Fractional Calculus and Applied Analysis (2022) 25:1166–1198
https://doi.org/10.1007/s13540-022-00056-w

ORIGINAL PAPER

Global solutions of nonlinear fractional diffusion equations with time-singular sources and perturbed orders

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Received: 29 November 2021 / Revised: 16 April 2022 / Accepted: 26 April 2022 / Published online: 3 June 2022

Abstract

In a Hilbert space, we consider a class of nonlinear fractional equations having the Caputo fractional derivative of the time variable t and the space fractional function of the self-adjoint positive unbounded operator. We consider various cases of global Lipschitz and local Lipschitz source with time-singular coefficient. These sources are generalized of the well–known fractional equations such as the fractional Cahn–Allen equation, the fractional Burger equation, the fractional Cahn–Hilliard equation, the fractional Kuramoto–Sivashinsky equation, etc. Under suitable assumptions, we investigate the existence, uniqueness of maximal solution, and stability of solution of the problems with respect to perturbed fractional orders. We also establish some global existence and prove that the global solution can be approximated by known asymptotic functions as $t \to \infty$.

Keywords Fractional diffusion (primary) · Caputo derivative · Initial value problem · Maximal solution · Global solution · Decay rate

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Mathematics Subject Classification Primary 26A33 · 35R11 · 35R30 · 35A01 · 35A02

1 Introduction

1.1 Statement of the problem

Let H be a Hilbert space, $A : D(A) \subset H \to H$ be a self-adjoint positive definite unbounded operator, and $f : [0, +\infty) \times D(A^s) \to H$ with $s \geq 0$. For $\alpha \in (0, 1]$, $\beta > 0$, we consider the problem to find a function $u : [0, T] \to H$ satisfying

$$D_t^\alpha u + A^\beta u = f(t, u(t)), \quad t > 0, \tag{1.1}$$

where D_t^α is the Caputo fractional derivative

$$D_t^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s)\, \mathrm{d}s, & \text{if } \alpha \in (0,1), \\ u'(t), & \text{if } \alpha = 1, \end{cases}$$

and the fractional power of the operator A^β will be defined later. The equation (1.1) is a general form of a lot of well-known equations such as the Ginzburg–Landau equation ($\alpha = 1, A = -\Delta, f(t,u) = au - bu^3$), Burger equation ($\alpha = 1, A = -\Delta, f(t,u) = uu_x$), and Kuramoto–Sivashinsky equation ($\alpha = 1, A = \Delta^2, f(t,u) = \nabla^2 u + (1/2)\|\nabla u\|^2$)). In the present paper, we will investigate the stability of solution of the initial value and the final value problems for (1.1).

The equation (1.1) subject to the initial data

$$u(0) = \zeta \tag{1.2}$$

is called the fractional initial value (or the Cauchy, the forward) problem (FIVP).

By the definition of the spectral resolution of the operator A and the Laplace transform, we can rewrite the FIVP as

$$\textbf{Problem } P_{\zeta,\alpha,\beta}: \quad u(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A, t, \tau) f(\tau, u(\tau))\, \mathrm{d}\tau, \tag{1.3}$$

where $E_{\alpha,\beta}(z, t, \tau) = (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-z^\beta (t-\tau)^\alpha)$ is expressed by the Mittag-Leffler function, and the operators $E_\alpha\big(-t^\alpha A^\beta\big)$, $E_{\alpha,\beta}(A, t, \tau)$ will be defined in Sect. 2.

A function u that satisfies Eq. (1.3) is called a mild solution of the FIVP and denote by $u = u_{\zeta,\alpha,\beta}$.

1.2 History and motivation

The abstract parabolic equation $u_t + Au = f$ was considered for the last thirty years in many works on this area. The readers can see the classical book by Cazenave and Haraux [5] and references therein. The FIVP was also studied by lot of researchers. Xing et al. [24] discussed the existence, uniqueness, analyticity and the long-time asymptotic behavior of solutions of space-time fractional reaction–diffusion equations in $\mathbb{R}^n$

$$D_t^\alpha u + (-\Delta)^\beta u = p(x)u,$$

subject to the initial condition $u(x, 0) = a(x)$. Existence and uniqueness of the maximal solutions of some linear and nonlinear fractional problems were investigated in [2, 6, 10, 12, 21]. The blow–up and global solution of time–fractional nonlinear diffusion–reaction equations were studied recently with some kind of nonlinear sources as $f(t, u) = au + u^p$ (see Cao et al. [4]), $f(t, u) = |u|^{p-1}u$ (see [27]), $f(t, u) = |u|^p$ (Zhang [26]), Asogwa et al. [3] studied finite time blow up results for a version of equation (1.1).

The first motivation of our paper is to study the existence and uniqueness of maximal solution of the initial value problem with respect to a singular source. In many practical situation, the source is often assumed to satisfy $\|f(t, u) - f(t, v)\| \le K(t, M)\|u - v\|$ for $\|u\|, \|v\| \le M$, where $K : (0, \infty) \times (0, \infty) \to \mathbb{R}$ is the Lipschitz coefficient. If $K(t, M)$ is dependent (independent) on t, we say that the coefficient is t-dependent (t-independent). Generally, in a lot of papers, the coefficient is assumed to be t-independent. A time-dependent coefficient which can be unbounded in a time interval $(0, T)$, i.e. $\sup_{0<t<T} K(t, M) = \infty$, is rarely studied. In the present paper, we consider a generalized form of singular source

$$\|f(t, u) - f(t, v)\|_H \le K(t, M)\|u - v\|_{D(A^s)} \text{ for } \|u\|_H, \|v\|_H \le M,$$

with $K(t, M) = \kappa_0(t)L(M)$ for $\lim_{t\to 0^+} \kappa_0(t) = \infty$. In the special case $\kappa_0(t) = t^{-\nu}\kappa(t)$, the condition of $K(t, M)$ is similar to the generalized Nagumo condition (see, e.g., [13], Ch. 7 or [11]). In particular, the time-dependent source promises to bring many interesting global properties to the solution of the problem.

The second motivation of our paper is of studying the continuity of solution with respect to the fractional orders α, β and the initial data ζ. In the papers mentioned above, the parameters α, β are assumed to be perfectly known. But in the real word of applications, the fractional orders can only be approximated from the mathematical model or statistical methods. In [1, 7], the Caputo derivatives can be identified approximately from observation data $u(x_0, t)$ with $t > 0$, or $u(x, T)$ with $x \in \Omega \subset \mathbb{R}^n$. Besides, Kateregga [15] used statistical methods as the quantiles, logarithmic moments method, maximum likelihood, and the empirical characteristic function method to identify the parameters of the Lévy process. In these examples, the fractional orders are obtained only as approximate values. Hence, a natural question is that whether the solutions of fractional equations is continuous with respect to the perturbed orders. The papers devoted to these questions are still rare. We can list here some papers. Li and Yamamoto [16] investigated the solution $u_{\gamma,D}$ of the problem

$$D_t^\gamma u = \frac{\partial}{\partial x}\left(D(x)\frac{\partial u}{\partial x}\right), \quad (x, t) \in (0, 1) \times (0, T),$$

subject to the Neumann condition $u_x(0,t) = u_x(1,t) = 0$ and the initial condition $u(x,0) = f(x)$. They proved that

$$\|u_{\gamma_1,D_1}(0,\cdot) - u_{\gamma_2,D_2}(0,\cdot)\|_{L^2(0,T)} \le C(|\gamma_1-\gamma_2| + \|D_1 - D_2\|_{C[0,1]}).$$

Dang et al. [8] studied the continuity of solutions of some linear fractional PDEs with perturbed orders. In the references [9, 19, 22], the authors considered the stability of solution of some class of nonlinear space–fractional diffusion problems taking into account the disturbance of parameters. To the best of our knowledge, until now, there is very few papers devoted to the stability of solutions of time-fractional problems with respect to the fractional time and space derivative parameters: α, β. In [20, 23], the stability was considered for the global Lipschitz source. The problem with the local time-singular Lipschitz source is still a topic of investigation. This stability result will serve as the foundation for numerical computation schemes for equations with imprecise fractional derivative parameters.

1.3 Outline of the paper

Summarizing the discussion of the FIVP, in the present paper, we will:

- Investigate the existence and uniqueness of the maximal solution of the nonlinear FIVP with respect to the singular nonlinear source on the maximal interval $[0, T_{\zeta,\alpha,\beta})$. To solve the problem, we have to establish an appropriate Gronwall-type inequality which also has a specific merit in investigating other fractional problems.
- Study the stability of the nonlinear FIVP with respect to the perturbed orders α, β. In fact, we establish

$$u_{\zeta',\alpha',\beta'} \to u_{\zeta,\alpha,\beta} \quad \text{as} \quad (\zeta',\alpha',\beta') \to (\zeta,\alpha,\beta)$$

  in an appropriate norm. Especially, for $\alpha \to 1^-$, we will prove that the solution of the nonlinear FIVP tends to that of classical nonlinear initial value parabolic problem.
- Study the existence of global solution on $[0,\infty)$ and prove decay estimates for the global solution. To illustrate for our results, we present an asymptotic result. Under some conditions, we will prove that

$$\lim_{t\to\infty} (1+t)^{\alpha} \|u_{\zeta,\alpha,\beta}(t)\|_s = \frac{1}{\Gamma(1-\alpha)} \|\zeta\|_{s-\beta}$$

  for every $\zeta \in D(A^s)$, $\zeta \ne 0$. The result shows that the decay of the FIVP is of polynomial order.

The rest of the paper is organized as follows. Section 2 gives the main results of our paper without proofs. In Sect. 3 the proofs of these results are presented.

2 Main results

2.1 Notations

To state our problem precisely, we will give some definitions. Firstly, in the paper, we always denote by C, C' generic constants which could be different from line to line. We denote the inner product in Hilbert space H by $\langle .,. \rangle$ and the associated norm by $\|.\|$.

We recall (see, e.g., [17,page 61, Ch.4]) that a resolution of the identity on a Hilbert space H is a one-parameter family $\{S_\lambda : \lambda \in R\}$ of orthogonal projections on H such that

(i) $S_\lambda \le S_{\lambda'}$ if $\lambda \le \lambda'$ (monotonicity),
(ii) $\lim_{\lambda' \to \lambda^+} S_{\lambda'}\zeta = S_\lambda \zeta$ for $\zeta \in H$ (strong right continuity),
(iii) $\lim_{\lambda \to -\infty} S_\lambda \zeta = 0$ and $\lim_{\lambda \to +\infty} S_\lambda \zeta = \zeta$ for $\zeta \in H$.

Assume that $\theta > 0$ is the lower bound of the spectrum of the operator A. Let us denote by $\{S_\lambda\}$ the spectral resolution of the identity associated to operator A such that $A = \int_\theta^\infty \lambda d S_\lambda$. We follow [25,page 29] (see also [17,page 92]) to define the power of the self-adjoint positive definite unbounded operator as

$$A^\beta u = \int_\theta^{+\infty} \lambda^\beta \, \mathrm{d}S_\lambda u, \quad \beta \in \mathbb{R}.$$

Generally, for a continuous function $h : \mathbb{R} \to \mathbb{R}$, we denote the domain of $h(A)$ to be

$$D(h(A)) := \left\{ w \in H : \int_\theta^{+\infty} |h(\lambda)|^2 \, \mathrm{d}\|S_\lambda w\|^2 < +\infty \right\}. \tag{2.1}$$

If $w \in D(h(A))$, we define the linear operator

$$h(A)w = \int_\theta^{+\infty} h(\lambda) \, \mathrm{d}S_\lambda w.$$

Particularly, if $h(z) = z^s$ for $z \ge 0$, $s \in \mathbb{R}$, we have the Hilbert space $D(A^s)$ with the norm $\|w\|_s = \left(\int_\theta^{+\infty} \lambda^{2s} \, \mathrm{d}\|S_\lambda w\|^2 \right)^{1/2}$. For $v \in C([0,T]; D(A^s))$ we denote $|v|_{s,t} = \sup_{0 \le \tau \le t} \|v(\tau)\|_s$. Let $0 \le s_* \le s^*$ and $s_1, s_2 \in [s_*, s^*]$, $s_2 \le s_1$. It is easy to see that

$$D(A^{s_1}) \subset D(A^{s_2}) \subset D(A^0) = H \text{ and } \|w\|_{s_2} \le \theta^{s_2 - s_1} \|w\|_{s_1}.$$

For $M, s > 0$, we put

$$B_s(M) = \{ w \in D(A^s) : \|w\|_s \le M \},$$
$$B_{s,T}(M) = \left\{ v \in C([0,T]; D(A^s)) : |v|_{s,T} \le M \right\}.$$

In this section, we also remind the Mittag–Leffler function and its properties which play important roles in the proof of main results of current paper. We recall also the Gamma and Beta functions,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \quad B(p,r) = \int_0^1 t^{p-1}(1-t)^{r-1}dt \ \text{ for } \ \mathrm{Re}(z),\, p, r > 0.$$

The Mittag–Leffler function with two parameters is defined as

$$E_{p,r}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(kp+r)}, \quad E_p(z) := E_{p,1}(z), \ z \in \mathbb{C} \ \text{ for } \ p, r > 0.$$

Definition 1 A function $u \in C([0,T); D(A^s))$ is a maximal solution of Problem $P_{\zeta,\alpha,\beta}$ if u satisfies $P_{\zeta,\alpha,\beta}$ on the interval $[0,T)$ such that $T = \infty$ or that $T < \infty$, $\limsup_{t\to T^-} \|u(t)\|_s = \infty$. In the case $T = \infty$, we say that u is a*global solution* of $P_{\zeta,\alpha,\beta}$. The global solution u is said to have

- the *sub-polynomial decay rate* if there are $\rho, C > 0$ such that

$$\|u(t)\|_s \le C(1+t)^{-\rho} \ \text{ for every } t > 0.$$

- the *asymptotically polynomial decay* if there is $\rho, C_\rho > 0$ such that

$$\lim_{t\to\infty} (1+t)^\rho \|u(t)\|_s = C_\rho.$$

2.2 The global Lipschitz source

Using the notations defined, we can state precisely the assumption for the singular source. In fact, we consider the source function of the problem satisfying the following assumptions.

Assumption F1(α) *Let* $T > 0$, $\alpha \in (0,1]$ *and* $f \in C((0,\infty) \times D(A^s); H)$. *We assume that*

$$m_{T,\alpha} := \sup_{0\le t\le T} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha) \|f(\tau,0)\|^2 \,\mathrm{d}\tau < \infty.$$

Assumption G1 *Let* $T > 0$, $s > 0$, $\nu \le \alpha/2$ *and* $f \in C((0,\infty) \times D(A^s); H)$, $\kappa \in C([0,\infty); [0,\infty))$. *We assume that*

$$\|f(t,w_1) - f(t,w_2)\| \le t^{-\nu}\kappa(t)\|w_1 - w_2\|_s \ \text{ for all } \ w_1, w_2 \in D(A^s). \tag{2.2}$$

Remark 1 Assumption **F1**(α) holds in many cases. For example, as shown in Lemma 2 (see the part of proofs), if $\|f(t,0)\| \le t^{-\nu_f}\kappa_f(t)$ for $2\nu_f < \alpha$, $\kappa_f \in C([0,\infty); \mathbb{R})$, then Assumption **F1**(α) holds.

Using the assumptions, we can obtain the following existence result.

Theorem 1 *Let $\alpha \in (0,1)$, $\beta > 0$, $s \in [0, \beta/2]$, $\zeta \in D(A^s)$ and let $f \in C((0,\infty) \times D(A^s); H)$. Assume that* ***Assumption F1****(α) and* ***Assumption G1*** *hold. For $\nu < \alpha/2$, the equation (1.3) has a unique solution $u = u_{\zeta,\alpha,\beta} \in C([0,\infty); D(A^s))$. Moreover, for a $T > 0$, if we put*

$$\kappa_{T,s} = \max_{0\le t\le T} \kappa(t), \quad g(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A,t,\tau) f(\tau,0)\,\mathrm{d}\tau,$$

then $g \in C\,([0,T], D(A^s))$ and there is a constant C independent of t such that

$$\|u(t)\|_s^2 \le 2C\Gamma(1-2\nu)|g|_{s,t}^2 E_{\alpha-2\nu,1-2\nu}\left(2\theta^{2s-\beta}\kappa_{T,s}^2 t^{\alpha-2\nu}\right) \ \text{ for any } \ t \in [0,T], \tag{2.3}$$

where we recall $|g|_{s,t} = \sup_{0\le\tau\le t} \|g(\tau)\|_s$.

For $\nu = \alpha/2$, $\kappa_{T,s} < \theta^{\beta/2-s}\,(\Gamma(1-\alpha))^{-1/2}$, the equation (1.3) has a unique solution $u \in C([0,T]; D(A^s))$.

Remark 2 We can use the Edelstein fixed point theorem (see, e.g., [13], Ch. 7) to obtain the desired result for the case $\nu = \alpha/2$, $\kappa_{T,s} = \theta^{\beta/2-s}\,(\Gamma(1-\alpha))^{-1/2}$. We note that if we put $u_{n+1} = \mathcal{F}(u_n)$, where $\mathcal{F}(u)$ is the right hand side of equation (1.3), then the sequence (u_n) converges to the solution u in $C([0,T], D(A^s))$.

From the existence result stated, we can obtain an interesting global existence result. Moreover, we also give an asymptotically polynomial result for decay estimates.

Theorem 2 *Let $\alpha \in (0,1)$, $\beta > 0$, $s \in [0,\beta/2]$, $\zeta \in D(A^s)$, $\nu, \nu_f > 0$, $\nu < \alpha/2$, $\nu_f < \alpha/2$ and let $f \in C((0,\infty) \times D(A^s); H)$. Assume:*

- **Assumption G1** *holds and there are $\kappa_1 > 0$, $\ell \in \mathbb{R}$ such that*

$$0 \le \kappa(t) \le \kappa_1(1+t)^\ell,$$

- *There exist $\kappa_f > 0$, $\nu_f > 0$, $\ell_f \in \mathbb{R}$ such that*

$$\|f(t,0)\|_s \le \kappa_f t^{-\nu_f}(1+t)^{\ell_f}.$$

With these assumptions, we have the following two results:

1. *If $\ell, \ell_f < \nu$, $\alpha > \frac{1}{2} - \nu$, then, for every $\zeta \in D(A^s)$, the problem $P_{\zeta,\alpha,\beta}$ has a unique global solution $u_{\zeta,\alpha,\beta} \in C([0,\infty); D(A^s))$ which has the sub-polynomial decay rate. More explicitly, for every ω satisfying $0 < \omega < \min\left\{\frac{1}{2}, \frac{1}{2} - \ell, \frac{1}{2} - \ell_f\right\}$, we can find $C_\omega > 0$ such that*

$$\|u_{\zeta,\alpha,\beta}(t)\|_s \le C(1+t)^{-\min\{\alpha,\omega\}} \quad \text{for } t > 0.$$

2. *If $\ell, \ell_f < \frac{1}{2} - \alpha$, $0 < \alpha < 1/2$ then, for every $\zeta \in D(A^s)$, the problem $P_{\zeta,\alpha,\beta}$ has a unique global solution $u_{\zeta,\alpha,\beta} \in C([0,\infty); D(A^s))$ which*

has the sub-polynomial decay rate. More explicitly, for every $\alpha < \omega < \min\left\{\frac{1}{2}, \frac{1}{2} - \ell, \frac{1}{2} - \ell_f\right\}$ *we can find* $C_{\alpha,\omega} > 0$ *such that*

$$\|u_{\zeta,\alpha,\beta}(t) - E_\alpha(-t^\alpha A^\beta)\zeta\|_s \leq C_{\alpha,\omega}\|\zeta\|_s(1+t)^{-\omega}.$$

In addition, the solution $u_{\zeta,\alpha,\beta}$ *has the asymptotically polynomial decay rate*

$$\lim_{t\to\infty}(1+t)^\alpha\|u_{\zeta,\alpha,\beta}(t)\|_s = \frac{1}{\Gamma(1-\alpha)}\|\zeta\|_{s-\beta}.$$

Remark 3 As mentioned, the set $D(A^s)$ is dense in H. Hence the polynomial decay rate of the solution holds for almost every $\zeta \in H$.

In the next two theorems, we state some results on stability of solution of the initial problem with respect to the initial data and the fractional orders. In the following stability results, we recall that $u_{\zeta,\alpha,\beta}$ is the solution of Problem $P_{\zeta,\alpha,\beta}$ in (1.3) corresponding to the initial data ζ and the orders α, β. We first have the classical stability with respect to the fixed orders α, β. More precisely, for $\alpha_*, \alpha^*, \beta_*, \beta^*$ satisfying $0 < \alpha_* < \alpha^* < 2\alpha_* \leq 2$, and $0 < \beta_* < \beta^*$, we consider

$$(\alpha, \beta) \in \Delta := [\alpha_*, \alpha^*] \times [\beta_*, \beta^*]. \tag{2.4}$$

Theorem 3 *Let* $\zeta, \xi \in D(A^s)$ *be two initial data with* $s \in [0, \beta/2]$*, and let* $T > 0$*,* $(\alpha, \beta) \in \Delta$ *be as in* (2.4)*. Let the source function* f *satisfy* ***Assumption F1***(α) *and* ***Assumption G1****. Then the problems* $P_{\xi,\alpha,\beta}, P_{\zeta,\alpha,\beta}$ *have the unique solutions* $u_{\xi,\alpha,\beta}, u_{\zeta,\alpha,\beta} \in C([0,T], D(A^s))$*, respectively. Moreover, there exists a positive constant* P_1 *independent of* ζ, ξ *such that*

$$|u_{\xi,\alpha,\beta} - u_{\zeta,\alpha,\beta}|_{s,T} \leq P_1\|\zeta - \xi\|_s.$$

Hence, letting $\zeta, \zeta_k \in D(A^s)$*,* $\zeta_k \to \zeta$ *in* $D(A^s)$*, we obtain*

$$\lim_{k\to\infty}|u_{\zeta_k,\alpha,\beta} - u_{\zeta,\alpha,\beta}|_{s,T} = 0.$$

Moreover, we also have the stability results with respect to the perturbed orders α, β. To investigate the stability of the solution of the problem (FIVP), we will restrict the value (α, β) in the bounded domain.

Theorem 4 *Let* $(\alpha, \beta), (\alpha_k, \beta_k) \in \Delta$ *(defined in* (2.4)*) such that* $(\alpha_k, \beta_k) \to (\alpha, \beta)$*, and let* $\zeta, \zeta_k \in D(A^s)$ *such that* $\zeta_k \to \zeta$ *in* $D(A^s)$ *as* $k \to \infty$*. Let the source function* f *satisfy* ***Assumption F1***(α_*) *and* ***Assumption G1****. Then the problems* $P_{\zeta,\alpha,\beta}, P_{\zeta_k,\alpha_k,\beta_k}$ *have the unique solutions*

$$u_{\zeta,\alpha,\beta} \in C([0,T], D(A^s)), \;\; u_{\zeta_k,\alpha_k,\beta_k} \in C([0,T], D(A^{\min\{s,\beta_k/2\}})),$$

respectively. In addition, the following results hold:

(i) If $s \in [0, \beta/2)$*, then*

$$\lim_{k\to\infty} |u_{\zeta_k,\alpha_k,\beta_k} - u_{\zeta,\alpha,\beta}|_{s,T} = 0. \tag{2.5}$$

(ii) If $s = \beta/2$ *and* $\beta_k \geq \beta$ *for* $k \to \infty$*, then*

$$\lim_{k\to\infty} |u_{\zeta_k,\alpha_k,\beta_k} - u_{\zeta,\alpha,\beta}|_{s,T} = 0. \tag{2.6}$$

(iii) If $\beta_k \leq \beta$ *as* $k \to \infty$*, then*

$$\lim_{k\to\infty} |u_{\zeta_k,\alpha_k,\beta_k} - u_{\zeta,\alpha,\beta}|_{\beta_k/2,T} = 0. \tag{2.7}$$

(iv) If we suppose further that $\zeta_k, \zeta \in D\left(A^{\beta^*/2+r_1}\right)$ *such that* $\zeta_k \to \zeta$ *in* $D\left(A^{\beta^*/2+r_1}\right)$ *for some* $r_1 > 0$*. We also suppose that* $s \in \left[\frac{\beta_*}{2}, \frac{\beta}{2}\right)$*,* $f(., u_{\zeta,\alpha,\beta}(.)) \in C([0, T], D(A^{r_2}))$ *for some* $r_2 > 0$*. Then, there exists constants* L_0, L_1 *independent of* ζ, ζ_k *such that*

$$\left|u_{\zeta_k,\alpha_k,\beta_k} - u_{\zeta,\alpha,\beta}\right|_{s,T} \leq L_0\|\zeta - \zeta_k\|_s + L_1(|\alpha - \alpha_k| + |\beta - \beta_k|)^{\frac{\gamma_2}{2(\gamma_1+\gamma_2+2)}}, \tag{2.8}$$

where L_0, L_1 *is depended on* $(\alpha_*, \alpha^*, \beta_*, \beta^*, T)$*,* $\gamma_1 = 2\beta^*$*, and* $\gamma_2 = \min\{\beta^* + 2(r_1 - s), 2r_2\}$*.*

Remark 4 Theorem 4 shows that if $\alpha \to 1^-$, $\beta \to 1$, then the solutions of the fractional equation (1.1)–(1.2) tend to the solution of the classical equation

$$u_t = Au + f(t, u(t)).$$

2.3 The local Lipschitz source

Assumption G2 *Let* $s \in [0, \beta/2]$*,* $\nu \leq \alpha/2$*,* $\kappa \in C([0, \infty); \mathbb{R})$*,* $\kappa(t) \geq 0$ for $t \geq 0$*, and* $f \in C((0, \infty) \times D(A^s); H)$*. For every* $T, M > 0$*, we assume that there is an* $L_T(M)$ *such that*

$$\|f(t, w_1) - f(t, w_2)\| \leq t^{-\nu}\kappa(t)L_T(M)\|w_1 - w_2\|_s \quad \text{for all } t \in [0, T],\ w_1, w_2 \in D(A^s),\ \|w_1\|_s, \|w_2\|_s \leq M.$$

Put $\Xi = (0, 1)$, $H = L^2(\Xi)$, we can directly check that some common sources of the following equations satisfy: the Ginzburg Landau equation and the Burger equation for $\nu = 0, s = 1$, the Cahn–Hilliard and Kuramoto–Sivashinsky equations for $\nu = 0, s = 2$.

In this section, we investigate the existence and uniqueness of the solution of the problem with local source defined in (2.2). In addition, we study the dependence of the solution with respect to the fractional order α, β and the initial data ζ. To emphasize the dependence of the solution u on these given data, let us write it by $u_{\zeta,\alpha,\beta}$. We have the following theorem.

Theorem 5 *Let $\alpha \in (0,1)$, $\beta \in (0,+\infty)$, $s \in [0,\beta/2]$, $\nu < \alpha/2$, and let ζ be the initial data defined in (1.2) such that $\zeta \in D(A^{\beta/2})$. Let the source function f satisfy* ***Assumption F1****(α) and* ***Assumption G2****.*

Then, for any $M > 2\|\zeta\|_{\beta/2}$, we have:

(i) (Local existence) There exists a $T_M > 0$ such that the FIVP has a unique mild solution $u_{\zeta,\alpha,\beta}$ which belongs to $C([0,T_M]; D(A^s))$.
(ii) (Uniqueness) If $V, W \in C([0,T]; D(A^s))$ are solutions of (1.3) on $[0,T]$, then $V = W$.
(iii) (Maximal existence) Let

$$T_{\zeta,\alpha,\beta} = \sup\{T > 0 : \text{(1.3)} \ \ \textit{has a unique solution on } [0,T]\}.$$

Then the equation (1.3) has a unique solution $u_{\zeta,\alpha,\beta} \in C([0,T_{\zeta,\alpha,\beta}); D(A^s))$. Moreover, we have either $T_{\zeta,\alpha,\beta} = +\infty$ or $T_{\zeta,\alpha,\beta} < +\infty$ and $\|u_{\zeta,\alpha,\beta}(t)\|_s \to \infty$ as $t \to T^-_{\zeta,\alpha,\beta}$. Besides, if $u_{\zeta,\alpha,\beta} \in B_{s,T}(M)$ then

$$\|u_{\zeta,\alpha,\beta}(t)\|_s^2 \le 2\Gamma(1-2\nu)|g|^2_{s,t} E_{\alpha-2\nu,1-2\nu}\left(2\theta^{2s-\beta} L_T^2(M) t^{\alpha-2\nu}\right),$$

for any $t \in [0,T]$.
(iv) Let $\zeta, \zeta_k \in D(A^s)$, $\alpha, \alpha_k \in [\alpha_*, \alpha^*]$, $\beta, \beta_k \in [\beta_*, \beta^*]$ satisfy

$$\zeta_k \to \zeta \ \text{ in } \ D(A^s), \ \ \alpha_k \to \alpha, \ \ \beta_k \to \beta \quad \textit{as} \ \ k \to \infty.$$

Assume in addition that ***Assumption F1****(α_*) hold and $s = \beta/2$, $\beta_k \ge \beta$ as $k \to \infty$ or $0 \le s < \beta/2$. Then for every $T \in (0, T_{\zeta,\alpha,\beta})$ we can find a $k_0 > 0$ such that $T_{\zeta_k,\alpha_k,\beta_k} > T$ for every $k > k_0$ and*

$$\lim_{k\to\infty} |u_{\zeta_k,\alpha_k,\beta_k} - u_{\zeta,\alpha,\beta}|_{T,s} = 0.$$

Moreover, we have

$$\liminf_{k\to\infty} T_{\zeta_k,\alpha_k,\beta_k} \ge T_{\zeta,\alpha,\beta}.$$

If $T_{\zeta,\alpha,\beta} = \infty$ then $\lim_{k\to\infty} T_{\zeta_k,\alpha_k,\beta_k} = \infty$.

Using Theorem 5, we can obtain global existence and polynomial decay results. To state precisely the theorem, we state the following

Assumption G3 *For $\nu > 0$, $f : C((0,\infty) \times D(A^s); H)$, $\kappa \in C([0,\infty); [0,\infty))$ we assume that there is an $L_\infty(M) > 0$ for every $M > 0$ such that*

$$\|f(t,w_1) - f(t,w_2)\| \le t^{-\nu}\kappa(t) L_\infty(M)\|w_1 - w_2\|_s \quad \text{for all } \ t \in [0,\infty),\ w_1, w_2 \in D(A^s),\ \|w_1\|_s, \|w_2\|_s \le M$$

and there are constants $\kappa_1, \eta > 0$, $\ell \in \mathbb{R}$ *such that*

$$\|f(t,v) - f(t,0)\| \le \kappa_1 t^{-\nu}(1+t)^{\ell}\|v\|_s^{1+\eta} \text{ for every } t > 0,\, v \in D(A^s).$$

Theorem 6 *Let* $(\alpha, \beta) \in \Delta$ *be as in* (2.4)*, let* $0 \le 2\nu \le \alpha < 1$*,* $\eta, \omega > 0$*,* $s \in [0, \beta/2]$*,* $\ell \in \mathbb{R}$ *and suppose that* ***Assumption G3*** *hold. Assume that*

(i) $\eta > \max\{2(\ell - \nu), 0\}$ *and* $\max\left\{\frac{\ell-\nu}{\eta}, 0\right\} < \omega < 1/2$,
(ii) $m^2_{\infty,\alpha,\omega} := \sup_{t\ge 0}(1+t)^{2\omega}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^{\beta}(t-\tau)^{\alpha})\|f(\tau,0)\|^2\,\mathrm{d}\tau < \infty$.

Then there exists $\delta_0 > 0$ *such that Problem* $P_{\zeta,\alpha,\beta}$ *has a unique solution* $u_{\zeta,\alpha,\beta} \in C([0,\infty); D(A^s))$ *which has the optimal decay rate for*

$$\|\zeta\|_s^2 + m^2_{\infty,\alpha,\omega} \le \delta_0^2.$$

Moreover, if $\frac{\ell-\nu}{\eta} < \alpha < 1/2$*, then*

$$\lim_{t\to\infty}(1+t)^{\alpha}\|u_{\zeta,\alpha,\beta}(t)\|_s = \frac{1}{\Gamma(1-\alpha)}\|\zeta\|_{s-\beta}.$$

Remark 5 From the condition (i), the global result holds for $\nu < \ell < 1/2$. In this case the Lipschitz coefficient can be unbounded as $t \to \infty$ since $\lim_{t\to\infty}\kappa_1 t^{-\nu}(1+t)^{\ell} = \infty$.

3 Proofs

3.1 Preliminary lemmas

Lemma 1 [see [18]] *Letting* $\lambda > 0$, $p > 0$ *and* $k \in \mathbb{N}$*, we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}E_p(-\lambda t^p) = -\lambda t^{p-k}E_{p,p-k+1}(-\lambda t^p),\;\; t \ge 0.$$

Lemma 2 *Let* $0 < p_* < p^* < 2$ *such that* $p^* < 2p_*$*, and* $r_* > 0$*. Then for any* $p, p_0 \in [p_*, p^*]$*, and* $r, r_0 \ge r_*$*, and* $\lambda \ge 0$*, we have*:

(a) There exists a constant $C = C(p_*, p^*, r_*) > 0$ *such that*

$$\left|E_{p,r}(-\lambda)\right| + \left|\frac{\partial E_{p,r}}{\partial p}(-\lambda)\right| + \left|\frac{\partial E_{p,r}}{\partial r}(-\lambda)\right| \le \frac{C}{1+\lambda}.$$

We also have

$$0 \le E_{\alpha}(-z) \le 1,\;\; 0 \le E_{\alpha,\alpha}(-z) \le \frac{1}{\Gamma(\alpha)}\;\; for\;\; z \ge 0. \tag{3.1}$$

(b) Let $0 < p_* < p^* < 1$. *There exist two constants* C_1, C_2 *which depend only on* p_*, p^* *such that*

$$\frac{1}{\Gamma(1-p)}\frac{C_1}{1+\lambda} \le E_p(-\lambda) \le \frac{1}{\Gamma(1-p)}\frac{C_2}{1+\lambda}.$$

Moreover, we have $\lim_{\lambda\to+\infty}\lambda E_p(-\lambda) = \frac{1}{\Gamma(1-p)}$.

(c) There exists a constant $C = C(p_*, p^*)$ *such that*

$$\left|E_p(-\lambda^r t^p) - E_{p_0}(-\lambda^{r_0}t^{p_0})\right| \le C\lambda^{r^*}(1+\ln\lambda)\left(|p-p_0|+|r-r_0|\right), \ \forall\lambda\ge 1.$$

(d) We denote

$$E_{a,b}(\lambda, t, \tau) = (t-\tau)^{a-1}E_{a,a}(-\lambda^b(t-\tau)^a).$$

Then, there exists a constant $C = C(p_*, p^*, r_*)$ *such that*

$$\int_0^t \left|E_{p,r}(\lambda, t, \tau) - E_{p_0,r_0}(\lambda, t, \tau)\right| \, d\tau \le C\left((1+\lambda^r)|p-p_0| + |\lambda^r - \lambda^{r_0}|\right).$$

(e) Put

$$I_{\alpha,\nu,\omega}(t) = \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\frac{\tau^{-2\nu}}{(1+\tau)^{2\omega}} \, d\tau.$$

For $t \ge 1$, $\nu + \omega' < 1/2$, $0 < \omega' \le \omega$, *there is a constant* $D_{\omega,\omega'} > 0$ *such that*

$$0 \le I_{\alpha,\nu,\omega}(t) \le D_{\omega,\omega'}t^{-2\nu-2\omega'}.$$

Proof We only prove (3.1) and (b), (e). The readers can see the proof of other cases in [8]. From the complete monotonicity of the Mittag-Leffler function $E_\alpha(-z)$ for $z \ge 0$ (see [14], Ch. 3) we have $(-1)^n \frac{d^n}{dz^n}E_\alpha(-z) \ge 0$ for $z \ge 0$. Hence we have $E_\alpha(-z)$, $E_{\alpha,\alpha}(-z)$ is decreasing which give $0 \le E_\alpha(-z) \le 1, 0 \le E_{\alpha,\alpha}(-z) \le \frac{1}{\Gamma(\alpha)}$ for $z \ge 0$.

We prove (b). Using the asymptotic expansion in [14,page 19, Ch. 3] we have $\lambda E_p(-\lambda) = \frac{1}{\Gamma(1-p)} + O(\lambda^{-1})$. Hence $\lim_{\lambda\to+\infty}\lambda E_p(-\lambda) = \frac{1}{\Gamma(1-p)}$.

We prove (e) next. In fact, noting that $\sup_{\tau\ge 0}\frac{\tau^{2\omega'}}{(1+\tau)^{2\omega}} \le 1$, we have

$$\begin{aligned} I_{\alpha,\nu,\omega} &= \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\frac{\tau^{-2\nu-2\omega'}\tau^{2\omega'}}{(1+\tau)^{2\omega}} \, d\tau \\ &\le \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\tau^{-2\nu-2\omega'} \, d\tau \\ &= t^{\alpha-2\nu-2\omega'}(J_1+J_2), \end{aligned}$$

where

$$J_1 = \int_0^{1/2} (1-s)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta t^\alpha (1-s)^\alpha) s^{-2\nu-2\omega'} \, \mathrm{d}s,$$
$$J_2 = \int_{1/2}^{1} (1-s)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta t^\alpha (1-s)^\alpha) s^{-2\nu-2\omega'} \, \mathrm{d}s.$$

Using Lemma 2 (a) and estimating directly J_1 gives

$$\begin{aligned} J_1 &\le \frac{2^{1-\alpha} E_{\alpha,\alpha}(-\theta^\beta t^\alpha 2^{-\alpha})}{(1-2\nu-2\omega')2^{1-2\nu-2\omega'}} \\ &\le \frac{2^{1-\alpha} C}{(1+\theta^\beta t^\alpha 2^{-\alpha})(1-2\nu-2\omega')2^{1-2\nu-2\omega'}} \le \frac{C'}{t^\alpha}. \end{aligned}$$

Similarly, by Lemma 1, we have

$$\begin{aligned} J_2 &\le 2^{2\nu+2\omega'} \int_{1/2}^{1} (1-s)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta t^\alpha (1-s)^\alpha) \, \mathrm{d}s \\ &= \frac{2^{2\nu+2\omega'}}{\theta^\beta \alpha t^\alpha} \int_{1/2}^{1} \frac{\mathrm{d}}{\mathrm{d}s} E_\alpha(-\theta^\beta t^\alpha (1-s)^\alpha) \, \mathrm{d}s \\ &= \frac{2^{2\nu+2\omega'}}{\theta^\beta \alpha t^\alpha} (1 - E_\alpha(-\theta^\beta t^\alpha 2^{-\alpha})) \le \frac{2^{2\nu+2\omega'}}{\theta^\beta \alpha t^\alpha}. \end{aligned}$$

From the estimation of J_1, J_2 we complete the proof of Part (e). □

In this paper, we also need the following useful inequality.

Lemma 3 *Let $\alpha, q \in \mathbb{R}$, $0 < \alpha \le 1$, $q < \alpha$, and let $v, g \in C[0,T]$. Then the equation*

$$u(t) = v(t) + g(t) \int_0^t (t-\tau)^{\alpha-1} \tau^{-q} u(\tau) \, \mathrm{d}\tau$$

has a unique solution $u \in C[0,T]$ which satisfies

$$|u(t)| \le \Gamma(1-q) \|v\|_{C[0,t]} E_{\alpha-q,1-q} \left(\|g\|_{C[0,t]} \Gamma(\alpha) t^{\alpha-q} \right) \tag{3.2}$$

for $t \in [0,T]$. As a consequence, if $w \in C[0,T]$ satisfies

$$0 \le w(t) \le v(t) + g(t) \int_0^t (t-\tau)^{\alpha-1} \tau^{-q} w(\tau) d\tau \quad \text{for } t \in [0,T],$$

and if $g(t) \ge 0$ for $t \in [0,T]$, then

$$w(t) \le C\Gamma(1-q) \|v\|_{C[0,t]} E_{\alpha-q,1-q} \left(\|g\|_{C[0,t]} \Gamma(\alpha) t^{\alpha-q} \right) \quad \text{for } t \in [0,T],$$

where $C = \max_{k\geq 1} d_k$ *with* $d_1 = \Gamma((\alpha-q)+1-q)/\Gamma(\alpha+1-q)$ *and* $d_k = \Gamma(k(\alpha-q)+1-q)/\Gamma(k(\alpha-q)+1)$.

Remark 6 We note that $\Gamma(a) \leq \Gamma(b)$ for any $2 \leq a \leq b$, therefore, $\Gamma(k(\alpha-q)+1-q)/\Gamma(k(\alpha-q)+1) \leq 1$ for k large enough or $d_{k+1} \leq d_k$. This implies that $C = \max_{k\geq 1} d_k < +\infty$.

Proof of Lemma 3 Put

$$Su(t) = v(t) + g(t)\int_0^t (t-\tau)^{\alpha-1}\tau^{-q}u(\tau)\,\mathrm{d}\tau.$$

Using the similar technique as in Theorem 1, we can prove that there exists $k_0 \in \mathbb{N}$ such that S^{k_0} is contraction in $C[0,T]$. Consequently, there exists a unique $u \in C[0,T]$ such that $u = Su$.

We put $u_0 = 0$, $u_{n+1} = Su_n$. The function can be represented by the series $u = \sum_{n=0}^{\infty}(u_{n+1}-u_n)$. The Weierstrass theorem shows that the series converges in $C[0,T]$ and

$$\begin{aligned}|u(t)| &\leq \|u_1-u_0\|_{C[0,t]}\sum_{k=0}^{\infty}\frac{\Gamma(1-q)(\|g\|_{C[0,t]}\Gamma(\alpha))^k t^{k(\alpha-q)}}{\Gamma(k(\alpha-q)-q+1)}\\ &= C\Gamma(1-q)\|v\|_{C[0,t]}E_{\alpha-q,1-q}\left(\|g\|_{C[0,t]}\Gamma(\alpha)t^{\alpha-q}\right).\end{aligned}$$

Now, we prove the final inequality. Put $w_0 = Sw$, $w_{n+1} = Sw_n$. Since $g(t) \geq 0$ for $t \in [0,T]$, we have $Sw_1(t) \leq Sw_2(t)$ for $w_1(t) \leq w_2(t)$, $t \in [0,T]$. We note that $w \leq w_0$, hence, by induction we obtain $w_n \leq w_{n+1}$. Using the contraction principle we obtain $\lim_{n\to\infty}\|w_n - u\|_{C[0,T]} = 0$. Since $w_n \leq w_{n+1}$ for every $n = 0, 1, \ldots$, we obtain $w(t) \leq w_0(t) \leq u(t)$ for $t \in [0,T]$. From (3.2) we obtain the desired inequality. □

We also need the following results.

Lemma 4 *Let* $T, \theta > 0, \alpha \in (0,1], \beta > 0$, $s \in [0, \beta/2]$, $r \geq 0$, $t \in (0,T]$, $w \in C([0,T]; D(A^r))$.

(i) For $\zeta \in D(A^{s+r})$, *we have* $E_\alpha(-t^\alpha A^\beta)\zeta \in D(A^{s+r})$ *and* $\|E_\alpha(-t^\alpha A^\beta)\zeta\|_{s+r} \leq \|\zeta\|_{s+r}$, *and*

$$\lim_{t\to\infty}(1+t)^\alpha\|E_\alpha(-t^\alpha A^\beta)\zeta\|_s = \frac{1}{\Gamma(1-\alpha)}\|\zeta\|_{s-\beta}. \tag{3.3}$$

For every $0 \leq \omega \leq \alpha/2$, *we also have*

$$\Lambda_\omega^2 := \sup_{\xi}\sup_{t>0}(1+t)^{2\omega}\|E_\alpha\big(-t^\alpha A^\beta\big)\xi\|_s^2 < \infty \text{ for } \xi \in D(A^s),\ \|\xi\|_s = 1 \tag{3.4}$$

and

$$\sup_{t>0}(1+t)^{2\omega}\|E_\alpha\big(-t^\alpha A^\beta\big)\zeta\|_s^2 \leq \Lambda_\omega^2\|\zeta\|_s^2. \tag{3.5}$$

(ii) Put

$$Q_{\alpha,\beta,A}(w)(t) = \int_0^t E_{\alpha,\beta}(A,t,\tau)w(\tau)\,\mathrm{d}\tau, \tag{3.6}$$

where

$$E_{\alpha,\beta}(\lambda,t,\tau) = (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda^\beta(t-\tau)^\alpha) = \frac{1}{\lambda^\beta}\frac{d}{d\tau}E_\alpha(-\lambda^\beta(t-\tau)^\alpha).$$

If $w \not\equiv 0$ *on* $[0,t]$*, then*

$$\left\|Q_{\alpha,\beta,A}(w)(t)\right\|^2_{s+r} < \sup_{\lambda\ge\theta}\lambda^{2s-\beta}H_0(\lambda,t)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\|w(\tau)\|_r^2\,\mathrm{d}\tau, \tag{3.7}$$

where

$$H_0(\lambda,t) := 1 - E_\alpha\big(-\lambda^\beta t^\alpha\big).$$

Proof We first prove (i). One has

$$\begin{aligned}\left\|E_\alpha(-t^\alpha A^\beta)\zeta\right\|^2_{s+r} &= \int_\theta^\infty \lambda^{2(s+r)}E_\alpha(-t^\alpha\lambda^\beta)\,\mathrm{d}\|S_\lambda\zeta\|^2\\ &\le \int_\theta^\infty \lambda^{2(s+r)}\,\mathrm{d}\|S_\lambda\zeta\|^2 = \|\zeta\|^2_{s+r}.\end{aligned}$$

We next prove (3.3). In fact, we have

$$(1+t)^{2\alpha}\left\|E_\alpha(-t^\alpha A^\beta)\xi\right\|_s^2 = (1+t)^{2\alpha}\int_\theta^\infty \lambda^{2s}E_\alpha^2(-t^\alpha\lambda^\beta)\,\mathrm{d}\|S_\lambda\xi\|^2.$$

Using Lemma 2 we have $\lim_{t\to\infty}(1+t)^\alpha E_\alpha(-t^\alpha\lambda^\beta) = \frac{1}{\lambda^\beta\Gamma(1-\alpha)}$ and

$$\sup_{t\ge0}|(1+t)^\alpha E_\alpha(-t^\alpha\lambda^\beta)| < \infty.$$

Hence, applying the Lebesgue dominated convergence theorem yields

$$\begin{aligned}\lim_{t\to\infty}(1+t)^{2\alpha}\int_\theta^\infty \lambda^{2s}E_\alpha^2(-t^\alpha\lambda^\beta)\,\mathrm{d}\|S_\lambda\xi\|^2 &= \frac{1}{\Gamma^2(1-\alpha)}\int_\theta^\infty \lambda^{2s-2\beta}\,\mathrm{d}\|S_\lambda\xi\|^2\\ &= \frac{1}{\Gamma^2(1-\alpha)}\|\xi\|^2_{s-\beta}.\end{aligned}$$

From (3.3), we deduce (3.4). Put $\xi = \zeta/\|\zeta\|_s$ we obtain (3.5).

Now, we consider Part (ii). We have $E_{\alpha,\alpha}(z) \ge 0$ (see [14], Ch. 3). Hence, Lemma 1 yields

$$\int_0^t |E_{\alpha,\beta}(\lambda, t, \tau)| \, d\tau = \int_0^t E_{\alpha,\beta}(\lambda, t, \tau) \, d\tau = \frac{1}{\lambda^\beta} H_0(\lambda, t). \tag{3.8}$$

By the Hölder inequality, Lemma 2 and (2.1), we obtain for $t \in (0, T]$

$$\begin{aligned}\left\| Q_{\alpha,\beta,A}(w)(t) \right\|_{s+r}^2 &= \left\| \int_0^t |E_{\alpha,\beta}(A, t, \tau)| w(\tau) \, d\tau \right\|_{s+r}^2 \\ &\leq \int_\theta^{+\infty} \lambda^{2(s+r)} \int_0^t |E_{\alpha,\beta}(\lambda, t, \tau)| \, d\tau \times \int_0^t |E_{\alpha,\beta}(\lambda, t, \tau)| \, d\|S_\lambda w(\tau)\|^2 \, d\tau \\ &\leq \int_0^t \int_\theta^{+\infty} \frac{\lambda^{2s}}{\lambda^\beta} H_0(\lambda, t) |E_{\alpha,\beta}(\lambda, t, \tau)| \lambda^{2r} \, d\|S_\lambda w(\tau)\|^2 \, d\tau.\end{aligned}$$

Noting that $E_{\alpha,\alpha}(-\lambda^\beta (t-\tau)^\alpha) \leq E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha)$ for $\lambda \geq \theta$, we obtain

$$\begin{aligned}&\left\| Q_{\alpha,\beta,A}(w)(t) \right\|_{s+r}^2 \\ &< \frac{1}{\Gamma(\alpha)} \sup_{\lambda \geq \theta} \lambda^{2s-\beta} H_0(\lambda, t) \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha) \|w(\tau)\|_r^2 \, d\tau.\end{aligned}$$

□

3.2 Proof of Theorem 1

For $w \in C([0, T], D(A^s))$, we put

$$F(w)(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A, t, \tau) f(\tau, w(\tau)) \, d\tau.$$

Choosing $r = 0$ in Lemma 4 (ii) gives $\sup_{\lambda \geq \theta} \lambda^{2s-\beta} H_0(\lambda, t_1, t_2) \leq \theta^{2s-\beta}$ and

$$\begin{aligned}&\|F(w_1)(t) - F(w_2)(t)\|_s^2 &(3.9)\\ &\leq \frac{1}{\Gamma(\alpha)} \theta^{2s-\beta} \int_0^t (t-\tau)^{\alpha-1} \|f(\tau, w_1(\tau)) - f(\tau, w_2(\tau))\|^2 \, d\tau \\ &\leq \frac{1}{\Gamma(\alpha)} \theta^{2s-\beta} \kappa_{T,s}^2 \int_0^t (t-\tau)^{\alpha-1} \tau^{-2\nu} \|w_1(\tau) - w_2(\tau)\|_s^2 \, d\tau. &(3.10)\end{aligned}$$

So we have

$$\begin{aligned}\|F(w_1)(t) - F(w_2)(t)\|_s^2 &\leq \frac{1}{\Gamma(\alpha)} \theta^{2s-\beta} \kappa_{T,s}^2 |w_1 - w_2|_{s,T}^2 \int_0^t (t-\tau)^{\alpha-1} \tau^{-2\nu} d\tau \\ &= \frac{1}{\Gamma(\alpha)} B(\alpha, 1-2\nu) \theta^{2s-\beta} \kappa_{T,s}^2 |w_1 - w_2|_{s,T}^2 t^{\alpha-2\nu} \\ &= d_1 \frac{\Gamma(1-2\nu)}{\Gamma(\alpha+1-2\nu)} \theta^{2s-\beta} \kappa_{T,s}^2 |w_1 - w_2|_{s,T}^2 t^{\alpha-2\nu},\end{aligned}$$

where $d_1 = \Gamma((\alpha - 2\nu) + 1 - 2\nu)/\Gamma(\alpha + 1 - 2\nu)$. We consider the case $\nu < \alpha/2$. For $w_1, w_2 \in C([0, T], D(A^s))$, using the similar technique as in [8], we can prove by induction that

$$\left\| F^k(w_1)(t) - F^k(w_2)(t) \right\|_s^2 \leq d_k \frac{\Gamma(1-2\nu)\left(\theta^{2s-\beta}\kappa_{T,s}^2\right)^k t^{k(\alpha-2\nu)}}{\Gamma(k(\alpha-2\nu)-2\nu+1)} |w_1 - w_2|_{s,T}^2$$
$$\leq C \frac{\Gamma(1-2\nu)\left(\theta^{2s-\beta}\kappa_{T,s}^2\right)^k t^{k(\alpha-2\nu)}}{\Gamma(k(\alpha-2\nu)-2\nu+1)} |w_1 - w_2|_{s,T}^2,$$

where $d_k = \Gamma(k(\alpha - 2\nu) + 1 - 2\nu)/\Gamma(k(\alpha - 2\nu) + 1)d_{k-1}$ with $k \geq 2$ and $C = \max_{k\geq 1} d_k$. This gives

$$\lim_{k\to\infty} C \frac{\Gamma(1-2\nu)\left(\theta^{2s-\beta}\kappa_{T,s}^2\right)^k T^{k(\alpha-2\nu)}}{\Gamma\left(k(\alpha-2\nu)-2\nu+1\right)} = 0.$$

Hence there is a $k_0 \in \mathbb{N}$ such that

$$C \frac{\Gamma(1-2\nu)\left(\theta^{2s-\beta}\kappa_{T,s}^2\right)^{k_0} T^{k_0(\alpha-2\nu)}}{\Gamma\left(k_0(\alpha-2\nu)-2\nu+1\right)} \leq \frac{1}{2}$$

which gives

$$|F^{k_0}(w_1) - F^{k_0}(w_2)|_{s,T}^2 \leq C \frac{\Gamma(1-2\nu)\left(\theta^{2s-\beta}\kappa_{T,s}^2\right)^{k_0} T^{k_0(\alpha-2\nu)}}{\Gamma\left(k_0(\alpha-2\nu)-2\nu+1\right)} |w_1 - w_2|_{s,T}^2$$
$$\leq \frac{1}{2}|w_1 - w_2|_{s,T}^2,$$

i.e., F^{k_0} is a contraction in $C([0, T], D(A^s))$. Hence, the exists a unique fixed point $u \in C([0, T], D(A^s))$ satisfying $u = F^{k_0}(u)$. We deduce that $Fu = F^{k_0}(Fu)$, i.e., Fu is also a fixed point of the operator F^{k_0}. Hence $u = Fu$.

We give the estimate of u. In fact, from (3.10) we obtain

$$\begin{aligned} \|u(t) - g\|_s^2 &= \|\mathcal{F}u(t) - \mathcal{F}(0)(t)\|_s^2 \\ &\leq \frac{1}{\Gamma(\alpha)}\theta^{2s-\beta}\kappa_{T,s}^2 \int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu}\|u(\tau) - 0\|_s^2 \,\mathrm{d}\tau. \end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned} \|u(t)\|_s^2 &\leq 2\|g(t)\|_s^2 + 2\|u(t) - g\|_s^2 \\ &\leq 2\|g(t)\|_s^2 + \frac{2}{\Gamma(\alpha)}\theta^{2s-\beta}\kappa_{T,s}^2 \int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu}\|u(\tau)\|_s^2 \,\mathrm{d}\tau. \end{aligned} \tag{3.12}$$

Using (3.2) of Lemma 3, we obtain the inequality of the theorem.

Finally, we consider the case $\nu = \alpha/2$. We can find a $\xi \in (0, T]$ such that $\|F(w_1)(\xi) - F(w_2)(\xi)\|_s^2 = \sup_{0\le t\le T} \|F(w_1)(t) - F(w_2)(t)\|_s^2$. Lemma 4 gives

$$\begin{aligned}\|F(w_1)(\xi)-F(w_2)(\xi)\|_s^2 &< \frac{1}{\Gamma(\alpha)}\theta^{2s-\beta}\int_0^\xi (\xi-\tau)^{\alpha-1}\|f(\tau, w_1(\tau))-f(\tau, w_2(\tau)\|^2 \,\mathrm{d}\tau \\ &\le \frac{1}{\Gamma(\alpha)}\theta^{2s-\beta}\kappa_{T,s}^2\int_0^\xi (\xi-\tau)^{\alpha-1}\tau^{-\alpha}\|w_1(\tau)-w_2(\tau)\|_s^2 \,\mathrm{d}\tau \\ &\le \frac{1}{\Gamma(\alpha)}B(\alpha, 1-\alpha)\theta^{2s-\beta}\kappa_{T,s}^2|w_1 - w_2|_{s,T}^2 \\ &= \Gamma(1-\alpha)\theta^{2s-\beta}\kappa_{T,s}^2|w_1 - w_2|_{s,T}^2.\end{aligned}$$

If $\kappa_{T,s} < \theta^{\beta/2-s}\,(\Gamma(1-\alpha))^{-1/2}$ then F is a contraction in $C([0, T], D(A^s))$. Consequently, the problem (1.3) has a unique solution in $C([0, T], D(A^s))$. □

3.3 Proof of Theorem 2

In the proof we denote $u = u_{\zeta,\alpha,\beta}$ for short. We verify that $\sup_{t\ge 0}(1+t)^\omega\|u(t)\|_s < \infty$. Assume by contradiction that $\sup_{t\ge 0}(1+t)^\omega\|u(t)\|_s = \infty$. For every $\lambda > 0$, we put

$$T_\lambda = \inf\{T > 0 : (1+t)^\omega\|u(t)\|_s \le \lambda \text{ for every } t \in [0, T]\}.$$

By the continuity of u, we have

$$(1+T_\lambda)^\omega\|u(T_\lambda)\|_s = \lambda, \quad \lim_{\lambda\to\infty} T_\lambda = \infty, \quad (1+t)^\omega\|u(t)\|_s \le \lambda,\ \forall t \in [0, T_\lambda].$$

We note that

$$\|f(t, u(t))\| \le \|f(t, u(t)) - f(t, 0)\| + \|f(t, 0)\| \le \kappa_1 t^{-\nu}(1+t)^\ell\|u(t)\|_s + \|f(t, 0)\|.$$

As in the proof of Theorem 1, choosing $r = 0$ in Lemma 4 gives

$$\sup_{\lambda\ge\theta}\lambda^{2s-\beta}H_0(\lambda, t_1, t_2) \le \theta^{2s-\beta}$$

and we have

$$\begin{aligned}&\|u(t)\|_s^2 \\ &\le 2\|E_\alpha\big(-t^\alpha A^\beta\big)\zeta\|_s^2 \\ &\quad + 2\theta^{2s-\beta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\|f(\tau, u(\tau))\|^2 \,\mathrm{d}\tau \\ &\le 2\|E_\alpha\big(-t^\alpha A^\beta\big)\zeta\|_s^2\end{aligned}$$

$$+4\theta^{2s-\beta}\int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu}\kappa_1^2(1+\tau)^{2\ell}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\|u(\tau)\|_s^2\,\mathrm{d}\tau$$
$$+4\theta^{2s-\beta}\int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\|f(\tau,0)\|^2\,\mathrm{d}\tau. \tag{3.13}$$

For every $\lambda > 0$, denoting $w(t) = (1+t)^\omega u(t)$ and using (3.13) yield

$$\|w(t)\|_s^2 \le 4\Lambda_\omega^2\|\zeta\|_s^2 + (1+t)^{2\omega}V_1^2 + (1+t)^{2\omega}V_2^2, \tag{3.14}$$

where $0 < t \le T_\lambda$ and

$$V_1^2 = 4\theta^{2s-\beta}\lambda^2\int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu}\kappa_1^2(1+\tau)^{2\ell-2\omega}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\,\mathrm{d}\tau,$$
$$V_2^2 = 4\theta^{2s-\beta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\theta^\beta(t-\tau)^\alpha)\kappa_f^2\tau^{-2\nu_f}(1+\tau)^{2\ell_f}\,\mathrm{d}\tau.$$

To prove the theorem, we will use two necessary inequalities. In fact, we can find two constants $\gamma_1, \gamma_2 > 0$ such that

$$(1+t)^{2\omega}V_1^2 \le Ct^{-2\gamma_1},\ \ (1+t)^{2\omega}V_2 \le Ct^{-2\gamma_2}. \tag{3.15}$$

The proof of these inequalities will be postponed to the end of the proof of the theorem.

(i) We first consider the case $\ell, \ell_f < \nu, \alpha > \frac{1}{2} - \nu$.
Using the inequalies (3.14), (3.15) yield

$$\|w(t)\|_s^2 \le 4\Lambda_\omega^2\|\zeta\|_s^2 + Ct^{-2\gamma_1} + Ct^{-2\gamma_2}.$$

Choosing $t = T_\lambda$, we obtain

$$\lambda^2 \le 4\Lambda_\omega^2\|\zeta\|_s^2 + C'\lambda^2T_\lambda^{-2\gamma_1} + C'T_\lambda^{-2\gamma_2}$$

which implies

$$4\lambda^{-2}\Lambda_\omega^2\|\zeta\|_s^2 + C'T_\lambda^{-2\gamma_1} + C'\lambda^{-2}T_\lambda^{-2\gamma_1} \ge 1. \tag{3.16}$$

Noting that $\lim_{\lambda\to\infty} T_\lambda = \infty$, we obtain in view of (3.16) that $0 \ge 1$ which is a contradiction.

(ii) We consider the case $\ell, \ell_f \le \frac{1}{2} - \alpha$. For the upper bound of $\|u(t)\|_s$ we can use the same argument as Part (i) with $\omega > \alpha$. We verify the asymptotic value for $(1+t)^\alpha\|u(t)\|_s$. Using Lemma 2 yields

$$\lim_{t\to\infty}(1+t)^{2\alpha}\left\|E_\alpha(-t^\alpha A^\beta)\zeta\right\|_s^2 = \frac{1}{\Gamma^2(1-\alpha)}\|\zeta\|_{s-\beta}^2.$$

Choose $\omega = \alpha$, denoting $w(t) = (1+t)^{\omega} u(t)$ and using (3.13) yield

$$\begin{aligned}
&\left\| w(t) - (1+t)^{\omega} E_{\alpha}\big(-t^{\alpha} A^{\beta}\big)\zeta \right\|_s \\
&\le \Bigg(4\theta^{2s-\beta} \int_0^t (t-\tau)^{\alpha-1} \tau^{-2\nu} \kappa_1^2 (1+\tau)^{2\ell-2\omega} E_{\alpha,\alpha}(-\theta^{\beta}(t-\tau)^{\alpha}) \|w(\tau)\|_s^2 \, \mathrm{d}\tau \\
&\quad + 4\theta^{2s-\beta}(1+t)^{2\omega} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\theta^{\beta}(t-\tau)^{\alpha}) \|f(\tau,0)\|^2 \, \mathrm{d}\tau \Bigg)^{1/2} \\
&\le \sqrt{(1+t)^{2\omega} V_1^2 + (1+t)^{2\omega} V_2^2}.
\end{aligned}$$

Combining the latter inequality with (3.15) yields

$$\left\| w(t) - (1+t)^{\omega} E_{\alpha}\big(-t^{\alpha} A^{\beta}\big)\zeta \right\|_s \le \sqrt{C t^{-2\gamma_1} + C t^{-2\gamma_2}}.$$

Hence

$$\lim_{t\to\infty} \|w(t)\|_s = \lim_{t\to\infty} \|(1+t)^{\alpha} E_{\alpha}\big(-t^{\alpha} A^{\beta}\big)\zeta\|_s = \frac{1}{\Gamma(1-\alpha)} \|\zeta\|_{s-\beta}.$$

Finally, we prove (3.15). Since the proof for the case $\ell \le 0$ is different from the one of the case $\ell > 0$, we divide the proof into two cases.

In the case $\ell \le 0$, since $0 < \omega < 1/2$, $\nu > 0$ we obtain $\omega - \nu < \min\{1/2 - \nu, \omega - \ell\}$. Hence we can choose ω' such that

$$\omega - \nu < \omega' < \min\{1/2 - \nu, \omega - \ell\}$$

which gives $\omega' < \omega - \ell$, $\nu + \omega' < 1/2$, $\gamma_1 := -\omega + \nu + \omega' > 0$. We can use (3.14) to obtain

$$V_1^2 \le \theta^{2s-\beta} \lambda^2 \kappa_1^2 I_{\alpha,\nu,\omega-\ell} \le C t^{-2\nu-2\omega'},$$

where $I_{\alpha,\nu,\omega}$ is defined in Lemma 2. Applying Lemma 2 (e) we can find a $C > 0$ such that

$$(1+t)^{2\omega} V_1^2 \le C t^{-2\gamma_1}.$$

Next, we consider the case $0 \le \ell < \nu$ we have $\omega + \ell - \nu < 1/2 - \nu$ and $\omega + \ell - \nu < \omega$. Hence we can find ω' such that $\omega + \ell - \nu < \omega' < \min\{1/2 - \nu, \omega\}$. It follows that $\omega' + \nu < 1/2$, $\omega' \le \omega$ and $\gamma_2 := -\omega - \ell + \nu + \omega' > 0$. Using Lemma (2) (e) gives

$$V_1^2 \le \theta^{2s-\beta} \lambda^2 \kappa_1^2 (1+t)^{2\ell} I_{\alpha,\nu,\omega} \le C t^{-2\gamma_1}.$$

In the second case, if $0 < \ell \le \frac{1}{2} - \alpha$, we can choose $\omega = \alpha$.

Similarly, we can prove that there is a $\gamma_2 > 0$ such that

$$(1+t)^{\omega} V_2^2 \le C t^{-2\gamma_2}.$$

This completes the proof of the theorem. □

3.4 Proof of Theorem 3

We denote

$$F_{\zeta,\alpha,\beta,A}(v)(t) = E_{\alpha}\big(-t^{\alpha} A^{\beta}\big)\zeta + \int_0^t E_{\alpha,\beta}(A,t,\tau) f(\tau, v(\tau))\, \mathrm{d}\tau,$$

where $E_{a,b}(.,t,\tau)$ defined in Lemma 4. Using (3.7) and direct computations, one has

$$\begin{aligned}
&\Big\| F_{\xi,\widetilde{\alpha},\widetilde{\beta},A}(w)(t) - F_{\zeta,\widetilde{\alpha},\widetilde{\beta},A}(v)(t)\Big\|_s^2 \\
&\le 2\Big\| E_{\widetilde{\alpha}}\big(-A^{\widetilde{\beta}} t^{\widetilde{\alpha}}\big)(\xi-\zeta)\Big\|_s^2 + 2\Big\| Q_{\widetilde{\alpha},\widetilde{\beta},A}(f(\cdot,w))(0,t) - Q_{\widetilde{\alpha},\widetilde{\beta},A}(f(\cdot,v))(0,t)\Big\|_s^2 \\
&\le 2\|\zeta-\xi\|_s^2 + \frac{2}{\Gamma(\widetilde{\alpha})}\theta^{2s-\widetilde{\beta}} \int_0^t (t-\tau)^{\widetilde{\alpha}-1} \|f(\tau,w) - f(\tau,v)\|^2\, \mathrm{d}\tau \\
&\le 2\|\zeta-\xi\|_s^2 + \frac{2}{\Gamma(\widetilde{\alpha})}\theta^{2s-\widetilde{\beta}} \kappa_{T,s}^2 \int_0^t (t-\tau)^{\widetilde{\alpha}-1}\tau^{-2\nu} \|w(\tau) - v(\tau)\|_s^2\, \mathrm{d}\tau.
\end{aligned}$$

Since $u_{\xi,\widetilde{\alpha},\widetilde{\beta}}$ and $u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}$ are solution of equations $F_{\xi,\widetilde{\alpha},\widetilde{\beta},A}(w) = w$ and $F_{\xi,\widetilde{\alpha},\widetilde{\beta},A}(v) = v$, respectively, by Lemma 3, we conclude that

$$\Big\| u_{\xi,\widetilde{\alpha},\widetilde{\beta}}(t) - u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t)\Big\|_s^2 \le 2\Gamma(1-2\nu) E_{\widetilde{\alpha}-2\nu,1-2\nu}\left(2\theta^{2s-\widetilde{\beta}}\kappa_{T,s}^2 t^{\widetilde{\alpha}-2\nu}\right)\|\zeta-\xi\|_s^2.$$

This leads to the result of Theorem 3. □

3.5 Proof of Theorem 4

We first state the following lemma necessary to prove the theorem. The proof of this lemma is postponed to the next subsection.

Lemma 5 *Let $T > 0$, $\zeta \in D(A^s)$, $\alpha, \widetilde{\alpha} \in [\alpha_*, \alpha^*]$, $\beta, \widetilde{\beta} \in [\beta_*, \beta^*]$. Let the source function f satisfy* **Assumption** *F1(α_*) and* **Assumption G1**. *Then the initial problems have the unique solutions $u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}, u_{\zeta,\alpha,\beta} \in C([0,T], D(A^s))$ with $s \in [0, \min\{\beta/2, \widetilde{\beta}/2\}]$. Then, for any $\epsilon > 0$, there exist two constants P, $P_\epsilon > 0$ which are independent of $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$ and t such that*

$$\Big\| u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t) - u_{\zeta,\alpha,\beta}(t)\Big\|_s \le P\left(\epsilon + P_\epsilon\left(|\alpha-\widetilde{\alpha}| + |\beta-\widetilde{\beta}|\right)\right)^{1/2}$$

for every $t \in [0,T]$. *Suppose further that* $\zeta \in D\left(A^{\beta^*/2+r_1}\right)$ *for some* $r_1 > 0$ *and* $f(.,u_{\zeta,\alpha,\beta}(.)) \in C([0,T], D(A^{r_2}))$. *Then, there exists a constant* $Q_0 > 0$ *which is independent of* $\alpha, \widetilde{\alpha}, \beta, \widetilde{\beta}$, N *and* t *such that*

$$\left\|u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t) - u_{\zeta,\alpha,\beta}(t)\right\|_s \leq Q_0\left(2^{\gamma_1+2}+1\right)\left(|\alpha-\widetilde{\alpha}|+|\beta-\widetilde{\beta}|\right)^{\frac{\gamma_2}{2(\gamma_1+\gamma_2+2)}}$$

for every $t \in [0,T]$. *Herein,* $\gamma_1 = 2\beta^*$, $\gamma_2 = \min\{\beta^*+2r_1-2s, 2r_2\}$.

Proof of Theorem 4 Using Theorem 3 and Lemma 5, we will prove the results of the theorem. Using the triangle inequality, we obtain

$$\begin{aligned}
&\left\|u_{\zeta_k,\alpha_k,\beta_k}(t) - u_{\zeta,\alpha,\beta}(t)\right\|_{\min\{\beta_k/2,\, s\}}\\
&\leq \left\|u_{\zeta_k,\alpha_k,\beta_k}(t) - u_{\zeta,\alpha_k,\beta_k}(t)\right\|_{\min\{\beta_k/2,\, s\}} + \left\|u_{\zeta,\alpha_k,\beta_k}(t) - u_{\zeta,\alpha,\beta}(t)\right\|_{\min\{\beta_k/2,\, s\}}\\
&\leq P_1\|\zeta-\zeta_k\|_s + P\left(\epsilon + P_\epsilon(|\alpha-\alpha_k|+|\beta-\beta_k|)\right)^{1/2},
\end{aligned}$$

for any $t \in [0,T]$. In addition, we note that

$$\|w\|_p \leq \theta^{p-q}\|w\|_q \ \text{ for any } \ 0 \leq p \leq q. \tag{3.17}$$

From the latter result, we can verify directly the main results (2.5), (2.6), (2.7) and (2.8) of the theorem. In fact, if $s \in [0, \beta/2)$ then with k large enough, we have $\beta_k/2 \geq s$. Hence, we can combine Lemma 3, Lemma 5 with (3.17) to obtain (2.5). We also use Lemmas 3 and 5 to deduce (2.6)-(2.7). Finally, combining Lemmas 3 with 5 (ii), we obtain (2.8). This completes the core of the proof. □

3.6 The proof of Lemma 5

By a direct computation, we have

$$\begin{aligned}
\left\|F_{\zeta,\widetilde{\alpha},\widetilde{\beta},A}(v)(t) - F_{\zeta,\alpha,\beta,A}(u)(t)\right\|_s^2 &\leq 2\left\|\left(E_{\widetilde{\alpha}}\left(-A^{\widetilde{\beta}}t^{\widetilde{\alpha}}\right) - E_\alpha\left(-A^\beta t^\alpha\right)\right)\zeta\right\|_s^2\\
&\quad +2\left\|Q_{\widetilde{\alpha},\widetilde{\beta},A}(v)(t) - Q_{\alpha,\beta,A}(u)(t)\right\|_s^2\\
&\leq 2I_1 + 4(I_2+I_3),
\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}
I_1 &= \left\|\left(E_{\widetilde{\alpha}}\left(-A^{\widetilde{\beta}}t^{\widetilde{\alpha}}\right) - E_\alpha\left(-A^\beta t^\alpha\right)\right)\zeta\right\|_s^2,\\
I_2 &= \left\|Q_{\widetilde{\alpha},\widetilde{\beta},A}(v)(t) - Q_{\widetilde{\alpha},\widetilde{\beta},A}(u)(t)\right\|_s^2,\\
I_3 &= \left\|Q_{\widetilde{\alpha},\widetilde{\beta},A}(u)(t) - Q_{\alpha,\beta,A}(u)(t)\right\|_s^2,
\end{aligned}$$

and the function Q is defined in (3.6). We will estimate $I_k (k=1,2,3)$ one by one.

Estimate for I_1. To give an estimation for I_1, we separate the sum I_1 into two sums as follows:

$$I_1 = I_{11}(N) + I_{12}(N), \tag{3.19}$$

where

$$I_{11}(N) = \int_\theta^N \lambda^{2s} \left| E_{\widetilde{\alpha}}\big(-\lambda^{\widetilde{\beta}} t^{\widetilde{\alpha}}\big) - E_\alpha\big(-\lambda^\beta t^\alpha\big)\right|^2 \, \mathrm{d}\|S_\lambda \zeta\|^2,$$
$$I_{12}(N) = \int_{\lambda > N} \lambda^{2s} \left| E_{\widetilde{\alpha}}\big(-\lambda^{\widetilde{\beta}} t^{\widetilde{\alpha}}\big) - E_\alpha\big(-\lambda^\beta t^\alpha\big)\right|^2 \, \mathrm{d}\|S_\lambda \zeta\|^2.$$

For convenience in estimating for $I_{11}(N)$, $I_{12}(N)$, let us assume $N > \max\{e, \theta\}$.

Estimate for $I_{11}(N)$. By Lemma 2, there exist two constants $C = C(\alpha_*, \alpha^*, \beta_*, \beta^*, T) > 0$, $C_0 = C_0(\alpha_*, \alpha^*, \beta_*, \beta^*, \theta, T) > 0$ such that

$$\begin{aligned} I_{11}(N) &\le C \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right)^2 \left(\int_\theta^N \lambda^{2(\beta^*+s)} (1 + |\ln \lambda|)^2 \, \mathrm{d}\|S_\lambda \zeta\|^2 \right) \\ &\le C_0 \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right)^2 N^{2\beta^*} \ln^2 N \int_\theta^N \lambda^{2s} \, \mathrm{d}\|S_\lambda \zeta\|^2 \\ &\le C_N \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right)^2, \end{aligned} \tag{3.20}$$

where $C_N = C_0 \|\zeta\|_s^2 N^{2\beta^*} \ln^2 N$.

Estimate for $I_{12}(N)$. We note that $0 \le E_\alpha(-x) \le 1$ for $x > 0$. This gives

$$I_{12}(N) \le \int_{\lambda > N} \lambda^{2s} \, \mathrm{d}\|S_\lambda \zeta\|^2. \tag{3.21}$$

Substituting (3.20) and (3.21) into (3.19), we obtain

$$I_1 \le C_N (|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|)^2 + \int_{\lambda > N} \lambda^{2s} \, \mathrm{d}\|S_\lambda \zeta\|^2, \tag{3.22}$$

where C_N is defined in (3.20).

Estimate for I_2. Similarly to the proof of Lemma 3, we get

$$I_2 \le \frac{1}{\Gamma(\widetilde{\alpha})} \theta^{2s - \widetilde{\beta}} \kappa_{T,s}^2 \int_0^t (t - \tau)^{\widetilde{\alpha} - 1} \tau^{-2\nu} \|v(\tau) - u(\tau)\|_s^2 \, \mathrm{d}\tau. \tag{3.23}$$

Estimate for I_3. Recall that Q is defined in (3.6) as follows:

$$Q_{\alpha,\beta,A}(f(\cdot, u))(t) = \int_0^t E_{\alpha,\beta}(A, t, \tau) f(\tau, u) \, \mathrm{d}\tau.$$

By the Hölder inequality and direct computation, we have

$$
\begin{aligned}
I_3 &\le \int_\theta^{+\infty} \lambda^{2s} \int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\tau \\
&\quad \times \int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2 \,\mathrm{d}\tau \\
&= I_{31}(N) + I_{32}(N),
\end{aligned} \tag{3.24}
$$

where

$$
\begin{aligned}
I_{31}(N) &= \int_\theta^{N} \lambda^{2s} \int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\tau \\
&\quad \times \int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2 \,\mathrm{d}\tau, \\
I_{32}(N) &= \int_{\lambda>N} \lambda^{2s} \int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\tau \\
&\quad \times \int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2 \,\mathrm{d}\tau.
\end{aligned} \tag{3.25}
$$

We will estimate $I_{31}(N)$ and $I_{32}(N)$ one by one.

Estimate for $I_{31}(N)$. By Lemma 2, we have

$$
\int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\tau \le C_1 \left((1+\lambda^\beta)|\alpha-\widetilde{\alpha}| + |\lambda^\beta - \lambda^{\widetilde{\beta}}| \right).
$$

By the mean value theorem, for $\lambda \le N$ with N large enough, we obtain

$$
\begin{aligned}
&\int_0^t \left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \,\mathrm{d}\tau \\
&\quad \le C_2 \lambda^{\beta^*} |\ln\lambda| \left(|\alpha-\widetilde{\alpha}| + |\beta-\widetilde{\beta}|\right).
\end{aligned} \tag{3.26}
$$

On the other hand, there exists $C_3 = C_3(\alpha_*, \alpha^*, \beta_*)$ such that

$$
\begin{aligned}
\left| E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| &\le C_3 \left((t-\tau)^{\alpha-1} + (t-\tau)^{\widetilde{\alpha}-1} \right) \\
&\le 2C_3 \left((t-\tau)^{\alpha_*-1} + (t-\tau)^{\alpha^*-1} \right).
\end{aligned} \tag{3.27}
$$

Plugging (3.26) and (3.27) into (3.25), we obtain

$$
\begin{aligned}
I_{31}(N) &\le C_4 N^{\beta^*+2s} \ln N \left(|\alpha-\widetilde{\alpha}| + |\beta-\widetilde{\beta}|\right) \\
&\quad \times \int_\theta^N \int_0^t \left((t-\tau)^{\alpha^*-1} + (t-\tau)^{\alpha_*-1} \right) \,\mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2 \,\mathrm{d}\tau
\end{aligned} \tag{3.28}
$$

for N large enough and $C_4 = 2C_2C_3$. Furthermore, thanks to the condition (2.2), we get that

$$\begin{aligned}
&\int_\theta^N \int_0^t \left((t-\tau)^{\alpha^*-1} + (t-\tau)^{\alpha_*-1}\right) \mathrm{d}\|S_\lambda f(\tau, u(\tau))\|^2 \,\mathrm{d}\tau \\
&\quad \le \int_\theta^{+\infty} \int_0^t \left((t-\tau)^{\alpha^*-1} + (t-\tau)^{\alpha_*-1}\right) \mathrm{d}\|S_\lambda f(\tau, u(\tau))\|^2 \,\mathrm{d}\tau \\
&\quad \le \int_0^t \left((t-\tau)^{\alpha^*-1} + (t-\tau)^{\alpha_*-1}\right) \left(\|f(\tau,0)\|^2 + \kappa_{T,s}^2 \tau^{-2\nu} \|u(\tau)\|_s^2\right) \mathrm{d}\tau \\
&\quad := C_5,
\end{aligned} \tag{3.29}$$

where $C_5 = C_5(\alpha_*, \alpha^*, \beta_*, M)$. Combining the inequality (3.28) with (3.29), we obtain

$$I_{31}(N) \le D_N \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right), \tag{3.30}$$

where $D_N = C_4C_5N^{\beta^*+2s} \ln N$.

Estimate for $I_{32}(N)$. Thanks to (3.8), one has

$$\begin{aligned}
&\int_0^t \left|E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \mathrm{d}\tau \\
&\quad \le \int_0^t E_{\alpha,\beta}(\lambda,t,\tau)\,\mathrm{d}\tau + \int_0^t E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\,\mathrm{d}\tau \\
&\quad \le \frac{1}{\lambda^\beta} + \frac{1}{\lambda^{\widetilde{\beta}}}.
\end{aligned}$$

Consequently,

$$\lambda^s \int_0^t \left|E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \le C_6,$$

where $C_6 = C_6(\beta_*, \beta^*, \theta)$, and that

$$\begin{aligned}
I_{32}(N) &\le C_6 \int_{\lambda>N} \int_0^t \left|E_{\alpha,\beta}(\lambda,t,\tau) - E_{\widetilde{\alpha},\widetilde{\beta}}(\lambda,t,\tau)\right| \mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2 \,\mathrm{d}\tau \\
&\le 2C_6C_3 \int_0^t \left((t-\tau)^{\alpha^*-1} + (t-\tau)^{\alpha_*-1}\right) \\
&\quad \times \int_N^{+\infty} \mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2 \,\mathrm{d}\tau.
\end{aligned} \tag{3.31}$$

From (3.22), (3.24), (3.30) and (3.31), for $|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}| \le 1$, we obtain

$$I_1 + I_3 \le E_N \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right) + 2\int_{\lambda>N} \lambda^{2s}\,\mathrm{d}\|S_\lambda \zeta\|^2$$

$$+4C_6C_3\int_0^t \left((t-\tau)^{\alpha^*-1}+(t-\tau)^{\alpha_*-1}\right) \times \int_N^{+\infty} \mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2\,\mathrm{d}\tau, \tag{3.32}$$

where $E_N = 2C_N + 6D_N$ with C_N defined in (3.20) and D_N defined in (3.30).

Let us mention (3.29) that

$$\int_0^t \left((t-\tau)^{\alpha^*-1}+(t-\tau)^{\alpha_*-1}\right)\int_\theta^{+\infty} \mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2\,\mathrm{d}\tau \\ \le \int_0^t \left((t-\tau)^{\alpha^*-1}+(t-\tau)^{\alpha_*-1}\right)\left(\|f(\tau,0)\|^2+\kappa_{T,s}^2\tau^{-2\nu}\|u(\tau)\|_s^2\right)\mathrm{d}\tau = C_5$$

and $\zeta \in D(A^s)$. This leads to the fact that there exists $N = N(\epsilon)$ independent of $\alpha, \widetilde{\alpha}$ and $\beta, \widetilde{\beta}$ such that

$$2\int_{\lambda>N}\lambda^{2s}\,\mathrm{d}\|S_\lambda\zeta\|^2 \\ +4C_6C_3\int_0^t \left((t-\tau)^{\alpha^*-1}+(t-\tau)^{\alpha_*-1}\right)\int_N^{+\infty} \mathrm{d}\|S_\lambda f(\tau,u(\tau))\|^2\,\mathrm{d}\tau \\ <\epsilon.$$

Combining (3.32) and the latter inequality, one gets

$$I_1 + I_3 \le \epsilon + P_\epsilon(|\alpha-\widetilde{\alpha}|+|\beta-\widetilde{\beta}|). \tag{3.33}$$

Substituting (3.23) and (3.33) into (3.18), we obtain

$$\left\|F_{\zeta,\widetilde{\alpha},\widetilde{\beta},A}(v)(t)-F_{\zeta,\alpha,\beta,A}(u)(t)\right\|_s^2 \\ \le 4(\epsilon+P_\epsilon(|\alpha-\widetilde{\alpha}| \\ +|\beta-\widetilde{\beta}|))+\frac{4}{\Gamma(\widetilde{\alpha})}\theta^{2s-\widetilde{\beta}}\kappa_{T,s}^2\int_0^t (t-\tau)^{\widetilde{\alpha}-1}\tau^{-2\nu}\|v(\tau)-u(\tau)\|_s^2\,\mathrm{d}\tau.$$

Since $u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}$ and $u_{\zeta,\alpha,\beta}$ are the solution of the equations $F_{\zeta,\widetilde{\alpha},\widetilde{\beta},A}(v) = v$ and $F_{\zeta,\alpha,\beta,A}(u) = u$, respectively. We conclude from Lemma 3 that

$$\left\|u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t)-u_{\zeta,\alpha,\beta}(t)\right\|_s^2 \\ \le P_0\left(\epsilon+P_\epsilon(|\alpha-\widetilde{\alpha}|+|\beta-\widetilde{\beta}|)\right)E_{\widetilde{\alpha}-2\nu,1-2\nu}\left(4\kappa_{T,s}^2\theta^{2s-\widetilde{\beta}}t^{\widetilde{\alpha}-2\nu}\right),$$

where $P_0 = 4\Gamma(1-2\nu)$. This completes the proof of the first part of Theorem 5.

Now we consider the proof of the second part of the theorem. From (3.32) in the proof of Theorem 5, we obtain

$$I_1 + I_3 \le CN^{2\beta^*} \ln^2 N \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right) + J_1 + J_2,$$

where

$$J_1 = 2 \int_{\lambda > N} \lambda^{2s} \, \mathrm{d}\|S_\lambda \zeta\|^2,$$
$$J_2 = 4C_6C_3 \int_0^t \left((t-\tau)^{\alpha^*-1} + (t-\tau)^{\alpha_*-1}\right) \int_N^{+\infty} \mathrm{d}\|S_\lambda f(\tau, u(\tau))\|^2 \, \mathrm{d}\tau.$$

In fact, using the assumption $\zeta \in D(A^{\beta^*/2+r_1})$, $f(., u(.)) \in C([0, T], D(A^{r_2}))$ yields

$$J_1 \le 2N^{-(\beta^*+2r_1-2s)} \int_{\lambda > N} \lambda^{\beta^*+2r_1} \, \mathrm{d}\|S_\lambda \zeta\|^2 \le 2N^{-(\beta^*+2r_1-2s)} \|\zeta\|^2_{\beta^*/2+r_1}$$

and $J_2 \le CN^{-2r_2}$. Hence, putting $\gamma_1 = 2\beta^*$, $\gamma_2 = \min\{\beta^* + 2r_1 - 2s, 2r_2\}$, we get

$$I_1 + I_3 \le CN^{-\gamma_2} + CN^{2\gamma_1} \ln^2 N \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right).$$

Hence, we can use Lemma 3 to prove that

$$\left\|u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t) - u_{\zeta,\alpha,\beta}(t)\right\|_s \le Q_0 \left(N^{-\gamma_2} + N^{\gamma_1} \ln^2 N \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right)\right)^{1/2},$$

where Q_0 is independent of N, α, $\widetilde{\alpha}$, β, $\widetilde{\beta}$.

Since $\ln N < N$, we obtain

$$\left\|u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t) - u_{\zeta,\alpha,\beta}(t)\right\|_s \le Q_0 \left(N^{-\gamma_2} + N^{\gamma_1+2} \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right)\right)^{1/2}. \quad (3.34)$$

Let us suppose that $|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}| \le 1$, and we can choose

$$N = \left[(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|)^{-1/(\gamma_1+\gamma_2+2)}\right] + 1.$$

It is easy to see that

$$(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|)^{-1/(\gamma_1+\gamma_2+2)} < N \le 2(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|)^{-1/(\gamma_1+\gamma_2+2)}.$$

Hence, by (3.34), we obtain

$$\left\|u_{\zeta,\widetilde{\alpha},\widetilde{\beta}}(t) - u_{\zeta,\alpha,\beta}(t)\right\|_s \le Q_0 \left(2^{\gamma_1+2} + 1\right) \left(|\alpha - \widetilde{\alpha}| + |\beta - \widetilde{\beta}|\right)^{\gamma_2/(2(\gamma_1+\gamma_2+2))}.$$

This completed the proof of Lemma 5. □

3.7 Proof of Theorem 5

Proof Before proving the theorem, we set up some notations. We will use Theorem 1 to prove Part (i). For $M > 0$, we put

$$f_M(t, v) = f\left(t, \frac{Mv}{\max\{M, \|v\|_s\}}\right) \text{ for } v \in D(A^s).$$

Verifying directly, we can prove that the function f_M is global Lipschitz with respect to the variable v, i.e.,

$$\|f_M(t, w_1) - f_M(t, w_2)\| \le \kappa_M t^{-\nu} \|w_1 - w_2\|_s \text{ for all } w_1, w_2 \in D(A^s),$$

where $\kappa_M > 0$ depends on M. We consider the problem of finding $U \in C([0, T], D(A^s))$ satisfying

$$U(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A, t, \tau) f_M(\tau, U(\tau))\, d\tau. \tag{3.35}$$

From Theorem 1, for any $T > 0$, the equation (3.35) has a unique solution

$$U_{M,T} \in C([0, T], D(A^s)).$$

(i) For any $m > 0$, we put $M = 2\|\zeta\| + m$. Since $U_T(0) = \zeta$, we can use the continuity of U_T to find a constant $T_M \in (0, T]$ such that $\sup_{0 \le t \le T_M} \|U_{M,T}(t)\|_s \le M$. In this case $f_M(t, U_{M,T}(t)) = f(t, U_{M,T}(t))$ for all $t \in [0, T_M]$ and $U_{M,T}(t)$ satisfies (1.3) for $t \in [0, T_M]$.

(ii) If $V, W \in C([0, T]; D(A^s))$ are solutions of (1.3), we denote

$$\mu = 1 + \max\left\{\sup_{0 \le t \le T} \|V(t)\|_s, \sup_{0 \le t \le T} \|W(t)\|_s\right\}$$

and consider the equation

$$U(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A, t, \tau) f_\mu(\tau, U(\tau))\, d\tau. \tag{3.36}$$

From Theorem 1, the equation (3.36) has a unique solution

$$U_{\mu,T} \in C([0, T]; D(A^s)).$$

Since $\|V(t)\|_s, \|W(t)\|_s \le \mu$ for $t \in [0, T]$, we have

$$f(t, V(t)) = f_\mu(t, V(t)), \ f(t, W(t)) = f_\mu(t, W(t)).$$

Hence, V, W satisfy (3.36). By Theorem 1, we have $V = U_{\mu,T} = W$.

(iii) For every $T \in (0, T_{\zeta,\alpha,\beta})$, the equation (1.3) has a unique solution $U_T \in C([0,T]; D(A^s))$. From Part (ii), for $T_1, T_2 \in (0, T_{\zeta,\alpha,\beta})$, $T_1 < T_2$, we have $U_{T_1}(t) = U_{T_2}(t)$ for $t \in [0, T_1]$. Hence, we can put $u_{\zeta,\alpha,\beta}(t) = U_T(t)$ for all $t \in [0,T]$, $T \in (0, T_{\zeta,\alpha,\beta})$. The function $u_{\zeta,\alpha,\beta}$ is the unique solution of (1.3) on $[0, T_{\zeta,\alpha,\beta})$.
We prove the second result of Part (iii). Assume by contradiction that $T_{\zeta,\alpha,\beta} < \infty$ and $\|u_{\zeta,\alpha,\beta}(t)\|_s \le M$ for every $t \in [0, T_{\zeta,\alpha,\beta})$. We consider the equation

$$U(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A,t,\tau) f_M(\tau, U(\tau))\, \mathrm{d}\tau. \tag{3.37}$$

From Theorem 1, the equation (3.37) has a unique solution $U_{M,\delta+T_{\zeta,\alpha,\beta}}$ with $\delta > 0$. From Part (ii) we have $u_{\zeta,\alpha,\beta}(t) = U_{M,\delta+T_{\zeta,\alpha,\beta}}(t)$ for every $t \in [0, T_{\zeta,\alpha,\beta})$. Since $U_{M,\delta+T_{\zeta,\alpha,\beta}} \in C([0, \delta+T_{\zeta,\alpha,\beta}]; D(A^s))$, we can find a constant $\delta' \in (0,\delta)$ such that $\|U_{M,\delta+T_{\zeta,\alpha,\beta}}(t)\|_s \le M$ for $t \in [0, \delta' + T_{\zeta,\alpha,\beta}]$. Hence the equation (1.3) has a unique solution on $[0, T_{\zeta,\alpha,\beta} + \delta']$. It follows that $T_{\zeta,\alpha,\beta} + \delta' \le T_{\zeta,\alpha,\beta}$, which is a contradiction.
Finally, the proof of the last inequality of the theorem is similar to the inequality (2.3). Hence we omit it.

(iv) Choose $T \in (0, T_{\zeta,\alpha,\beta})$ and $M = 1 + |u_{\zeta,\alpha,\beta}|_{s,T}$ and consider the problem (3.37) and

$$U(t) = E_{\alpha_k}\big(-t^{\alpha_k} A^{\beta_k}\big)\zeta_k + \int_0^t E_{\alpha_k,\beta_k}(A,t,\tau) f_M(\tau, U(\tau))\, \mathrm{d}\tau. \tag{3.38}$$

Denote the solution of (3.37), (3.38) by $U_{\zeta,\alpha,\beta}$ and $U_{\zeta_k,\alpha_k,\beta_k}$ respectively. From the stability result, we obtain

$$U_{\zeta_k,\alpha_k,\beta_k} \to U_{\zeta,\alpha,\beta} \ \text{ in } \ C([0,T]; D(A^s)) \ \text{ as } \ k \to \infty. \tag{3.39}$$

Since $|u_{\zeta,\alpha,\beta}|_{s,T} < M$, we have $u_{\zeta,\alpha,\beta} = U_{\zeta,\alpha,\beta}$. From (3.39), there is a $k_0 \in \mathbb{N}$ such that $|U_{\zeta_k,\alpha_k,\beta_k}|_{s,T} < M$ which gives $U_{\zeta_k,\alpha_k,\beta_k} = u_{\zeta_k,\alpha_k,\beta_k}$ is the solution of Problem $P_{\zeta_k,\alpha_k,\beta_k}$. It follows that $T < T_{\zeta_k,\alpha_k,\beta_k}$ for $k > k_0$ which implies $\liminf_{k\to\infty} T_{\zeta_k,\alpha_k,\beta_k} \ge T_{\zeta,\alpha,\beta}$. Using (3.39) yields

$$\lim_{k\to\infty} |u_{\zeta_k,\alpha_k,\beta_k} - u_{\zeta,\alpha,\beta}|_{s,T} = 0.$$

This completes the proof of the theorem.

□

3.8 Proof of Theorem 6

Proof $\omega' \le \omega(\eta+1) - \ell$, $\omega' < 1/2 - \nu$, $\omega - \nu - \omega' < 0$. We have $\omega - \nu < \omega' \le \omega(\eta+1) - \ell$ which gives $-\nu < \omega\eta - \ell$ or $\omega > \frac{\ell-\nu}{\eta}$. We also need $\omega - \nu < 1/2 - \nu$

which gives $\omega < 1/2$. Hence $\frac{\ell-\nu}{\eta} < 1/2$ Since $\eta > \max\{2(\ell-\nu), 0\}$ we can choose ω such that

$$\max\left\{\frac{\ell-\nu}{\eta}, 0\right\} < \omega < 1/2$$

which gives

$$\max\{\omega-\nu, 0\} < \min\{\omega(\eta+1)-\ell, 1/2-\nu\}.$$

Choosing ω' such that

$$\max\{\omega-\nu, 0\} < \omega' < \min\{\omega(\eta+1)-\ell, 1/2-\nu\}$$

we obtain

$$\omega' < \min\{\omega(\eta+1)-\ell, 1/2-\nu\}, \omega-\nu-\omega' < 0. \tag{3.40}$$

The maximal solution $u = u_{\zeta,\alpha,\beta} \in C([0, T_{\zeta,\alpha,\beta}), D(A^s))$ satisfies

$$u(t) = E_\alpha\big(-t^\alpha A^\beta\big)\zeta + \int_0^t E_{\alpha,\beta}(A, t, \tau) f(\tau, u(\tau))\, \mathrm{d}\tau \quad \text{for all } \ t \in [0, T_{\zeta,\alpha,\beta}).$$

We claim that $T_{\zeta,\alpha,\beta} = \infty$ and $\sup_{t>0}(1+t)^\omega \|u(t)\|_s < \infty$. We note that

$$\begin{aligned} \|f(t, u(t))\| &\le \|f(t, u(t)) - f(t, 0)\| + \|f(t, 0)\| \\ &\le \kappa_1 t^{-\nu}(1+t)^\ell \|u(t)\|_s^{1+\eta} + \|f(t, 0)\|. \end{aligned}$$

As in the proof of Theorem 1, choosing $r = 0$ in Lemma 4 gives

$$\sup_{\lambda \ge \theta} \lambda^{2s-\beta} H_0(\lambda, t_1, t_2) \le \theta^{2s-\beta}$$

and we have

$$\begin{aligned} &\|u(t)\|_s^2 \\ &\le 2\|E_\alpha\big(-t^\alpha A^\beta\big)\zeta\|_s^2 \\ &\quad + 2\theta^{2s-\beta} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha) \|f(\tau, u(\tau))\|^2\, \mathrm{d}\tau \\ &\le 2\|E_\alpha\big(-t^\alpha A^\beta\big)\zeta\|_s^2 \\ &\quad + 4\theta^{2s-\beta} \int_0^t (t-\tau)^{\alpha-1} \tau^{-2\nu} \kappa_1^2 (1+\tau)^{2\ell} E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha) \|u(\tau)\|_2^{2(\eta+1)}\, \mathrm{d}\tau \\ &\quad + 4\theta^{2s-\beta} \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha) \|f(\tau, 0)\|^2\, \mathrm{d}\tau. \end{aligned} \tag{3.41}$$

Putting $w(t) = (1+t)^{\omega} u(t)$ and, for $\Lambda > \|\zeta\|_s$, denoting

$$T_{\text{bound}} = \sup\{T_0 \in [0, T_{\zeta,\alpha,\beta}) : \ \|w(t)\|_s < \Lambda \text{ for } 0 \le t \le T_0\}.$$

We claim that there are $\Lambda > 0, \delta_0 > 0$ such that $T_{\text{bound}} = T_{\zeta,\alpha,\beta}$ for every $\|\zeta\|_s^2 + m^2_{\infty,\alpha,\omega} < \delta_0$. Assume by contradiction that $T_{\text{bound}} < T_{\zeta,\alpha,\beta}$. Then, we obtain $\|w(T_{\text{bound}})\|_s = \Lambda$. The inequality (3.41) yields

$$\|w(t)\|_s^2 \le K_1 + K_2 \ \text{ for } 0 \le t < T_{\text{bound}},$$

where

$$\begin{aligned} K_1 &= 2(1+t)^{2\omega} \|E_\alpha\big(-t^\alpha A^\beta\big)\zeta\|_s^2 + 4\theta^{2s-\beta} m^2_{\infty,\alpha,\omega}, \\ K_2 &= 4\theta^{2s-\beta}(1+t)^{2\omega}\kappa_1^2 \int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu} E_{\alpha,\alpha}(-\theta^\beta (t-\tau)^\alpha) \\ &\quad \times (1+\tau)^{2\ell-2\omega(\eta+1)} \|w(\tau)\|_s^{2(\eta+1)} d\tau. \end{aligned}$$

From (3.4) we obtain

$$K_1 \le 2\Lambda_0 \|\zeta\|_s^2 + 4\theta^{2s-\beta} m^2_{\infty,\alpha,\omega} < \zeta_0^2.$$

Next, we consider K_2. For $t \ge 1$, we obtain in view of Lemma 2 and (3.40) that

$$\begin{aligned} K_2 &\le 4\Lambda^{2(\eta+1)}\theta^{2s-\beta}(1+t)^{2\omega}\kappa_1^2 I_{\alpha,\nu,\omega(\eta+1)-\ell} \\ &\le 4C\Lambda^{2(\eta+1)}\theta^{2s-\beta}(1+t)^{2\omega}\kappa_1^2 t^{-2\nu-2\omega'} \le C"\Lambda^{2(\eta+1)}, \end{aligned}$$

where $\omega' \le \omega(\eta+1) - \ell$, $\omega' < 1/2 - \nu$, $\omega - \nu - \omega' < 0$. We estimate for the case $0 < t < 1$. We have

$$\begin{aligned} K_2 &\le 4\Lambda^{2(\eta+1)}\theta^{2s-\beta}(1+t)^{2\omega})\kappa_1^2 \int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu} E_{\alpha,\alpha}(-\theta^\alpha (t-\tau)^\alpha) \\ &\quad \times (1+\tau)^{2\ell-2\omega(\eta+1)} \, \mathrm{d}\tau \\ &\le \frac{4\Lambda^{2(\eta+1)}}{\Gamma(\alpha)} \theta^{2s-\beta} 2^{2\omega}\kappa_1^2 \int_0^t (t-\tau)^{\alpha-1}\tau^{-2\nu} \, \mathrm{d}\tau \\ &\le \frac{4\Lambda^{2(\eta+1)}}{\Gamma(\alpha)} \theta^{2s-\beta} 2^{2\omega}\kappa_1^2 t^{\alpha-2\nu} B(\alpha, 1-2\nu) \\ &\le \frac{4\Lambda^{2(\eta+1)}}{\Gamma(\alpha)} \theta^{2s-\beta} 2^{2\omega}\kappa_1^2 B(\alpha, 1-2\nu). \end{aligned}$$

Combining the two cases gives

$$K_2 \le C\Lambda^{2(\eta+1)}.$$

From the estimates for K_1, K_2 we obtain

$$\Lambda^2 = \|w(T_{\text{bound}})\|_s^2 = \sup_{0 \le t < T_{\text{bound}}} \|w(t)\|_s^2 \le \zeta_0^2 + C\Lambda^{2(\eta+1)}.$$

Since $2(\eta + 1) > 2$ we can choose $\Lambda, \zeta_0 > 0$ such that $1 > C\Lambda^{2\eta}, \zeta_0^2 < \Lambda^2 - C\Lambda^{2(\eta+1)}$ and

$$\delta_0^2 = \zeta_0^2 \left(2\Lambda_\omega + \frac{4}{\Gamma(\alpha)} \theta^{2s-\beta} \right)^{-1}.$$

In this case we get $\Lambda^2 = \|w(T_{\text{bound}})\|_s^2 \le \zeta_0^2 + C\Lambda^{2(\eta+1)} < \Lambda^2$ which is a contradiction. Hence, we have to obtain $T_{\text{bound}} = T_{\zeta,\alpha,\beta}$. So we have

$$\|u(t)\|_s \le (1+t)^{-\omega}\Lambda \quad \text{for every } t \in [0, T_{\zeta,\alpha,\beta}).$$

Using the continuation result in Theorem 5 leads to $T_{\zeta,\alpha,\beta} = \infty$. Finally, we consider the case $\frac{\ell-\nu}{\eta} < \alpha < 1/2$. In this case, we can choose $\omega = \alpha$ and obtain the desired result by a similar argument as in Theorem 2. This completes the proof of the theorem. □

Acknowledgements The authors thank an anonymous referee and the editor for helpful comments that improved the quality and presentation of the paper. The research was supported by Vietnam National University of Hochiminh City [Grant No. B2021-18-02]. The research of Erkan Nane was partially supported by the Simons Foundation Collaboration Grants for Mathematicians.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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