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GENERAL AND APPLIED PHYSICS

Wigner Function and Non-classicality for Oscillator Systems

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Abstract
The Schrödinger equation in phase space is used to calculate the Wigner function for oscillator systems. The first is two oscillators including dissipative effects. In this case, the non-classicality of the states is studied by the non-classicality indicator of the Wigner function, which is calculated as a function of the dissipation parameter. The second oscillator system is non-linear pendulum. Analytical results are derived and the Wigner function is analyzed.

Keywords Moyal product · Phase space · Oscillator systems

1 Introduction

Oscillator systems in quantum mechanics have raised interest since long time ago [1, 2], due to the possibility to study some important physical properties as dissipation and non-classicallity effects [3–9]. Such studies, in most of the cases, shed light into the very structure of matter, considering, in particular, experimental apparatus with atoms where the neighborhood effect is the cause of dissipation or non-linearity [10, 11].

Both experiments with fermions and those designed to detect Bose-Einstein condensates include some degree of dissipation. For instance laser cooling, magnetic and magnetic-optic traps [12, 13]. The fermion counterpart has been accomplished by considering a degenerate Fermi gas as well as condensates of rare isotopes [13–17]. In addition, there is a great deal of interest in the entangled of multipartite (fermion or bose) states for quantum communication [18, 19]. A simple but intricate example of such a fermion system, at the level of electronic structure, is the Helium atom considered as a few-fermion system taken in an external field, which can in turn be considered as a dissipative effect [20, 21].

From a theoretical point of view, the Schrödinger equation has no known exact solution for such a system. The main results are based on approximative methods or variational formalisms. For instance, due to the resemblance between the Gaussian wave function of the spherically symmetric harmonic oscillator and the $1s$ state of the hydrogen atom, some models are used to study solutions of the Schrödinger equation for Helium atom. It consists in changing the Coulomb interactions by a harmonic-like oscillator potential. In particular, the electron-nuclei interactions can be replaced by a harmonic oscillator-like potential while the electron-electron interaction is Coulombic. Even considering such a drastic aproximation, the energy values are near to the experimental data [22, 23]. Motived by these results with this harmonic-like interacting systems, here we study the non-classicality of such states, analyzing the phase -space Wigner function [24–29].

The analysis in phase space is important in order to track the statistical nature of quantum states [30]. In this case, the Wigner formalism is physically appealing, in particular in experiments for the reconstruction of quantum states, in quantum tomography, and for the direct measurement of the Wigner function [31–38].

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When non-linearity is present, however, the derivation of the Wigner function through the density matrix is an intricate procedure. This is a result associated directly to the fact that the Wigner function is a real quantity; and as such, preventing the improvement of gauge symmetry: a necessary condition for introducing generalised interaction. This aspect has implications as a lack of a direct way to study superposition effects. This type of problem has led to the search for the analysis of the Wigner formalism following different perspectives [39–52], and advancements have been reached, including the study of representations of symmetry for quantum equations directly in phase space [53]. After some preliminary attempts [54, 55], exploring a representation of the Schrödinger equation in phase space [56–60], a consistent formalism has been introduced [61], by using the notion of quasi-amplitude of probability, which is associated with the Wigner function by the Moyal (or star) product [63, 64]. This notion of symplectic structure and Weyl product has been explored to study unitary representations of Galilei group, leading to a symplectic representation of the Schrödinger equation [61]. This approach provides an interesting procedure to derive the Wigner function, by using consistently the gauge invariance and superposition effects [62]. This symplectic representation has been applied in kinetic theory and extended to the relativistic contexts, giving rise to the Klein-Gordon and the Dirac equations in phase space [65–69]. Here, we use this symplectic quantum mechanics to analyze the behavior of the Wigner function for oscillators, considering dissipation and non-linearity. In this case, we study the non-classicality (negativity) indicator of the Wigner function [70] as a function of the dissipation parameter.

The paper is organized in the following way. In Section 2, we present an outline of the symplectic representation of the Schrödinger equation in phase space and the connection between phase space quasi-amplitudes and the Wigner function. In Section 3, we solve the Schrödinger equation in phase space for the Helium atom in the two-oscillator approximation. In Section 4, a quantum damped oscillator is studied. Finally, some closing comments are given in Section 5.

2 Outline on Symplectic Schrödinger Equation

In this section, we present a brief outline of the construction of the Schrödinger equation in phase space, emphasizing the association of phase space amplitude of probability with the Wigner function. We consider initially a one-particle system described by the Hamiltonian $H = \widehat{p}^2/2m$, where m and $\widehat{p}$ are the mass and the momentum, respectively, of the particle. The Wigner formalism for such a system is constructed from the Liouville-von Neumann equation [24–27]

$$i\hbar\partial_t\rho(t) = [H, \rho],$$

where $\rho(t)$ is the density matrix. The Wigner function, $f_W(q, p)$, is defined by

$$f_W(q, p) = (2\pi\hbar)^{-1}\int dz \exp\left(\frac{ipz}{\hbar}\right)\left\langle q - \frac{z}{2}|\rho|q + \frac{z}{2}\right\rangle, \tag{1}$$

where $|q + z/2 >$ is a translated ket in coordinate representation which is solution of the traditional Schrödinger equation. It satisfies the equation of motion

$$i\hbar\partial_t f_W(q, p, t) = \{H_W, f_W\}_M, \tag{2}$$

where H_W is the Weyl transform of the Hamiltonian operator. It is related to H by means the expression $H_W(q, p) = \int dz\langle q - z/2|H(q, p)|q + z/2\rangle$. Here, $\{a, b\}_M = a \star b - b \star a$ is the Moyal bracket, such that the star operator [27] is given by

$$\star \equiv e^{\frac{i\hbar\Lambda}{2}}$$

with $\Lambda = \overleftarrow{\partial}_q\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_q$. The functions $a(q, p)$ are defined in a manifold Γ, using the basis (q, p) with the physical content of the phase space. In this formalism, an operator, say A, defined in the Hilbert space $\mathcal{H}$, is represented by the function

$$A(q, p) = \int dz \exp\left(\frac{ipz}{\hbar}\right)\left\langle q - \frac{z}{2}|A|q + \frac{z}{2}\right\rangle,$$

such that the product of two operators, AB, reads

$$(AB)(q, p) = A(q, p)e^{\frac{i\hbar\Lambda}{2}}B(q, p) = A(q, p) \star B(q, p).$$

The average of the operator A in a state $\psi \in \mathcal{H}$ is given by

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \int dqdpA(q, p)f_W(q, p) = Tr\rho A.$$

Now, we proceed in order to introduce the symplectic representation of quantum mechanics in phase space. First, we introduce a Hilbert space associated to the phase space Γ, by considering the set of function $\phi(q, p)$ in Γ, such that

$$\int dpdq\psi^*(q, p)\psi(q, p) < \infty$$

is a bilinear real form. This Hilbert space is denoted by $\mathcal{H}(\Gamma)$. Unitary mappings, $U(\alpha)$, in $\mathcal{H}(\Gamma)$ are naturally introduced by using the star product, i.e.,

$$U(\alpha) = \exp(\alpha\widehat{A}),$$

where

$$\widehat{A} = A(q,p)\star = A(q,p)\exp\left[\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p} - \frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right)\right]$$
$$= A\left(q + \frac{i\hbar}{2}\partial_p, p - \frac{i\hbar}{2}\partial_p\right). \qquad (3)$$

Let us consider some examples. For the basic functions q and p (3-dimensional Euclidian vectors), we have

$$\widehat{q_i} = q_i\star = q_i + \frac{i\hbar}{2}\partial_{p_i}, \qquad (4)$$

$$\widehat{p_i} = p_i\star = p_i - \frac{i\hbar}{2}\partial_{q_i}, \qquad (5)$$

where $i = 1, 2, 3$. These operators satisfy the Heisenberg relations $\left[\widehat{q}_j, \widehat{p}_l\right] = i\hbar\delta_{jl}$. Then we introduce a Galilei boost by defining the boost generator $\widehat{k_i} = mq_i \star - tp_i\star = m\widehat{q_i} - t\widehat{p_i}$, retirar $i = 1, 2, 3$, such that

$$\exp\left(-i\mathbf{v}\cdot\widehat{\mathbf{k}}/\hbar\right)\widehat{q}_j \exp\left(i\mathbf{v}\cdot\widehat{\mathbf{k}}/\hbar\right) = \widehat{q}_j + v_j t,$$
$$\exp\left(-i\mathbf{v}\cdot\widehat{\mathbf{k}}/\hbar\right)\widehat{p}_j \exp\left(i\mathbf{v}\cdot\widehat{\mathbf{k}}/\hbar\right) = \widehat{p}_j + mv_j.$$

These results, with the commutation relations, show that $\widehat{q}$ and $\widehat{p}$ are physically the position and momentum operators, respectively.

We introduce the operators $\overline{Q}$ and $\overline{P}$, such that $[\overline{Q}, \overline{P}] = 0$, $\overline{Q}|q,p\rangle = q|q,p\rangle$ and $\overline{P}|q,p\rangle = p|q,p\rangle$, with

$$\langle q,p|q',p'\rangle = \delta(q-q')\delta(p-p'),$$

and $\int dqdp|q,p\rangle\langle q,p| = 1$. From a physical point of view, we observe the transformation rules:

$$\exp\left(-iv\frac{\widehat{k}}{\hbar}\right)2\overline{Q}\exp\left(iv\frac{\widehat{k}}{\hbar}\right) = 2\overline{Q} + vt\mathbf{1},$$

and

$$\exp\left(-iv\frac{\widehat{k}}{\hbar}\right)2\overline{P}\exp\left(iv\frac{\widehat{k}}{\hbar}\right) = 2\overline{P} + mv\mathbf{1}.$$

Then $\overline{Q}$ and $\overline{P}$ are transformed, under the Galilei boost, as position and momentum, respectively. Therefore, the manifold defined by the set of eigenvalues (q,p) has the content of a phase space. However, the operators $\overline{Q}$ and $\overline{P}$ are not observables, since they commute with each other.

Considering a homogeneous systems satisfying the Galilei symmetry, the commutations relation between $\widehat{k}$ and $\widehat{H}$ is $[\widehat{k}_j, \widehat{H}] = i\widehat{P}_j$. Explicitly, we have

$$\left[mq_j + i\hbar\frac{\partial}{\partial p_j}, H(q,p)\star\right] = ip_j + \frac{\hbar}{2}\frac{\partial}{\partial q_j}.$$

A solution, providing a general form to $\widehat{H} = H(q,p)\star$, is

$$\widehat{H} = \frac{p^2\star}{2m} + V(q)\star$$
$$= \frac{p^2}{2m} - \frac{\hbar^2}{8m}\frac{\partial^2}{\partial q^2} - \frac{i\hbar p}{2m}\frac{\partial}{\partial q} + V(q\star). \qquad (6)$$

This is the Hamiltonian of a one-body system in an external field.

Consider the time evolution of a state $\psi(q,p;t)$, that is given by $\psi(q,p;t) = U(t,t_0)\psi(q,p;t_0)$, where $U(t,t_0) = \exp(-i\hbar(t-t_0)\widehat{H})$. This result leads to a Schrödinger-like equation written in phase space, i.e., [61]

$$i\hbar\partial_t\psi(q,p,t) = \widehat{H}\psi(q,p,t) \qquad (7)$$

Now, physical meaning of the state $\psi(q,p,t)$ has to be identified. This is done by associating $\psi(q,p,t)$ with the Wigner function. From (7), one can prove that $g(q,p) = \psi(q,p,t)\star\psi^\dagger(q,p,t)$ satisfies (2) [61, 62, 65]. In addition, using the associative property of the Moyal product and the relation

$$\int dqdp\psi(q,p,t)\star\psi^\dagger(q,p,t) = \int dqdp\psi(q,p,t)\psi^\dagger(q,p,t),$$

we have

$$\langle A\rangle = \langle\psi|A|\psi\rangle$$
$$= \int dqdp\psi(q,p,t)\widehat{A}(q,p)\psi^\dagger(q,p,t)$$
$$= \int dqdpf_W(q,p,t)A(q,p,t),$$

where $\widehat{A}(q,p) = A(q,p)\star$ is an observable. Thus, the Wigner function can be calculated by using

$$f_W(q,p) = \psi(q,p)\star\psi^\dagger(q,p). \qquad (8)$$

It is to be noted also that the eigenvalue equation,

$$H(q,p)\star\psi = E\psi, \qquad (9)$$

results in $H(q,p)\star f_W = Ef_W$. Therefore, $\psi(q,p)$ and $f_W(q,p)$ satisfy the same differential equation. These results show that (7) is a fundamental starting point for the description of quantum physics in phase space, fully compatible with the Wigner formalism.

3 Coupled Oscillators in Phase Space

In this section, we consider two coupled harmonic oscillators defined by the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2(x_1^2 + x_2^2) - \frac{\xi}{4}(x_1 - x_2)^2. \qquad (10)$$

This system describes, for instance, a Helium-like system such that the electron-nuclei interaction is replaced by Hooke-like forces [22, 23, 71, 72]. This is a crude approximation, although providing good enough spectrum of energy. Then, we use the Schrödinger equation in phase space to obtain the Wigner function for such a system. Using the variables,

$$u = \frac{x_1 + x_2}{\sqrt{2}},$$
$$v = \frac{x_1 - x_2}{\sqrt{2}},$$
$$p_u = \frac{p_1 + p_2}{\sqrt{2}},$$
$$p_v = \frac{p_1 - p_2}{\sqrt{2}},$$

Equation (10) is written as

$$H = \frac{p_u^2}{2m} + m\omega^2 u^2 + \frac{p_v^2}{2m} + \frac{(1-\xi)}{2} m\omega^2 v^2. \tag{11}$$

It is convenient to write $H = H_u + H_v$, where

$$H_u = \frac{p_u^2}{2m} + m\omega^2 u^2,$$

and

$$H_v = \frac{p_v^2}{2m} + \frac{(1-\xi)}{2} m\omega^2 v^2.$$

The time-independent Schrödinger equation in phase space is written as

$$H \star \psi(u, v, p_u, p_v) = E\psi(u, v, p_u, p_v). \tag{12}$$

In order to solve this equation, we take

$$\psi(u, v, p_u, p_v) = \varphi(u, p_u)\chi(v, p_v),$$

and

$$E = E_u + E_v.$$

It is important to consider the relations,

$$u\star = u + \frac{i\hbar}{2}\frac{\partial}{\partial p_u},$$
$$p_u\star = p_u - \frac{i\hbar}{2}\frac{\partial}{\partial u},$$
$$v\star = v + \frac{i\hbar}{2}\frac{\partial}{\partial p_v},$$
$$p_v\star = p_v - \frac{i\hbar}{2}\frac{\partial}{\partial v},$$

which are obtained from the star product in (3). In this sense, the resulting equations are solved by starting from the equation for the coordinates u and p_u, i.e.,

$$\left(\frac{p_u^2\star}{2m} + m\omega^2 u^2\star\right)\varphi_n = E_u\varphi_n. \tag{13}$$

Writing

$$H_u\star = \frac{m\omega^2}{2}\left(u\star + \frac{i}{m\omega}p_u\star\right)\left(u\star - \frac{i}{m\omega}p_u\star\right) - \hbar\omega, \tag{14}$$

we then introduce the operators

$$a_u\star = \sqrt{\frac{m\omega}{2\hbar}}\left(u\star + \frac{i}{m\omega}p_u\star\right), \tag{15}$$

$$a_u^\dagger\star = \sqrt{\frac{m\omega}{2\hbar}}\left(u\star - \frac{i}{m\omega}p_u\star\right), \tag{16}$$

satisfying the relations,

$$[a_u\star, a_u^\dagger\star] = 1,$$
$$a_u\star\varphi_n \propto \varphi_{n-1},$$
$$a_u^\dagger\star\varphi_n \propto \varphi_{n+1},$$

where $n = 0, 1, 2, \ldots$, and $a_u\star\varphi_0 = 0$.

The Hamiltonian given in (14) is then written as

$$H_u\star = \hbar\omega\left(a_u^\dagger\star a_u\star + \frac{1}{2}\right). \tag{17}$$

In this way, we have

$$\sqrt{\frac{m\omega}{2\hbar}}\left(u\star + \frac{i}{m\omega}p_u\star\right)\varphi_0 = 0. \tag{18}$$

Substituting $u\star = u + \frac{i\hbar}{2}\frac{\partial}{\partial p_u}$ and $p_u\star = p_u - \frac{i\hbar}{2}\frac{\partial}{\partial_u}$ in (18) we obtain

$$\sqrt{\frac{m\omega}{2\hbar}}\left[u + \frac{i\hbar}{2}\frac{\partial}{\partial_{p_u}} + \frac{i}{m\omega}\left(p_u - \frac{i\hbar}{2}\frac{\partial}{\partial_u}\right)\right]\varphi_0 = 0. \tag{19}$$

Separating the real and imaginary part of (19), and considering $\varphi_0(u, p_u) = \varphi_0^a(u)\varphi_0^b(p_u)$, we can show that real part satisfies the differential equation

$$u\varphi_0 + \frac{\hbar}{2m\omega}\frac{\partial\varphi_0^a}{\partial u} = 0, \tag{20}$$

with a solution given by

$$\varphi_0^a(u) = \exp\left(-\frac{m\omega}{\hbar}u^2\right). \tag{21}$$

For the imaginary part, we have

$$\frac{\hbar}{2}\frac{\partial\varphi_0}{\partial p_u} + \frac{p_u}{m\omega}\varphi_0^b = 0, \tag{22}$$

with the solution

$$\varphi_0^b(p_u) = \exp\left(-\frac{1}{\hbar m\omega}p_u^2\right). \tag{23}$$

Then, we get

$$\varphi_0(u, p_u) \sim \exp\left(-\frac{m\omega}{\hbar}u^2 - \frac{1}{\hbar m\omega}p_u^2\right). \tag{24}$$

Similarly, the solution of the equation for χ is obtained, i.e.,

$$\left(\frac{p_v^2\star}{2m} + \frac{(1-\xi)}{2}m\omega^2 v^2\star\right)\chi_n = E_v\chi_n. \tag{25}$$

This leads to

$$H_v\star = \frac{(1-\xi m\omega^2)}{2}\left(v\star+\frac{i}{m\omega(1-\xi)^{1/2}}p_v\star\right)\times\left(v\star-\frac{i}{m\omega(1-\xi)^{1/2}}p_v\star\right)-\frac{\hbar\omega}{2}(1-\xi)^{1/2}. \tag{26}$$

Then, we define

$$a_v\star=\sqrt{\frac{m\omega(1-\xi)}{2\hbar}}\left(v\star+\frac{i}{m\omega(1-\xi)^{1/2}}p_v\star\right), \tag{27}$$

and

$$a_v^\dagger\star=\sqrt{\frac{m\omega(1-\xi)}{2\hbar}}\left(v\star-\frac{i}{m\omega(1-\xi)^{1/2}}p_v\star\right). \tag{28}$$

These operators satisfy the relations,

$$[a_v\star, a_v^\dagger\star] = 1,$$
$$a_v\star\chi_n \propto \chi_{n-1},$$
$$a_v^\dagger\star\chi_n \propto \chi_{n+1},$$

where $n = 0, 1, 2, \ldots$.

The operator given in (26) has the form

$$H_v\star=\hbar\omega\left(a_v^\dagger\star a_v\star-\frac{(1-\xi)^{1/2}}{2}\right). \tag{29}$$

We can show that $a_v\star\chi_0 = 0$, such that

$$\sqrt{\frac{m\omega(1-\xi)}{2\hbar}}\left(v\star+\frac{i}{m\omega(1-\xi)^{1/2}}p_v\star\right)\chi_0=0. \tag{30}$$

Substituting $v\star = v + \frac{i\hbar}{2}\frac{\partial}{\partial p_v}$ and $p_v\star = p_v - \frac{i\hbar}{2}\frac{\partial}{\partial v}$ in (30), we obtain

$$\sqrt{\frac{m\omega(1-\xi)}{2\hbar}}\left[v+\frac{i\hbar}{2}\frac{\partial}{\partial_{p_v}}+\frac{i}{m\omega(1-\xi)^{1/2}}\left(p_v-\frac{i\hbar}{2}\frac{\partial}{\partial_v}\right)\right]\chi_0=0. \tag{31}$$

Separating the real and imaginary part of (31), and considering $\chi_0(u, p_u) = \chi_0^a(v)\varphi_0^b(p_v)$, we can show that real part satisfies the differential equation

$$v\chi_0^a+\frac{\hbar}{2m\omega(1-\xi)^{1/2}}\frac{\partial\chi_0^a}{\partial v}=0, \tag{32}$$

with a solution given by

$$\chi_0^a(v)=\exp\left(-\frac{m\omega(1-\xi)^{1/2}}{\hbar}v^2\right). \tag{33}$$

For the imaginary part, we have

$$\frac{\hbar}{2}\frac{\partial\chi_0}{\partial p_v}+\frac{p_v}{m\omega(1-\xi)^{1/2}}\chi_0^b=0, \tag{34}$$

with the solution

$$\chi_0^b(p_v)=\exp\left(-\frac{1}{\hbar m\omega(1-\xi)^{1/2}}p_v^2\right). \tag{35}$$

Then, we get

$$\chi_0(v, p_v)\sim\exp\left(-\frac{m\omega(1-\xi)^{1/2}}{\hbar}u^2-\frac{1}{\hbar m\omega(1-\xi)^{1/2}}p_u^2\right). \tag{36}$$

Therefore, the zero order solution of the Schrödinger equation is

$$\psi_0(u, v, p_u, p_v)=\frac{2}{\pi\hbar}\exp\left(-\frac{m\omega}{\hbar}[u^2+(1-\xi)^{1/2}v^2]\right)\times\exp\left(-\frac{1}{m\omega\hbar}[p_u^2+(1-\xi)^{-1/2}p_v^2]\right),$$

where we have used the normalization condition

$$\int dudvdp_udp_v\psi_n^\dagger(u, v, p_u, p_v)\star\psi_n(u, v, p_u, p_v)=1.$$

To obtain higher order wave functions, we use the relation

$$\psi_n(u, v, p_u, p_v)=(a_u^\dagger\star a_v^\dagger\star)^n\psi_0(u, v, p_u, p_v). \tag{37}$$

The Wigner function is found from

$$f_W^{(n)}(u, v, p_u, p_v)=\psi_n(u, v, p_u, p_v)\star\psi_n^\dagger(u, v, p_u, p_v).$$

In particular for $n = 0$, we obtain

$$f_W^{(0)}(q_1, q_2, p_1, p_2) = \left(\frac{2}{\pi\hbar}\right)\exp\left(-\frac{m\omega}{\hbar}\frac{(x_1+x_2)^2}{2}\right)\times\exp\left(-\frac{m\omega}{\hbar}(1-\xi)^{1/2}\frac{(x_1-x_2)^2}{2}\right)\times\exp\left(-\frac{1}{m\omega\hbar}\frac{(p_1+p_2)^2}{2}\right)\times\exp\left(-\frac{1}{m\omega\hbar}(1-\xi)^{-1/2}\frac{(p_1-p_2)^2}{2}\right).$$

Hence, the energy of the fundamental state is given by $E_0 = \hbar\omega(1-\frac{\xi}{4})$.

These results are interesting in a double sense. First, we have calculated analytically the Wigner function for Helium-like atom. Second, the Wigner function has many applications, among them one stands out: quantum computing. So, such a procedure to study the Wigner function for Helium-like atom opens up new possibilities for analyzing entanglement. In this context, in experiments, the dissipation due to the effect of external fields, is a crucial factor. In the next section, in order to consider the Helium atom in a non-conservative external field, we add to the Hooke-like force, a linear dissipation.

4 Damped Quantum Oscillator in Phase Space

In order to analyze dissipation effect of neighborhood, here, we solve the Schrödinger equation in phase space for

Hooke-like system with a damped interaction. This stands for the Helium atom in a dissipative field. We consider a one-dimensional system, where the Hamiltonian with a dissipative term is (see a different treatment for such a model in Refs [10, 11])

$$\widehat{H} = \frac{1}{2}\left(\widehat{P}^2 + \widehat{Q}^2\right) - \frac{\lambda}{2}\left(\widehat{Q}\widehat{P} + \widehat{P}\widehat{Q}\right), \tag{38}$$

where $\lambda < 1$.

Using the operators given in (4) and (5) with $\hbar = 1$,

$$\widehat{Q} = q + \frac{i}{2}\partial_p, \tag{39}$$

and

$$\widehat{P} = p - \frac{i}{2}\partial_q, \tag{40}$$

Equation (38) becomes

$$\begin{aligned}\widehat{H} &= \frac{1}{2}\left(p^2 + q^2 - ip\partial_q + iq\partial_p - \frac{1}{4}\partial_q^2 - \frac{1}{4}\partial_p^2\right) \\ &-\frac{\lambda}{2}\left(2qp - iq\partial_q + ip\partial_p + \frac{1}{2}\partial_q\partial_p\right).\end{aligned}$$

Applying this Hamiltonian in the eigenvalue equation $\widehat{H}\psi(q,p) = E\psi(q,p)$, we obtain

$$\begin{aligned}&(p^2+q^2)\psi(q,p) - \frac{1}{4}\partial_q^2\psi(q,p) - \frac{1}{4}\partial_p^2\psi(q,p) \\ &-\frac{\lambda}{2}\partial_q\partial_p\psi(q,p) - 2\lambda qp\psi(q,p) - 2E\psi(q,p) = 0.\end{aligned}$$

Introducing the new variable

$$z = \frac{1}{2}(p^2+q^2) - \lambda qp, \tag{41}$$

we obtain

$$\frac{1}{2}(\lambda^2-1)z\partial_z^2\psi(z) + \frac{1}{2}(\lambda^2-1)\partial_z\psi(z) + 2(z-E)\psi(z) = 0. \tag{42}$$

Taking $a = \frac{1}{2}(1-\lambda^2)$ and using the ansatz

$$\psi(z) = e^{-\frac{z}{\sqrt{a/2}}}\omega(z), \tag{43}$$

we have, after the changing of variables $y = 2\sqrt{\frac{2}{a}}z$, the following expression

$$y\partial_y^2\omega(y) + (1-y)\partial_y\omega(y) - \left[\frac{1}{2} - \frac{E/2}{\sqrt{a/2}}\right]\omega(y) = 0. \tag{44}$$

The solution of (44) is given by the Kummer function (a confluent hypergeometric function, i.e.),

$$\omega(z) = F\left(\frac{1}{2} - \frac{E/2}{\sqrt{a/2}}; 1; 2\sqrt{\frac{2}{a}z}\right). \tag{45}$$

In this way, we have the solution

$$\psi(z) = e^{z\frac{1}{\sqrt{a/2}}}F\left(\frac{1}{2} - \frac{E/2}{\sqrt{a/2}}; 1; 2\sqrt{\frac{2}{a}z}\right), \tag{46}$$

where z is given in (41). The confluent hypergeometric function condition is such that

$$\frac{1}{2} - \frac{E/2}{\sqrt{a/2}} = -n,$$

where $n \in \mathbb{Z}$. This relation gives

$$E_n = (1-\lambda^2)^{1/2}\left[n + \frac{1}{2}\right]. \tag{47}$$

Note that if $\lambda = 0$, we obtain the result $E_n = (n + 1/2)$.

The Wigner function can be calculated by

$$f_W(q,p,t) = \psi \star \psi^*. \tag{48}$$

In this case, we calculate the Wigner functions given in (48) using a MAPLE routine. The behavior of the stationary Wigner function for $\lambda = 0.1$ is shown in Figs. 1, 2, 3, and 4 and for $\lambda = 0.9$ are shown in Fig. 5, 6, 7, and 8.

A measure of non-classicality of quantum states is defined on the volume of the negative part of Wigner function, which may be interpreted as a signature of quantum interference. In this sense, the non-classicality (negativity) indicator is given by [70]

$$\begin{aligned}\eta(\psi) &= \int\int [|f_W(q,p)| - f_W(q,p)]dqdq \\ &= \int\int |f_W(q,p)|dqdq - 1.\end{aligned} \tag{49}$$

This indicator represents the doubled volume of the integrated part of the Wigner function. In sequence, we calculated numerically this indicator for damped oscillator. The results of this calculation are shown in Tables 1 and 2 below. We note that parameter $\eta(\psi)$ depends of the damped constant λ.

In Fig. 9, the dependence of non-classicality indicator $\eta(\psi)$ as a function of the order n of the Wigner function for damped oscillator for case $\lambda = 0.9$ is plotted.

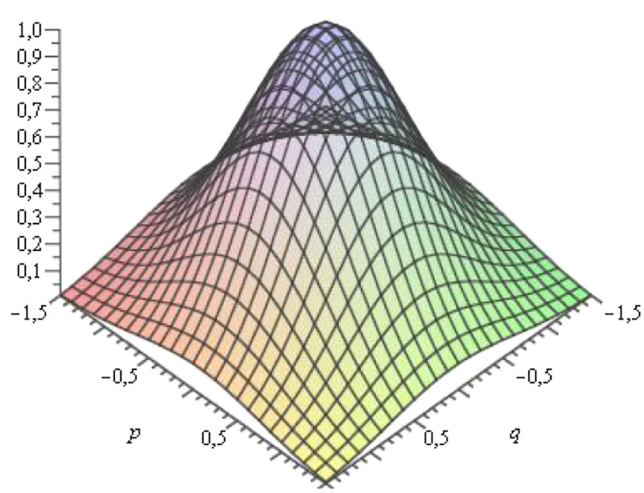

Fig. 1 Wigner function, $n = 0$, $\lambda = 0.1$

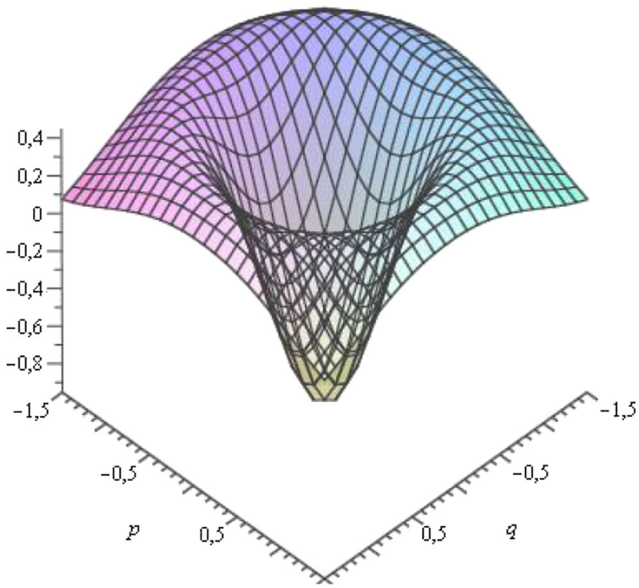

Fig. 2 Wigner function, $n = 1$, $\lambda = 0.1$

5 Non-linear Pendulum in Phase Space

The Hamiltonian for a non-linear pendulum is given by

$$\widehat{H} = \frac{\widehat{L}^2}{8ml^2} + mgl(1 - \cos(2\widehat{\theta}), \tag{50}$$

leading to the steady Schroedinger equation

$$\widehat{H}\psi(L, \theta) = E\psi(L\theta). \tag{51}$$

Using the operators

$$\widehat{\theta} = \theta + \frac{i\hbar}{2}\frac{\partial}{\partial L}, \tag{52}$$

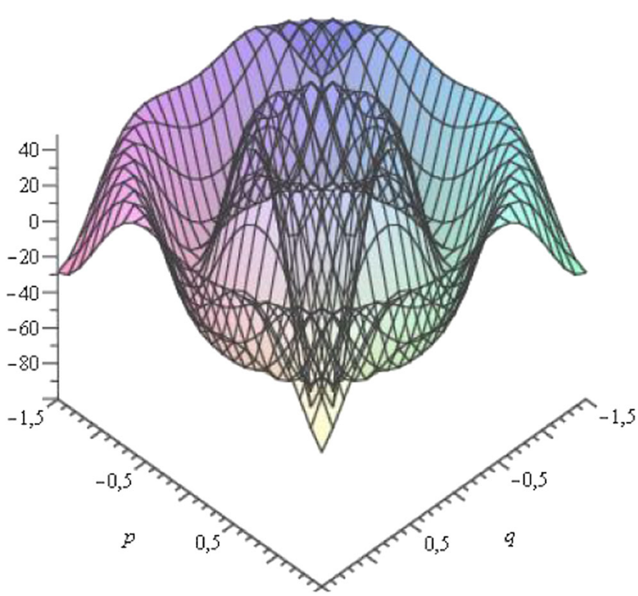

Fig. 3 Wigner function, $n = 5$, $\lambda = 0.1$

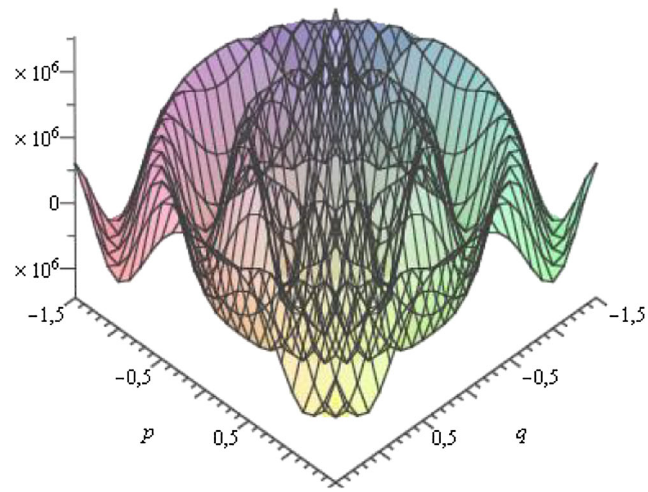

Fig. 4 Wigner function, $n = 10$, $\lambda = 0.1$

and

$$\widehat{L} = L - \frac{i\hbar}{2}\frac{\partial}{\partial\theta}, \tag{53}$$

that satisfy

$$[\widehat{\theta}, \widehat{L}] = i\hbar,$$

Equation (51) is written as

$$\frac{1}{8ml^2}\left(L^2 - i\hbar\frac{\partial}{\partial\theta} - \frac{\hbar^2}{4}\frac{\partial^2}{\partial\theta^2}\right)\psi \\ +mgl\left(1 - \cos\left(2\theta + i\hbar\frac{\partial}{\partial L}\right)\right)\psi = E\psi. \tag{54}$$

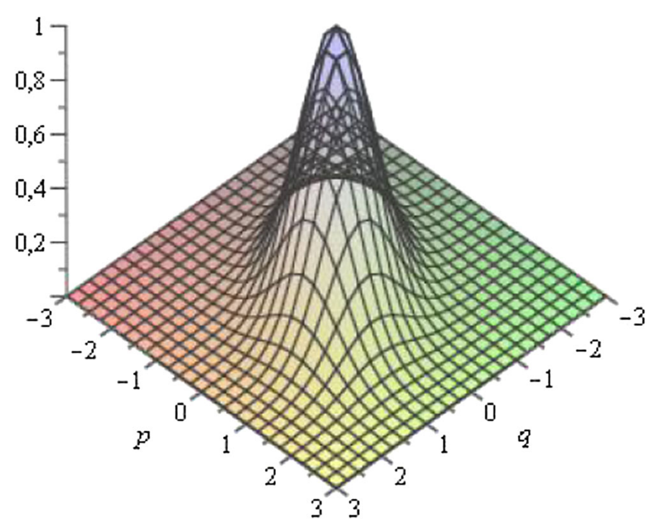

Fig. 5 Wigner function, $n = 0$, $\lambda = 0.9$

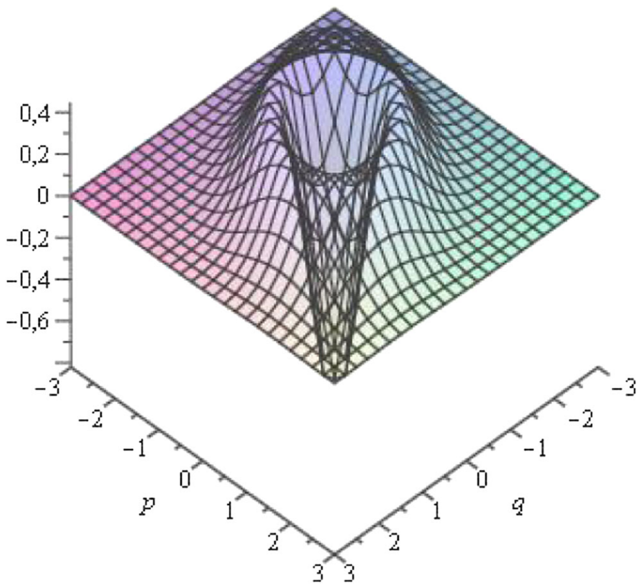

Fig. 6 Wigner function, $n = 1$, $\lambda = 0.9$

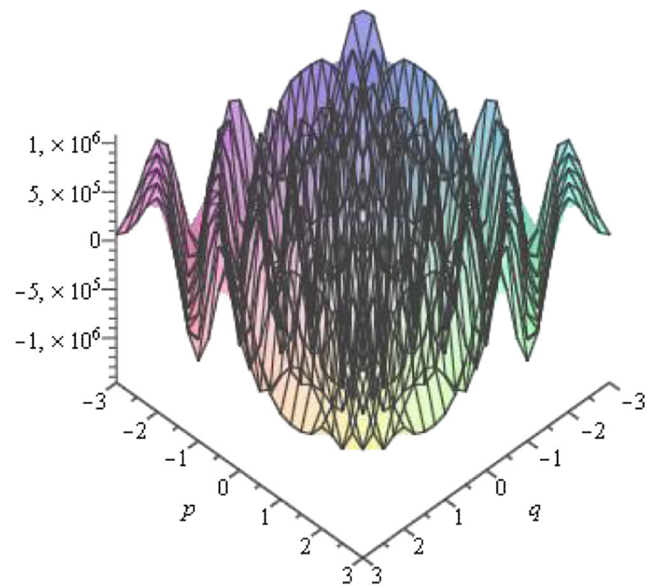

Fig. 8 Wigner function, $n = 10$, $\lambda = 0.9$

Using the relation $\cos(a+b) = \cos a \cos b - \sin a \sin b$, we obtain

$$\left(L^2 - i\hbar\frac{\partial}{\partial\theta} - \frac{\hbar^2}{4}\frac{\partial^2}{\partial\theta^2}\right)\psi + 8m^2gl^3\left(1 - \cos 2\theta \cos\left(i\hbar\frac{\partial}{\partial\theta}\right) + \sin 2\theta \sin\left(i\hbar\frac{\partial}{\partial\theta}\right)\right)\psi = 8m^2gl^3\psi. \tag{55}$$

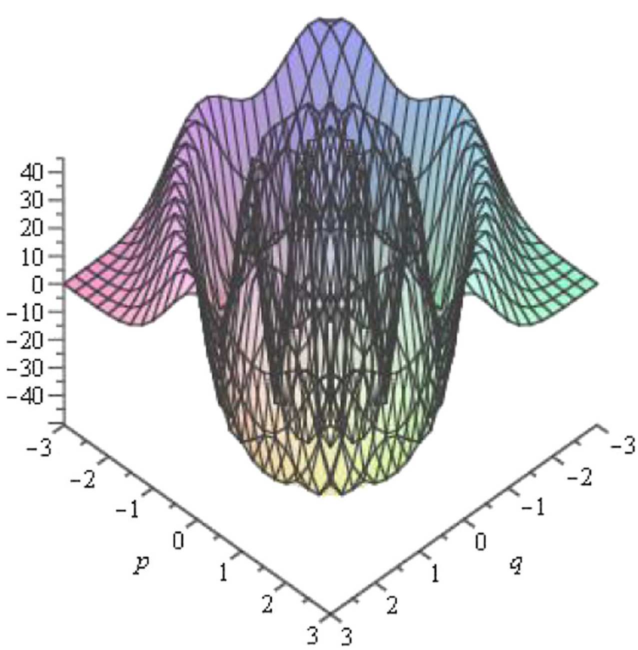

Fig. 7 Wigner function, $n = 5$, $\lambda = 0.9$

Using the expansion of $\cos\left(i\hbar\frac{\partial}{\partial\theta}\right)$ and $\sin\left(i\hbar\frac{\partial}{\partial\theta}\right)$ up to the second order in $\hbar$, (55) reads

$$-4\hbar^2m^2gl^3\cos 2\theta\frac{\partial^2\psi}{\partial L^2} - \frac{\hbar^2}{4}\frac{\partial^2\psi}{\partial\theta^2} + 8im^2g\hbar l^3\sin 2\theta\frac{\partial\psi}{\partial L} - i\hbar L\frac{\partial\psi}{\partial\theta} + (L^2 + 8m^2gl^3 - 8m^2gl^3\cos 2\theta - 8ml^2E)\psi = 0. \tag{56}$$

Taking $\lambda = \frac{L}{\hbar}$, $\frac{\hbar^2}{m^2gl^3} = a^2$ and $\frac{E}{mgl} = \epsilon$, (56) assumes the form

$$-4\cos 2\theta\frac{\partial^2\psi}{\partial\lambda^2} - \frac{a^2}{4}\frac{\partial^2\psi}{\partial\theta^2} + 8i\sin 2\theta\frac{\partial\psi}{\partial\lambda} - i\lambda a^2\frac{\partial\psi}{\partial\theta} + (\lambda^2a^2 + 8 - 8\cos 2\theta - 8\epsilon)\psi = 0. \tag{57}$$

Table 1 The non-classicality indicator as a function of the order of the Wigner function, the parameter n, for $\lambda = 0.9$

n	$\eta(\psi)$, $\lambda = 0.9$
0	0
1	0.4261226344263795
2	0.7289892587057898
3	0.9766730799293403
4	1.1913424288065964
5	1.3834384856692004
6	1.5588521972493026
7	1.7212933835545317
8	1.873265816082318
9	2.016572434609475

Table 2 The non-classicality indicator as a function of the order of the Wigner function, the parameter n, for $\lambda = 0.1$

n	$\eta(\psi)$, $\lambda = 0.1$
0	0
1	0.5367235498765983
2	0.8298745655538895
3	0.9957478374530056
4	1.21045749693345208
5	1.45987546307934771
6	1.62345098621689579
7	1.77980374568037570
8	1.95688276504037623
9	2.13987564392078583

We analyze (57) to particular values of λ. To λ constant (57) becomes

$$-\frac{a^2}{4}\frac{\partial^2\psi}{\partial\theta^2} - i\lambda a^2\frac{\partial\psi}{\partial\theta} + (\lambda^2 a^2 + 8 - 8\cos 2\theta - 8\epsilon)\psi = 0. \quad (58)$$

This equation has a solution in the following general form,

$$\psi(\theta) = C_1\cos(2\theta)\mathcal{C}(a, b, \theta) + C_2\cos(2\theta)\mathcal{S}(a, b, x), \quad (59)$$

where C_1, C_2, a, b are constant parameters, and $\mathcal{C}(a, b, \theta)$, and $\mathcal{S}(a, b, \theta)$ are Mathieu functions. The behavior of solutions given in (59) for constant values of the angular momentum $\lambda = 1$ and different energies ϵ are presented in Figures below.

From Figs. 10, 11, and 12, the behavior of non-linear pendulum in phase space for constant angular momentum is periodic and positive. We notice also that with the increase of energy, the oscillation frequency becomes smaller.

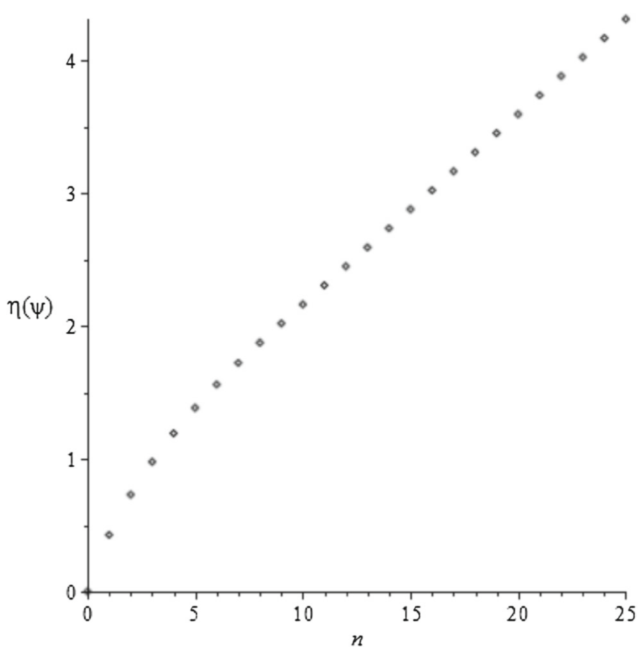

Fig. 9 The non-classicality indicator versus quantum number for damped oscillator $n \leq 25$

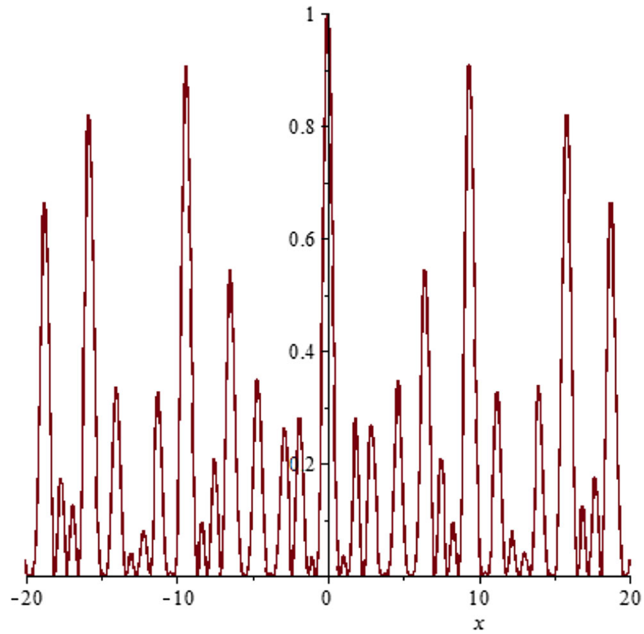

Fig. 10 $\psi(\theta)$ for $\epsilon = 1$

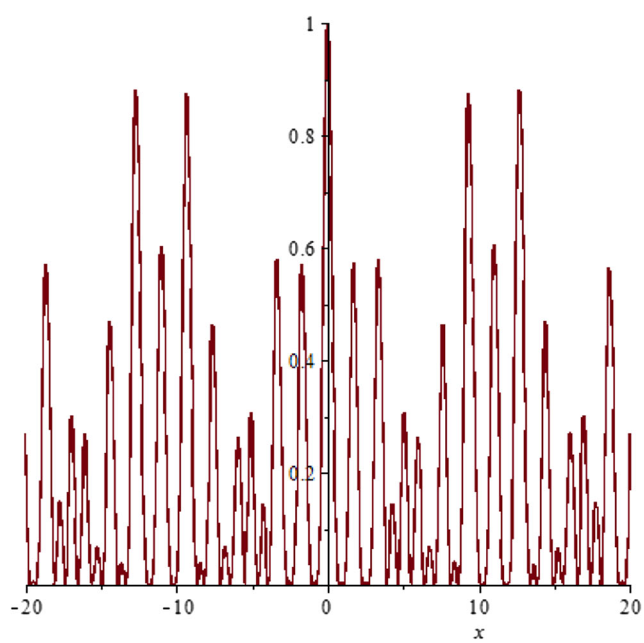

Fig. 11 $\psi(\theta)$ for $\epsilon = 5$

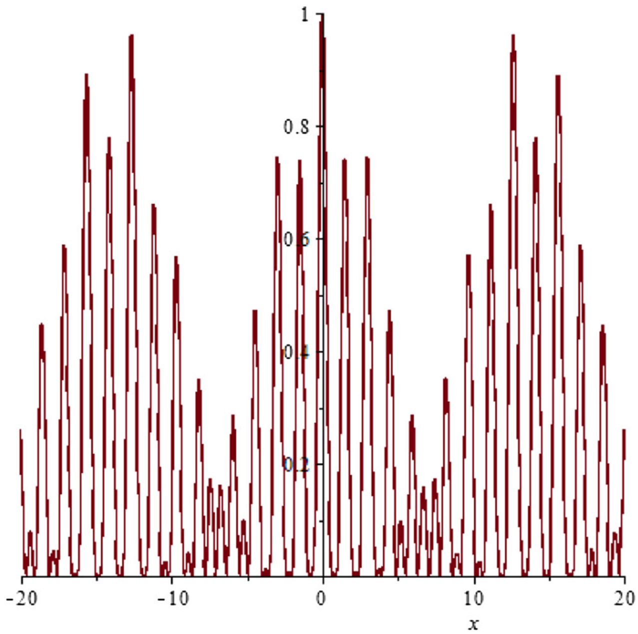

Fig. 12 $\psi(\theta)$ for $\epsilon = 10$

6 Concluding Remarks

In this work, the Wigner function for the Helium-like atom is calculated in the approximation of two-harmonic oscillators, considering also dissipation. Regarding the value of energy, this approximation has provided satisfactory results with the experiments [22, 23, 71]. Here, we have considered the statistical nature of such quantum states, by analyzing the non-classicality through the Wigner function. We have proceeded by formulating the problem with the Schrödinger equation in phase space, such that the state, called a quasi-amplitude of probability, is associated with the Wigner function by the Moyal product. In this context, we study a damped as well as a non-linear oscillator in phase space. Using Wigner functions, a non-classicality indicator is calculated as a function of the dissipation parameter. In addition the quasi-amplitude for the non-linear oscillator is positive defined [73, 74].

Data Availability No experimental data were used in this article.

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