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ORIGINAL PAPER

A proof of Chen-Malešević's conjecture

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Abstract
In this paper, we obtain a new inequality between the inverse hyperbolic tangent and inverse sine functions, which is a conjecture of Chen-Malešević [(Chen in Rev Real Acad Cienc. Exactas Fis. Nat. Ser. A-Mat 114:105, 2020) conjecture 2.1]spsCM.

Keywords Generalized trigonometric functions · Inverse hyperbolic tangent function · Inverse sine function

Mathematics Subject Classification 26D05 · 26D15

1 Introduction

In 2010, Masjed-Jamei [1] studied the relation of inverse tangent function arctan x and inverse hyperbolic sine function $\sinh^{-1}(x)$ and proved an inequality as follows.

$$(\arctan x)^2 \leqslant \frac{x \sinh^{-1}(x)}{\sqrt{1+x^2}}, \quad x \in (-1, 1). \tag{1.1}$$

The study related to (1.1) attracted much attention in last decade. At first, Zhu and Malešević [3] proved that (1.1) holds for any $x \in (-\infty, +\infty)$. They also obtained some refinements of (1.1).

Proposition 1.1 [3, Theorem 1.3] For any $x \in (-\infty, +\infty)$, we have

$$-\frac{1}{45}x^6 \leqslant (\arctan x)^2 - \frac{x \sinh^{-1} x}{\sqrt{1+x^2}} \leqslant -\frac{1}{45}x^6 + \frac{4}{105}x^8, \tag{1.2}$$

$$\begin{aligned} -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} &\leqslant (\arctan x)^2 - \frac{x \sinh^{-1} x}{\sqrt{1+x^2}} \\ &\leqslant -\frac{1}{45}x^6 + \frac{4}{105}x^8 - \frac{11}{225}x^{10} + \frac{586}{10395}x^{12}. \end{aligned} \tag{1.3}$$

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Define

$$v_n = \frac{1}{n}\left[\frac{n!2^{n-1}}{(2n-1)!!} - \left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right)\right], n \geqslant 3. \tag{1.4}$$

By using flexible analysis tools, Zhu and Malešević [4] extended (1.2) and (1.3) to general form as follows.

Proposition 1.2 [4, Theorem 1.1] For any $x \in (-\infty, +\infty)$, we have

$$\sum_{n=3}^{2m+1} (-1)^n v_n x^{2n} \leqslant (\arctan x)^2 - \frac{x \sinh^{-1} x}{\sqrt{1+x^2}} \leqslant \sum_{n=3}^{2m+2} (-1)^n v_n x^{2n}. \tag{1.5}$$

Proposition 1.3 [5, Theorem 2.1] The double inequality

$$\frac{x \sinh^{-1}(x)}{\sqrt{1 + x^2 + \frac{1}{45}x^2}} < (\arctan x)^2 < \frac{x \sinh^{-1}(x)}{\sqrt{1+x^2}} \tag{1.6}$$

holds for any $x \in (0, +\infty)$ with best constants 0 and $1/45$.

Please see [6, 7] for more generalizations.

Motivated by (1.1)-(1.6), Zhu and Malešević [3] also studied the relation of inverse hyperbolic tangent function $\tanh^{-1}(x)$ and inverse sine function $\arcsin x$ as follows.

Proposition 1.4 [3, Theorem 1.4] The inequality

$$\left[\tanh^{-1}(x)\right]^2 < \frac{x \arcsin x}{\sqrt{1-x^2}} \tag{1.7}$$

holds for any $x \in (0, 1)$ with the the best power number 2.

Proposition 1.5 [3, Theorem 1.6] The inequality

$$\frac{x \arcsin x}{\sqrt{1-x^2}} - \left[\tanh^{-1}(x)\right]^2 < \sum_{n=3}^{N} v_n x_{2n} \tag{1.8}$$

holds for any $x \in (0, 1)$.

Moreover, by investigating the power series of the following function

$$\frac{\left[\tanh^{-1}(x)\right]^2}{\dfrac{\arcsin x}{\sqrt{1-x^2}}} = x - \frac{1}{45}x^5 - \frac{22}{945}x^7 - \frac{61}{2835}x^9 + O(x^{10}),$$

L. Zhu [2] obtained the following interesting double inequality of Masjed-Jamei type.

Proposition 1.6 [2, Theorem 1] The double inequality

$$\frac{(x - x^5)\arcsin x}{\sqrt{1-x^2}} < \left[\tanh^{-1}(x)\right]^2 < \frac{\left(x - \frac{1}{45}x^5\right)\arcsin x}{\sqrt{1-x^2}} \tag{1.9}$$

holds for any $x \in (0, 1)$ with best constants -1 and $-\frac{1}{45}$.

We gave a now proof of (1.9) in [8] and provided a refinement in [9].

Proposition 1.7 [9, Theorem 1] The double inequality

$$\frac{\left(x-\frac{1}{45}x^5-\frac{44}{45}x^7\right)\arcsin x}{\sqrt{1-x^2}} < \left[\tanh^{-1}(x)\right]^2 < \frac{\left(x-\frac{1}{45}x^5-\frac{22}{945}x^7\right)\arcsin x}{\sqrt{1-x^2}} \tag{1.10}$$

holds for any $x \in (0, 1)$ with best constants $-\frac{44}{45}$ and $-\frac{22}{945}$.

The goal of this paper is to prove a new lower bound of $\left[\tanh^{-1}(x)\right]^2$, which is a conjecture of Chen-Malešević [5, Conjecture 2.1].

Theorem 1.8 *If* $x \in (0, 1)$*, then*

$$\frac{x \arcsin x}{\left(1-\frac{41}{45}x^2\right)^{45/82}} < \left[\tanh^{-1}(x)\right]^2. \tag{1.11}$$

Remark 1.9 (1) Numerical experiments show that inequality (1.11) is stronger than the left-hand side inequality of (1.10).

(2) L. Zhu [10] claimed that he have proved Chen-Malešević's conjecture. Unfortunately, his proof in [10] is false.

2 Proof of theorem 1.8

Lemma 2.1 *Let*

$$f(x) = \frac{\left[\tanh^{-1}(x)\right]^2\left(1-\frac{41}{45}x^2\right)^{45/82}}{x} - \arcsin x,$$

then $f(x)$ *is strictly increasing on* $(0, 1)$.

Proof **Step 1**: Let $t = \tanh^{-1}(x) \in (0, +\infty)$, then $x = \tanh(t)$. Define

$$F(t) := f(\tanh t) = \frac{t^2\left(1-\frac{41}{45}\tanh^2 t\right)^{45/82}}{\tanh t} - \arcsin(\tanh t).$$

In order to prove that $f(x)$ is strictly increasing on $(0, 1)$, we only need to prove $F(t)$ is strictly increasing on $(0, +\infty)$.

Step 2: By direct computation, we have

$$\begin{aligned}
F'(t)\cdot\tanh^2 t &= \left[2t\left(1-\frac{41}{45}\tanh^2 t\right)^{45/82} + \frac{45}{82}t^2\left(1-\frac{41}{45}\tanh^2 t\right)^{-37/82}\right.\\
&\quad \left.\cdot\left(-\frac{82}{45}\tanh t\cdot\frac{1}{\cosh^2 t}\right)\right]\tanh t - t^2\left(1-\frac{41}{45}\tanh^2 t\right)^{45/82}\\
&\quad \cdot\frac{1}{\cosh^2 t} - \frac{1}{\cosh t}\cdot\tanh^2 t\\
&= 2t\left(1-\frac{41}{45}\tanh^2 t\right)^{45/82}\tanh t - t^2\left(1-\frac{41}{45}\tanh^2 t\right)^{-37/82}\cdot\tanh^2 t\cdot\frac{1}{\cosh^2 t}\\
&\quad - t^2\left(1-\frac{41}{45}\tanh^2 t\right)^{45/82}\cdot\frac{1}{\cosh^2 t} - \frac{1}{\cosh t}\cdot\tanh^2 t.
\end{aligned}$$

Then

$$\begin{aligned}\varphi(t):&=F'(t)\cdot\tanh^2 t\cdot\left(1-\frac{41}{45}\tanh^2 t\right)^{37/82}\cdot\cosh^4 t\\&=2t\left(\sinh t\cosh^3 t-\frac{41}{45}\sinh^3 t\cosh t\right)-t^2\left(\cosh^2 t+\frac{4}{45}\sinh^2 t\right)\\&\quad-\sinh^2 t\cosh t\left(1-\frac{41}{45}\tanh^2 t\right)^{37/82}.\end{aligned}$$

In order to prove $F(t)$ is strictly increasing on $(0,+\infty)$, it is suffice to prove $F'(t)>0$ for $t\in(0,+\infty)$, which is equivalent to $\varphi(t)>0$ for $t\in(0,+\infty)$.

Step 3: Denote

$$\begin{aligned}\varphi_1(t)&=2t\left(\sinh t\cosh^3 t-\frac{41}{45}\sinh^3 t\cosh t\right)-t^2\left(\cosh^2 t+\frac{4}{45}\sinh^2 t\right),\\\varphi_2(t)&=\sinh^2 t\cosh t\left(1-\frac{41}{45}\tanh^2 t\right)^{37/82},\end{aligned}$$

then

$$\varphi(t)=\varphi_1(t)-\varphi_2(t).$$

Obviously, $\varphi_2(t)>0$ on $(0,+\infty)$. And for any $t\in(0,+\infty)$,

$$\begin{aligned}\varphi_1(t)&=\frac{1}{45}t\left(\sinh(4t)+43\sinh(2t)\right)-\frac{1}{90}t^2\left(49\cosh(2t)+41\right)\\&=\frac{1}{90}\left[2t\left(\sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!}+43\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!}\right)-t^2\left(49\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n)!}+41\right)\right]\\&=t^2+\frac{1}{90}\sum_{n=1}^{\infty}\left(\frac{2\cdot 4^{2n+1}+86\cdot 2^{2n+1}}{(2n+1)!}-\frac{49\cdot 2^{2n}}{(2n)!}\right)t^{2n+2}\\&>0,\end{aligned}$$

since for any $n\geqslant 1$, we have

$$\frac{2\cdot 4^{2n+1}+86\cdot 2^{2n+1}}{(2n+1)!}-\frac{49\cdot 2^{2n}}{(2n)!}>0.$$

Step 4: Define

$$\psi_1(t)=\ln\varphi_1(t),\quad \psi_2(t)=\ln\varphi_2(t)$$

and

$$\psi(t)=\psi_1(t)-\psi_2(t),$$

then $\varphi(t)>0$ on $(0,+\infty)$ is equivalent to $\psi(t)>0$ on $(0,+\infty)$. Since

$$\lim_{t\to 0^+}\psi(t)=\lim_{t\to 0^+}\ln\frac{\varphi_1(t)}{\varphi_2(t)}=0,$$

it is suffice to prove that $\psi'(t)>0$ on $(0,+\infty)$.

Step 5: From

$$\begin{aligned}\psi_1(t) &= \ln \varphi_1(t)\\ &= \ln\left[2t\left(\sinh t\cosh^3 t - \frac{41}{45}\sinh^3 t\cosh t\right) - t^2\left(\cosh^2 t + \frac{4}{45}\sinh^2 t\right)\right]\\ &= \ln\left[\frac{1}{45}t\left(\sinh(4t)+43\sinh(2t)\right) - \frac{1}{90}t^2\left(49\cosh(2t)+41\right)\right]\end{aligned}$$

and

$$\psi_2(t) = \ln\varphi_2(t) = 2\ln\sinh t + \ln\cosh t + \frac{37}{82}\ln\left(1-\frac{41}{45}\tanh^2 t\right),$$

we get

$$\begin{aligned}\psi_1'(t) &= \frac{\frac{1}{45}\left(\sinh(4t)+43\sinh(2t)\right)+\frac{1}{45}t\left(4\cosh(4t)+86\cosh(2t)\right)-\frac{1}{45}t\left(49\cosh(2t)+41\right)-\frac{4}{45}t^2\sinh(2t)}{\frac{1}{45}t\left(\sinh(4t)+43\sinh(2t)\right)-\frac{1}{90}t^2\left(49\cosh(2t)+41\right)}\\ &= \frac{\left(\sinh(4t)+43\sinh(2t)\right)+t\left(4\cosh(4t)+37\cosh(2t)-41\right)-49t^2\sinh(2t)}{t\left(\sinh(4t)+43\sinh(2t)\right)-\frac{1}{2}t^2\left(49\cosh(2t)+41\right)}\end{aligned}$$

and

$$\begin{aligned}\psi_2'(t) &= \frac{2\cosh t}{\sinh t} + \frac{\sinh t}{\cosh t} + \frac{37}{82}\cdot\frac{-\frac{82}{45}\tanh t\cdot\frac{1}{\cosh^2 t}}{\left(1-\frac{41}{45}\tanh^2 t\right)}\\ &= \frac{2\cosh t}{\sinh t} + \frac{\sinh t}{\cosh t} - \frac{37\tanh t}{4\cosh^2 t + 41}\\ &= \frac{\left(2\cosh^2 t+\sinh^2 t\right)\left(4\cosh^2 t+41\right) - 37\sinh^2 t}{\sinh t\cosh t\left(4\cosh^2 t+41\right)}\\ &= \frac{\left(\cosh(2t)+1+\frac{\cosh(2t)-1}{2}\right)\left(2\cosh(2t)+43\right) - 37\frac{\cosh(2t)-1}{2}}{\sinh t\cosh t\left(2\cosh(2t)+43\right)}\\ &= \frac{\left(3\cosh(2t)+1\right)\left(2\cosh(2t)+43\right)-37\cosh(2t)+37}{\sinh(4t)+43\sinh(2t)}\\ &= \frac{3\cosh(4t)+94\cosh(2t)+83}{\sinh(4t)+43\sinh(2t)}.\end{aligned}$$

Then

$$\begin{aligned}\psi'(t) &= \psi_1'(t) - \psi_2'(t)\\ &= \frac{A_2t^2 + A_1t + A_0}{45\varphi_1(t)\left[\sinh(4t)+43\sinh(2t)\right]},\end{aligned}$$

where

$$\begin{aligned}A_0 &= \left(\sinh(4t)+43\sinh(2t)\right)^2\\ &= \sinh^2(4t) + 86\sinh(4t)\sinh(2t) + 1849\sinh^2(2t)\\ &= \frac{\cosh(8t)-1}{2} + 43\left(\cosh(6t)-\cosh(2t)\right) + 1849\cdot\frac{\cosh(4t)-1}{2}\\ &= \frac{\cosh(8t)}{2} + 43\cosh(6t) + \frac{1849}{2}\cosh(4t) - 43\cosh(2t) - 925\end{aligned}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(8t)^{2n}}{(2n!)} + 43\sum_{n=0}^{\infty}\frac{(6t)^{2n}}{(2n!)} + + \frac{1849}{2}\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n!)} - 43\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n!)} - 925,$$

$$\begin{aligned}
A_1 &= (\sinh(4t) + 43\sinh(2t))\,[(4\cosh(4t) + 37\cosh(2t) - 41) \\
&\quad - (3\cosh(4t) + 94\cosh(2t) + 83)] \\
&= (\sinh(4t) + 43\sinh(2t))\,(\cosh(4t) - 57\cosh(2t) - 124) \\
&= \frac{\sinh(8t)}{2} - \frac{2451}{2}\sinh(4t) - \frac{57}{2}(\sinh(6t) + \sinh(2t)) \\
&\quad + \frac{43}{2}(\sinh(6t) - \sinh(2t)) - 124\,(\sinh(4t) + 43\sinh(2t)) \\
&= \frac{\sinh(8t)}{2} - 7\sinh(6t) - \frac{2699}{2}\sinh(4t) - 5382\sinh(2t) \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(8t)^{2n+1}}{(2n+1)!} - 7\sum_{n=0}^{\infty}\frac{(6t)^{2n+1}}{(2n+1)!} - \frac{2699}{2}\sum_{n=0}^{\infty}\frac{(4t)^{2n+1}}{(2n+1)!} - 5382\sum_{n=0}^{\infty}\frac{(2t)^{2n+1}}{(2n+1)!},
\end{aligned}$$

and

$$\begin{aligned}
A_2 &= \frac{1}{2}(49\cosh(2t) + 41)\,(3\cosh(4t) + 94\cosh(2t) + 83) \\
&\quad - 49\sinh(2t)\,(\sinh(4t) + 43\sinh(2t)) \\
&= \frac{49}{4}\cosh(6t) + \frac{319}{2}\cosh(4t) + \frac{16087}{4}\cosh(2t) + \frac{7813}{2} \\
&= \frac{49}{4}\sum_{n=0}^{\infty}\frac{(6t)^{2n}}{(2n!)} + \frac{319}{2}\sum_{n=0}^{\infty}\frac{(4t)^{2n}}{(2n!)} + \frac{16087}{4}\sum_{n=0}^{\infty}\frac{(2t)^{2n}}{(2n!)} + \frac{7813}{2}.
\end{aligned}$$

Step 6: Let

$$\theta(t) := A_2t^2 + A_1t + A_0 = \sum_{n=1}^{\infty} a_{2n+2}t^{2n+2},$$

where

$$\begin{aligned}
a_{2n+2} &= \frac{49}{4}\cdot\frac{6^{2n}}{(2n)!} + \frac{319}{2}\cdot\frac{4^{2n}}{(2n!)} + \frac{16087}{4}\cdot\frac{2^{2n}}{(2n!)} \\
&\quad + \frac{1}{2}\cdot\frac{8^{2n+1}}{(2n+1)!} - 7\cdot\frac{6^{2n+1}}{(2n+1)!} - \frac{2699}{2}\cdot\frac{4^{2n+1}}{(2n+1)!} - 5382\cdot\frac{2^{2n+1}}{(2n+1)!} \\
&\quad + \frac{1}{2}\frac{8^{2n+2}}{(2n+2)!} + 43\frac{6^{2n+2}}{(2n+2)!} + \frac{1849}{2}\frac{4^{2n+2}}{(2n+2)!} - 43\cdot\frac{2^{2n+2}}{(2n+2)!}.
\end{aligned}$$

It is easy to check that

$$a_{2n+2} > 0, \quad 1 \leqslant n \leqslant 5$$

and

$$
\begin{aligned}
a_{2n+2} >& \frac{6^{2n}}{(2n)!} \cdot \left(\frac{49}{4} - \frac{42}{2n+1}\right) \\
&+ \frac{1}{2} \cdot \frac{4^{2n+1}}{(2n+1)!} \cdot \left(2^{2n+1} - 2699\right) \\
&+ \frac{2^{2n}}{(2n)!} \cdot \left(\frac{16087}{4} - \frac{10764}{2n+1}\right) \\
>& 0
\end{aligned}
$$

for any $n \geqslant 6$. Therefore, $\theta(t) > 0$ for any $t \in (0, +\infty)$, which implies

$$\psi'(t) > 0 \;\text{ for any }\; t \in (0, +\infty).$$

The proof of Lemma 2.1 is completed. □

Proof of theorem 1.8 By Lemma 2.1 and

$$\lim_{x \to 0^+} f(x) = 0,$$

we get $f(x) > 0$ for any $x \in (0, 1)$, which implies

$$\frac{x \arcsin x}{\left(1 - \frac{41}{45}x^2\right)^{45/82}} < \left[\tanh^{-1}(x)\right]^2 .$$

The proof is completed. □

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