Brazilian Journal of Physics (2021) 51:1043–1068
https://doi.org/10.1007/s13538-021-00913-8

GENERAL AND APPLIED PHYSICS

Abundant exact closed-form solutions and solitonic structures for the double-chain deoxyribonucleic acid (DNA) model

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Received: 2 February 2021 / Accepted: 20 April 2021 / Published online: 27 May 2021

Abstract

In this work, the abundant exact closed-form solutions and dynamics of solitons for the double-chain deoxyribonucleic acid (DNA) model is obtained by utilizing the generalized exponential rational function (GERF) method. Deoxyribonucleic acid (DNA) retains the genetic information that creatures need to live and reproduce themselves. We obtained several novel exact soliton and exponential rational functional solutions in the shapes of dynamics of solitons like multi-solitons, breather-type solitons, abundant elastic interactions between multi-solitons, and nonlinear waves, oscillating multi-solitons, and Lump solitons. These derived solutions were never reported in the literature. The dynamical structures of some exact solitons are exhibited graphically by assigning suitable values to the free parameters via 3D figures. The generated solutions can be more useful and help to explain the internal interactions of the double-chain DNA model. The symbolic computational work and the obtained solutions show that the present proposed GERF method is effective, robust, and straightforward. Moreover, these types of higher-order NLEEs can be solved using the current technique.

Keywords Double-chain DNA model · GERF method · Exact solutions · Solitons

1 Introduction

Deoxyribonucleic acid, or DNA, is the central information storage system in every cell. It is an extremely large linear molecule that contains genetic instructions that make each specie unique in the form of a sequence of its constituent elements, that is, the nucleotides. The dynamics of DNA molecules is one of the most interesting problems in the field of modern biophysics as it is related to the basis of life. DNA is double stranded, that is, it is made up of two complementary chains or strands. These two chains run opposite to each other and connected by hydrogen bonds, and therefore known as anti-parallel chains. The word "deoxyribo" in DNA describes the sugar and the "nucleic acid" refers to the phosphate molecules.

The study of qualitative and quantitative dynamics of mathematical modeling became an important research topic in the field of mathematical biology and applied mathematics. In the literature, many methods have been developed and extensively used by many researchers to solve these mathematical models. For instance, (G'/G)-expansion method [1], the reproducing kernel Hilbert space method [2], the Laplace homotopy perturbation method [3], Lie symmetry method [4–9], the sine-Gordon expansion method [10], GERF method [8, 11], and other mathematical techniques [12–19].

The dynamics of the DNA model have been analyzed by many mathematicians and biologists [20–24]. It has been observed that modeling of DNA system is very complicated beacuse of its characteristic multiplicity and the chaoticity which is relevant for genome evolution [25]. Watson and Crick [26] described different structure for the salt of deoxyribose nucleic acid and then double helix structure of DNA has become an interesting topic in the field of biology and applied mathematics. Englander et al. [27] demonstrated the existence of open states in DNA system and double helices by hydrogen exchange measurements. They proved the

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consistency of the hydrogen exchange open state. Yomosa [23] presented a soliton theory for the open states in DNA and synthetic polynucleotide double helices. The dynamics of a simple lattice model is investigated by Peyrard and Bishop [28] for the denaturation of the DNA system. Muto et al. [29] studied the ring-shaped DNA molecules with longitudinal wave propagation. Later, Qian and Lou [30] analyzed the double-chain model of DNA by using two different expansions, namely, the Conte's Painlevé truncation expansion and the Pickering's truncation expansion. The method of dynamical systems is applied by Ouyang and Zheng [31] to study the nonlinear DNA system. Also, Mabrouk [32] investigated (2+1)-dimensional double-chain DNA model by using (G'/G)–expansion method and obtained numerous soliton solutions. Recently, Saleh et al. [33] obtained various soliton solutions for double-chain DNA model by using Lie transformation method and the singular manifold method (SMM).

The dynamics of exact explicit solutions of nonlinear dynamical systems play a crucial role in the theory of solitons and the formations of exact analytical solutions perform an important role in the area of nonlinear sciences and applied mathematics. Moreover, these analytical solutions may ensure dynamical and physical behavior of the system which assist us about the mechanism of considered complex nonlinear systems. The solitary waves, in the theory of solitons, interact with each other without losing its identities, for instance, its amplitude, velocity, and shape do not vary after interactions.

In this article, our main objective is to construct the numerous solitary wave solutions together with the dynamics of solitons for the double-chain deoxyribonucleic acid (DNA) model by employing the generalized exponential rational function method (GERFM). The GERF method is very effective and straightforward as it is described by the wave transformations via symbolic computations. Abundant new exact explicit solutions and exponential rational function solutions of the DNA system are investigated analytically and physically through 3D graphics via mathematical software, *Wolfram Mathematica*. The generated solutions are obtained in the forms of multi-solitons, various elastic interactions between multi-solitons, nonlinear waves, Lump solitons and oscillating multi-solitons. These derived solutions are entirely new and never reported in the literature. Furthermore, the wide diversity of features and physical parameters of these generated solutions are elucidated with the support of three-dimensional graphical representations, considering the appropriate choice of involved functional and other constant parameters. This type of investigation is highly recommended in the areas of progressive research and development.

The remaining part of the article is summarized as follows:

- Section 2 deals with the formulation of the double-chain DNA model.
- In section 3, we describe the methodology of the GERF method.
- Section 4 is related to the application of GERF method to main equation in which we obtained various exact explicit solutions together with the graphical representation of the generated solutions.
- Section 5 explains the physical explanations of some obtained solutions for the considered DNA model.
- Finally, the article is concluded in section 6.

2 Mathematical formulation

The double-chain DNA model in (2+1)-dimensional coupled partial differential Equation [29, 32, 33]

$$u_{tt} - c_1^2 u_{xx} - c_1^2 u_{yy} = \lambda_1 u + \gamma_1 uv + \mu_1 u^3 + \beta_1 uv^2 \tag{1}$$

$$v_{tt} - c_2^2 v_{xx} - c_2^2 v_{yy} = \lambda_2 v + \gamma_2 u^2 + \mu_2 u^2 v + \beta_2 v^3 + c_0, \tag{2}$$

where $u(x, y, t)$ is the difference of the longitude displacements of the bottom and top strands, and $v(x, y, t)$ is the difference of the transverse displacements of the bottom and top strands. The constants $c_1,\ c_2,\ \lambda_1,\ \lambda_2,\ \gamma_1,\ \gamma_2,\ \mu_1,\ \mu_2,\ \beta_1,\ \beta_2$, and c_0 are defined as:

$$\begin{aligned} &c_1 = \pm\frac{\epsilon}{\rho},\ c_2 = \pm\frac{F}{\rho}, \lambda_1 = -\frac{2\mu}{\rho\sigma h}(c - l_0),\\ &\beta_1 = \beta_2 = \frac{4\mu l_0}{\rho\sigma h^3}, c_0 = \frac{\sqrt{2}\mu}{\rho\sigma}(h - l_0 5), \lambda_2 = -\frac{2\mu}{\rho\sigma},\\ &\gamma_1 = 2\gamma_2 = \frac{2\sqrt{2}\mu l_0}{\rho\sigma h^2},\ \mu_1 = \mu_2 = -\frac{2\mu l_0}{\rho\sigma h^3}, \end{aligned} \tag{3}$$

where ρ represents the mass density, σ shows the area of the transverse cross section, ϵ is the Young's modulus, F is the tension density of the strands, μ is the rigidity of the elastic membrane, h is the distance between two strands and l_0 is the height of the membrane in the equilibrium position. Now, we have the following transformation

$$v(x, y, t) = b_1 u(x, y, t) + b_2 \tag{4}$$

which reduces Equation (1) into the following form

$$\begin{aligned} u_{tt} - c_1^2 u_{xx} - c_1^2 u_{yy} =& \left(\mu_1 + \beta_1 b_1^2\right)\\ &u^3 + \left(\gamma_1 b_1 + 2\beta_1 b_1 b_2\right) u^2\\ &+ \left(\lambda_1 + \gamma_1 b_2 + \beta_1 b_2^2\right) u, \end{aligned} \tag{5}$$

and Equation (2) to

$$b_1 u_{tt} - b_1 c_2^2 u_{xx} - b_1 c_2^2 u_{yy} = \left(b_1 \mu_2 + \beta_2 b_1^3\right) u^3 + \left(\gamma_2 + \mu_2 b_2 + 3\beta_2 b_1^2 b_2\right) u^2 + \left(b_1 \lambda_2 + 3\beta_2 b_1 b_2^2\right) u + \lambda_2 b_2 + \beta_2 b_2^3 + c_0. \tag{6}$$

Equations (5) and (6) are same for

$$b_2 = \frac{h}{\sqrt{2}}, \quad F = \epsilon. \tag{7}$$

Consequently, the system of Equation (1) and (2) is reduced to a single partial differential equation as

$$u_{tt} - c_1^2 u_{xx} - c_1^2 u_{yy} - A u^3 + B u^2 + C u = 0, \tag{8}$$

where $A = \frac{\Omega}{h^3}(4b_1^2 - 2)$, $B = \frac{6\sqrt{2}b_1}{h^3}\Omega$, $C = \frac{6\Omega}{h} - \frac{2\Omega}{l_0}$, $\Omega = \frac{\mu l_0}{\rho\sigma}$ and $c_1 = c_2$.

3 Methodology of the GERF method

In this section, the methodology of the generalized exponential rational function method (GERFM) is discussed. This method is a quite novel technique to nonlinear PDEs given by

$$\mathscr{F}(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xy}, u_{xt}, u_{yt} \ldots) = 0. \tag{9}$$

This method was first introduced by Ghanbari and Inc [34] for obtaining the soliton solutions of the nonlinear PDEs. The following are the steps of GERF method:

Step 1: By employing the transformations, the nonlinear PDE will be reduced into the nonlinear ODE given as:

$$\mathscr{F}(U, U', U'', \ldots) = 0. \tag{10}$$

Step 2: Let us assume that Equation (10) has the explicit invariant solution having form

$$\varphi(X) = \frac{\delta_1 e^{\xi_1 X} + \delta_2 e^{\xi_2 X}}{\delta_3 e^{\xi_3 X} + \delta_4 e^{\xi_4 X}}, \tag{11}$$

where $X = kx + vy + \omega t$ with k, v and ω as arbitrary constants, $\delta_1, \delta_2, \delta_3, \delta_4$ and $\xi_1, \xi_2, \xi_3, \xi_4$ are real (or complex) numbers such that the exact solitary wave solutions of Equation (10) can be furnished as follows:

$$G(X) = E_0 + \sum_{n=1}^{N} E_n \varphi(X)^n + \sum_{n=1}^{N} \frac{F_n}{\varphi(X)^n}. \tag{12}$$

The values of arbitrary coefficients $E_0, E_n, F_n (1 \le n \le N)$ and constants $\delta_k, \xi_k (1 \le k \le 4)$ satisfy the Equation (10). Here, it has been observed that the positive integer N can be found by the Balancing principle.

Step 3: Substituting (12) into Equation (10) and choosing the coefficient of $\varphi(X)$ and then its derivative equating to zero. Meanwhile, we employ the symbolic computation to obtain a set of algebraic equations for $\delta_n, \xi_n (1 \le n \le 4)$ and E_0, E_n, F_n.

Step 4: The algebraic equations in Step 3 are solved with the assistance of *Wolfram Mathematica*, and then, by substituting non-trivial solutions in (12), one can get the exact solitonic solutions of nonlinear PDEs.

4 Exact closed-form solutions for DNA equations

For establish more new exact solutions of Equation (8), we consider the following wave transformation,

$$u(x, y, t) = U(X), \quad \text{with} \quad X = kx + vy + \omega t, \tag{13}$$

where k, v and ω are arbitrary constants. On inserting $u(x, y, t)$ along with X from Equation (13) into Equation (8), a nonlinear ordinary differential equation is obtained as

$$(\omega^2 - (k^2 + v^2)c_1^2)U'' - AU^3 + BU^2 + CU = 0. \tag{14}$$

To solve Equation (14), Balancing principle used on terms U^3 and U'' in Equation (14) which yield $N + 2 = 3N$ implies $N = 1$. Using $N = 1$ along with Equations (11) and (12) Equation (14), one finds

$$U(X) = E_0 + E_1 \varphi(X) + \frac{F_1}{\varphi(X)}. \tag{15}$$

In the continuation, we will apply the GERF method to get the following exact solitary wave solutions as:

Family 1: For $[\delta_1; \delta_2; \delta_3; \delta_4] = [1; -1; 1; 1]$ and $[\xi_1; \xi_2; \xi_3; \xi_4] = [1; -1; 1; -1]$, consequently, Equation (11) becomes

$$\varphi(X) = \tanh(X).$$

The values of constants can be obtained by solving the algebraic equations through symbolic computation via Mathematica 11, a mathematical software, which yields following set of solutions

Case 1.1

$$E_0 = \frac{B}{3A}, \quad E_1 = \pm E_0, \quad F_1 = 0, \quad \omega = \pm\frac{\sqrt{18Ac_1^2(k^2 + v^2) + B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and} \quad C = -\frac{2B^2}{9A}. \tag{16}$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A}(1 \pm \tanh(X)). \tag{17}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A}\left(1 \pm \tanh\left(kx + vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{18}$$

Substituting (18) into (4), the transversal reads

$$v(x,y,t) = \frac{Bb_1}{3A}\left(1 \pm \tanh\left(kx + vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)\right) + b_2. \tag{19}$$

Case 1.2

$$E_0 = \frac{B}{3A},\quad E_1 = \pm\frac{E_0}{2},\quad F_1 = \pm\frac{E_0}{2},\quad \omega = \pm\frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}} \text{ and } C = -\frac{2B^2}{9A}. \tag{20}$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A} \pm \frac{B\coth(X)}{6A} \pm \frac{B\tanh(X)}{6A}. \tag{21}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A} \pm \frac{B}{6A}\coth\left(kx + vb \pm \frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right) \pm \frac{B}{6A}\tanh\left(kx + vb \pm \frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right). \tag{22}$$

Substituting (22) into (4), the transversal reads

$$v(x,y,t) = \frac{b_1B}{6A}\left(2 \pm \coth\left(kx + vb \pm \frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right)\right) \pm b_1\frac{B}{6A}\tanh\left(kx + vb \pm \frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right) + b_2. \tag{23}$$

Case 1.3

$$E_0 = \frac{B}{3A},\quad E_1 = \pm\frac{iB}{3\sqrt{2A}},\quad F_1 = \mp E_1,\quad \omega = \pm\frac{\sqrt{36Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{A}} \text{ and } C = -\frac{2B^2}{9A}. \tag{24}$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A} \pm \frac{iB\tanh(X)}{3\sqrt{2A}} \mp \frac{iB\coth(X)}{3\sqrt{2A}}. \tag{25}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A} \pm \frac{iB}{3\sqrt{2A}}\tanh\left(kx + vy \pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{A}}\right) \mp \frac{iB}{3\sqrt{2A}}\coth\left(kx + vy \pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{A}}\right). \tag{26}$$

Substituting Equation (26) into Equation (4), the transversal reads

$$v(x,y,t) = \frac{b_1B}{3A}\left(1 \pm \frac{i}{\sqrt{2}}\tanh\left(kx + vy \pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{A}}\right)\right) \mp \frac{iBb_1}{3\sqrt{2A}}\coth\left(kx + vy \pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{A}}\right) + b_2. \tag{27}$$

Case 1.4

$$E_0 = \frac{B}{3A},\quad E_1 = 0,\quad F_1 = \pm E_0,\quad \omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}} \text{ and } C = -\frac{2B^2}{9A}. \tag{28}$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A}(1 \pm \coth(X)). \tag{29}$$

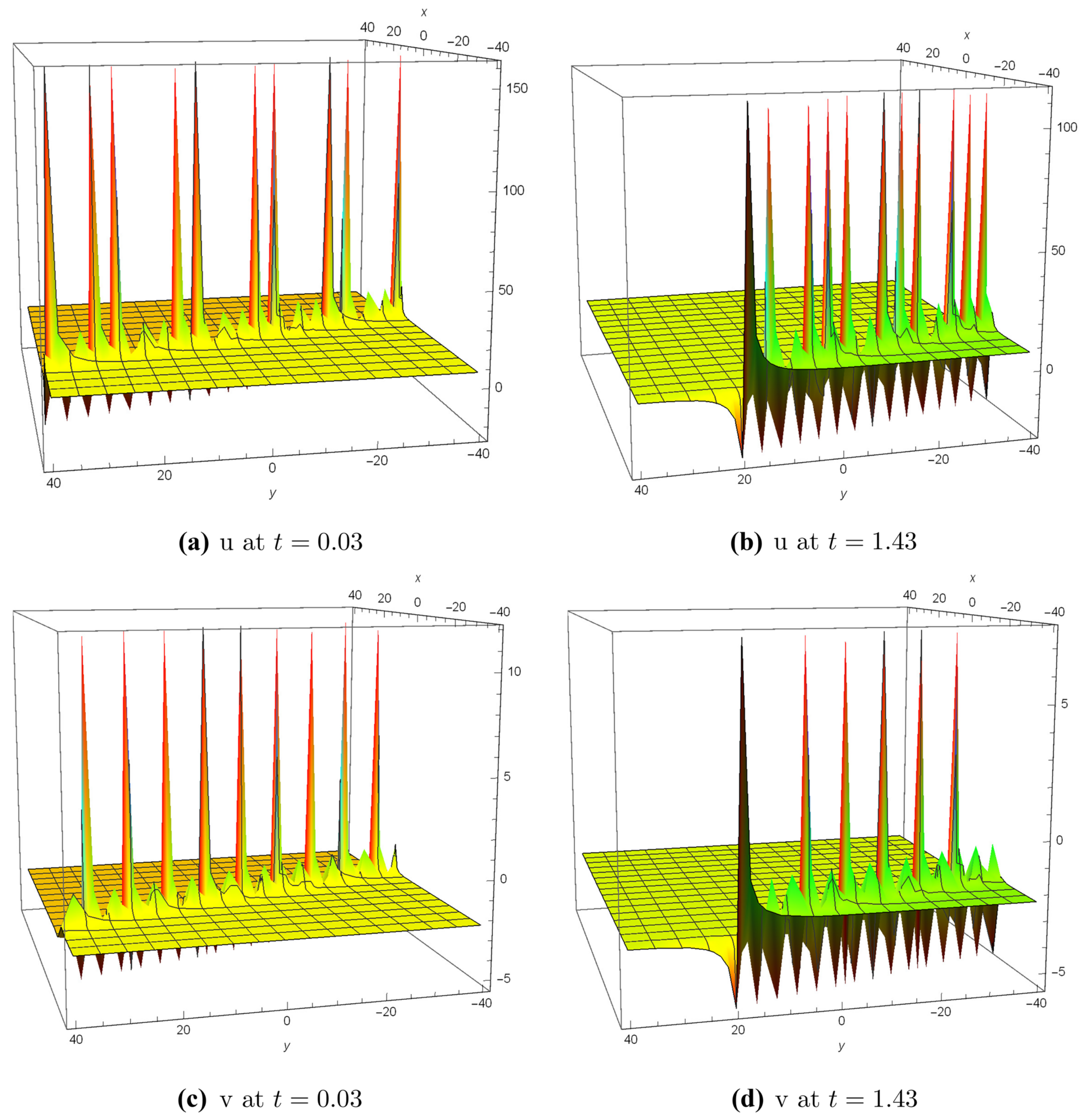
Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A}\left(1 \pm \coth\left(kx + vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{30}$$

Substituting Equation (30) into Equation (4), the transversal reads

$$v(x,y,t) = \frac{Bb_1}{3A}\left(1 \pm \coth\left(kx + vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)\right) + b_2. \tag{31}$$

Family 2: We obtain $[\delta_1; \delta_2; \delta_3; \delta_4] = [i; -i; 1; 1]$ and $[\xi_1; \xi_2; \xi_3; \xi_4] = [i; -i; i; -i]$, and hence the expression (11) becomes



(a) u at $t = 0.03$ **(b)** u at $t = 1.43$

(c) v at $t = 0.03$ **(d)** v at $t = 1.43$

Fig. 1 Four distinct solitonic structures of breather-type waves for Equations (30)-(31) with arbitrary constants $B = 99$, $A = 9$, $k = 0.33$, $v = 0.11$, $c_1 = 0.3$, $b_1 = 0.093$, $b_2 = 2$

$\varphi(X) = -\tan(X)$.

Case 2.1

$$E_0 = \frac{B}{3A},\quad E_1 = \pm\frac{iB}{6A},\quad F_1 = \mp E_1,$$
$$\omega = \pm\frac{\sqrt{72Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}} \quad \text{and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A} \mp \frac{iB\tan(X)}{6A} \pm \frac{iB\cot(X)}{6A}. \tag{32}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A} \mp \frac{iB}{6A}\tan\left(kx+vy\pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}}\right) \pm \frac{iB}{6A}\cot\left(kx+vy\pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}}\right). \tag{33}$$

Substituting Equation (33) into Equation (4), the transversal reads

$$v(x,y,t) = \frac{b_1B}{6A}\left(2\pm i\tan\left(kx+vy\pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}}\right)\right) \mp \frac{iBb_1}{6A}\cot\left(kx+vy\pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}}\right) + b_2. \tag{34}$$

Case 2.2

$$E_0 = \frac{B}{3A},\quad E_1 = \pm iE_0,\quad F_1 = 0,$$
$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A}(1 \mp \tan(X)). \tag{35}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A}\left(1 \mp i\tan\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{36}$$

Substituting Equation (36) into Equation (4), the transversal reads

$$v(x,y,t) = \frac{Bb_1}{3A}\left(1 \mp i\tan\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right) + b_2. \tag{37}$$

Case 2.3

$$E_0 = \frac{B}{3A},\quad E_1 = \pm\frac{E_0}{\sqrt{2}},\quad F_1 = \pm\frac{E_0}{\sqrt{2}},$$
$$\omega = \pm\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{A}} \quad \text{and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{3A} \pm \frac{B\tan(X)}{3\sqrt{2}A} \pm \frac{B\cot(X)}{3\sqrt{2}A}. \tag{38}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{3A}\left(1 \pm \frac{1}{\sqrt{2}}\tan\left(kx+vy\pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{A}}\right)\right) \pm \frac{B}{3\sqrt{2}A}\cot\left(kx+vy\pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{A}}\right). \tag{39}$$

Substituting Equation (39) into Equation (4), the transversal reads

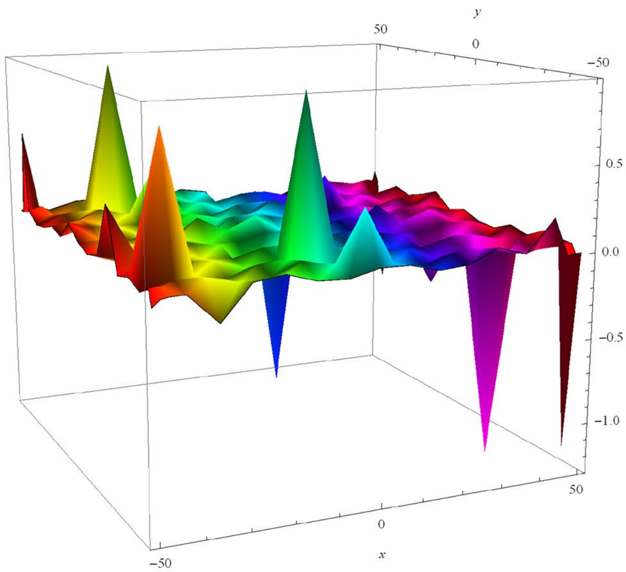
$$v(x,y,t) = \frac{Bb_1}{3A}\left(1 \pm \frac{1}{\sqrt{2}} \tan\left(kx + vy \pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{A}}\right)\right) \pm \frac{Bb_1}{3\sqrt{2}A}\cot\left(kx + vy \pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{A}}\right) + b_2. \tag{40}$$

Case 2.4

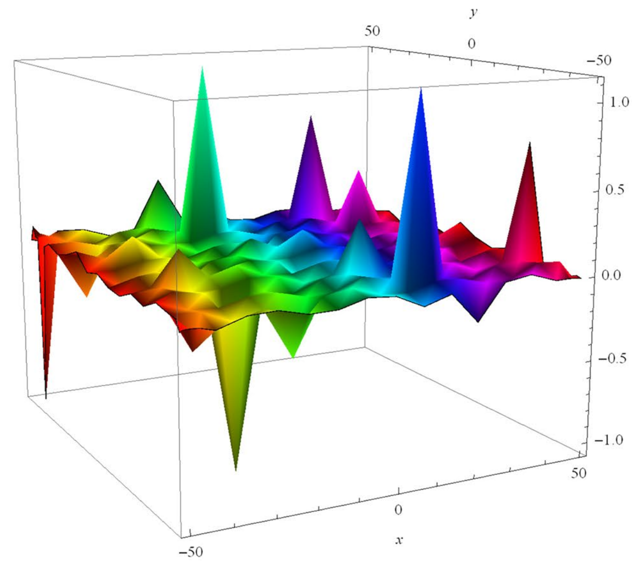
$$E_0 = \frac{B}{3A},\quad E_1 = 0,\quad F_1 = \pm iE_0,$$

$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}$$ and

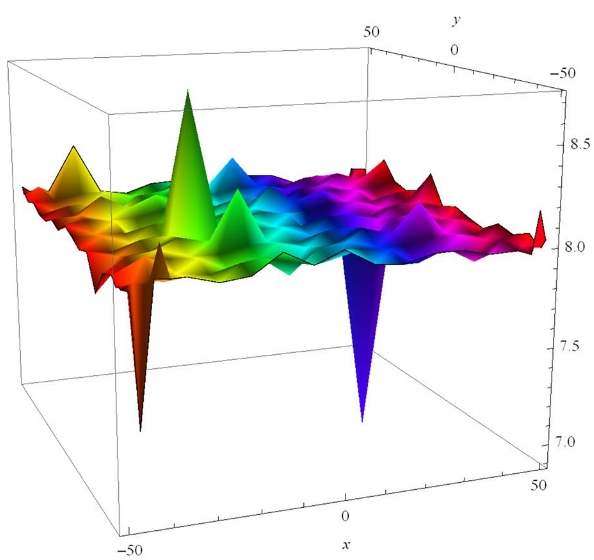
$$C = -\frac{2B^2}{9A}.$$



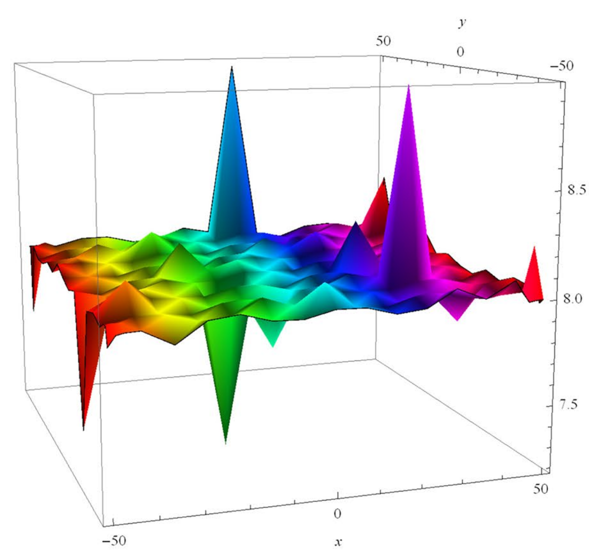
(a) u at $t = 0.022$



(b) u at $t = 1.8$



(c) v at $t = 0.022$



(d) v at $t = 1.8$

Fig. 2 Four distinct solitonic structures of Lump waves for Equations (39)-(40) with parameter $A = 49$, $B = 1.4$, $k = 0.1$, $v = 5$, $p = 12$, $b_1 = 0.81$, $b_2 = 8$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{B}{3A}(1\mp i\cot(X)). \tag{41}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{B}{3A}\left(1\mp i\cot\left(kx+vy\pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{42}$$

Substituting (42) into (4), the transversal reads

$$v(x,y,t)=\frac{Bb_1}{3A}\left(1\mp i\cot\left(kx+vy\pm t\frac{\sqrt{36Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)\right)+b_2. \tag{43}$$

Family 3: We attain $[\delta_1;\delta_2;\delta_3;\delta_4]=[1+i;1-i;1;1]$ and $[\xi_1;\xi_2;\xi_3;\xi_4]=[i;-i;i;-i]$, and then the expression (11) becomes

$$\varphi(X)=1-\tan(X).$$

Case 3.1

$$E_0=\frac{B}{A}\left(\frac{1}{3}\mp\frac{i}{3}\right),\quad E_1=0,\quad F_1=\pm\frac{2iB}{3A},$$
$$\omega=\pm\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\quad\text{and}$$
$$C=-\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{B}{A}\left(\frac{1}{3}\mp\frac{i}{3}\right)\pm\frac{2iB}{3A(1-\tan(X))}. \tag{44}$$

Also, the Equation (8) has explicit soliton solutions given by

$$\begin{aligned}u(x,y,t)=&\frac{B}{A}\left(\frac{1}{3}\mp\frac{i}{3}\right)\\&\pm\frac{2iB}{3A\left(1-\tan\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right)}.\end{aligned} \tag{45}$$

Substituting (45) into (4), the transversal reads

$$\begin{aligned}v(x,y,t)=&\frac{Bb_1}{A}\left(\frac{1}{3}\mp\frac{i}{3}\right)\\&\pm\frac{2iBb_1}{3A\left(1-\tan\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right)}+b_2.\end{aligned} \tag{46}$$

Case 3.2

$$E_0=\frac{2B}{3A},\quad E_1=-\frac{E_0}{2},\quad F_1=-E_0,$$
$$\omega=\pm\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\quad\text{and}$$
$$C=\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$\begin{aligned}U(X)=&\frac{2B}{3A}-\frac{B(1-\tan(X))}{3A}\\&-\frac{2B}{3A(1-\tan(X))}.\end{aligned} \tag{47}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{B\sec^2\left(kx+vy\pm\frac{t\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)}{3A\left(\tan\left(kx+vy\pm\frac{t\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)-1\right)}. \tag{48}$$

Substituting Equation (48) into Equation (4), the transversal reads

$$v(x,y,t)=\frac{Bb_1\sec^2\left(kx+vy\pm\frac{t\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)}{3A\left(\tan\left(kx+vy\pm\frac{t\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)-1\right)}+b_2. \tag{49}$$

Case 3.3

$$\begin{aligned}v(x,y,t)=&\frac{Bb_1}{A}\left(\frac{1}{3}\mp\frac{2i}{3}\right)\\&\pm\frac{5iBb_1}{3A\left(2+\tan\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}}\right)\right)}+b_2.\end{aligned}$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{B}{A}\left(\frac{1}{3}\mp\frac{i}{3}\right)\pm\frac{iB}{3A}(1-\tan(X)). \tag{50}$$

Also, the Equation (8) has explicit soliton solutions given by

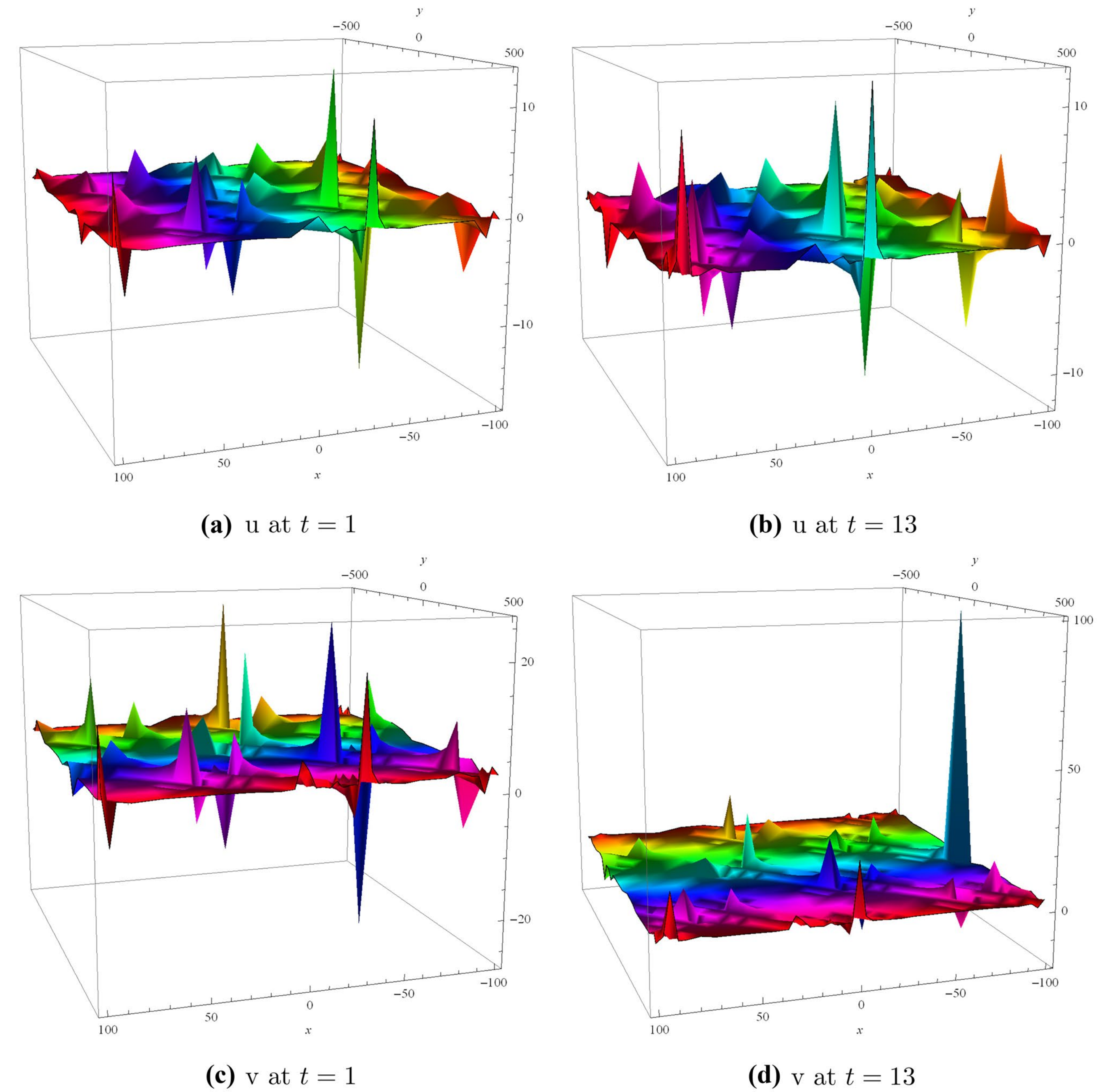


Fig. 3 Four distinct solitonic structures of elastic interactions between periodic multi-soliton waves for Equations (48)-(49) with arbitrary constants $B = 3,\ A = 13,\ k = 13,\ v = 7,\ c_1 = 12,\ b_1 = 1.7,\ b_2 = 3$

$$u(x, y, t) = \frac{B}{A}\left(\frac{1}{3} \mp \frac{i}{3}\right) \pm \frac{iB}{3A}\left(1 - \tan\left(kx + vy \pm t\frac{\sqrt{18Ac_1^2(k^2 + v^2) - B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{51}$$

$$v(x, y, t) = \frac{Bb_1}{A}\left(\frac{1}{3} \mp \frac{i}{3}\right) \pm \frac{iB}{3A}\left(1 - \tan\left(kx + vy \pm t\frac{\sqrt{18Ac_1^2(k^2 + v^2) - B^2}}{3\sqrt{2}\sqrt{A}}\right)\right) + b_2. \tag{52}$$

Substituting Equation (51) into Equation (4), the transversal reads

Case 3.4

$$E_0 = \frac{\sqrt{17}B + 9B}{24A}, \quad E_1 = -\frac{B}{24A}(1+\sqrt{17}),$$
$$F_1 = 2E_1, \quad C = \frac{B^2}{288A}(5\sqrt{17} - 51) \text{ and}$$
$$\omega = \pm\frac{1}{24}\sqrt{\frac{B^2}{A}(\sqrt{17}+9) + 576c_1^2(k^2+v^2)}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{\sqrt{17}B + 9B}{24A} - \frac{B(1+\sqrt{17})(1-\tan(X))}{24A} - \frac{(1+\sqrt{17})B}{12A(1-\tan(X))}. \tag{53}$$

Also, the Equation (8) has explicit soliton solutions given by

$$\begin{aligned} u(x,y,t) =& -\frac{(1+\sqrt{17})B}{24A}\left((1-\tan\left(kx+vy\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(\sqrt{17}+9)+576c_1^2(k^2+v^2)}\right)\right) \\ &+ \frac{\sqrt{17}B+9B}{24A} - \frac{(1+\sqrt{17})B}{12A\left(1-\tan\left(kx+vy\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(\sqrt{17}+9)+576c_1^2(k^2+v^2)}\right)\right)}. \end{aligned} \tag{54}$$

Substituting Equation (54) into Equation (4), the transversal reads

$$\begin{aligned} v(x,y,t) =& b_2 - \frac{Bb_1(1+\sqrt{17})}{24A}\left((1-\tan\left(kx+vy\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(\sqrt{17}+9)+576c_1^2(k^2+v^2)}\right)\right) \\ &+ \frac{b_1(\sqrt{17}B+9B)}{24A} - \frac{Bb_1(1+\sqrt{17})}{12A\left(1-\tan\left(kx+vy\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(\sqrt{17}+9)+576c_1^2(k^2+v^2)}\right)\right)}. \end{aligned} \tag{55}$$

Case 3.5

$$E_0 = \frac{B}{24A}(9-\sqrt{17}), \quad E_1 = \frac{B}{24A}(\sqrt{17}-1),$$
$$F_1 = 2E_1, \quad C = -\frac{B^2}{288A}(51 + 5\sqrt{17}) \text{ and}$$
$$\omega = \pm\frac{1}{24}\sqrt{\frac{B^2}{A}(9-\sqrt{17}) + 576c_1^2(k^2+v^2)}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{24A}(9-\sqrt{17}) + \frac{B}{24A}(\sqrt{17}-1)(1-\tan(X)) + \frac{B}{12A}\frac{(\sqrt{17}-1)}{(1-\tan(X))}. \tag{56}$$

Also, the Equation (8) has explicit soliton solutions given by

$$\begin{aligned} u(x,y,t) =& \frac{B}{24A}(\sqrt{17}-1)\left(1-\tan\left(kx+vy+\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(9-\sqrt{17})+576c_1^2(k^2+v^2)}\right)\right) \\ &+ \frac{B}{24A}(9-\sqrt{17}) + \frac{B}{12A}\frac{(\sqrt{17}-1)}{\left(1-\tan\left(kx+vy+\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(9-\sqrt{17})+576c_1^2(k^2+v^2)}\right)\right)}. \end{aligned} \tag{57}$$

Substituting (57) into (4), the transversal reads

$$\begin{aligned} v(x,y,t) =& b_2 + \frac{Bb_1}{24A}(\sqrt{17}-1)\left(1-\tan\left(kx+vy+\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(9-\sqrt{17})+576c_1^2(k^2+v^2)}\right)\right) \\ &+ \frac{Bb_1}{24A}(9-\sqrt{17}) + \frac{Bb_1}{12A}\frac{(\sqrt{17}-1)}{\left(1-\tan\left(kx+vy+\pm t\frac{1}{24}\sqrt{\frac{B^2}{A}(9-\sqrt{17})+576c_1^2(k^2+v^2)}\right)\right)}. \end{aligned} \tag{58}$$

Family 4: By setting $[\delta_1; \delta_2; \delta_3; \delta_4] = [2+i; 2-i; 1; 1]$ and $[\xi_1; \xi_2; \xi_3; \xi_4] = [-i; i; -i; i]$, and then the expression (11) gives

$\phi(X) = 2 + \tan(X)$.

Case 4.1

$$E_0 = \frac{B}{A}\left(\frac{1}{3} \mp \frac{2i}{3}\right), \quad E_1 = 0, \quad F_1 = \pm\frac{5iB}{3A},$$

$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2) - B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and}$$

$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{B}{A}\left(\frac{1}{3} \mp \frac{2i}{3}\right) \pm \frac{5iB}{3A(2+\tan(X))}. \tag{59}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B}{A}\left(\frac{1}{3} \mp \frac{2i}{3}\right) \pm \frac{5iB}{3A\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right)}. \tag{60}$$

Substituting Equation (60) into Equation (4), the transversal reads

$$v(x,y,t) = \frac{Bb_1}{A}\left(\frac{1}{3} \mp \frac{2i}{3}\right) \pm \frac{5iBb_1}{3A\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{6\sqrt{2}\sqrt{A}}\right)\right)} + b_2. \tag{61}$$

Case 4.2

$$E_0 = \frac{2B}{3A}, \quad E_1 = -\frac{B}{6A}, \quad F_1 = -\frac{5B}{6A},$$

$$\omega = \pm\frac{\sqrt{72Ac_1^2(k^2+v^2) + B^2}}{6\sqrt{2}\sqrt{A}} \quad \text{and}$$

$$C = \frac{B^2}{18A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{2B}{3A} - \frac{B}{6A}(2+\tan(X)) - \frac{5B}{6A(2+\tan(X))}. \tag{62}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{2B}{3A} - \frac{B}{6A}\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right)\right) - \frac{5B}{6A\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right)\right)}. \tag{63}$$

Substituting (63) into (4), the transversal reads

$$v(x,y,t) = \frac{2Bb_1}{3A} - \frac{Bb_1}{6A}\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right)\right) - \frac{5Bb_1}{6A\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{72Ac_1^2(k^2+v^2)+B^2}}{6\sqrt{2}\sqrt{A}}\right)\right)} + b_2. \tag{64}$$

Case 4.3

$$E_0 = \frac{B(1\mp 2i)}{3A}, \quad E_1 = \pm\frac{iB}{3A}, \quad F_1 = 0,$$

$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2) - B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and}$$

$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

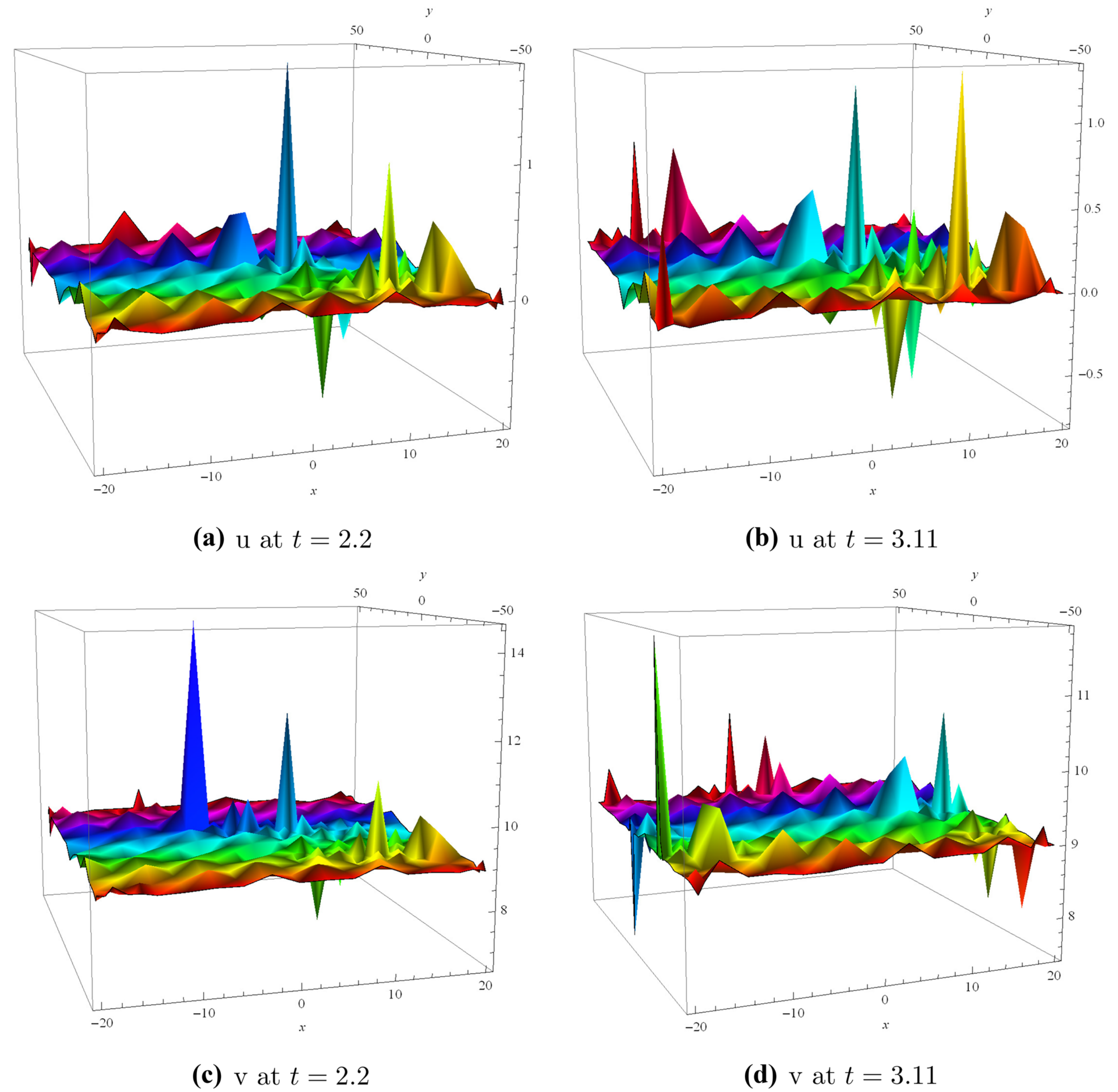
$$U(X) = \frac{B(1\mp 2i)}{3A} \pm \frac{iB}{3A}(2+\tan(X)). \tag{65}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{B(1\mp 2i)}{3A} \pm \frac{iB}{3A}\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{66}$$

Substituting Equation (66) into Equation (4), the transversal reads

$$v(x,y,t) = \frac{Bb_1(1\mp 2i)}{3A} \pm \frac{iBb_1}{3A}\left(2+\tan\left(kx+vy \pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right) + b_2. \tag{67}$$



(a) u at $t = 2.2$ **(b)** u at $t = 3.11$

(c) v at $t = 2.2$ **(d)** v at $t = 3.11$

Fig. 4 Four distinct solitonic structures of multiple-solitons wave profiles for Equations (63)-(64) with arbitrary constants $B = 0.91,\ A = 17,\ k = 3,\ v = 9,\ c_1 = 18,\ b_1 = 1.9,\ b_2 = 9$

Case 4.4

$$E_0 = \frac{\left(6+\sqrt{11}\right)B}{15A},\quad E_1 = -\frac{\left(1+\sqrt{11}\right)B}{30A},$$

$$F_1 = 5E_1,\quad C = \frac{\left(7\sqrt{11}-33\right)B^2}{225A} \text{ and}$$

$$\omega = \pm\frac{1}{30}\sqrt{900c_1^2(k^2+v^2) + \frac{\left(6+\sqrt{11}\right)B^2}{A}}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{\left(6+\sqrt{11}\right)B}{15A} - \frac{\left(1+\sqrt{11}\right)B}{30A}(2+\tan(X)) - \frac{\left(1+\sqrt{11}\right)B}{6A(2+\tan(X))}. \tag{68}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = -\frac{\left(1+\sqrt{11}\right)B}{30A}\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6+\sqrt{11}\right)B^2}{A}}\right)\right) + \frac{\left(6+\sqrt{11}\right)B}{15A} - \frac{\left(1+\sqrt{11}\right)B}{6A\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6+\sqrt{11}\right)B^2}{A}}\right)\right)}. \tag{69}$$

Substituting Equation (69) into Equation (4), the transversal reads

$$v(x,y,t) = -\frac{\left(1+\sqrt{11}\right)Bb_1}{30A}\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6+\sqrt{11}\right)B^2}{A}}\right)\right) + \frac{\left(6+\sqrt{11}\right)Bb_1}{15A} - \frac{\left(1+\sqrt{11}\right)Bb_1}{6A\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6+\sqrt{11}\right)B^2}{A}}\right)\right)} + b_2. \tag{70}$$

Case 4.5

$$E_0 = \frac{\left(6-\sqrt{11}\right)B}{15A},\quad E_1 = \frac{\left(\sqrt{11}-1\right)B}{30A},\quad F_1 = \frac{\left(\sqrt{11}-1\right)B}{6A},$$
$$\omega = \pm\frac{1}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6-\sqrt{11}\right)B^2}{A}}\quad \text{and}\quad C = -\frac{\left(33+7\sqrt{11}\right)B^2}{225A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

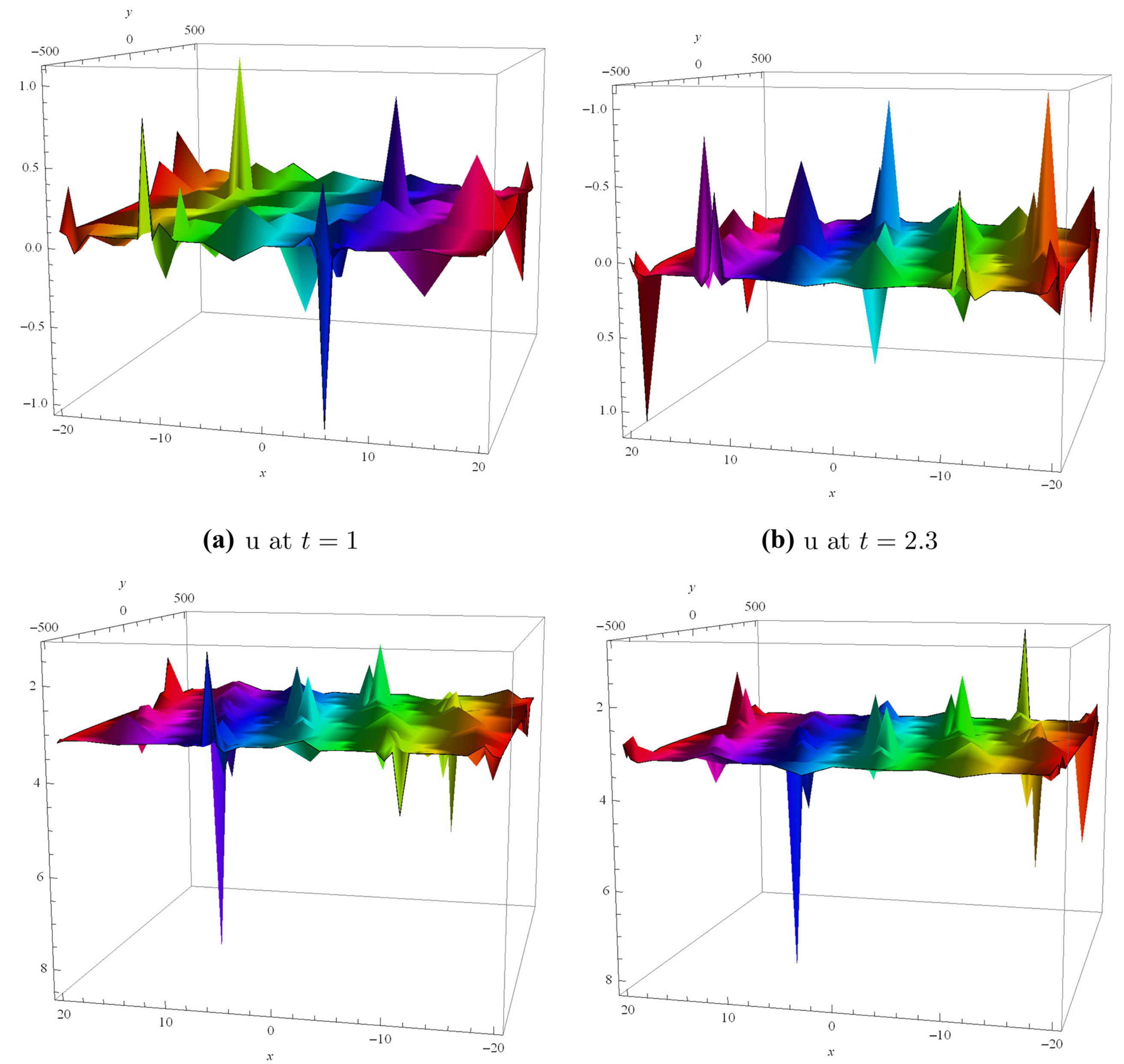
$$U(X) = \frac{\left(6-\sqrt{11}\right)B}{15A} + \frac{\left(\sqrt{11}-1\right)B}{30A}(2+\tan(X)) + \frac{\left(\sqrt{11}-1\right)B}{6A(2+\tan(X))}. \tag{71}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{\left(\sqrt{11}-1\right)B}{30A}\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6-\sqrt{11}\right)B^2}{A}}\right)\right) + \frac{\left(6-\sqrt{11}\right)B}{15A} + \frac{\left(\sqrt{11}-1\right)B}{6A\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6-\sqrt{11}\right)B^2}{A}}\right)\right)}. \tag{72}$$

Substituting (72) into (4), the transversal reads

$$\begin{aligned} v(x,y,t) =& \frac{\left(\sqrt{11}-1\right)Bb_1}{30A}\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6-\sqrt{11}\right)B^2}{A}}\right)\right) \\ &+\frac{\left(6-\sqrt{11}\right)Bb_1}{15A}+\frac{\left(\sqrt{11}-1\right)Bb_1}{6A\left(2+\tan\left(kx+vy\pm\frac{t}{30}\sqrt{900c_1^2(k^2+v^2)+\frac{\left(6-\sqrt{11}\right)B^2}{A}}\right)\right)}+b_2. \end{aligned} \tag{73}$$



(a) u at $t = 1$

(b) u at $t = 2.3$

(c) v at $t = 1.3$

(d) v at $t = 2.7$

Fig. 5 Four distinct solitonic structures of sharp Lump soliton waves for Equations (72)-(73) with arbitrary constants $B = 1,\ A = 7,\ k = 15,\ v = 49,\ c_1 = 1.1,\ b_1 = 91,\ b_2 = 3$

Family 5: By setting $[\delta_1; \delta_2; \delta_3; \delta_4] = [1-i; -1-i; -1; 1]$ and $[\xi_1; \xi_2; \xi_3; \xi_4] = [i; -i; i; -i]$, and hence the expression (11) shows

$$\phi(X) = -1 + \cot(X).$$

Case 5.1

$$E_0 = \frac{(1\pm i)B}{3A},\quad E_1 = 0,\quad F_1 = \pm\frac{2iB}{3A},$$
$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{(1\pm i)B}{3A} \pm \frac{2iB}{3A(-1+\cot(X))}. \tag{74}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{(1\pm i)B}{3A} \pm \frac{2iB}{3A\left(-1+\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right)}. \tag{75}$$

Substituting (75) into (4), the transversal reads

$$v(x,y,t) = \frac{(1\pm i)Bb_1}{3A} \pm \frac{2iBb_1}{3A\left(-1+\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right)} + b_2. \tag{76}$$

Case 5.2

$$E_0 = \frac{(1\pm i)B}{3A},\quad E_1 = \pm\frac{iB}{3A},\quad F_1 = 0,$$
$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{(1\pm i)B}{3A} \pm \frac{2iB}{3A}(-1+\cot(X)). \tag{77}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{(1\pm i)B}{3A} \pm \frac{2iB}{3A}\left(-1+\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right). \tag{78}$$

Substituting (78) into (4), the transversal reads

$$v(x,y,t) = \frac{(1\pm i)Bb_1}{3A} \pm \frac{2iBb_1}{3A}\left(-1+\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)-B^2}}{3\sqrt{2}\sqrt{A}}\right)\right) + b_2. \tag{79}$$

Case 5.3

$$E_0 = \frac{2B}{3A};,\quad E_1 = \frac{E_0}{2},\quad F_1 = E_0,$$
$$\omega = \pm\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}} \quad \text{and}$$
$$C = \frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

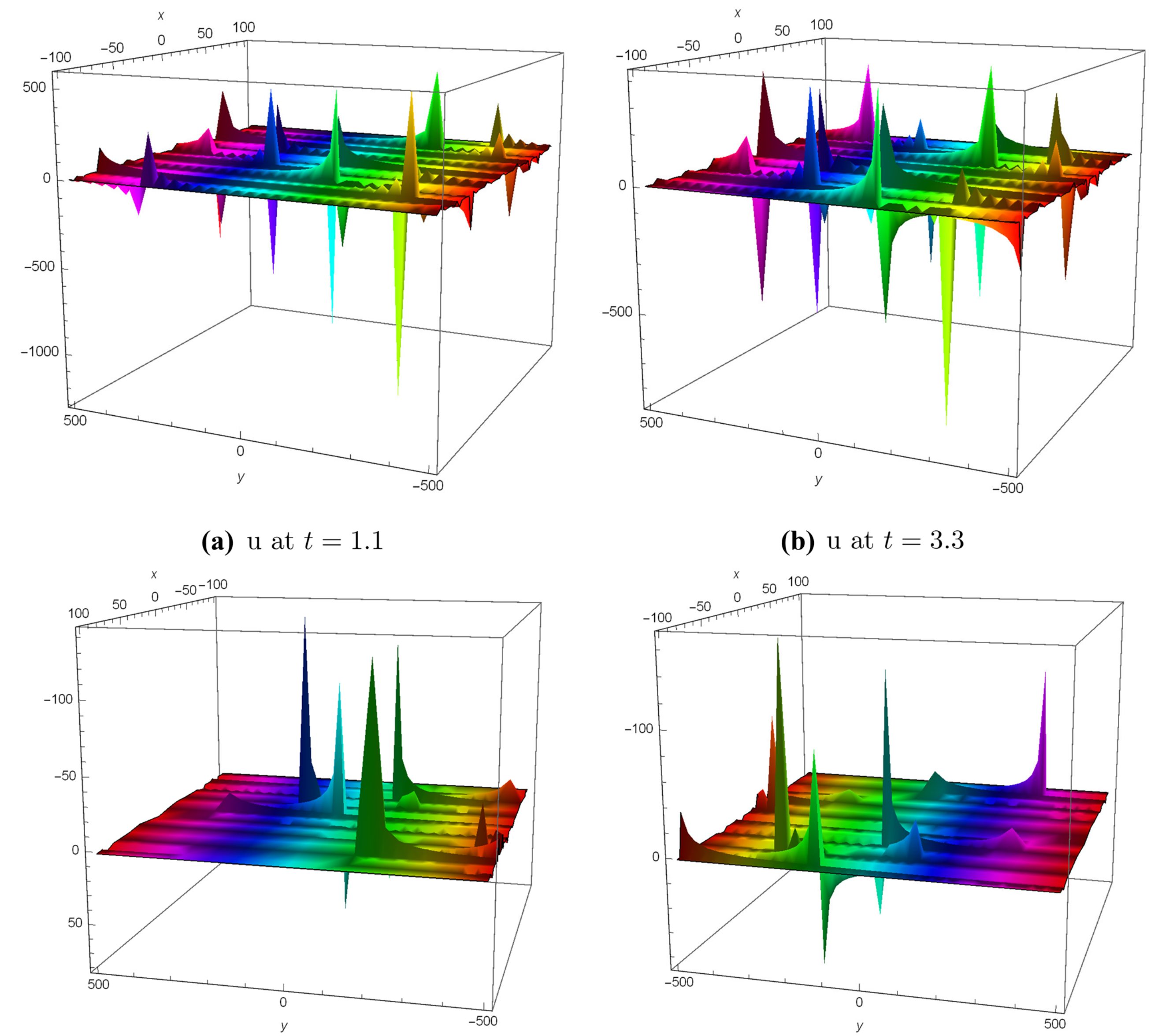
$$U(X) = \frac{2B}{3A} + \frac{B}{3A}(\cot(X)-1) + \frac{2B}{3A(\cot(X)-1)}. \tag{80}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{2B}{3A} + \frac{B}{3A}\left(\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)-1\right) + \frac{2B}{3A\left(\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)-1\right)}. \tag{81}$$

Substituting (81) into (4), the transversal reads

$$v(x,y,t) = \frac{2Bb_1}{3A} + \frac{Bb_1}{3A}\left(\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)-1\right) + \frac{2Bb_1}{3A\left(\cot\left(kx+vy\pm t\frac{\sqrt{18Ac_1^2(k^2+v^2)+B^2}}{3\sqrt{2}\sqrt{A}}\right)-1\right)} + b_2. \tag{82}$$



(a) u at $t = 1.1$ **(b)** u at $t = 3.3$

(c) v at $t = 1.1$ **(d)** v at $t = 3.3$

Fig. 6 Four distinct solitonic structures of interactions between periodic Lump solitons and anti-kink wave profiles for Equations (81)-(82) with arbitrary constants $B = 1,\ A = 7,\ k = 15,\ v = 49,\ c_1 = 1.1,\ b_1 = 91,\ b_2 = 3$

Case 5.4

$$U(X) = \frac{\left(9-\sqrt{17}\right)B}{24A} + \frac{\left(1-\sqrt{17}\right)B}{24A}(\cot(X)-1) + \frac{\left(1-\sqrt{17}\right)B}{12A(\cot(X)-1)}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{\left(9-\sqrt{17}\right)B}{24A} + \frac{\left(1-\sqrt{17}\right)B}{24A}(\cot(X)-1) + \frac{\left(1-\sqrt{17}\right)B}{12A(\cot(X)-1)}. \tag{83}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{\left(1-\sqrt{17}\right)B}{24A}\left(\cot\left(kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)-\frac{\left(\sqrt{17}-9\right)B^2}{A}}\right)-1\right)+\frac{\left(9-\sqrt{17}\right)B}{24A}+\frac{\left(1-\sqrt{17}\right)B}{12A\left(\cot\left(kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)-\frac{\left(\sqrt{17}-9\right)B^2}{A}}\right)-1\right)}. \tag{84}$$

Substituting (84) into (4), the transversal reads

$$v(x,y,t)=\frac{\left(1-\sqrt{17}\right)Bb_1}{24A}\left(\cot\left[kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)-\frac{\left(\sqrt{17}-9\right)B^2}{A}}\right]-1\right)+\frac{\left(9-\sqrt{17}\right)Bb_1}{24A}+\frac{\left(1-\sqrt{17}\right)Bb_1}{12A\left(\cot\left[kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)-\frac{\left(\sqrt{17}-9\right)B^2}{A}}\right]-1\right)}+b_2. \tag{85}$$

Case 5.5

$$E_0=\frac{\left(9+\sqrt{17}\right)B}{24A},\quad E_1=\frac{\left(1+\sqrt{17}\right)B}{24A},$$
$$F_1=2E_1,\quad C=\frac{\left(5\sqrt{17}-51\right)B^2}{288A}\ \text{and}$$
$$\omega=\pm\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)+\frac{\left(9+\sqrt{17}\right)B^2}{A}}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

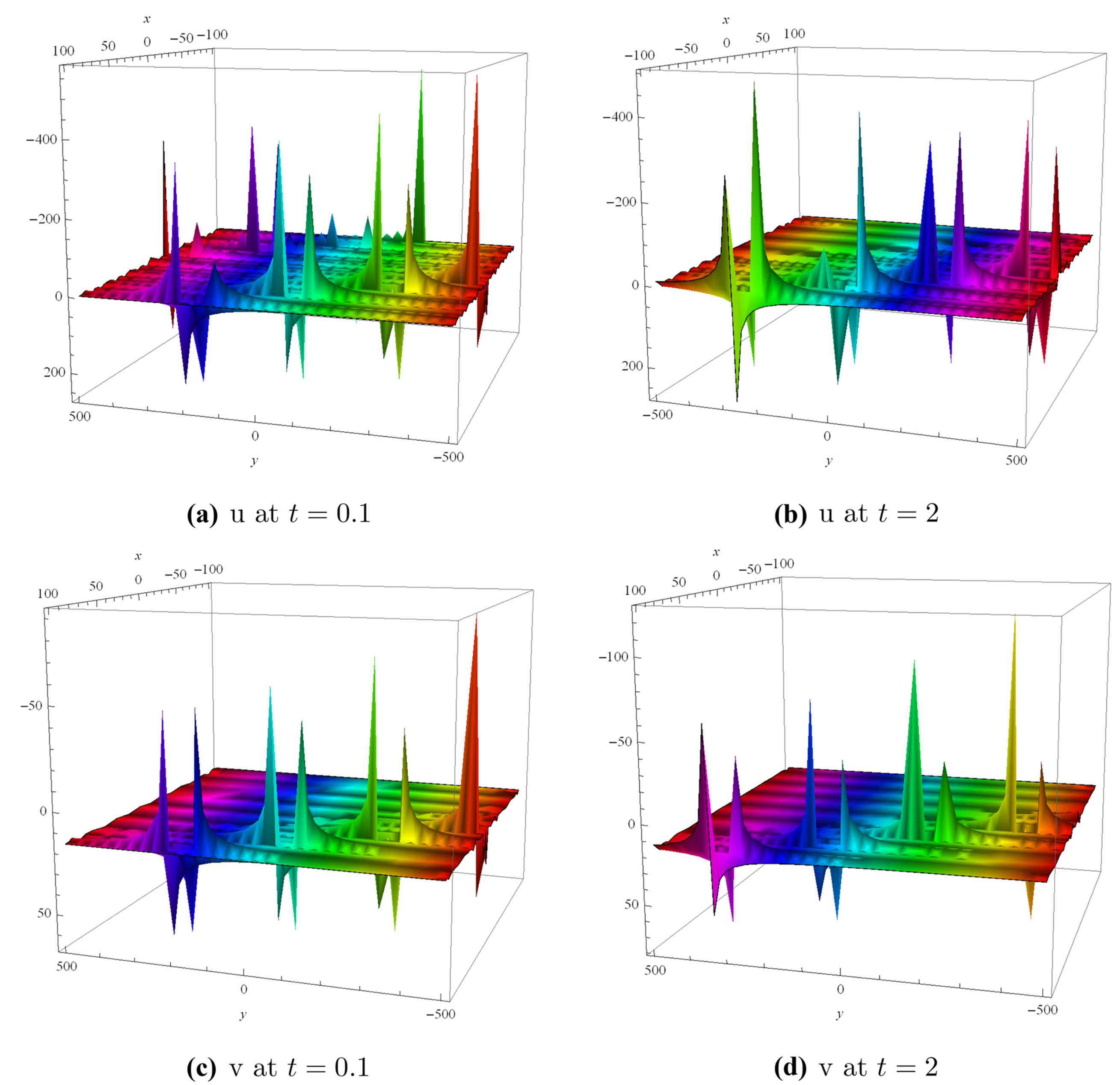
$$U(X)=\frac{\left(9+\sqrt{17}\right)B}{24A}+\frac{\left(1+\sqrt{17}\right)B}{24A}(\cot(X)-1)+\frac{\left(1+\sqrt{17}\right)B}{24A(\cot(X)-1)}. \tag{86}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{\left(1+\sqrt{17}\right)B}{24A}\left(\cot\left(kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)+\frac{\left(\sqrt{17}+9\right)B^2}{A}}\right)-1\right)+\frac{\left(9+\sqrt{17}\right)B}{24A}+\frac{\left(1+\sqrt{17}\right)B}{24A\left(\cot\left(kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)+\frac{\left(\sqrt{17}+9\right)B^2}{A}}\right)-1\right)}. \tag{87}$$

Substituting (87) into (4), the transversal reads

$$v(x,y,t)=\frac{\left(1+\sqrt{17}\right)Bb_1}{24A}\left(\cot\left(kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)+\frac{\left(\sqrt{17}+9\right)B^2}{A}}\right)-1\right)+\frac{\left(9+\sqrt{17}\right)Bb_1}{24A}+\frac{\left(1+\sqrt{17}\right)Bb_1}{24A\left(\cot\left(kx+vy\pm t\frac{1}{24}\sqrt{576c_1^2\left(k^2+v^2\right)+\frac{\left(\sqrt{17}+9\right)B^2}{A}}\right)-1\right)}+b_2. \tag{88}$$



(a) u at $t = 0.1$ **(b)** u at $t = 2$

(c) v at $t = 0.1$ **(d)** v at $t = 2$

Fig. 7 Four distinct solitonic structures of vertical Lump-type solitons wave profiles for Equations (87)-(88) with arbitrary constants $B = 4,\ A = 1,\ k = 32,\ v = 19,\ c_1 = 0.16,\ b_1 = 0.19,\ b_2 = 16$

x

Family 6: By setting $[\delta_1; \delta_2; \delta_3; \delta_4] = [2; 1; 1; 1]$ and $[\xi_1; \xi_2; \xi_3; \xi_4]$= [1; 0;1; 0], and so expression (11) shows

$$\phi(X) = \frac{2e^X + 1}{e^X + 1}.$$

Case 6.1

$$E_0 = \frac{4B}{3A}, \quad E_1 = 0, \quad F_1 = -E_0,$$

$$\omega = \pm\frac{\sqrt{9Ac_1^2(k^2 + v^2) + 2B^2}}{3\sqrt{A}} \quad \text{and}$$

$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{4B}{3A}-\frac{4B(e^{X}+1)}{3A(2e^{X}+1)}. \tag{89}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{4B}{3A}-\frac{4B\left(exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}. \tag{90}$$

Substituting (90) into (4), the transversal reads

$$v(x,y,t)=\frac{4Bb_1}{3A}-\frac{4Bb_1\left(exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}+b_2. \tag{91}$$

Case 6.2

$$E_0=\frac{2B}{3A},\quad E_1=-\frac{E_0}{3},\quad F_1=-\frac{2E_0}{3},$$
$$\omega=\pm\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\quad\text{and}$$
$$C=-\frac{2B^2}{81A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{4B}{3A}-\frac{2B(2e^{X}+1)}{9A(e^{X}+1)}-\frac{4B(e^{X}+1)}{9A(2e^{X}+1)}. \tag{92}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{4B}{3A}-\frac{2B\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}{9A\left(exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}-\frac{4B\left(exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}{9A\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}. \tag{93}$$

Substituting (93) into (4), the transversal reads

$$v(x,y,t)=\frac{4Bb_1}{3A}-\frac{2Bb_1\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}{9A\left(exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}-\frac{4Bb_1\left(exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}{9A\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{81Ac_1^2(k^2+v^2)+2B^2}}{9\sqrt{A}}\right)+1\right)}+b_2. \tag{94}$$

Case 6.3

$$E_0=-\frac{2B}{3A},\quad E_1=-E_0,\quad F_1=0,$$
$$\omega=\pm\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\quad\text{and}$$
$$C=-\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=-\frac{2B}{3A}+\frac{2B(2e^{X}+1)}{3A(e^{X}+1)}. \tag{95}$$

Also, the Equation (8) has explicit soliton solutions given by

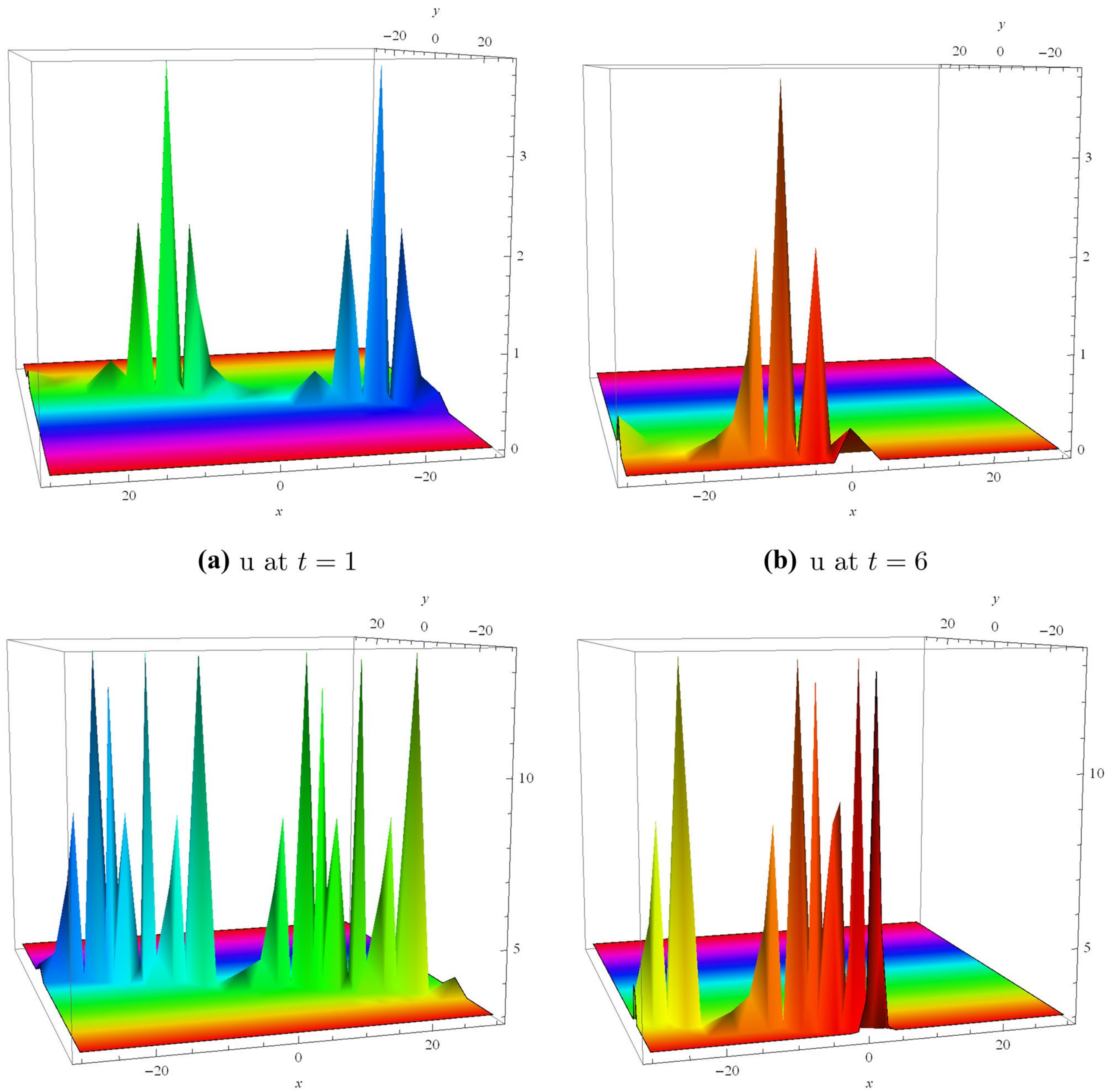
$$u(x,y,t)=-\frac{2B}{3A}-\frac{2B\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}. \tag{96}$$

Substituting (96) into (4), the transversal reads

$$v(x,y,t)=-\frac{2Bb_1}{3A}+\frac{2Bb_1\left(2\ exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}+b_2. \tag{97}$$

Family 7: By setting $[\delta_1;\delta_2;\delta_3;\delta_4]=[1;2;1;1]$ and $[\xi_1;\xi_2;\xi_3;\xi_4]=[1;0;1;0]$, and so expression (11) shows

$$\phi(X)=\frac{e^{X}+2}{e^{X}+1}$$



(a) u at $t = 1$ **(b)** u at $t = 6$

(c) v at $t = 1$ **(d)** v at $t = 6$

Fig. 8 Four distinct solitonic structures of oscillating waves for Equations (93)-(94) with arbitrary constants $B = 431,\ A = 4.1,\ k = 3,\ v = 7,\ c_1 = 1.001,\ b_1 = 2.7,\ b_2 = 3$

Case 7.1

$$E_0 = -\frac{2B}{3A},\ \ E_1 = 0,\ \ F_1 = -2E_0,$$

$$\omega = \pm\frac{\sqrt{9Ac_1^2(k^2 + v^2) + 2B^2}}{3\sqrt{A}} \ \text{ and}$$

$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = -\frac{2B}{3A} + \frac{4B(e^X + 1)}{3A(e^X + 2)}. \tag{98}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = -\frac{2B}{3A} + \frac{4B\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)}. \tag{99}$$

Substituting (99) into (4), the transversal reads

$$v(x,y,t) = -\frac{2Bb_1}{3A} + \frac{4Bb_1\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)} + b_2. \tag{100}$$

Case 7.2

$$E_0 = \frac{4B}{3A},\ E_1 = -\frac{E_0}{2},\ F_1 = 0,$$
$$\omega = \pm\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}} \text{ and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{4B}{3A} - \frac{2B(e^X+2)}{3A(e^X+1)}. \tag{101}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t) = \frac{4B}{3A} - \frac{2B\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)}{3A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}. \tag{102}$$

Substituting (102) into (4), the transversal reads

$$v(x,y,t) = \frac{4Bb_1}{3A} - \frac{2Bb_1\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)}{3A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)} + b_2. \tag{103}$$

Case 7.3

$$E_0 = \frac{2B}{3A},\ E_1 = -\frac{E_0}{3},\ F_1 = -\frac{2E_0}{3},$$
$$\omega = \pm\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}} \text{ and}$$
$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = \frac{4B}{3A} - \frac{2B(e^X+2)}{9A(e^X+1)} - \frac{4B(e^X+1)}{9A(e^X+2)}. \tag{104}$$

Also, the Equation (8) has explicit soliton solutions given by

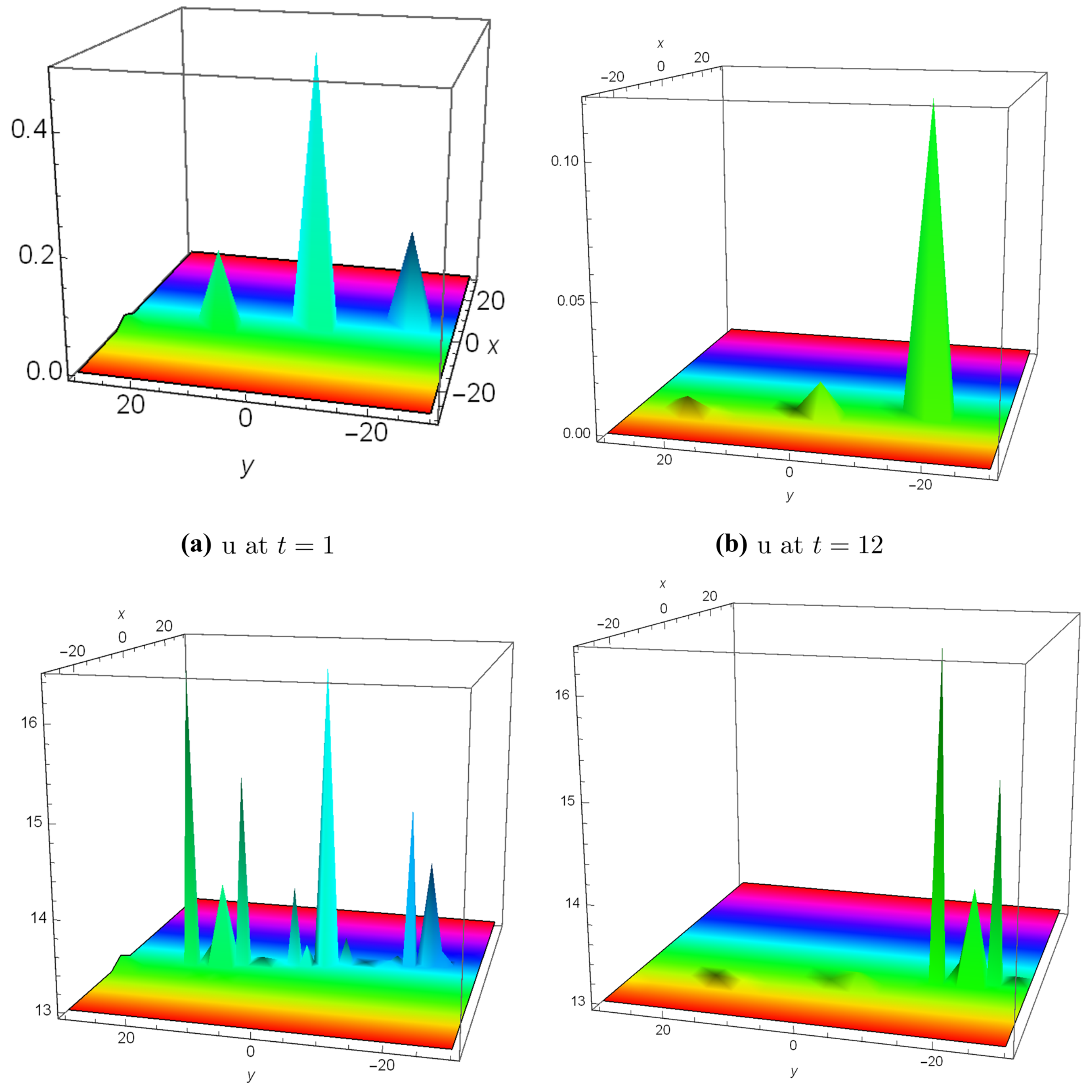
$$u(x,y,t) = \frac{4B}{3A} - \frac{2B\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)}{9A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)} - \frac{4B\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{9A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)}. \tag{105}$$

Substituting (105) into (4), the transversal reads

$$v(x,y,t) = \frac{4Bb_1}{3A} - \frac{2Bb_1\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)}{9A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)} - \frac{4Bb_1\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{9A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+2\right)} + b_2. \tag{106}$$

Family 8: By setting $[\delta_1; \delta_2; \delta_3; \delta_4] = [2; 3; 1; 1]$ and $[\xi_1; \xi_2; \xi_3; \xi_4] = [1; 0;1; 0]$, and so expression (11) shows

$$\phi(X) = \frac{2e^X+3}{e^X+1}$$



(a) u at $t = 1$ **(b)** u at $t = 12$

(c) v at $t = 1$ **(d)** v at $t = 12$

Fig. 9 Four distinct solitonic structures of oscillating waves for Equations (105)-(106) with arbitrary constants $B = 13$, $A = 1.01$, $k = 13$, $v = 3.11$, $c_1 = 1.001$, $b_1 = 7$, $b_2 = 13$

Case 8.1

$$E_0 = -\frac{4B}{3A}, \ \ E_1 = 0, \ \ F_1 = -3E_0,$$

$$\omega = \pm\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}} \ \text{ and}$$

$$C = -\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X) = -\frac{2B}{3A} + \frac{4B(e^X+1)}{A(2e^X+3)}. \tag{107}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=-\frac{4B}{3A}+\frac{4B\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{3A\left(2\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+3\right)}. \tag{108}$$

Substituting (108) into (4), the transversal reads

$$v(x,y,t)=-\frac{4Bb_1}{3A}+\frac{4Bb_1\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}{A\left(2\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+3\right)}+b_2. \tag{109}$$

Case 8.2

$$E_0=\frac{2B}{A},\quad E_1=-\frac{E_0}{3},\quad F_1=0,$$
$$\omega=\pm\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}} \quad \text{and}$$
$$C=-\frac{2B^2}{9A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{2B}{A}-\frac{2B(2e^X+3)}{3A(e^X+1)}. \tag{110}$$

Also, the Equation (8) has explicit soliton solutions given by

$$u(x,y,t)=\frac{2B}{A}-\frac{2B\left(2\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+3\right)}{3A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}. \tag{111}$$

Substituting (111) into (4), the transversal reads

$$v(x,y,t)=\frac{2Bb_1}{A}-\frac{2Bb_1\left(2\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+3\right)}{3A\left(\exp\left(kx+vy\pm t\frac{\sqrt{9Ac_1^2(k^2+v^2)+2B^2}}{3\sqrt{A}}\right)+1\right)}+b_2. \tag{112}$$

Case 8.3

$$E_0=\frac{2B}{3A},\quad E_1=-\frac{E_0}{5},\quad F_1=-\frac{6E_0}{5},$$
$$\omega=\pm\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}} \quad \text{and}$$
$$C=-\frac{2B^2}{225A}.$$

Considering all these values in Equation (15), then the solution of Equation (14) is obtained as

$$U(X)=\frac{2B}{3A}-\frac{2B(2e^X+3)}{15A(e^X+1)}-\frac{4B(e^X+1)}{5A(2e^X+3)}. \tag{113}$$

Also, the Equation (8) has explicit soliton solutions given by

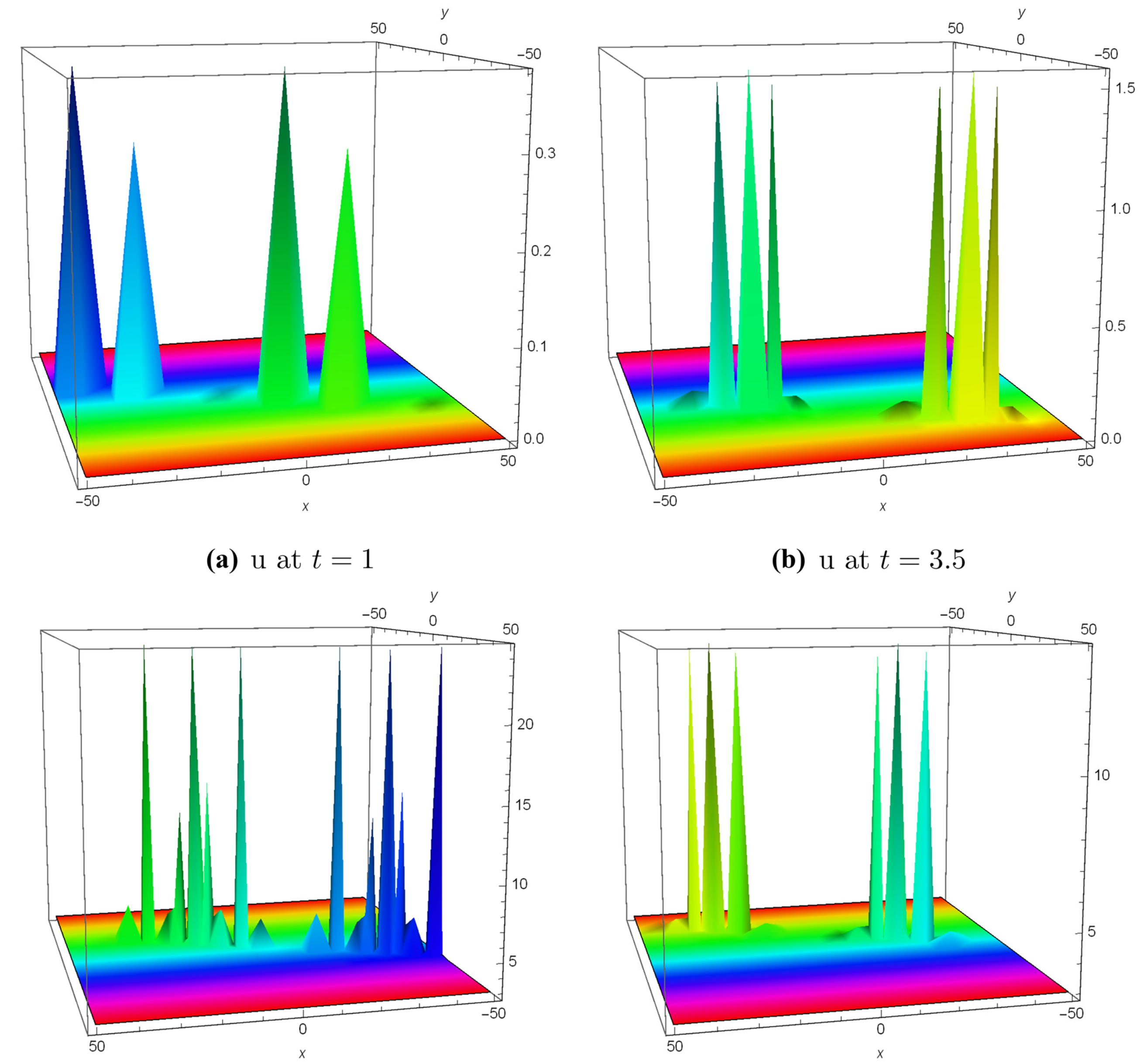
$$u(x,y,t)=\frac{2B}{3A}-\frac{2B\left(2\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+3\right)}{15A\left(\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+1\right)}-\frac{4B\left(\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+1\right)}{5A\left(2\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+3\right)}. \tag{114}$$

Substituting (114) into (4), the transversal reads

$$v(x,y,t)=\frac{2Bb_1}{3A}-\frac{2Bb_1\left(2\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+3\right)}{15A\left(\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+1\right)}-\frac{4Bb_1\left(\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+1\right)}{5A\left(2\exp\left(kx+vy\pm t\frac{\sqrt{225Ac_1^2(k^2+v^2)+2B^2}}{15\sqrt{A}}\right)+3\right)}+b_2. \tag{115}$$

5 Physical explanations for some solutions

It has been demonstrated that obtained closed-form exact soliton solutions by applying the current proposed generalized exponential rational function (GERF) method for the



(a) u at $t=1$ (b) u at $t=3.5$

(c) v at $t=1$ (d) v at $t=3.5$

Fig. 10 Four distinct solitonic structures of oscillating waves for Equations (114)-(115) with arbitrary constants $B=481$, $A=2.1$, $k=3$, $v=7$, $c_1=1$, $b_1=7$, $b_2=3$

double-chain deoxyribonucleic acid (DNA) model, are new and different from other used methods. As the proposed method has accomplished the distinct wave solutions forms of solutions like exponential functions, trigonometric functions, hyperbolic trigonometric functions, and rational functions through the Equation (8) employing the different values of constants parameters. The observable features for obtaining solutions of the mentioned method are the novel structure of solitonic solutions and new equations which derived different form of solutions. For a physical description of achieved solutions, graphics are presented to elaborate the several novel exact soliton and exponential rational functional solutions in the shapes of dynamics of solitons like multi-solitons, breather-type solitons, abundant elastic interactions between multi-solitons, and nonlinear waves, oscillating multi-solitons, and Lump solitons, etc. The graphical structures given in Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are plotted in 3-dimensional by Mathematica software.

Figure 1 reveals the breather-type waves for solutions (30)-(31) for $t=0.03$ and $t=1.43$. For best visibility of the figure, we sketch the graph for suitable choice of arbitrary constants as $B = 99,\ A = 9,\ k = 0.33,\ v = 0.11,\ c_1 = 0.3,\ b_1 = 0.093$, and $b_2 = 2$. In Fig. 2, the dynamics of the closed-form solutions (39)-(40) in various shapes are traced with different value of t. The dynamical structures of achieved Lump wave soliton solutions are demonstrated by changing the distinct values of free parameter constants $A = 49,\ B = 1.4,\ k = 0.1\ v = 5,\ p = 12,\ b_1 = 0.81,\ b_2 = 8$. In Fig. 3, the physical aspect of the solutions (48)-(49) is investigated at $t = 1$ and $t = 13$. The graphical structure has shown the elastic interactions between periodic multi-soliton waves for u and v. For Sketching this profile, we have consider the appropriate choice of arbitrary constants $B = 3,\ A = 13,\ k = 13,\ v = 7,\ c_1 = 12,\ b_1 = 1.7,\ b_2 = 3$. Multiple-solitons wave profiles for Equations (63)-(64) are exhibited in this Fig. 4. The graphical behavior of multi-soliton for solutions u , v is noticed at $t = 2.2$ and $t = 3.11$. These graphs are sketched by considering arbitrary constants $B = 0.91, A = 17, k = 3, v = 9, c_1 = 18, b_1 = 1.9, b_2 = 9$. In Fig. 5, the dynamics of the solitonic solutions (72)-(73) in distinct shapes are plotted at the value of $t = 1$ and $t = 2.3$. The dynamical structures of achieved sharp Lump soliton waves for solutions are demonstrated by changing the distinct values of free parameter constants $B = 1,\ A = 7,\ k = 15,\ v = 49,\ c_1 = 1.1,\ b_1 = 91,\ b_2 = 3$. Distinct solitonic structures of interactions between periodic Lump solitons and anti-kink wave profiles for solutions (81)-(82) are exhibited in this Fig. 6. The geometrical behavior of multisoliton for solutions u , v is recorded at $t = 1.1$ and $t = 3.3$. These graphs are illustrated by choosing parameters $B = 1,\ A = 7,\ k = 15,\ v = 49,\ c_1 = 1.1,\ b_1 = 91,\ b_2 = 3$. In Fig. 7, solitonic structures of vertical Lump-type solitons wave profiles for Equations (87)-(88) has been analyzed at $t = 0.1$ and $t = 2$. The arbitrary constants are considered as $B = 4, A = 1, k = 32, v = 19, c_1 = 0.16, b_1 = 0.19, b_2 = 16$ for ambient visibility. In Fig. 8, solitonic structures of oscillating waves are revealed for solutions u and v given by Equations (93)-(94), respectively. Profile is plotted for the values of constants chosen as $B = 431, A = 4.1, k = 3, v = 7, c_1 = 1.001, b_1 = 2.7, b_2 = 3$. In Fig. 9, Equations (105)-(106) illustrated multisoliton wave profile with the variation of t. For numerical simulation, we have taken arbitrary constants as $B = 13,\ A = 1.01,\ k = 13,\ v = 3.11,\ c_1 = 1.001,\ b_1 = 7,\ b_2 = 13$. In Fig. 10, Equations (114)-(115) represented multisoliton wave profile with the variation of t. For numerical simulation, the arbitrary constants are considered as $B = 481,\ A = 2.1,\ k = 3,\ v = 7,\ c_1 = 1,\ b_1 = 7,\ b_2 = 3$.

6 Conclusion

In summary, we successfully performed the generalized rational function method to acquire the closed-form exact solutions of the considered double-chain DNA model described by Equations (1) and (2). This double-chain model of DNA consists of two long elastic homogeneous chains which depict two polynucleotide strands of the DNA molecule, connected with each other by a linear springs describing the hydrogen bonds between the base pair of the two strands. The nonlinear dynamical equations are considered and some soliton solutions are constructed. Moreover, this method concerning novel closed-form exact soliton solutions like rational functional, trigonometric functional, and hyperbolic functional solutions. The physical significance of the established soliton solutions has been demonstrated via the 3D shapes. Different forms of well-known dynamical wave structures of soliton solutions are examined: the breather-type wave shape, Lump solitons shape, periodic multi-solitons shape, multiple solitons wave shape, and the singular periodic waves. Apart from this, the generated soliton solutions showed that the current method is efficient, straightforward, robust, and influential which can be used to obtain closed-form exact soliton solutions to NLEEs arising in different physical models of nonlinear dynamic systems. In the future, we will modify the computerized symbolic method used here to deal with various NLEEs, when their coefficients are variables, for introducing more exact solitary wave solutions. From our generated closed-form solutions, it is remarkable to note that the method is highly reliable, beneficial and very efficient for evaluating various new solitonic structures of several nonlinear development equations. Some of the families of exact explicit solutions exhibit the effectiveness, trustworthiness, relevance and straightforwardness of the method.

6.1 Future scope

All the new findings obtained in this article have a large number of applications in the areas of mathematical biology, dynamical systems, dynamics of solitons, nonlinear dynamics, and various related fields of mathematical sciences and applied mathematics. Basically, solitary waves or solitons are used to describe any optical field that does not vary due to a balance between nonlinear outcomes and dispersive. In future, solitonic structures of solitons create a comprehensive theory to predict the dealing of various physical and biological systems. The development of successful soliton solutions will lead to tremendous research in different fields for various dynamical systems.

Declarations

Disclosure statement No potential conflict of interest was reported by the authors.

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