ and $y(t_k)$. $e_{\tilde{u}_\ell}(t_k)$ and $e_{\tilde{y}}(t_k)$ stand for discrete-time noise affecting, respectively, the input and the output measurements. They are defined, respectively, by the following equations:

$$e_{\tilde{u}_\ell}(t_k) = H_{\tilde{u}_\ell}(q^{-1})e^0_{\tilde{u}_\ell}(t_k), \ \ell = 1, ..., L, \tag{2.2}$$

$$e_{\tilde{y}}(t_k) = H_{\tilde{y}}(q^{-1})e^0_{\tilde{y}}(t_k), \tag{2.3}$$

q^{-1} represent the backward shift operator and $e^0_{\tilde{u}_\ell}(t_k)$, $e^0_{\tilde{y}}(t_k)$ are zero-mean white Gaussian noises.

The SISO fractional differential equation, relating the noise-free input $\tilde{u}_\ell(t)$ and the noise-free output $\tilde{y}_\ell(t)$, is:

$$\left(1 + \sum_{n=1}^{N_\ell} a_{n,\ell} p^{\alpha_{n,\ell}}\right) \tilde{y}_\ell(t) = \left(\sum_{m=0}^{M_\ell} b_{m,\ell} p^{\beta_{m,\ell}}\right) \tilde{u}_\ell(t), \tag{2.4}$$

$(a_{n,\ell}, b_{m,\ell}) \in \mathbb{R}^2$ are the linear coefficients. $p = D = \left(\dfrac{d}{dt}\right)$ designates the differential operator in the time-domain and the differentiation orders $\alpha_{n,\ell}$ and $\beta_{m,\ell}$ are positive real numbers.

Definition 1 (**Grünwald-Leitnikov derivative approximation**) The υth ($\upsilon \in \mathbb{R}^+$) derivative of a causal function $f(t)$ ($f(t) = 0$ for $t \leq 0$), is computed by using the following equation [14]:

$$D^\upsilon f(t) \simeq \frac{1}{h^\upsilon} \sum_{k=0}^{K} (-1)^k \binom{\upsilon}{k} f(t - kh), \tag{2.5}$$

h and K denote, respectively, the sampling period and the number of samples. $\binom{\upsilon}{k}$ is the Newton binomial generalized to fractional orders:

$$\binom{\upsilon}{k} = \frac{\upsilon(\upsilon - 1)(\upsilon - 2)...(\upsilon - k + 1)}{k!}. \tag{2.6}$$

Definition 2 (**Laplace transform**) The Laplace transform of the υth derivative of a causal function $f(t)$ is defined as the rational case:

$$\mathcal{L}(D^\upsilon f(t)) = s^\upsilon F(s), \tag{2.7}$$

s denotes the Laplace variable.

Applying Definition 2 to equation (2.4), leads to the fractional SISO transfer function:

$$H_\ell(s) = \frac{\sum\limits_{m=0}^{M_\ell} b_{m,\ell} s^{\beta_{m,\ell}}}{1 + \sum\limits_{n=1}^{N_\ell} a_{n,\ell} s^{\alpha_{n,\ell}}}. \tag{2.8}$$

To ensure the MISO model identifiability, all differentiation orders $\alpha_{n,\ell}$ and $\beta_{m,\ell}$ are allowed to be real positive numbers and are ordered as following:

$$0 < \alpha_{1,\ell} < ... < \alpha_{N_\ell,\ell} \;\; ; \;\; 0 \le \beta_{0,\ell} < ... < \beta_{M_\ell,\ell}.$$

Definition 3 (**Local commensurability**) An ℓth SISO subsystem has a local commensurate order, denoted υ_ℓ, if all its differentiation orders are successive integer multiples of υ_ℓ. In that case, equation (2.8) is rewritten as follows:

$$H_{\upsilon_\ell}(s) = \frac{\sum\limits_{i=0}^{n_{b,\ell}} \tilde{b}_i s^{i\upsilon_\ell}}{1 + \sum\limits_{j=1}^{n_{a,\ell}} \tilde{a}_j s^{j\upsilon_\ell}}, \tag{2.9}$$

where $n_{b,\ell} = \dfrac{\beta_{M_\ell,\ell}}{\upsilon_\ell}$ and $n_{a,\ell} = \dfrac{\alpha_{N_\ell,\ell}}{\upsilon_\ell}$ are integers and:

$$\begin{cases} \tilde{b}_{i,\ell} = b_{m,\ell} \text{ if } \exists m \in \{0, 1, \ldots, M_\ell\} \text{ such that } i\upsilon_\ell = \beta_{m,\ell} \\ \tilde{b}_{i,\ell} = 0 \text{ otherwise} \\ \tilde{a}_{j,\ell} = a_{n,\ell} \text{ if } \exists n \in \{1, \ldots, N_\ell\} \text{ such that } j\upsilon_\ell = \alpha_{n,\ell} \\ \tilde{a}_{j,\ell} = 0 \text{ otherwise.} \end{cases} \tag{2.10}$$

Example 1 Let us take, as an example, the following SISO transfer function:

$$H_1(s) = \frac{0.5 + s^{0.3}}{1 + 2s^{0.3} + s^{0.6}}. \tag{2.11}$$

The local commensurate order is $\upsilon_1 = 0.3$. Indeed, according to Definition 3, equation (2.11) is rewritten as follows:

$$H_{\upsilon_1}(s) = \frac{0.5 + s^{1\times0.3}}{1 + 2s^{1\times0.3} + s^{2\times0.3}}. \tag{2.12}$$

Definition 4 (**Global commensurability**) Consider a fractional MISO system such that all SISO transfer functions have local commensurate orders υ_ℓ. If these are integer multiples of the same order, then there is a global commensurability. The order is called global commensurate order and is denoted υ_G.

For each ℓth SISO subsystem, a vector containing the fractional differentiation orders can be defined:

$$\eta_\ell = \left[\alpha_{1,\ell}, ..., \alpha_{N_\ell,\ell}, \beta_{0,\ell}, ..., \beta_{M_\ell,\ell}\right]^T. \tag{2.13}$$

The integers multiplying the global commensurate order are grouped in a vector as follows:

$$\chi_\ell = \left[\gamma_{1,\ell}, ..., \gamma_{N_\ell,\ell}, \delta_{0,\ell}, ..., \delta_{M_\ell,\ell}\right]^T, \tag{2.14}$$

where

$$\begin{cases} \text{if } a_{n,\ell} \neq 0 \quad \text{then} \quad \gamma_{n,\ell} = \dfrac{\alpha_{n,\ell}}{\upsilon_G} \text{ else } \gamma_{n,\ell} = 0, \quad n = 1, ..., N_\ell, \\[3mm] \text{if } b_{m,\ell} \neq 0 \quad \text{then} \quad \delta_{m,\ell} = \dfrac{\beta_{m,\ell}}{\upsilon_G} \text{ else } \delta_{m,\ell} = 0, \quad m = 0, ..., M_\ell. \end{cases} \tag{2.15}$$

Example 2 Take the following example of MISO fractional system:

$$\begin{cases} H_1(s) = \dfrac{1}{1 + 0.5s^{0.4} + 2s^{0.8}} \\[3mm] H_2(s) = \dfrac{1.5}{1 + 2s^{0.6} + s^{1.2}} \\[3mm] H_3(s) = \dfrac{3}{1 + 1.5s^{0.8} + 0.5s^{1.6}}. \end{cases} \tag{2.16}$$

According to Definition 3, the local commensurate orders are:

$$\begin{aligned} \upsilon_1 &= 0.4 = 2 \times 0.2 \\ \upsilon_2 &= 0.6 = 3 \times 0.2 \\ \upsilon_3 &= 0.8 = 4 \times 0.2. \end{aligned} \tag{2.17}$$

So, the global commensurate order is $\upsilon_G = 0.2$. Equation (2.16) can be rewritten as follows:

$$\begin{cases} H_1(s) = \dfrac{1}{1 + 0.5s^{1 \times \upsilon_1} + 2s^{2 \times \upsilon_1}} = \dfrac{1}{1 + 0.5s^{2 \times \upsilon_G} + 2s^{4 \times \upsilon_G}} \\[3mm] H_2(s) = \dfrac{1.5}{1 + 2s^{1 \times \upsilon_2} + s^{2 \times \upsilon_2}} = \dfrac{1.5}{1 + 2s^{3 \times \upsilon_G} + s^{6 \times \upsilon_G}} \\[3mm] H_3(s) = \dfrac{3}{1 + 1.5s^{1 \times \upsilon_3} + 0.5s^{2 \times \upsilon_3}} = \dfrac{3}{1 + 1.5s^{4 \times \upsilon_G} + 0.5s^{8 \times \upsilon_G}}. \end{cases} \tag{2.18}$$

Theorem 1 (Bounded-Input-Bounded-Output stability of fractional MISO systems) *A fractional MISO system is stable in the Bounded-Input-Bounded-Output (BIBO) sense if and only if all its SISO subsystems are stable in the BIBO sense.*

Consider an ℓth SISO fractional subsystem represented by equation (2.9). The latter can be rewritten as follows:

$$H_{\upsilon_\ell}(s) = \frac{Z_{\upsilon_\ell}(s)}{P_{\upsilon_\ell}(s)}. \tag{2.19}$$

Solving equation $P_{\upsilon_\ell}(s) = 0$ leads to its poles, denoted $s_{k,\ell}, \ k = 1, ..., n_{a,\ell}$.

Thus, the ℓth SISO subsystem, is BIBO stable if and only if [23]:

$$0 < \upsilon_\ell < 2, \tag{2.20}$$

and

$$\forall s_{k,\ell} \in \mathbb{C}, \quad P_{\upsilon_\ell}(s_{k,\ell}) = 0 \text{ such as } | \arg(s_{k,\ell}) | > \upsilon_\ell \frac{\pi}{2}. \tag{2.21}$$

3 Problem formulation

The aim is to estimate the parameters of the fractional MISO model ($\mathcal{H}$) described by equation (2.1) using available noisy data measured at discrete-time intervals. Two different cases are studied in this paper: the first case assumes that all differentiation orders of all sub-models $H_\ell(p)$ ($\ell = 1, ..., L$) are known a priori and only the linear coefficients are estimated. In the second case, all differentiation orders are assumed to be unknown and estimated at the same time as the linear coefficients.

When only the linear coefficients are estimated, the parameter vector is defined by:

$$\Theta = [\theta_1, ..., \theta_\ell, ..., \theta_L], \tag{3.1}$$

where

$$\theta_\ell = \left[a_{1,\ell}, a_{2,\ell}, ..., a_{N_\ell,\ell}, b_{0,\ell}, b_{1,\ell}, ..., b_{M_\ell,\ell}\right]^T. \tag{3.2}$$

The dimension of this vector is $\sum_{\ell=1}^{L} (N_\ell + M_\ell + 1)$.

When all parameters are estimated: coefficients and differentiation orders, there are three sub-cases to be distinguished:

- **Sub-case 1:** If the fractional MISO system has a global commensurate order, then the parameter vector is rewritten as follows:

$$\Theta = [\theta_1, ..., \theta_\ell, ..., \theta_L, \upsilon_G]. \tag{3.3}$$

The dimension of this vector equals $1 + \sum_{\ell=1}^{L} (N_\ell + M_\ell + 1)$.

- **Sub-case 2:** If the MISO system has not a global commensurate order, or the latter is too small, then the local commensurate order of all subsystems must be estimated. The parameter vector is rewritten as follows:

$$\Theta = [\theta_1, ..., \theta_\ell, ..., \theta_L, \upsilon], \tag{3.4}$$

where

$$\upsilon = [\upsilon_1, ..., \upsilon_\ell, ..., \upsilon_L]. \tag{3.5}$$

$\upsilon_\ell \{\ell = 1, ..., L\}$ denotes the local commensurate order of the subsystem H_ℓ.

The dimension of this vector equals $\sum_{\ell=1}^{L} (N_\ell + M_\ell + 2)$.

- **Sub-case 3:** If the MISO system is non-commensurate, then the parameter vector is rewritten as follows:

$$\Theta = [\theta_1, ..., \theta_\ell, ..., \theta_L, \eta],$$ (3.6)

where

$$\eta = [\eta_1, ..., \eta_\ell, ..., \eta_L],$$ (3.7)

and

$$\eta_\ell = \left[\alpha_{1,\ell}, ..., \alpha_{N_\ell,\ell}, \beta_{0,\ell}, ..., \beta_{M_\ell,\ell}\right]^T.$$ (3.8)

The dimension of this vector equals $2 \sum_{\ell=1}^{L} (N_\ell + M_\ell + 1)$.

The fractional MISO system identification problem in the EIV context consists in estimating the parameter vector Θ (equation (3.3), (3.4) or (3.6)) using N_t samples of noisy input and output data $\{u_1(t_k), ...u_\ell(t_k), ..., u_L(t_k), y(t_k)\}_{k=1}^{N_t}$. In this work, all sub-cases are handled.

The proposed idea consists in decomposing the fractional MISO system into L SISO subsystems. The parameters of the ℓth subsystem are estimated while assuming that the parameters of all others q^{th} ($q \neq \ell$) are known.

The noisy output signal of the ℓth subsystem, denoted $x_\ell(t_k)$, is obtained as follows:

$$x_\ell(t_k) = y(t_k) - \sum_{q=1,q\neq\ell}^{L} \tilde{y}_q(t_k).$$ (3.9)

As is well-known for rational and fractional systems, classical system identification methods cannot be applied in the EIV framework. Indeed, the available input and output data are noisy, and this often leads to biased estimates. In this context, a consistent estimation may be obtained either by using identification methods based on second-order statistics or by using identification methods based on Higher-Order Statistics (such as third- and fourth-order cumulants based methods).

In this work, the use of fourth-order cross-cumulants (*foc*) is proposed because of the advantage of being immune to colored Gaussian noise, as well as white Gaussian noise. Definitions and properties related to the *foc* are given in [3].

The following definition will be used in the developed algorithms.

Definition 5 (**Fourth-order cross-cumulant estimation**) The fourth-order cross-cumulant is estimated from N_t samples of processes $\chi_i(t_k)$, $i = 1, ..., 4$, by replacing the mathematical expectations by sample averages:

$$\hat{C}_{\chi_1\chi_2\chi_3\chi_4}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$$

$$= \frac{N_t + 2}{N_t(N_t - 1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k + \mathcal{T}_1)\chi_3(t_k + \mathcal{T}_2)\chi_4(t_k + \mathcal{T}_3)$$

$$- \frac{3}{N_t(N_t - 1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k + \mathcal{T}_1) \sum_{k=1}^{N_t} \chi_3(t_k + \mathcal{T}_2)\chi_4(t_k + \mathcal{T}_3)$$

$$-\frac{3}{N_t(N_t-1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k+\mathcal{T}_2) \sum_{k=1}^{N_t} \chi_3(t_k+\mathcal{T}_1)\chi_4(t_k+\mathcal{T}_3)$$

$$-\frac{3}{N_t(N_t-1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k+\mathcal{T}_3) \sum_{k=1}^{N_t} \chi_3(t_k+\mathcal{T}_1)\chi_4(t_k+\mathcal{T}_2).$$

(3.10)

$\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are the discrete-time gaps.

Remark 1 The fourth-order cumulants (respectively cross-cumulants) depend on several variables, namely N_t, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$. These variables contain a big number of informations which may, as a consequence, be redundant in the cumulants estimation. The proposed idea is to use:

- some parts of the time space[1]: these parts are dependent on a single variable and are called cumulant lines. For example, a fourth-order cumulant line can be obtained by fixing $\mathcal{T}_2$ and $\mathcal{T}_3$ as a constant chosen arbitrarily and $\mathcal{T}_1$ variable [5, 30].
- some part of sampled time data: a part of N_t samples of data is suffisant to obtain unbiased cumulants (or cross-cumulants) estimates. This part is denoted N_h.

When using the same signal χ, $C_{\chi\chi\chi\chi}(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ is called cumulant. To simplify writing, $(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ will be replaced by $\mathcal{T}$ in the rest of the paper.

Remark 2

- Taking into account Remark 1 and choosing $\mathcal{T}_2 = \mathcal{T}_3 = 0$ leads to the following fourth-order cross-cumulant estimates:

$$\hat{C}_{\chi_1\chi_2\chi_3\chi_4}(\mathcal{T}) = \frac{N_t+2}{N_t(N_t-1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k+\mathcal{T})\chi_3(t_k)\chi_4(t_k)$$

$$-\frac{3}{N_t(N_t-1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k+\mathcal{T}) \sum_{k=1}^{N_t} \chi_3(t_k)\chi_4(t_k)$$

$$-\frac{3}{N_t(N_t-1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k) \sum_{k=1}^{N_t} \chi_3(t_k+\mathcal{T})\chi_4(t_k)$$

$$-\frac{3}{N_t(N_t-1)} \sum_{k=1}^{N_t} \chi_1(t_k)\chi_2(t_k) \sum_{k=1}^{N_t} \chi_3(t_k+\mathcal{T})\chi_4(t_k).$$

(3.11)

$\mathcal{T} = \mathcal{T}_1$ which varies from 1 to N_h.
- The refining of the part of sampled time data N_h depends essentially on the system, which is supposed to be unknown. The optimal value of N_h can therefore be chosen by experimental trials.

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[1] The time space contains the lines of cumulants obtained for N_t samples of data

4 Linear coefficients estimation

With regard to the MISO fractional systems in the EIV framework, the following assumptions can be made.

Assumptions 1

- **A1:** The structures of all SISO subsystems are supposed to be known and the same.
- **A2:** The real orders $\alpha_{n,\ell}(n = 1, ..., N_\ell)$ and $\beta_{m,\ell}(m = 0, ..., M_\ell)$ for $\{\ell = 1, ..., L\}$ are supposed to be known.
- **A3:** The noise-free input signals $\tilde{u}_\ell$ $\{\ell = 1, ..., L\}$ are zero mean stationary stochastic processes such that their fourth-order cumulants are non-zero. Their probability density function (pdf) cannot therefore be Gaussian.
- **A4:** $e_{\tilde{u}_\ell}(t_k)$, $\{\ell = 1, ..., L\}$, $e_{\tilde{y}}(t_k)$ are stationary zero-mean random variables independent of $\tilde{u}_\ell$ $\{l = 1, ..., L\}$ and $\tilde{y}$ and having a Gaussian pdf.

4.1 Fourth-order cumulants based-simplified and refined instrumental variable algorithm

Under assumptions **A1-A4**, the estimation of fourth-order cross-cumulants is consistent. Indeed, additive noise has a Gaussian pdf, so their fourth-order cross-cumulants equal zero (assumption **A4**), and the estimates of fourth-order cross-cumulants of inputs and output, therefore, contain only unbiased data [3]. Hence fourth-order cross-cumulants of inputs and output will be used instead of noisy data.

Like the SISO fractional systems, the ℓth $(\ell = 1, ..., L)$ subsystem is described by the following model [2, 4]:

$$C_{u_\ell x_\ell u_\ell u_\ell}(\tau) = \frac{\sum_{m=0}^{M_\ell} b_{m,\ell} p^{\beta_{m,\ell}}}{1 + \sum_{n=1}^{N_\ell} a_{n,\ell} p^{\alpha_{n,\ell}}} C_{u_\ell u_\ell u_\ell u_\ell}(\tau), \tag{4.1}$$

where τ is the continuous-time gap.

The use of fourth-order cross-cumulants estimates leads to:

$$\hat{C}_{u_\ell x_\ell u_\ell u_\ell}(\tau) = \frac{\sum_{m=0}^{M_\ell} b_{m,\ell} p^{\beta_{m,\ell}}}{1 + \sum_{n=1}^{N_\ell} a_{n,\ell} p^{\alpha_{n,\ell}}} \hat{C}_{u_\ell u_\ell u_\ell u_\ell}(\tau) + \epsilon_\ell(\tau). \tag{4.2}$$

An ℓth output error can therefore be defined as:

$$\epsilon_\ell(\tau) = \hat{C}_{u_\ell x_\ell u_\ell u_\ell}(\tau) - \frac{\mathcal{Z}_\ell(p)}{\mathcal{P}_\ell(p)} \hat{C}_{u_\ell u_\ell u_\ell u_\ell}(\tau). \tag{4.3}$$

The polynomials $\mathcal{Z}_\ell(p)$ and $\mathcal{P}_\ell(p)$ are defined respectively by:

$$\begin{cases} \mathcal{Z}_\ell(p) = \sum_{m=0}^{M_\ell} b_{m,\ell} \, p^{\beta_{m,\ell}} \\ \mathcal{P}_\ell(p) = 1 + \sum_{n=1}^{N_\ell} a_{n,\ell} \, p^{\alpha_{n,\ell}}. \end{cases} \tag{4.4}$$

Equation (4.3) can be rewritten as follows:

$$\begin{aligned} \epsilon_\ell(\tau) &= \mathcal{P}_\ell(p) \left(\frac{\hat{C}_{u_\ell x_\ell u_\ell u_\ell}(\tau)}{\mathcal{P}_\ell(p)} \right) - \mathcal{Z}_\ell(p) \left(\frac{\hat{C}_{u_\ell u_\ell u_\ell u_\ell}(\tau)}{\mathcal{P}_\ell(p)} \right) \\ &= \mathcal{P}_\ell(p) \hat{C}_{u_\ell x_\ell, f u_\ell u_\ell}(\tau) - \mathcal{Z}_\ell(p) \hat{C}_{u_\ell u_\ell, f u_\ell u_\ell}(\tau) \\ &= \hat{C}_{u_\ell x_\ell, f u_\ell u_\ell}(\tau) - \hat{\varphi}_{\ell, f}^T(\tau) \theta_\ell, \end{aligned} \tag{4.5}$$

θ_ℓ is defined by equation (3.2) and $\hat{\varphi}_{\ell, f}^T(\tau)$ contains the fractional derivatives of the filtered fourth-order cross-cumulants:

$$\begin{aligned} \hat{\varphi}_{\ell, f}^T(\tau) = \Big[&-p^{\alpha_{1,\ell}} \hat{C}_{u_\ell x_\ell, f u_\ell u_\ell}(\tau), \ldots, -p^{\alpha_{N_\ell,\ell}} \hat{C}_{u_\ell x_\ell, f u_\ell u_\ell}(\tau), \\ &p^{\beta_{0,\ell}} \hat{C}_{u_\ell u_\ell, f u_\ell u_\ell}(\tau), \ldots, p^{\beta_{M_\ell,\ell}} \hat{C}_{u_\ell u_\ell, f u_\ell u_\ell}(\tau) \Big], \end{aligned} \tag{4.6}$$

$\hat{C}_{u_\ell x_\ell, f u_\ell u_\ell}(\tau)$ and $\hat{C}_{u_\ell u_\ell, f u_\ell u_\ell}(\tau)$ are computed using the following filter:

$$F_\ell(p) = \frac{1}{\mathcal{P}_\ell(p)}. \tag{4.7}$$

It contains the true parameters of the ℓth subsystem which are unknown but can be estimated. Thus, the proposed idea is to generate the filter iteratively by using the estimates obtained at each iteration i,

$$F_\ell^i(p) = \frac{1}{\hat{\mathcal{P}}_\ell^{i-1}(p)} = \frac{1}{1 + \sum_{n=1}^{N_\ell} \hat{a}_{n,\ell}^{i-1} \, p^{\alpha_{n,\ell}}}. \tag{4.8}$$

The optimal parameter vector, obtained at each iteration i, is the solution of the following problem:

$$\hat{\theta}_\ell^i = \arg \min_{\theta_\ell^i} \left(J(\theta_\ell^i) \right), \tag{4.9}$$

obtained by minimizing the criterion:

$$J\left(\theta_\ell^i\right) = \frac{1}{N_h} \sum_{\tau=1}^{N_h} \frac{1}{2} \left(\left(\epsilon_\ell^i(\tau)\right)^T \epsilon_\ell^i(\tau) \right), \tag{4.10}$$

$\epsilon_\ell^i(t_k)$ denotes the output error of the ℓth ($\ell = 1, ..., L$) subsystem computed at each iteration i according to the estimates computed at iteration $i - 1$.

The optimal solution define the *frac-foc-sriv* estimator:

$$\hat{\theta}_\ell^i = \left(\frac{1}{N_h} \sum_{\tau=1}^{N_h} \hat{\xi}_{\ell,f}^i(\tau) \hat{\varphi}_{\ell,f}^{i,T}(\tau) \right)^{-1} \cdot \left(\frac{1}{N_h} \sum_{\tau=1}^{N_h} \hat{\xi}_{\ell,f}^i(\tau) \hat{C}_{u_\ell x_{\ell,f} u_\ell u_\ell}(\tau) \right), \quad (4.11)$$

for $\ell = 1, ..., L$. $\hat{\varphi}_{\ell,f}^{i,T}(\tau)$ and $\hat{\xi}_{\ell,f}^i(\tau)$ are defined, respectively, by equations (4.12) and (4.13):

$$\hat{\varphi}_{\ell,f}^{i,T}(\tau) = \left[-\hat{C}_{u_\ell \kappa^{i,1} u_\ell u_\ell}(\tau), ..., -\hat{C}_{u_\ell \kappa^{i,N_\ell} u_\ell u_\ell}(\tau), \right.$$
$$\left. \hat{C}_{u_\ell \varrho^{i,0} u_\ell u_\ell}(\tau), ..., \hat{C}_{u_\ell \varrho^{i,M_\ell} u_\ell u_\ell}(\tau) \right], \quad (4.12)$$

where:

- $\kappa^{i,n} = x_{\ell,f}^{i,\alpha_{n,\ell}}$, $\{n = 1, ..., N_\ell\}$ is the fractional derivative of the ℓth filtered output signal,
- $\varrho^{i,m} = u_{\ell,f}^{\beta_{m,\ell}}$, $\{m = 0, ..., M_\ell\}$ is the fractional derivative of the ℓth filtered input signal.

$$\hat{\xi}_{\ell,f}^T(\tau) = \left[-\hat{C}_{u_\ell \hat{\kappa}^{i,1} u_\ell u_\ell}(\tau), ..., -\hat{C}_{u_\ell \hat{\kappa}^{i,N_\ell} u_\ell u_\ell}(\tau), \right.$$
$$\left. \hat{C}_{u_\ell \varrho^{i,0} u_\ell u_\ell}(\tau), ..., \hat{C}_{u_\ell \varrho^{i,M_\ell} u_\ell u_\ell}(\tau) \right], \quad (4.13)$$

$\hat{\kappa}^{i,n} = \hat{x}_{\ell,f}^{i,\alpha_{n,\ell}}$, $\{n = 1, ..., N_\ell\}$ is the fractional derivative of the ℓth estimated and filtered output signal.

The *frac-foc-sriv* algorithm is summarized in three steps:

---

frac-foc-sriv algorithm

- **Step 1:** iteration $i = 0$

  ⋄ compute the first estimate $\hat{\Theta}^0 = \left[\hat{\theta}_1^0, ..., \hat{\theta}_\ell^0, ..., \hat{\theta}_L^0 \right]$ by applying the *frac-foc-ls* algorithm [3]:

$$\hat{\theta}_\ell^0 = \left(\frac{1}{N_h} \sum_{\tau=1}^{N_h} \hat{\varphi}_{\ell,f}(\tau) \hat{\varphi}_{\ell,f}^T(\tau) \right)^{-1} \cdot \left(\frac{1}{N_h} \sum_{\tau=1}^{N_h} \hat{\varphi}_{\ell,f}(\tau) \hat{C}_{u_\ell x_{\ell,f} u_\ell u_\ell}(\tau) \right), \quad (4.14)$$

for $\ell = 1, ..., L$.

As long as the first estimate is not available, the following state variable filter (SVF) is used in this step:

$$F_\ell(p) = \frac{1}{\left(1 + \omega_{f,\ell} p\right)^{N_{f,\ell}}}, \quad (4.15)$$

$N_{f,\ell}$ and $\omega_{f,\ell}$ represent its order and cut-off frequency respectively. The SVF parameters tuning is detailed in [3].

Fractional derivatives of x_ℓ and u_ℓ signals are computed as following:

$$
\begin{cases}
\kappa^{0,n} = x_{\ell,f}^{\alpha_{n,\ell}} = \dfrac{p^{\alpha_{n,\ell}}}{\left(1 + \omega_{f,\ell} p\right)^{N_{f,\ell}}} x_\ell, \ \{n = 1, ..., N_\ell\} \\[4mm]
\varrho^{0,m} = u_{\ell,f}^{\beta_{m,\ell}} = \dfrac{p^{\beta_{m,\ell}}}{\left(1 + \omega_{f,\ell} p\right)^{N_{f,\ell}}} u_\ell, \ \{m = 0, ..., M_\ell\}.
\end{cases}
\tag{4.16}
$$

$\diamond$ compute the estimated output of the fractional MISO system $\hat{y}^0(t_k)$
- **Step 2:** iteration $i = i + 1$
  for each subsystem $\ell = 1, ..., L$:
  $\diamond$ compute the ℓth output $x_\ell^i(t_k)$ and the estimated output $\hat{x}_\ell^i(t_k)$
  $\diamond$ compute the fractional derivatives of the filtered input and output signals

$$
\begin{cases}
\kappa^{i,n} = \dfrac{p^{\alpha_{n,\ell}}}{1 + \sum\limits_{n=1}^{N_\ell} \hat{a}_{n,\ell}^{i-1} p^{\alpha_{n,\ell}}} x_\ell \\[6mm]
\hat{\kappa}^{i,n} = \dfrac{p^{\alpha_{n,\ell}}}{1 + \sum\limits_{n=1}^{N_\ell} \hat{a}_{n,\ell}^{i-1} p^{\alpha_{n,\ell}}} \hat{x}_\ell^i \\[6mm]
\varrho^{i,m} = \dfrac{p^{\beta_{m,\ell}}}{1 + \sum\limits_{n=1}^{N_\ell} \hat{a}_{n,\ell}^{i-1} p^{\alpha_{n,\ell}}} u_\ell \\[4mm]
n = 1, ..., N_\ell, \ m = 0, ..., M_\ell.
\end{cases}
\tag{4.17}
$$

$\diamond$ compute the fractional derivatives of fourth-order cross-cumulants estimates according to equation (3.11) [3].
$\diamond$ built the regression and instrument vectors by using equations (4.12) and (4.13).
$\diamond$ estimate the parameter vector using equation (4.11).
- **Step 3:** go back to **Step 2** until:

$$
\frac{\left\| \hat{\Theta}^i - \hat{\Theta}^{i-1} \right\|}{\left\| \hat{\Theta}^{i-1} \right\|} < \delta, \text{ for example } \delta = 10^{-5}
$$

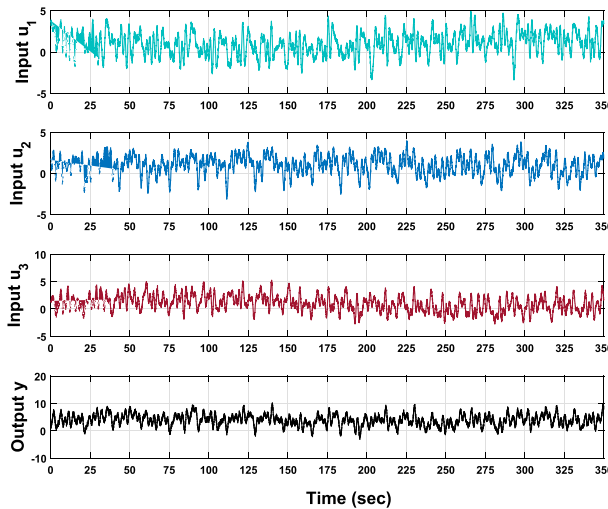or a maximum number of iterations is reached ($i > i_{max}$).

Fig. 2 Three input and single output signals used for system identification

4.2 Example 1

The following fractional system with three inputs and single output is used to analyze the performance of the developed *frac-foc-sriv* algorithm:

$$(\mathcal{H}_1): \begin{cases} \tilde{y}(t) = H_1(p)\tilde{u}_1(t) + H_2(p)\tilde{u}_2(t) + H_3(p)\tilde{u}_3(t) \\ H_1(p) = \dfrac{b_{0,1}}{1 + a_{1,1}p^{\alpha_{1,1}}} = \dfrac{1.5}{1 + 3p^{0.8}} \\ H_2(p) = \dfrac{b_{0,2}}{1 + a_{1,2}p^{\alpha_{1,2}}} = \dfrac{2}{1 + p^{0.2}} \\ H_3(p) = \dfrac{b_{0,3}}{1 + a_{1,3}p^{\alpha_{1,3}}} = \dfrac{1}{1 + 0.5p^{1.2}} \\ u_\ell(t_k) = \tilde{u}_\ell(t_k) + e_{\tilde{u}_\ell}(t_k), \ell = 1, 2, 3 \\ y(t_k) = \tilde{y}(t_k) + e_{\tilde{y}}(t_k). \end{cases} \tag{4.18}$$

The noise-free inputs are chosen as an uncorrelated multisine signals with a non Gaussian pdf. The available data, plotted in Figure 2, are sampled uniformly with a sampling period $h = 0.05$ sec. The number of samples is $N_t = 7000$.

The performance is assessed through $N_{mc} = 100$ runs of Monte Carlo simulations with different noise realizations. Different additive white noise corrupt the input signals with a Signal-to-Noise-Ratio (SNR) denoted SNR_u and the output signal with a SNR denoted SNR_y. They are defined, respectively, by:

$$SNR_u = 10 \log \left(\frac{\text{var}(u_\ell)}{\text{var}(e_{\tilde{u}_\ell})} \right), \quad \ell = 1, ..., L, \tag{4.19}$$

$$SNR_y = 10 \log \left(\frac{\text{var}(y)}{\text{var}(e_{\tilde{y}})} \right), \tag{4.20}$$

var is the variance.

Table 1 Parameter estimation obtained by applying the *frac-foc-ls* and the *frac-foc-sriv* algorithms in presence of white noise ($SNR_u=SNR_y$=20dB and 10dB)

| True | $SNR_u=SNR_y$=20dB | | | | $SNR_u=SNR_y$=10dB | | | |
| | frac-foc-ls | | frac-foc-sriv | | frac-foc-ls | | frac-foc-sriv | |
	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$
$a_{1,1}=3$	2.9961	0.0557	3.0010	0.0498	2.9249	0.1641	3.0105	0.1534
$b_{0,1}=1.5$	1.5001	0.0061	1.0052	0.0060	1.4941	0.0195	1.4955	0.0195
$a_{1,2}=1$	1.0060	0.0401	1.0042	0.0257	0.9595	0.1410	1.0002	0.0896
$b_{0,2}=2$	2.0043	0.0239	2.0033	0.0175	1.9748	0.0908	1.9960	0.0653
$a_{1,3}=0.5$	0.5002	0.0238	0.50006	0.0043	0.5118	0.1147	0.5016	0.0139
$b_{0,3}=1$	1.0006	0.0049	1.0005	0.0049	0.9970	0.0167	0.9969	0.0163
$NRQE$	0.0018		0.0014		0.0066		0.0046	

In this section, all the structures of the SISO subsystems making up the MISO system are assumed to have prior knowledge (**A1** satisfied).

4.2.1 Known differentiation orders

Assuming that all differentiation orders are known (assumption $A2$ satisfied), the performance of the *frac-foc-sriv* algorithm is compared with the one obtained with the *frac-foc-ls* algorithm developed in [3] and used in the initialization step. The SVF cut-off frequency used in the *frac-foc-ls* algorithm is set to 5rad/sec and N_h equals 1000.

Both *frac-foc-ls* and *frac-foc-sriv* algorithms are applied for each noise realization. Table 1 provides the obtained results for two levels of noise $SNR_u = SNR_y = 20$dB and $SNR_u = SNR_y = 10$dB. It contains the means of estimates ($\bar{\Theta}$), their standard deviations ($\hat{\sigma}(\Theta)$) and the Normalized Relative Quadratic Error (NRQE)

$$NRQE = \frac{1}{N_{mc}}\sqrt{\frac{\left\|\hat{\Theta}-\Theta_0\right\|^2}{\|\Theta_0\|^2}},\qquad(4.21)$$

Θ_0 denotes the true parameter vector.

Thanks to the iterative strategy of the instrumental variable estimator and the use of fourth-order cumulants, the *frac-foc-sriv* algorithm yields unbiased estimates. Indeed, the means of the estimated parameters are close to the real ones, and the variances are very low. The consistency of the *frac-foc-sriv* algorithm is confirmed by comparing it with the *frac-foc-ls* algorithm. The latter can also provide unbiased estimates in presence of noise with high levels but with larger variances.
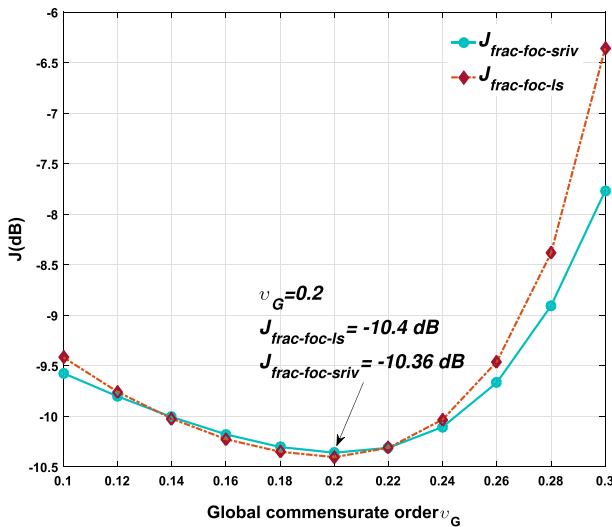
Fig. 3 Cost function versus the global commensurate order

4.2.2 Unknown differentiation orders

If the differentiation orders are unknown (assumption **A2** not satisfied), which is the case of real systems, the estimation of the global commensurate order is crucial. It facilitates the calculation of all differentiation orders if the structures of all SISO sub-models are known (assumption **A1** satisfied). Hence, the influence of the global commensurate order on the parameter estimation is analyzed. The *frac-foc-ls* and *frac-foc-sriv* algorithms are applied for different values of υ_G chosen in a range between 0.1 and 0.3 (for $SNR_u = SNR_y = 10$dB).

The following ℓ_2-norm (in dB) of the normalized output error is evaluated for each value of υ_G:

$$J = 10\log\left(\frac{\|y(t) - \hat{y}(t)\|^2}{\|y(t)\|^2}\right), \tag{4.22}$$

where y and $\hat{y}$ denote, respectively, the measured and the estimated output. The minimum value of J must be obtained at the true global commensurate order and would have the value of $NSR = -SNR_y = -10$dB.

The variation of the cost function J versus the global commensurate order is plotted in Figure 3. The plot shows that the minimum value of J is obtained, by both algorithms, at the same value $\upsilon_G = 0.2$.

The modeling error obtained by the *frac-foc-ls* algorithm equals 0.4dB ($J_{frac-foc-ls} = -10.4$dB) and that obtained by the *frac-foc-sriv* algorithm equals 0.36dB ($J_{frac-foc-sriv} = -10.36$dB).

The *frac-foc-sriv* algorithm gives the lowest modeling error for $\upsilon_G = 0.2$.

5 All parameters estimation

5.1 *frac-foc-sriv* combined with optimization orders algorithm

In this section, only assumption **A2** is not satisfied and all fractional orders are estimated together with linear coefficients. It is therefore a matter of dealing with the three sub-cases discussed in Section 3.

The parameter vector is given by equation (3.3), (3.4) or (3.6) and obtained by minimizing the ℓ_2-norm of output error:

$$J(\Theta) = \frac{1}{2}\|\varepsilon(t)\|_2, \tag{5.1}$$

where

$$\varepsilon(t) = y(t) - \hat{y}(t), \tag{5.2}$$

$\hat{y}(t)$ is the estimated output.

This error is linear with respect to the MISO model coefficients but it is nonlinear with respect to the differentiation orders. So, the proposed idea is to estimate the linear coefficients using the *frac-foc-sriv* algorithm and the differentiation orders using a nonlinear optimization technique (such as Gauss-Newton or Levenberg-Marquardt) [3, 24].

Denote by ϑ the differentiation orders vector to be optimized:

- **Case 1:** global commensurate order optimization:

$$\vartheta = \upsilon_G. \tag{5.3}$$

- **Case 2:** local commensurate orders optimization:

$$\vartheta = \nu = [\upsilon_1, ..., \upsilon_\ell, ..., \upsilon_L]. \tag{5.4}$$

- **Case 3:** all differentiation orders optimization:

$$\vartheta = \mu = [\mu_1, ..., \mu_\ell, ..., \mu_L]. \tag{5.5}$$

The differentiation orders vector is estimated iteratively according to:

$$\vartheta^{i+1} = \vartheta^i - \lambda \left[\tilde{H}^{-1}G\right]_{\vartheta=\vartheta^i}, \tag{5.6}$$

where $\tilde{H}$ is the approximated Hessian [6]:

$$\tilde{H} = \frac{\partial \varepsilon^T}{\partial \vartheta} \frac{\partial \varepsilon}{\partial \vartheta}, \tag{5.7}$$

G denotes the gradient, which is defined as following:

$$G = \frac{\partial J}{\partial \vartheta} = \frac{\partial \varepsilon^T}{\partial \vartheta} \varepsilon, \qquad (5.8)$$

and λ is a weighting positive constant which is used to ensure the decrease of J.

The analytic compute of the differentiation orders error sensitivity functions are detailed in [24].

The *frac-foc-sriv* combined with optimization orders algorithm (*frac-foc-oosriv*) is summarized in three steps:

---

frac-foc-oosriv algorithm

- **Step 1:** iteration $i = 0$
  - ◇ initialize ϑ^0 and Λ
  - ◇ initialize $\hat{\theta}_\ell^0$, $\ell = 1, ..., L$ by applying the *frac-foc-sriv* algorithm
  - ◇ form the parameter vector $\Theta^0 = \left[\hat{\theta}_1^0, ..., \hat{\theta}_L^0, \vartheta^0 \right]$
  - ◇ compute J^0 (equation (5.1))
- **Step 2:** iteration $i = i + 1$
  do
  $\lambda = \Lambda$
  do
  - ◇ refine the differentiation orders vector using equation (5.6)
  - ◇ estimate the linear coefficients $\left(\hat{\theta}_\ell^i, \ell = 1, ..., L \right)$ using the *frac-foc-sriv* algorithm
  - ◇ form the parameter vector $\Theta^i = \left[\hat{\theta}_1^i, ..., \hat{\theta}_L^i, \vartheta^i \right]$
  - ◇ evaluate the criterion J^i (equation (5.1))
  - ◇ $\lambda = \dfrac{\lambda}{2}$
  while $J^i > J^{i-1}$
  while

$$\frac{\left\| \hat{\Theta}^i - \hat{\Theta}^{i-1} \right\|}{\left\| \hat{\Theta}^{i-1} \right\|} > \delta, \text{ for example } \delta = 10^{-5}$$

or a maximum number of iterations is reached ($i > i_{max}$).

5.2 Example 2

The consistency of the developed *frac-foc-oosriv* algorithm is studied in this paragraph. The same data generating the fractional MISO model, represented by equation (4.18), are used. Firstly, in accordance with Definition 4, the global commensurate order is estimated together with linear coefficients. The *frac-foc-oosriv* performance is compared with that obtained by applying the output optimization (*oe*) based algorithm. The latter is developed in [24] to deal with fractional MISO systems identification when only the output measurements are noisy. It is used in this paragraph to prove the efficiency of the *frac-foc-oosriv* algorithm when input-output measurements are noisy. Secondly, according to Definition 3, the local commensurate orders of all SISO sub-models are estimated along with linear coefficients.

5.2.1 Global commensurate order estimation

To demonstrate the relevance of the *oe* and *frac-foc-oosriv* algorithms, a Monte Carlo simulation is performed for two different Gaussian white noise levels affecting both input and output measurements: $SNR_u = SNR_y = 20$dB and $SNR_u = SNR_y = 10$dB. The parameters which must be initialized are chosen arbitrarily as follows:

– for the *oe* algorithm:

$$\Theta^0 = \left[a_{1,1}^0, b_{0,1}^0, a_{1,2}^0, b_{0,2}^0, a_{1,3}^0, b_{0,3}^0, \upsilon_G^0 \right] = [0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.3],$$

– for the *frac-foc-oosriv* algorithm:

$$\vartheta^0 = \upsilon_G^0 = 0.3.$$

A Monte Carlo simulation with different noise realizations with $N_{mc} = 50$ runs is carried out. Table 2 summarizes the Monte Carlo simulation results obtained by the *oe* and *frac-foc-oosriv* algorithms. The *oe* algorithm gives good results for $SNR_u = SNR_y = 20$dB. Indeed, the means of the estimates are close to the true ones, the variances are minimal and the $NRQE$ is low. However, for $SNR_u = SNR_y = 10$dB, it gives biased estimates. Consequently, the *oe* algorithm can be used in the context of EIV, but in presence of low noise levels. On the other side, the *frac-foc-oosriv* produces unbiased estimates with low $NRQE$ and low values of standard deviation. The obtained results confirm the consistency of the *frac-foc-oosriv* algorithm in the EIV framework.

For a run of Monte Carlo simulation, the cost function (equation (4.22)) is evaluated and plotted in Figure 4. The modeling error equals 0.0419dB for $SNR_u = SNR_y = 20$dB and 0.4044dB for $SNR_u = SNR_y = 10$dB. Even when the noise level is high, the modeling error is small. As shown in Figure 5, the global commensurate order estimate converges to the true value.

Table 2 Parameter estimation obtained by applying the *oe* and the *frac-foc-oosriv* algorithms in presence of white noise ($SNR_u = SNR_y = 20$dB and 10dB): coefficients and global commensurate order estimation

| True	$SNR_u = SNR_y = 20$dB				$SNR_u = SNR_y = 10$dB			
	oe		*frac-foc-oosriv*		*oe*		*frac-foc-oosriv*	
	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$
$a_{1,1} = 3$	2.9776	0.0560	3.0233	0.0507	2.7521	0.1650	3.0435	0.1290
$b_{0,1} = 1.5$	1.4920	0.0140	1.4958	0.0069	1.3435	0.0484	1.4970	0.0176
$a_{1,2} = 1$	1.0719	0.0348	0.9770	0.0239	1.9358	0.1930	0.9968	0.0892
$b_{0,2} = 2$	2.0518	0.0316	1.9771	0.0192	2.7351	0.1845	1.9950	0.0669
$a_{1,3} = 0.5$	0.4993	0.0054	0.4929	0.0049	0.4864	0.0156	0.4971	0.0149
$b_{0,3} = 1$	0.9967	0.0072	0.9992	0.0053	0.9784	0.0179	0.9984	0.0152
$v_G = 0.2$	0.2003	0.0011	0.2046	0.0008	0.2070	0.0025	0.2020	0.0023
$NRQE$	0.0028		0.0017		0.0302		0.0042	
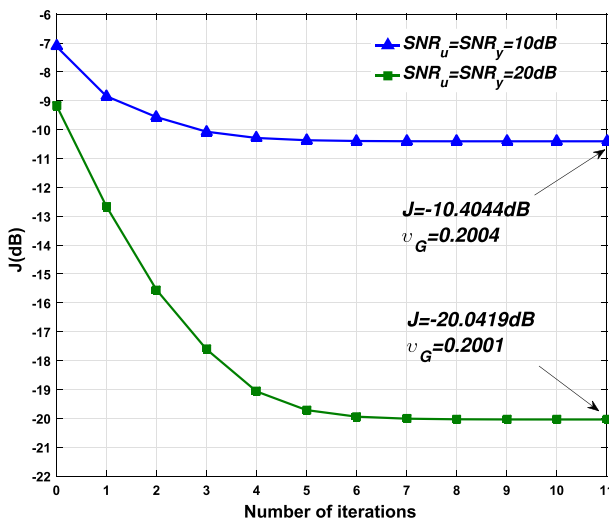
Fig. 4 Cost function variation versus number of iterations: coefficients and global commensurate order estimation

5.2.2 Local commensurate orders estimation

In this paragraph, the local commensurate orders of all sub-models composing the MISO model (equation (4.18)) are estimated by applying the *frac-foc-oosriv* algorithm for:

– white noise affecting input and output measurements ($SNR_u = SNR_y = 20$dB and $SNR_u = SNR_y = 10$dB);
– white and colored noise affecting, respectively, input and output measurements ($SNR_u = SNR_{e_{\tilde{y}}^0} = 5$dB).
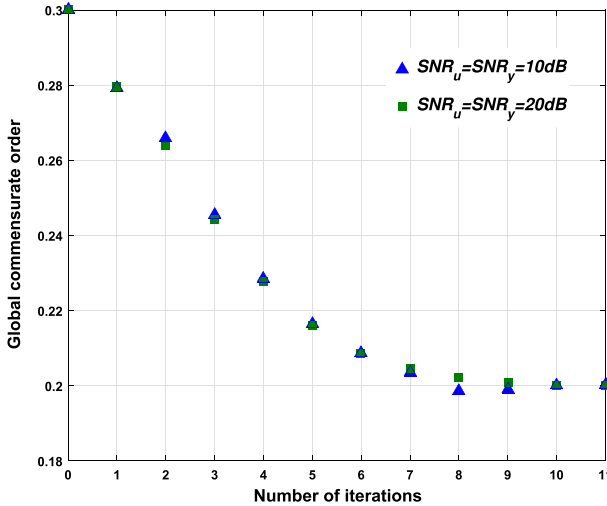
Fig. 5 Global commensurate order versus number of iterations

Table 3 Parameter estimation obtained by applying the *frac-foc-oosriv* algorithm in presence of white and colored noise: coefficients and local commensurate orders estimation

True	White noise		White noise		Colored noise	
	$SNR_u=SNR_y = 20$dB		$SNR_u=SNR_y = 10$dB		$SNR_u=SNR_{e_{\bar{y}}^0} = 5$dB	
	$\bar{\Theta}$	$\hat{\sigma}(\Theta)$	$\bar{\Theta}$	$\hat{\sigma}(\Theta)$	$\bar{\Theta}$	$\hat{\sigma}(\Theta)$
$a_{1,1} = 3$	3.0071	0.0393	3.0353	0.1260	3.0232	0.1855
$b_{0,1} = 1.5$	1.5003	0.0051	1.5006	0.0148	1.5017	0.0156
$a_{1,2} = 1$	0.9973	0.0447	1.0069	0.1590	1.0800	0.2772
$b_{0,2} = 2$	1.9978	0.0387	2.0034	0.1393	2.0694	0.2608
$a_{1,3} = 0.5$	0.5001	0.0045	0.5001	0.0143	0.5020	0.0121
$b_{0,3} = 1$	0.9997	0.0046	0.9994	0.0142	0.9996	0.0130
$\upsilon_1 = 0.8$	0.8000	0.0038	0.7999	0.0124	0.8023	0.0120
$\upsilon_2 = 0.2$	0.2006	0.0044	0.2027	0.0140	0.1984	0.0239
$\upsilon_3 = 1.2$	1.2001	0.0030	1.1999	0.0094	1.1993	0.0072
$NRQE$	0.0023		0.0088		0.0178	

The initial parameter vector is chosen arbitrarily as follows:

$$\vartheta^0 = [\upsilon_1, \upsilon_2, \upsilon_3] = [0.5, 0.5, 0.5].$$

The Monte Carlo simulation results are recapitalized in Table 3. The *frac-foc-oosriv* algorithm accurately identifies local commensurate orders and linear coefficients. The means of estimates are close to the true ones with a very low standard deviations and $NRQE$. These results show that even with a large number of parameters, it is possible to consistently estimate the linear coefficients as well as all local commensurate orders.
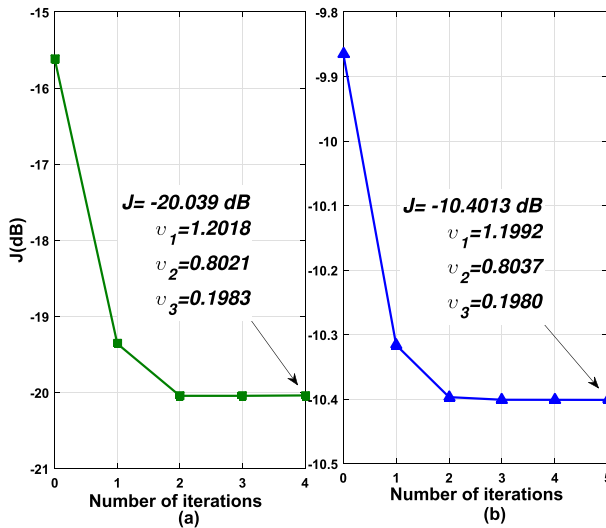
Fig. 6 Cost function variation versus the number of iterations in presence of white noise (for $SNR_u = SNR_y = 20$dB (a) and for $SNR_u = SNR_y = 10$dB (b)): coefficients and local commensurate order estimation

Figures 6 and 7 respectively represent the cost function and the local commensurate orders variations versus the number of iterations for a Monte Carlo simulation. All local commensurate orders converge to the true orders with a very low modeling error which equals 0.039dB for $SNR_u = SNR_y = 20$dB and 0.4013dB for $SNR_u = SNR_y = 10$dB.

5.3 Example 3

In this example, data generating system is not commensurate and all fractional differentiation orders are estimated along with linear coefficients.

Consider the following two-input-single-output fractional system:

$$(\mathcal{H}_2): \begin{cases} \tilde{y}(t) = G_1(p)\tilde{u}_1(t) + G_2(p)\tilde{u}_2(t) \\ G_1(p) = \dfrac{b_{0,1}}{1 + a_{1,1}p^{\alpha_{1,1}} + a_{2,1}p^{\alpha_{2,1}}} = \dfrac{1.5}{1 + 1.5p^{0.8} + 2p^{1.8}} \\ G_2(p) = \dfrac{b_{0,2}}{1 + a_{1,2}p^{\alpha_{1,2}} + a_{2,2}p^{\alpha_{2,2}}} = \dfrac{2}{1 + 0.5p^{0.6} + p^{1.5}} \\ u_1(t_k) = \tilde{u}_1(t_k) + e_{\tilde{u}_1}(t_k) \\ u_2(t_k) = \tilde{u}_2(t_k) + e_{\tilde{u}_2}(t_k) \\ y(t_k) = \tilde{y}(t_k) + e_{\tilde{y}}(t_k). \end{cases} \qquad (5.9)$$

The noise-free input signals are chosen as uncorrelated multisine signals. The input-output data are uniformly sampled with a sampling time $h = 0.05$ sec. They are represented in Figure 8. The number of samples is $N_t = 9000$ and the part of these samples which is used for system identification is set to $N_h = 1000$.

Fig. 7 Local commensurate orders variation versus the number of iterations in presence of white noise (for $SNR_u = SNR_y = 20$dB (a) and for $SNR_u = SNR_y = 10$dB (b))
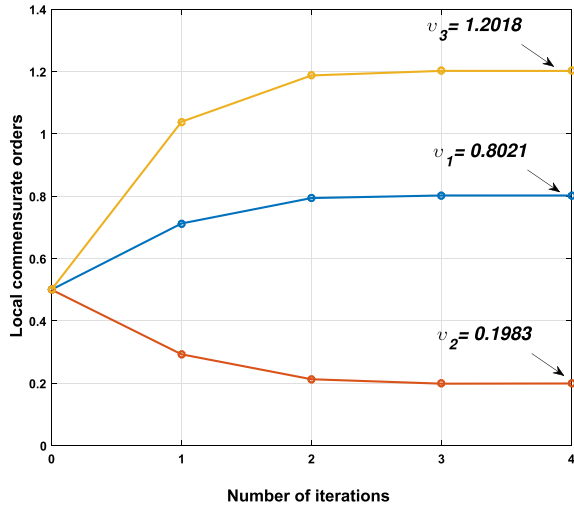
(a)
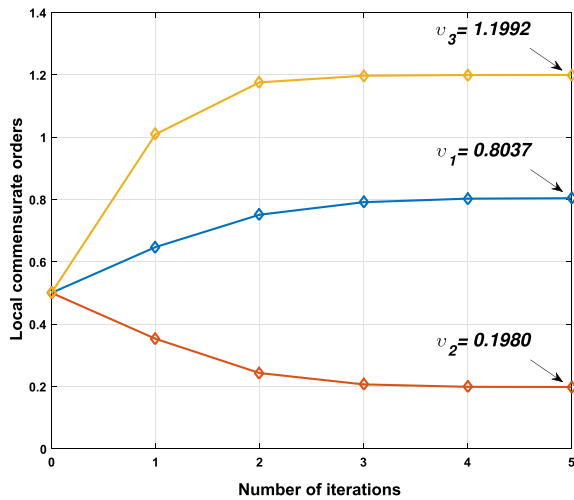
(b)

To provide representative results, a Monte Carlo simulation of $N_{mc} = 50$ runs is performed. For each Monte Carlo run, a new noise realization on input and output signals is generated and the *frac-foc-oosriv* algorithm is applied. The latter is iterative, so the initialization step is necessary. Indeed, the coefficients are initialized by applying the *frac-foc-ls* estimator ($\omega_f = 2$ rad/sec) and the differentiation orders are initialized, arbitrarily, as follows:
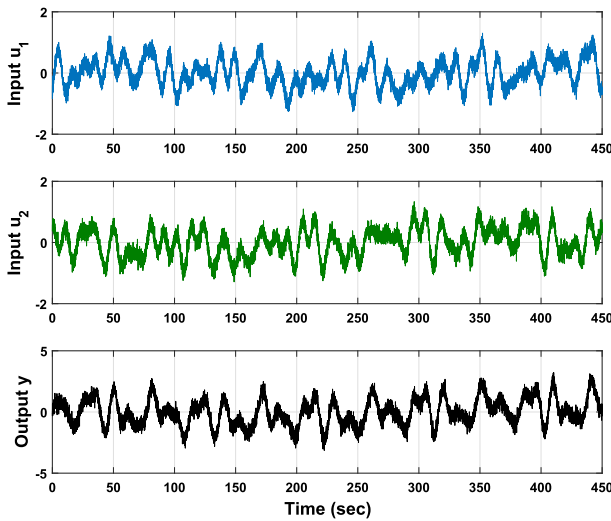
$$\vartheta^0 = [0.9, 0.4, 0.75, 0.3].$$

Fig. 8 Two input and single output signals used for system identification

Table 4 Parameter estimation obtained by applying the *frac-foc-oosriv* algorithm in presence of white and colored noise: coefficients and all differentiation orders estimation

True	White noise		White noise		Colored noise	
	$SNR_u = SNR_y = 20$dB		$SNR_u = SNR_y = 10$dB		$SNR_u = SNR_{e_{\tilde{y}}^0} = 5$dB	
	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$	Θ	$\hat{\sigma}(\Theta)$
$a_{2,1} = 2$	1.9951	0.0147	1.9776	0.0638	1.9823	0.1576
$a_{1,1} = 1.5$	1.5101	0.0341	1.4757	0.1349	1.4856	0.1821
$b_{0,1} = 1.5$	1.5013	0.0047	1.5027	0.0149	1.4937	0.0729
$a_{2,2} = 1$	0.9995	0.0089	1.0092	0.0338	1.0273	0.0531
$a_{1,2} = 0.5$	0.5039	0.0143	0.4867	0.0584	1.5103	0.0526
$b_{0,2} = 2$	2.0001	0.0062	1.9967	0.0161	2.0238	0.864
$\alpha_{2,1} = 1.8$	1.8036	0.0160	1.7875	0.0628	1.7866	0.0853
$\alpha_{1,1} = 0.8$	0.8016	0.0071	0.7944	0.0279	0.8050	0.0337
$\alpha_{2,2} = 1.5$	1.5030	0.0133	1.4896	0.0524	1.4847	0.0771
$\alpha_{1,2} = 0.6$	0.6012	0.0053	0.5958	0.0209	0.5899	0.0448
$NRQE$	0.0020		0.0077		0.0183	

The obtained results are presented in Table 4 for two levels of white noise affecting the input and output signals, and for one level of colored noise. The *frac-foc-oosriv* gives very accurate results. All estimated parameters are unbiased. Variances are very low and $NRQE$ remain low even in presence of a high noise level.

For a run of additive noise with $SNR_u = SNR_y = 10$dB, the cost function and all orders variations versus the number of iterations are, respectively, plotted in Figure 9 and Figure 10. All fractional orders converge to the real values with a small modeling error (0.4244dB).
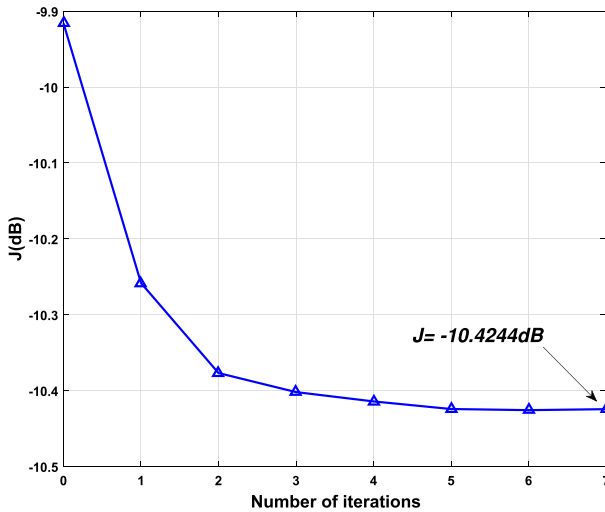
Fig. 9 Cost function versus the number of iterations in presence of white noise ($SNR_u = SNR_y = 10$dB): coefficients and all orders estimation

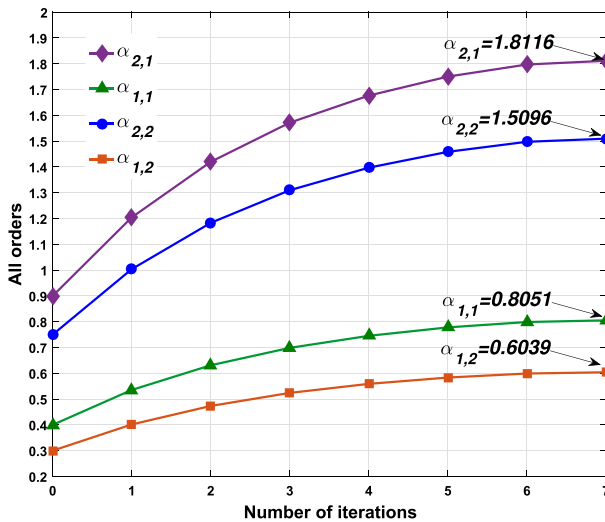**Fig. 10** All orders variation versus the number of iterations in presence of white noise ($SNR_u = SNR_y = 10$dB)

6 Conclusions

Continuous-time MISO systems identification with fractional models, when the available input-output data are noisy or contain measurement errors, has been studied in this paper. A new extension of the simplified and refined instrumental variable algorithm has been proposed. This involves using fourth-order cross-cumulants estimates, which are insensitive to Gaussian noise, instead of noisy input and output signals. So,

a new algorithm has been developed, it was called fractional-fourth-order cumulants based-simplified and refined instrumental variable (*frac-foc-sriv*). Assuming that all the structures and differentiations orders of the hole system are known a priori, this estimator allows to estimate only the linear coefficients. When only the structures are known and there is no prior knowledge of differentiation orders, a combination of the *frac-foc-sriv* estimator with a nonlinear optimization technique has been proposed. Three cases have been highlighted : coefficients and global commensurate order estimation, coefficients and local commensurate orders estimation and coefficients and all differentiation orders estimation. The latter case, has been proposed when the commensurability constraint is released. The performance of the developed algorithms has been analyzed by comparing them with the fractional-fourth-order cumulants based-least squares (*frac-foc-ls*) and the output error (*oe*) based optimization algorithms. The *frac-foc-sriv* and *frac-foc-oosriv* algorithms gave unbiased estimates in presence of white and colored noise. In the future work, it will be interesting to study the problem of unstable MISO systems identification. It will also be worthwhile developing a technique for selecting the unknown structures of SISO subsystems.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Abuaisha, T., Kertzscher, J.: Fractional-order modelling and parameter identification of electrical coils. Fractional Calculus and Applied Analysis **22**(1), 193–216 (2019). https://doi.org/10.1515/fca-2019-0013
2. Chetoui, M., Aoun, M.: Instrumental variables based methods for linear systems identification with fractional models in the EIV context. In: 2019 16th International Multi-Conference on Systems, Signals & Devices (SSD), pp. 90–95 (2019)
3. Chetoui, M., Aoun, M.: Fourth-order cumulants based-least squares methods for fractional Multiple-Input-Single-Output Errors-In-Variables system identification. Fractional Calculus and Applied Analysis **26**(4), 1868–1893 (2023). https://doi.org/10.1007/s13540-023-00174-z
4. Chetoui, M., Malti, R., Aoun, M., Thomassin, M., Abdelkrim, M., Oustaloup, A.: Continuous-time system identification with fractional models from noisy input and output data using fourth-order cumulants. Systems, Automation, and Control 2017 **7**, 125 (2017)
5. Chetoui, M., Thomassin, M., Malti, R., Aoun, M., Najar, S., Abdelkrim, M.N., Oustaloup, A.: New consistent methods for order and coefficient estimation of continuous-time errors-in-variables fractional models. Computers & Mathematics with Applications **66**(5), 860–872 (2013)
6. Dennis Jr, J.E., Schnabel, R.B.: Numerical Methods for Unconstrained Optimization and Nonlinear Equations. SIAM (1996)
7. Du, W., Miao, Q., Tong, L., Tang, Y.: Identification of fractional-order systems with unknown initial values and structure. Physics Letters A **381**(23), 1943–1949 (2017)
8. Gabano, J.D., Poinot, T.: Estimation of thermal parameters using fractional modelling. Signal Processing **91**(4), 938–948 (2011)
9. Garnier, H.: Direct continuous-time approaches to system identification. overview and benefits for practical applications. European Journal of Control **24**, 50–62 (2015)
10. Garnier, H., Gilson, M., Zheng, W.: A bias-eliminated least-squares method for continuous-time model identification of closed-loop systems. International Journal of Control **73**(1), 38–48 (2000)
11. Garnier, H., Wang, L., Young, P.C.: Direct Identification of Continuous-time Models from Sampled Data: Issues, Basic Solutions and Relevance, pp. 1–29. Springer London, London (2008)

12. Gilson, M., Garnier, H., Young, P., Van den Hof, P.: A refined IV method for closed-loop system identification. IFAC Proceedings Volumes **39**(1), 903–908 (2006)
13. González, R.A., Rojas, C.R., Pan, S., Welsh, J.S.: Refined instrumental variable methods for unstable continuous-time systems in closed-loop. International Journal of Control **96**(10), 2527–2541 (2022)
14. Grünwald, A.: Ueber begrenzte derivationen und deren anwendung. Zeitschrift fur Mathematik und Physik **12**(6), 441–480 (1867)
15. Hajjem, M., Victor, S., Melchior, P., Lanusse, P., Thomas, L.: Wind turbulence modeling for real-time simulation. Fractional Calculus and Applied Analysis **26**(4), 1632–1662 (2023). https://doi.org/10.1007/s13540-023-00165-0
16. Ionescu, C.M., De Keyser, R.: Relations between fractional-order model parameters and lung pathology in chronic obstructive pulmonary disease. IEEE Transactions on Biomedical Engineering **56**(4), 978–987 (2008)
17. Jiang, Y., Xia, B., Zhao, X., Nguyen, T., Mi, C., de Callafon, R.A.: Data-based fractional differential models for non-linear dynamic modeling of a lithium-ion battery. Energy **135**, 171–181 (2017)
18. Laurain, V., Gilson, M., Garnier, H., Young, P.C.: Refined instrumental variable methods for identification of hammerstein continuous-time box-jenkins models. In: 2008 47th IEEE Conference on Decision and Control, pp. 1386–1391 (2008)
19. Laurain, V., Gilson, M., Tóth, R., Garnier, H.: Refined instrumental variable methods for identification of LPV box-jenkins models. Automatica **46**(6), 959–967 (2010)
20. Linder, J., Enqvist, M.: Identification of systems with unknown inputs using indirect input measurements. International Journal of Control **90**(4), 729–745 (2017)
21. Magin, R.L.: Fractional calculus models of complex dynamics in biological tissues. Computers & Mathematics with Applications **59**(5), 1586–1593 (2010)
22. Malti, R., Aoun, M., Sabatier, J., Oustaloup, A.: Tutorial on system identification using fractional differentiation models. IFAC Proceedings Volumes **39**(1), 606–611 (2006)
23. Matignon, D.: Stability properties for generalized fractional differential systems. In: ESAIM: Proceedings, vol. 5, pp. 145–158. EDP Sciences (1998)
24. Mayoufi, A., Victor, S., Chetoui, M., Malti, R., Aoun, M.: Output error MISO system identification using fractional models. Fractional Calculus and Applied Analysis **24**(5), 1601–1618 (2021). https://doi.org/10.1515/fca-2021-0067
25. Moussa, N.B., Chetoui, M., Amairi, M.: MISO fractional systems identification with fractional models in the EIV context. In: 18th International Multi-Conference on Systems, Signals & Devices (SSD), pp. 942–947 (2021)
26. Nasser-Eddine, A., Huard, B., Gabano, J.D., Poinot, T., Martemianov, S., Thomas, A.: Fast time domain identification of electrochemical systems at low frequencies using fractional modeling. Journal of Electroanalytical Chemistry **862**, 113957 (2020)
27. Pan, S., González, R.A., Welsh, J.S., Rojas, C.R.: Consistency analysis of the simplified refined instrumental variable method for continuous-time systems. Automatica **113**, 108767 (2020)
28. Patil, N., Datar, R.G., Patil, D.: System identification of a temperature control process using open loop and closed loop methods. In: 2018 Second International Conference on Computing Methodologies and Communication (ICCMC), pp. 240–246 (2018)
29. Salem, T., Chetoui, M., Aoun, M.: Instrumental variable based methods for continuous-time linear parameter varying system identification with fractional models. In: 2016 24th Mediterranean Conference on Control and Automation (MED), pp. 640–645 (2016)
30. Thil, S., Garnier, H., Gilson, M.: Third-order cumulants based methods for continuous-time errors-in-variables model identification. Automatica **44**(3), 647–658 (2008)
31. Thomassin, M., Malti, R.: Subspace method for continuous-time fractional system identification. In: 15th IFAC Symposium on System Identification, pp. 880–885. Saint-Malo, France (2009)
32. Victor, S., Malti, R., Garnier, H., Oustaloup, A.: Parameter and differentiation order estimation in fractional models. Automatica **49**(4), 926–935 (2013)
33. Victor, S., Mayoufi, A., Malti, R., Chetoui, M., Aoun, M.: System identification of MISO fractional systems: Parameter and differentiation order estimation. Automatica **141**, 110268 (2022)
34. Victor, S., Melchior, P., Pellet, M., Oustaloup, A.: Lung thermal transfer system identification with fractional models. IEEE Transactions on Control Systems Technology **28**(1), 172–182 (2018)
35. Wang, J., Zheng, W.X., Chen, T.: Identification of linear dynamic systems operating in a networked environment. Automatica **45**(12), 2763–2772 (2009)

36. Wang, Y., Chen, Y., Liao, X.: State-of-art survey of fractional order modeling and estimation methods for lithium-ion batteries. Fractional Calculus and Applied Analysis **22**(6), 1449–1479 (2019). https://doi.org/10.1515/fca-2019-0076

37. Wang, Y., Li, M., Chen, Z.: Experimental study of fractional-order models for lithium-ion battery and ultra-capacitor: Modeling, system identification, and validation. Applied Energy **278**, 115736 (2020)

38. Wang, Z., Wang, C., Ding, L., Wang, Z., Liang, S.: Parameter identification of fractional-order time delay system based on legendre wavelet. Mechanical Systems and Signal Processing **163**, 108141 (2022)

39. Yakoub, Z., Amairi, M., Aoun, M., Chetoui, M.: On the fractional closed-loop linear parameter varying system identification under noise corrupted scheduling and output signal measurements. Transactions of the Institute of Measurement and Control **41**(10), 2909–2921 (2019)

40. Yakoub, Z., Amairi, M., Chetoui, M., Aoun, M.: On the closed-loop system identification with fractional models. Circuits, Systems, and Signal Processing **34**(12), 3833–3860 (2015)

41. Yakoub, Z., Aoun, M., Amairi, M., Chetoui, M.: Identification of continuous-time fractional models from noisy input and output signals. In: S. Cham (ed.) Fractional Order Systems Control Theory and Applications, pp. 181–216 (2022)

42. Yakoub, Z., Chetoui, M., Amairi, M., Aoun, M.: A comparison between the direct and the indirect fractional order closed-loop bias eliminated least squares method. In: 2016 17th International Conference on Sciences and Techniques of Automatic Control and Computer Engineering (STA), pp. 271–282 (2016)

43. Young, P.C., Garnier, H., Gilson, M.: Refined instrumental variable identification of continuous-time hybrid box-jenkins models. pp. 91–131. Identification of continuous-time models from sampled data, Springer (2008)