Braz J Phys (2015) 45:369–375
DOI 10.1007/s13538-015-0309-z

PARTICLES AND FIELDS

Coherent States of Systems with Quadratic Hamiltonians

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Received: 8 February 2015 / Published online: 13 March 2015

Abstract Different families of generalized coherent states (CS) for one-dimensional systems with general time-dependent quadratic Hamiltonian are constructed. In principle, all known CS of systems with quadratic Hamiltonian are members of these families. Some of the constructed generalized CS are close enough to the well-known due to Schrödinger and Glauber CS of a harmonic oscillator; we call them simply CS. However, even among these CS, there exist different families of complete sets of CS. These families differ by values of standard deviations at the initial time instant. According to the values of these initial standard deviations, one can identify some of the families with semiclassical CS. We discuss properties of the constructed CS, in particular, completeness relations, minimization of uncertainty relations and so on. As a unknown application of the general construction, we consider different CS of an oscillator with a time dependent frequency.

Keywords Coherent states · Quadratic systems

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1 Introduction

1.1 General

Coherent states (CS) play an important role in modern quantum theory as states that provide a natural relation between quantum mechanical and classical descriptions. They have a number of useful properties and, as a consequence, a wide range of applications, e.g., in semiclassical description of quantum systems, in quantization theory, in condensed matter physics, in radiation theory, in quantum computations, in loop quantum gravity, and so on, see, e.g., refs. [1–9]. Despite the fact that there exist a great number of publications devoted to constructing CS of different systems, a universal definition of CS and a constructive scheme of their constructing for arbitrary physical system is not known. However, it seems that for systems with quadratic Hamiltonians, there exist, at present, a common point of view on this problem.[1] Starting the works [7, 8, 14, 15], CS are defined as eigenvectors of some annihilation operators that are at the same time integrals of motion, see also [16–21, 27]. Of course, such defined CS have to satisfy the corresponding Schrödinger equation. In the frame of such a definition, one can, in principle, construct CS for a general quadratic system. This construction is based on solutions of some classical equations, their analysis represent a nontrivial part of the CS construction.

In this article, we, following, the integral of motion method, construct different families of generalized CS for one-dimensional systems with general time-dependent quadratic Hamiltonian. Analyzing these families, we see that some of them are more close to the well known

[1] In this article, we do not discuss the so-called generalized CS [10–13].

due to Schrödinger and Glauber CS (see [24, 25]) of a harmonic oscillator, we call them simply CS. However, among the latter CS, there exist still different families of complete sets of CS. These families differ by values of standard deviations at the initial time instant. According to the values of these initial standard deviations, one can identify some of the families with semiclassical CS, as was demonstrated by us in the free-particle case [25]. We discuss properties of the constructed CS, in particular, completeness relations, minimization of uncertainty relations, and so on. As an application of the general construction, we consider CS of an oscillator with a time dependent frequency.

1.2 Basic Equations

Consider quantum motion of a one-dimensional system with the generalized coordinate x on the whole real axis, $x \in \mathbb{R} = (-\infty, \infty)$, supposing that the corresponding quantum Hamiltonian $\hat{H}_x$ is given by a quadratic form of the operator x and the momentum operator $\hat{p}_x = -i\hbar\partial_x$,

$$\hat{H}_x = r_1 \hat{p}_x^2 + r_2 x^2 + r_3 (x\hat{p}_x + \hat{p}_x x) + r_4 x + r_5 \hat{p}_x + r_6, \quad (1)$$

where $r_s = r_s(t)$, $s = 1, ..., 6$ are some given functions of the time t. We suppose that these functions are real and both $\hat{H}_x$ and $\hat{p}_x$ are self-adjoint on their natural domains D_{H_x} and D_{p_x} respectively, see, e.g., [28, 29].

Quantum states of the system under consideration are described by a wave function $\Psi(x,t)$ which satisfies the Schrödinger equation

$$i\hbar\partial_t \Psi(x,t) = \hat{H}_x \Psi(x,t). \quad (2)$$

In what follows, we restrict ourselves by a physically reasonable case $r_1(t) > 0$. In this case, we introduce dimensionless variables, a coordinate q and a time τ as follows

$$q = x l^{-1}, \tau = \int_0^t \frac{ds}{T(s)} = \frac{2\hbar}{l^2}\int_0^t r_1(s)\, ds,$$
$$T(t) = \frac{l^2}{2\hbar r_1(t)}, \quad (3)$$

where l is an arbitrary constant of the dimension of the length. The new momentum operator $\hat{p}$ and the new wave function $\psi(q,\tau)$ read

$$\hat{p} = \frac{l}{\hbar}\hat{p}_x = -i\partial_q, \quad \psi(q,\tau) = \sqrt{l}\,\Psi\left(lq, \frac{ml^2}{\hbar}\tau\right), \quad (4)$$

so that $|\Psi(x,t)|^2 dx = |\psi(q,\tau)|^2 dq$.

In the new variables, (2) takes the form

$$\hat{S}\psi(q,\tau) = 0, \ \hat{S} = i\partial_\tau - \hat{H}, \quad (5)$$

where the new Hamiltonian reads

$$\hat{H} = \frac{\hat{p}^2}{2} + \alpha\hat{q}^2 + \beta\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right) + \varrho\hat{q} + \nu\hat{p} + \varepsilon. \quad (6)$$

Here, $\alpha = \alpha(\tau)$, $\beta = \beta(\tau)$, $\varrho = \varrho(\tau)$, $\nu = \nu(\tau)$ and $\varepsilon = \varepsilon(\tau)$,

$$\alpha(\tau) = \frac{l^4}{2\hbar^2}\frac{r_2(t)}{r_1(t)}, \ \beta(\tau) = \frac{l^2}{2\hbar}\frac{r_3(t)}{r_1(t)}, \ \varrho(\tau) = \frac{l^3}{2\hbar^2}\frac{r_4(t)}{r_1(t)},$$
$$\nu(\tau) = \frac{l}{2\hbar}\frac{r_5(t)}{r_1(t)}, \ \varepsilon(\tau) = \frac{l^2}{2\hbar^2}\frac{r_6(t)}{r_1(t)}, \quad (7)$$

are dimensionless real functions on τ if t is expressed via τ by the help of (3). In what follows, we call $\hat{S}$ the equation operator.

2 Constructing Time-Dependent Generalized CS

2.1 Integrals of Motion Linear in Canonical Operators $\hat{q}$ and $\hat{p}$

First, we construct an integral of motion $\hat{A}(\tau)$ linear in $\hat{q}$ and $\hat{p}$. The general form of such an integral of motion reads

$$\hat{A}(\tau) = f(\tau)\hat{q} + ig(\tau)\hat{p} + \varphi(\tau), \quad (8)$$

where $f(\tau)$, $g(\tau)$, and $\varphi(\tau)$ are some complex functions on τ. The operator $\hat{A}(\tau)$ is an integral of motion if it commutes with equation operator (5),

$$\left[\hat{S}, \hat{A}(\tau)\right] = 0. \quad (9)$$

In the case if the Hamiltonian is self-adjoint, the adjoint operator $\hat{A}^\dagger(\tau)$ is also an integral of motion, i.e.,

$$\left[\hat{S}, \hat{A}^\dagger(\tau)\right] = 0. \quad (10)$$

The commutator $\left[\hat{A}(\tau), \hat{A}^\dagger(\tau)\right]$ reads

$$\left[\hat{A}(\tau), \hat{A}^\dagger(\tau)\right] = \delta = 2\,\mathrm{Re}\left[g^*(\tau) f(\tau)\right]. \quad (11)$$

Substituting representation (8) into (9), we obtain the following equations for the functions $f(\tau)$, $g(\tau)$, and $\varphi(\tau)$:

$$\dot{f}(\tau) + 2\beta(\tau) f(\tau) - 2i\alpha(\tau) g(\tau) = 0,$$
$$\dot{g}(\tau) - if(\tau) - 2\beta(\tau) g(\tau) = 0,$$
$$\dot{\varphi}(\tau) + \nu(\tau) f(\tau) - i\varrho(\tau) g(\tau) = 0. \quad (12)$$

It is enough to find the functions $f(\tau)$ and $g(\tau)$, then the function $\varphi(\tau)$ can be found by a simple integration. In addition, without loss of the generality, we can set $\varphi(0) = 0$.

Equations (12) imply that δ is a real integral of motion, $\delta = \text{const}$. In what follows, we suppose that $\delta = 1$, which means

$$\text{Re}\left[g^*(\tau) f(\tau)\right] = \text{Re}\left[g^*(0) f(0)\right] = 1/2. \quad (13)$$

Any nontrivial solution of the two first equations (12) consists of two nonzero functions $f(\tau)$ and $g(\tau)$. That is why we can chose arbitrary integration constants in these equations as

$$f(0) = c_1 = |c_1| e^{i\mu_1},$$
$$g(0) = c_2 = |c_2| e^{i\mu_2}, \quad |c_2| \neq 0, \quad |c_1| \neq 0. \quad (14)$$

In terms of the introduced constants, condition (13) yields

$$|c_2| |c_1| \cos(\mu_1 - \mu_2) = 1/2. \quad (15)$$

Under the choice $\delta = 1$, operators $\hat{A}(\tau)$ and $\hat{A}^\dagger(\tau)$ become annihilation and creation operators,

$$\left[\hat{A}(\tau), \hat{A}^\dagger(\tau)\right] = 1. \quad (16)$$

It follows from (8) and (13) that

$$\hat{q} = g^*(\tau)\left[\hat{A}(\tau) - \varphi(\tau)\right] + g(\tau)\left[\hat{A}^\dagger(\tau) - \varphi^*(\tau)\right],$$
$$i\hat{p} = f^*(\tau)\left[\hat{A}(\tau) - \varphi(\tau)\right] - f(\tau)\left[\hat{A}^\dagger(\tau) - \varphi^*(\tau)\right]. \quad (17)$$

I We note that the two first (12) can be reduced to a one second-order differential equation for the function $g(\tau)$, such an equation has the form of the oscillator equation with a time-dependent frequency $\omega^2(\tau)$,

$$\ddot{g}(\tau) + \omega^2(\tau) g(\tau) = 0, \ \omega^2(\tau) = 2\alpha - 4\beta^2 - 2\dot{\beta}. \quad (18)$$

If we have an exact solution $g(\tau)$ for a given function $\omega^2(\tau)$, then the function $f(\tau)$ can be found via the function $g(\tau)$ as

$$f(\tau) = 2i\beta(\tau) g(\tau) - i\dot{g}(\tau). \quad (19)$$

One can chose the functions $\alpha(\tau)$ and $\beta(\tau)$ such that

$$\omega^2(\tau) = 2\alpha(\tau) - 4\beta^2(\tau) - 2\dot{\beta}(\tau). \quad (20)$$

For example, if we chose

$$\alpha(\tau) = \frac{1}{2}\omega^2(\tau), \ \beta = \varrho = \nu = \varepsilon = 0, \quad (21)$$

then we are dealing with Hamiltonian of the form

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\omega^2(\tau)}{2} q^2. \quad (22)$$

II. In addition, the one-dimensional Schrödinger equation

$$-d_q^2 \Psi(q) + V(q)\Psi(q) = E\Psi(q), \quad (23)$$

can be identified with (18) if $q \to \tau$, $\Psi(q) \to g(\tau)$, $V(q) - E \to \omega^2(\tau)$.

III. It should be also noted that the two first equations (12) can be identified with a particular form of the so-called spin equation, see [28],

$$i\dot{V} = (\boldsymbol{\sigma}\mathbf{F}) V, \quad V = \begin{pmatrix} f \\ g \end{pmatrix}, \quad (24)$$

with

$$\mathbf{F}(\tau) = -\frac{1}{2}(2\alpha + 1, i(2\alpha - 1), 4i\beta).$$

2.2 Time-Dependent Generalized CS

Let us consider eigenvectors $|z, \tau\rangle$ of the annihilation operator $\hat{A}(\tau)$ corresponding to the eigenvalue z,

$$\hat{A}(\tau)|z, \tau\rangle = z|z, \tau\rangle. \quad (25)$$

In the general case, z is a complex number.

It follows from (17) and (25) that

$$q(\tau) \equiv \langle z, \tau|\hat{q}|z, \tau\rangle = g^*(\tau)[z - \varphi(\tau)] + g(\tau)\left[z^* - \varphi^*(\tau)\right],$$
$$ip(\tau) \equiv \langle z, \tau|\hat{p}|z, \tau\rangle = f^*(\tau)[z - \varphi(\tau)] - f(\tau)\left[z^* - \varphi^*(\tau)\right],$$
$$z = f(\tau) q(\tau) + ig(\tau) p(\tau) + \varphi(\tau). \quad (26)$$

Using (12), one can easily verify that the functions $q(\tau)$ and $p(\tau)$ satisfy the Hamilton equations

$$\dot{q}(\tau) = \frac{\partial H}{\partial p}, \quad \dot{p}(\tau) = -\frac{\partial H}{\partial q},$$

where $H = H(q, p)$ is the classical Hamiltonian that corresponds to the quantum Hamiltonian (6). Thus, the pair $q(\tau)$ and $p(\tau)$ represents a classical trajectory in the phase space of the system under consideration. All such trajectories can be parameterized by the initial data, $q_0 = q(0)$ and $p_0 = p(0)$.

Being written in the q representation, (25) reads

$$\left[f(\tau) q + g(\tau)\partial_q + \varphi(\tau)\right]\langle q|z, \tau\rangle = z\langle q|z, \tau\rangle. \quad (27)$$

General solution of this equation has the form

$$\langle q|z, \tau\rangle = \Phi_z^{c_1c_2}(q, \tau) = \exp\left[-\frac{f(\tau)}{g(\tau)}\frac{q^2}{2} + \frac{z - \varphi(\tau)}{g(\tau)} q + \chi(\tau)\right], \quad (28)$$

where $\chi(\tau)$ is an arbitrary function on τ.

One can see that the functions $\Phi_z^{c_1c_2}(q, \tau)$ can be written in terms of the mean values $q(\tau)$ and $p(\tau)$ given by (26),

$$\Phi_z^{c_1c_2}(q, \tau) = \exp\left\{ip(\tau) q - \frac{f(\tau)}{2g(\tau)}[q - q(\tau)]^2 + \tilde{\chi}(\tau)\right\}. \quad (29)$$

where $\tilde{\chi}(\tau)$ is again an arbitrary function on τ.

The functions $\Phi_z^{c_1c_2}$ satisfy the following equation

$$\hat{S}\Phi_z^{c_1c_2}(q,\tau) = \lambda(\tau)\,\Phi_z^{c_1c_2}(q,\tau)\,, \tag{30}$$

where

$$\lambda(\tau) = i\partial_\tau\tilde{\chi}(\tau) + \alpha q^2(\tau) - \frac{1}{2}\left[p^2(\tau) + \frac{f}{g}\right] - i\nu p(\tau) - \beta - i\varepsilon. \tag{31}$$

If we wish the functions (29) satisfies the Schrödinger equation (5), we have to fix $\tilde{\chi}(\tau)$ from the condition $\lambda(\tau) = 0$. Thus, we obtain for the function $\tilde{\chi}(\tau)$ the following result:

$$\tilde{\chi}(\tau) = \phi(\tau) + \ln N,$$
$$\phi(\tau) = \int_0^\tau \left\{i\alpha q^2(\tau) - \frac{i}{2}\left[p^2(\tau) + \frac{f}{g}\right] - i\nu p(\tau) - \beta - i\varepsilon\right\} d\tau, \tag{32}$$

were N is a normalization constant, which we suppose to be real.

The probability densities generated by the wave functions (29) have the form

$$\rho_z^{c_1c_2}(q,\tau) = \left|\Phi_z^{c_1c_2}(q,\tau)\right|^2 = N^2\exp\left\{-\frac{[q-q(\tau)]^2}{2|g(\tau)|^2} + 2\,\mathrm{Re}\,\phi(\tau)\right\}. \tag{33}$$

Considering the normalization integral, we find the constant N,

$$\int_{-\infty}^{\infty}\rho_z^{c_1c_2}(q,\tau)\,dq = 1 \Rightarrow N = \frac{\exp(-\mathrm{Re}\,\phi(\tau))}{\sqrt{\sqrt{2\pi}\,|g(\tau)|}}. \tag{34}$$

Thus, normalized solutions of the Schrödinger equation that at the same time are eigenfunctions of the annihilation operator $\hat{A}(\tau)$ have the form

$$\Phi_z^{c_1c_2}(q,\tau) = \frac{1}{\sqrt{\sqrt{2\pi}\,|g(\tau)|}} \exp\left\{ip(\tau)q - \frac{f(\tau)}{g(\tau)}\frac{[q-q(\tau)]^2}{2} + i\,\mathrm{Im}\,\phi(\tau)\right\} \tag{35}$$

and the corresponding probability densities read

$$\rho_z^{c_1c_2}(q,\tau) = \frac{1}{\sqrt{2\pi}\,|g(\tau)|}\exp\left\{-\frac{[q-q(\tau)]^2}{2|g(\tau)|^2}\right\}. \tag{36}$$

In what follows, we call the solutions (35) the time-dependent generalized CS.

3 Time-Dependent CS of Quadratic Systems

Using (17) and (25) we can calculate standard deviations $\sigma_q(\tau)$, $\sigma_p(\tau)$, and the quantity $\sigma_{qp}(\tau)$, in the generalized CS,

$$\sigma_q(\tau) = \sqrt{\langle(\hat{q}-\langle q\rangle)^2\rangle} = \sqrt{\langle q^2\rangle - \langle q\rangle^2} = |g(\tau)|\,,$$
$$\sigma_p(\tau) = \sqrt{\langle(\hat{p}-\langle p\rangle)^2\rangle} = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = |f(\tau)|\,,$$
$$\sigma_{qp}(\tau) = \frac{1}{2}\langle(\hat{q}-\langle q\rangle)(\hat{p}-\langle p\rangle) + (\hat{p}-\langle p\rangle)(\hat{q}-\langle q\rangle)\rangle = i\left[1/2 - g(\tau)f^*(\tau)\right]. \tag{37}$$

One can easily see that the generalized CS (35) minimize the Robertson-Schrödinger uncertainty relation [29, 30],

$$\sigma_q^2(\tau)\,\sigma_p^2 - \sigma_{qp}^2(\tau) = 1/4. \tag{38}$$

This means that the generalized CS are squeezed states [8].

Let us analyze the Heisenberg uncertainty relation in the generalized CS taking into account restriction (13),

$$\sigma_q(\tau)\,\sigma_p(\tau)\big|_{2\,\mathrm{Re}(c_1^*c_2)} = \frac{1}{2}\sqrt{1 + 4\,(\mathrm{Im}\,(gf^*))^2} \geq \frac{1}{2}. \tag{39}$$

Then using (37), we find $\sigma_q(0) = \sigma_q = |c_2|$ and $\sigma_p(0) = \sigma_p = |c_1|$, such that at $\tau = 0$, this relation reads

$$\sigma_q\sigma_p\big|_{2\,\mathrm{Re}(c_1^*c_2)} = \sqrt{\frac{1}{4} + [|c_2|\,|c_1|\sin(\mu_2-\mu_1)]^2}. \tag{40}$$

Taking into account (14), we see that if $\mu_1 = \mu_2 = \mu$, the left hand side of (39) is minimal, such that

$$\sigma_q\sigma_p = 1/2, \quad \sigma_{qp} = 0. \tag{41}$$

One can see that the constant μ does not enter CS (35). Then, in what follows, we consider generalized CS with the restriction $\mu_1 = \mu_2 = \mu = 0$. Namely, such states we call simply CS.

Now restriction (13) takes the form $c_1 = |c_1|$, $c_2 = |c_2|$, $2c_1 = c_2^{-1}$, such that

$$g(0) = |c_2| = \sigma_q, \quad f(0) = |c_1| = \sigma_p = \frac{1}{2\sigma_q}. \tag{42}$$

Thus, $\sigma_{qp} = \sigma_{qp}(0) = i\,[1/2 - g(0)\,f(0)] = 0$, which is consistent with (41).

With account taken of (35), (37), and (42), we obtain the following expression for the CS:

$$\Phi_z^{\sigma_q}(q,\tau) = \frac{1}{\sqrt{\sqrt{2\pi}\sigma_q(\tau)}} \exp\left\{ip(\tau)q - \frac{f(\tau)}{g(\tau)}\frac{[q-q(\tau)]^2}{2} + i\,\mathrm{Im}\,\phi(\tau)\right\},$$
$$\phi(\tau) = \int_0^\tau i\left\{\alpha q^2(\tau) - \frac{i}{2}\left[p^2(\tau) + \frac{f(\tau)}{g(\tau)}\right] - i\nu p(\tau) - \beta - i\varepsilon\right\}d\tau. \quad (43)$$

In fact, we have a family of CS parameterized by one real parameter—the initial standard deviation $\sigma_q > 0$. Each set of CS in the family has its specific initial standard deviations σ_q. Different CS from a family with a given σ_q have different quantum numbers z, which are in one to one correspondence with trajectory initial data q_0 and p_0. It follows from (26) that

$$z = \frac{q_0}{2\sigma_q} + i\sigma_q p_0,\ q_0 = 2\sigma_q\,\mathrm{Re}\,z,\ p_0 = \frac{\mathrm{Im}\,z}{\sigma_q}. \quad (44)$$

The probability density that corresponds to the CS (43) reads

$$\rho_z^{\sigma_q}(q,\tau) = \frac{1}{\sqrt{2\pi}\sigma_q(\tau)}\exp\left\{-\frac{[q-q(\tau)]^2}{2\sigma_q^2(\tau)}\right\}. \quad (45)$$

One can prove that for any fixed σ_q states (43) form an over complete set of functions with the following orthogonality and completeness relations

$$\int \overline{\Phi_{z'}^{\sigma_q}(q,\tau)}\Phi_z^{\sigma_q}(q,\tau)\,dq = \exp\left(z'^*z - \frac{|z'|^2+|z|^2}{2}\right),\quad \forall\tau,$$
$$\int\int \Phi_z^{\sigma_q}(q,\tau)\,\overline{\Phi_z^{\sigma_q}(q',\tau)}d^2z = \pi\delta\left(q-q'\right),$$
$$d^2z = d\,\mathrm{Re}\,z\ d\,\mathrm{Im}\,z, \forall\tau. \quad (46)$$

4 An Exact Solution of Oscillator Equation with Time-Dependent Frequency and Related CS

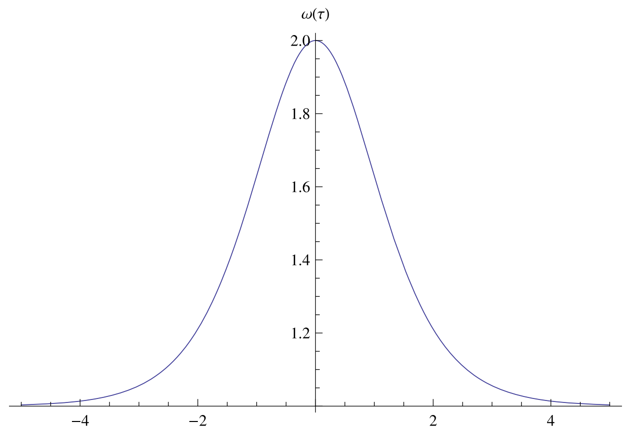
Let us consider the following function $\omega^2(\tau)$,

$$\omega^2(\tau) = \omega^2 + \frac{2\omega_0^2}{\cosh^2\omega_0\tau},\ \omega^2 \le \omega^2(\tau) \le \omega_{\max}^2,\ \omega(\pm\infty) = \omega^2, \quad (47)$$

where ω and ω_0 are some positive constants, $\omega_{\max} \ge \omega$. The function $\omega^2(\tau)$ is an even function, which decreases monotonically as $|\tau|$ changes from 0 to ∞,

$$\omega^2(\pm\infty) = \omega^2,\quad \omega^2(0) = \omega_{\max}^2 = \omega^2 + 2\omega_0^2. \quad (48)$$

For $\omega = 1$ and $\omega_0 = 2^{-1/2}$, the plot of the function $\omega(\tau)$ has the form



The general solution of equation (18) can be written as

$$g(\tau) = \left[\frac{iA\,\omega_0\tanh(\omega_0\tau)}{\omega^2+\omega_0^2} + B\right]\cos(\omega\tau) + \left[\frac{iA\,\omega^2}{\omega^2+\omega_0^2} - B\,\omega_0\tanh(\omega_0\tau)\right]\frac{\sin(\omega\tau)}{\omega}. \quad (49)$$

The restriction (13), (14), and (42) that set the CS from the entire set of generalized CS lead to the following relations for the constants A and B:

$$B = g(0) = |g(0)| = \sigma_q,$$
$$A = f(0) = |f(0)| = 1/2\sigma_q. \quad (50)$$

Using (49) and (50), we calculate the mean trajectories $q(\tau)$ and $p(\tau)$ according (26),

$$q(\tau) = 2\,\mathrm{Re}\left[g(\tau)z^*\right] = g(\tau)|_{A=\bar{A},\ B=\bar{B}},$$
$$p(\tau) = \dot{g}(\tau)|_{A=\bar{A},\ B=\bar{B}} = \dot{q}(\tau),$$
$$\overline{A} = -\sigma_q^{-1}\,\mathrm{Im}\,z,\ \overline{B} = 2\sigma_q\,\mathrm{Re}\,z. \quad (51)$$

For $\omega_{\min} > 0$, the mean trajectory $q(\tau)$ can be presented as

$$q(\tau) = R_0R(\tau)\sin[\omega\tau + \Theta(\tau) + \Theta_0], \quad (52)$$

where functions $R(\tau)$ and $\Theta(\tau)$ and constants R_0 and Θ_0 are

$$R(\tau) = \sqrt{1+\frac{\omega_0^2}{\omega^2}\tanh^2\omega_0\tau}\,,\quad 1\leqslant R(\tau)<\omega^{-1}\sqrt{\omega^2+\omega_0^2}\,;$$
$$\Theta(\tau) = \arctan\left[\frac{\omega_0}{\omega}\tanh\omega_0\tau\right],\quad -\Delta<\Theta(\tau)<\Delta\,,$$
$$\Delta = \arctan\left(\frac{\omega_0}{\omega}\right);$$
$$R_0 = \frac{\sqrt{p_0^2\omega^2+q_0^2(\omega^2+\omega_0^2)^4}}{(\omega^2+\omega_0^2)^2},\quad \sin\Theta_0=\frac{q_0}{R_0},$$
$$\cos\Theta_0 = \frac{p_0\,\omega}{R_0(\omega^2+\omega_0^2)^2}\,. \quad (53)$$

Thus, we deal with a quasiharmonic motion with the frequency ω and an amplitude that is changing in time in finite limits and with a time-dependent phase that is slowly changing in also finite limits.

Let us derive the case of a harmonic oscillator with a fixed frequency $\omega>0$ from the above formulas. To this end, we have to set $\omega_0=0$ and $\alpha=\omega^2/2$, $\nu=\beta=\varepsilon=0$ such that $\omega^2(\tau)=\omega^2$. Then,

$$g(\tau) = \sigma_q\cos\omega\tau+\frac{i\sin\omega\tau}{2\sigma_q\omega},$$
$$f(\tau) = \frac{\cos\omega\tau}{2\sigma_q}+i\sigma_q\omega\sin\omega\tau,$$
$$q(\tau) = q_0\cos\omega\tau+\frac{p_0}{\omega}\sin\omega\tau,$$
$$p(\tau) = p_0\cos\omega\tau-\omega q_0\sin\omega\tau, \quad (54)$$

and $z=\sigma_p q_0+i\sigma_q p_0$. Taking all that into account in (43), we obtain the following representation for CS (in the above given definition) of the harmonic oscillator:

$$\Phi_z^{\sigma_q}(q,\tau) = \frac{1}{\sqrt{\sqrt{2\pi}\,g(\tau)}}\exp\left\{-\frac{1}{2}\frac{f(\tau)}{g(\tau)}\left[q-\frac{z}{f(\tau)}\right]^2 + \frac{f^*(\tau)}{f(\tau)}\frac{z^2}{2}-\frac{|z|^2}{2}\right\}. \quad (55)$$

For these CS

$$\sigma_q(\tau) = \sigma_q\sqrt{1+\frac{\left(1-4\sigma_q^4\omega^2\right)}{4\sigma_q^4\omega^2}\sin^2\omega\tau},$$
$$\sigma_p(\tau) = \sigma_p\sqrt{1-\left(1-4\sigma_q^4\omega^2\right)\sin^2\omega\tau}, \quad (56)$$

and the corresponding probability density reads

$$\rho_z^{\sigma_q}(q,\tau) = \frac{1}{\sqrt{2\pi}\sigma_q(\tau)}\exp\left\{-\frac{[q-q(\tau)]^2}{2\sigma_q^2(\tau)}\right\}. \quad (57)$$

One has to consider the following three cases:

a) $\sigma_q\sqrt{2\omega}=1$, then

$$\sigma_q(\tau)=\sigma_q,\quad \sigma_p(\tau)=\sigma_p,$$
$$\sigma_q(\tau)\sigma_p(\tau)=\sigma_q\sigma_p=1/2,\ \forall\tau.$$

b) $\sigma_q\sqrt{2\omega}<1$, then

$$\sigma_q(\tau)\big|_{\min} = \sigma_q(\tau)\big|_{\tau=\frac{\pi n}{\omega}}=\sigma_q,$$
$$\sigma_q(\tau)\big|_{\max} = \sigma_q(\tau)\big|_{\tau=\frac{2n+1}{2}\frac{\pi}{\omega}}=\frac{1}{2\sigma_q\omega},$$

$$\sigma_p(\tau)\big|_{\min} = \sigma_p(\tau)\big|_{\tau=\frac{2n+1}{2}\frac{\pi}{\omega}}=\sigma_q\omega,$$
$$\sigma_p(\tau)\big|_{\max} = \sigma_p(\tau)\big|_{\tau=\frac{\pi n}{\omega}}=\sigma_p,$$
$$\sigma_q(\tau)\sigma_p(\tau)\big|_{\min} = \sigma_q(\tau)\sigma_p(\tau)\big|_{\tau=\frac{\pi n}{\omega}}=1/2,$$
$$\sigma_q(\tau)\sigma_q(\tau)\big|_{\max} = \sigma_q(\tau)\sigma_q(\tau)\big|_{\tau=\frac{2n+1}{4}\frac{\pi}{\omega}} = \frac{1+4\sigma_q^4\omega^2}{8\sigma_q^2\omega},$$
$$n\in\mathbb{N}=0,1,2,\ldots\,.$$

c) $\sigma_q\sqrt{2\omega}>1$, then

$$\sigma_q(\tau)\big|_{\min} = \sigma_q(\tau)\big|_{\tau=\frac{2n+1}{2}\frac{\pi}{\omega}}=\frac{1}{2\sigma_q\omega},$$
$$\sigma_q(\tau)\big|_{\max} = \sigma_q(\tau)\big|_{\tau=\frac{n\pi}{\omega}}=\sigma_q,$$
$$\sigma_p(\tau)\big|_{\min} = \sigma_p(\tau)\big|_{\tau=\frac{n\pi}{\omega}}=\sigma_p,$$
$$\sigma_p(\tau)\big|_{\max} = \sigma_p(\tau)\big|_{\tau=\frac{2n+1}{2}\frac{\pi}{\omega}}=\sigma_q\omega,$$
$$\sigma_q(\tau)\sigma_p(\tau)\big|_{\min} = \sigma_q(\tau)\sigma_p(\tau)\big|_{\tau=\frac{n}{2}\frac{\pi}{\omega}}=1/2,$$
$$\sigma_q(\tau)\sigma_q(\tau)\big|_{\max} = \sigma_q(\tau)\sigma_q(\tau)\big|_{\tau=\frac{2n+1}{4}\frac{\pi}{\omega}} = \frac{1+4\sigma_q^4\omega^2}{8\sigma_q^2\omega},\quad n\in\mathbb{N}\,.$$

We can see that in case (a), the Heisenberg uncertainty relation is minimized in the CS (55). In the same case, these CS coincide (up to a phase factor) with the well-known Schrödinger-Glauber CS [23]. CS with σ_q obeying (b) and (c) minimize the Heisenberg uncertainty relation periodically, but the product $\sigma_q(\tau)\sigma_q(\tau)$ is always restricted by the limits 1/2 and $\frac{1+4\sigma_q^4\omega^2}{8\sigma_q^2\omega}$.

Setting $\omega_0=\omega=\alpha=\nu=\beta=\varepsilon=0$, and taking into account the limits

$$\lim_{\omega_0\to 0} g(\tau) = B\cos(\omega\tau)+\frac{iA}{\omega}\sin(\omega\tau)\,,$$
$$\lim_{\omega_0,\omega\to 0} g(\tau) = B+iA\tau,$$
$$\lim_{\omega\to 0} g(\tau) = (iA-B\omega_0^2\tau)\frac{\tanh(\omega_0\tau)}{\omega_0}+B,$$

we obtain from (43) CS of a free particle studied by us in the [25].

Acknowledgments Bagrov thanks FAPESP for support and IF USP for hospitality; Gitman thanks CNPq and FAPESP for permanent support; The work of Bagrov and Gitman is also partially supported by

Tomsk State University A. S. Competitiveness Improvement Program; Pereira thanks FAPESP for support.

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