Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. (2024) 118:140
https://doi.org/10.1007/s13398-024-01636-6

SURVEY

The Banach Fixed Point Theorem: selected topics from its hundred-year history

Jacek Jachymski[1] · Izabela Jóźwik[2] · Małgorzata Terepeta[1,2]

Received: 23 December 2023 / Accepted: 14 June 2024

Abstract

On June 24, 1920 Stefan Banach presented his doctoral dissertation titled *O operacjach na zbiorach abstrakcyjnych i ich zastosowaniach do równañ całkowych* (*On operations on abstract sets and their applications to integral equations*) to the Philosophy Faculty of Jan Kazimierz University in Lvov. He passed his PhD examinations in mathematics, physics and philosophy, and in January 1921 he became a doctor. A year later, he published the results of his doctorate in Fundamenta Mathematicae. Among them there was the theorem known today as the Banach Fixed Point Theorem or the Banach Contraction Principle. It is one of the most famous theorems in mathematics, one of many under the name of Banach. It concerns certain mappings (called contractions) of a complete metric space into itself and it gives the conditions sufficient for the existence and uniqueness of a fixed point of such mapping. In 2022 we had a centenary of publishing this theorem. In the paper, we want to present its most important modifications and generalizations, several contractive conditions, the converse theorems and some applications. It is not possible to provide complete information about what has been written during the last hundred years about the Banach Fixed Point Theorem and we are just trying to touch on some breakthrough moments in the development of the metric fixed point theory. The main purpose of this article is to organize the knowledge on this subject and to elaborate a broad bibliography which all interested persons can refer to.

Keywords Banach fixed point theorem · Contraction · Complete metric space · Banach space · Semimetric space · Applications of the contraction principle

Mathematics Subject Classification 47H09 · 47H10 · 01A60 · 01A61 · 54H25

Jacek Jachymski, Izabela Jóźwik, and Małgorzata Terepeta have contributed equally to this work.

✉ Małgorzata Terepeta
malgorzata.terepeta@p.lodz.pl

Jacek Jachymski
jacek.jachymski@p.lodz.pl

Izabela Jóźwik
izabela.jozwik@p.lodz.pl

1 Institute of Mathematics, Lodz University of Technology, Żeromskiego 116, Lodz 90-924, Poland

2 Centre of Mathematics and Physics, Lodz University of Technology, Żeromskiego 116, Lodz 90-924, Poland

Published online: 09 July 2024

1 Introduction

There are many notions and theorems with *Banach* in their names such as: the Banach space, the Banach integral, the Banach generalized limit, the Banach algebra, the Banach–Mazur distance, the Hahn–Banach theorem, the Banach–Steinhaus theorem, the Banach closed graph theorem, the Banach open mapping principle, the Banach–Alaoglu Theorem, the Banach–Tarski theorem and some others (the list of things named after Banach is available on Wikipedia). Among them there is also the Banach Fixed Point Theorem which will be the hero of our article.

It is considered that functional analysis has emerged as a discipline in 1932 with publishing the book *Théorie des Opérations Linéaires* written by Stefan Banach [5]. But the beginning of this part of mathematics goes back to the earlier years and it is strictly associated (as Banach wrote in his PhD thesis) with such names as: Vito Volterra, Maurice Fréchet, Jacques Hadamard, Frigyes Riesz, Salvatore Pincherle, Hugo Steinhaus, Hermann Weyl, Henri Lebesgue, David Hilbert. They and some other mathematicians contributed to the conception of *a function space*, i.e., such topological space in which functions are the points. In [12] (the article which is the wide survey of how the functional analysis was established) we can read:

> To make precise the idea of a function space, one must first have clear definition of the words "function" and "space".

The article [12] starts with the concept of "function" and it ends with the last section devoted to the book [5].

With the appearance of the Fundamenta Mathematicae journal in 1920, Poland became an important center in Europe related to the set theory, topology and functional analysis.

The year 1922, when Banach published the results of his doctorate [4], is considered a breakthrough in the history of mathematics [88]. Lech Maligranda wrote in [85]:

> Praca doktorska Banacha była prapoczątkiem analizy funkcjonalnej, natomiast książka Banacha *Théorie des opérations linéaires* [5] z 1932 roku z trzema fundamentalnymi twierdzeniami była początkiem liniowej analizy funkcjonalnej.[1]

The 100th anniversary of the publication in Fundamenta Mathematicae of the Banach Fixed Point Theorem was for us a stimulus for preparing this paper. In our article we are going to present its significance and impact. It is impossible to present the complete survey of all that has been written about the Banach Fixed Point Theorem and we are just trying to touch on some breakthrough moments in the development of the metric fixed point theory.

The Banach Fixed Point Theorem was not the first theorem connected with fixed points. One of the first theorems were formulated by Henri Poincaré in 1886. In 1909 Luitzen E. J. Brouwer proved that any continuous function f from a closed ball in $\mathbb{R}^n$ into itself has at least one fixed point. This proof was presented for $n = 3$. A year later Hadamard showed the generalization of this theorem for arbitrary n. In 1912 Brouwer [21] gave another proof for this generalization. The Brouwer theorem was a non-constructive result—it was only existential. It said about the existence of such point but did not explain how to obtain it. The proof of the Banach Fixed Point Theorem was constructive. Its important feature is that

[1] Banach's doctoral thesis was the very beginning of functional analysis, while Banach's book *Théorie des opérations linéaires* [5] from 1932 with three fundamental theorems was the beginning of linear functional analysis.

it gives the existence and uniqueness of a fixed point and convergence of the sequence of successive approximations to a solution of the problem.

2 The Banach Fixed Point Theorem

Stefan Banach was born in Cracow (Kraków in Polish) in 1892 on March 30. He moved to Lvov in 1920 to take up his job at the Lvov Polytechnic. The doctorate of Banach was defended in the same year. It was described in many books with several misleading stories. The most popular story is the following:

> The story goes that Banach could not be bothered with writing a thesis, since he was interested mainly in solving problems not necessarily connected to a possible doctoral dissertation. After some time, the university authorities became impatient. It is said that another university assistant (instructed by Stanisław Ruziewicz) wrote down Banach's theorems and proofs, and those notes were accepted as a superb dissertation. However, an exam was also required, and Banach was unwilling to take it. So one day, Banach was accosted in the corridor by a colleague, who asked him to join him in a meeting with some mathematicians who were visiting the university in order to clarify certain mathematical details, since Banach would certainly be able to answer their questions. Banach agreed and eagerly answered the questions, not realizing that he was being examined by a special commission that had arrived from Warsaw for just this purpose (by Danuta Ciesielska and Krzysztof Ciesielski in [29]).

The story is attractive, but not true (check in [29]). On June 24, 1920 Banach presented his doctoral dissertation titled *O operacjach na zbiorach abstrakcyjnych i ich zastosowaniach do równañ całkowych*[2] to the Philosophy Faculty of Jan Kazimierz University in Lvov. He passed his PhD examinations in mathematics, physics and philosophy and became a doctor in January 1921. A year later he published the results of his doctorate in [4]. Precise and very interesting information about Banach's doctorate and his students one can find not only in the mentioned paper [29] but in [85, 86] as well.

The article [4] of Banach has 49 pages and is divided into some parts. In the first section Banach presented axioms of a space which is now known under the name Banach space. Those axioms (on the page 135) are grouped in the following way:

I. Axioms of linear space (additivity of vectors, multiplication by a number),
II. Axioms of the norm,
III. Axiom of completeness (formulated in the way: if $\{X_n\}$ is a sequence of elements of a space E such that $\lim_{r,p\to\infty} \|X_r - X_p\| = 0$, then there exists element $X \in E$ such that $\lim_{n\to\infty} \|X - X_n\| = 0$).

In some papers Banach called the spaces which fulfill all above axioms *espaces (B)*, i.e., *the B-spaces* (the name was introduced by Hugo Steinhaus).

He proved some results on such spaces and some theorems on linear operators.

The history of Banach spaces is presented in [84]. For some time those spaces were called *Banach–Wiener* spaces because, independently, very similar results were obtained by Norbert Wiener. However, the simplicity and brevity of Banach's definitions turned out to be decisive and Wiener withdrew from the work on this subject. As he wrote later [117]:

[2] *On operations on abstract sets and their applications to integral equations.*

For a short while I kept publishing a paper or two on this topic, but I gradually left the field. At present these spaces are quite justly named after Banach alone.

In the second part of the article [4] Banach proved the theorem known now as *the Banach Fixed Point Theorem*. With the assumption that a space E fulfills all the axioms I, II, III, it was formulated in the following way:

Theorem 1 [4, Th. 6, p. 160] *If*

(a) $T\colon E \to E$ *is a continuous operator,*
(b) *there exists a number* $0 < M < 1$ *such that for any* $x_1, x_2 \in E$

$$\|T(x_1) - T(x_2)\| \leq M\|x_1 - x_2\|,$$

then there exists a unique element x *such that* $x = T(x)$.

Proof (The original Banach's proof.) We take an arbitrary $y \in E$ and we define a sequence $(x_n)_{n\in\mathbb{N}}$ in the following way:

$$x_1 = y \quad \text{and} \quad x_{n+1} = T(x_n) \text{ for } n \geq 1.$$

Observe that for $n > 1$

$$\|x_{n+1} - x_n\| \leq \|T(x_n) - T(x_{n-1})\| \leq M\|x_n - x_{n-1}\|.$$

From the above inequality we have $\|x_{n+1} - x_n\| \leq M^{n-1}\|x_2 - x_1\|$.[3] We assumed that $M < 1$, so the series $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|$ is convergent which implies the convergence of $x_1 + \sum_{n=1}^{\infty}(x_{n+1} - x_n)$ to some element $x \in E$. Because $x_1 + \sum_{k=1}^{n-1}(x_{k+1} - x_k) = x_n$, so $\lim_{n\to\infty} x_n = x$. From the continuity of T we obtain $\lim_{n\to\infty} T(x_n) = T(x)$. Since $x_n = T(x_{n-1})$ we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T(x_{n-1})$ and $x = T(x)$ which finished the proof. □

The operator fulfilling condition (b) from Theorem 1 is called a contraction, therefore the above theorem is also known as *the Banach Contraction Principle*. Observe that Banach contractions are continuous, since they satisfy the Lipschitz condition, so actually assumption (a) is superfluous.

The Banach theorem is simple in its formulation, the fixed point is always unique and it is obtained by an explicit calculation. Its disadvantage is that the condition that the mapping being a contraction is a somewhat severe restriction. Despite that, in the famous book [51] one can read:

> The Banach contraction principle is the simplest and one of the most versatile elementary results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map: it produces approximations of any required accuracy, and moreover, even the number of iterations needed to get a specified accuracy can be determined.

Nowadays the above theorem is formulated in a slightly different way.

Theorem 2 [13] *Let* (X, d) *be a complete metric space and* $T\colon X \to X$ *be a contraction mapping with the Lipschitz constant* $\alpha \in [0, 1)$. *Then:*

[3] In the original Banach's proof this inequality is written in the form $\|x_n - x_{n-1}\| \leq M^{n-1}\|x_2 - x_1\|$. It seems to be a mistake, as for $n = 2$ we would infer that $x_1 = x_2$.

(1) *T has a unique fixed point x_* in X.*
(2) *For an arbitrary point $x_0 \in X$, the sequence $(x_n)_{n\in\mathbb{N}}$ generated by the Picard iteration process (defined by $x_{n+1} = T(x_n)$, $n \in \mathbb{N} \cup \{0\}$) converges to x_*.*
(3) *$d(x_n, x_*) \le \frac{\alpha^n}{1-\alpha} d(x_0, x_1)$ for all $n \in \mathbb{N}$.*

In the proof of the above version of the Banach theorem it is shown that the sequence $(x_n)_{n\in\mathbb{N}}$ defined above is a Cauchy sequence. The completeness of X implies that this sequence converges to some element x_*. Then it is proved that x_* is a fixed point of T and that it is unique. By inequality (3) from the above theorem we can estimate the accuracy with which we approach the fixed point.

Note that one immediate corollary from the proof is the following: if T is a contraction mapping, then each iterate T^n possesses exactly one (the same) fixed point. Let us mention that the study of iterates of selfmappings is part of the dynamical systems theory.

Definition 1 [82] We say that $x_* \in X$ is a contractive fixed point (abbr. CFP) of T if $x_* = T(x_*)$ and the Picard iterates $T^n(x)$ converge to x_* as $n \to \infty$ for all $x \in X$.

Later, the operator T from Definition 1 became known as a Picard operator (shortly PO), see, e.g., [106].

In 1930 similar results were obtained independently by Renato Caccioppoli [26], who rediscovered and generalized Banach's theorem for complete metric space. Due to this fact for many mathematicians the theorem is known under the name *Banach–Caccioppoli theorem* (see for instance [119]).

One of the first extensions of the Banach Contraction Principle is the following:

Theorem 3 [74, 116] *Let (X, d) be a complete metric space and let $T\colon X \to X$ be a mapping such that for each $n \ge 1$, there exists a constant c_n such that $d(T^n(x), T^n(y)) \le c_n d(x, y)$ for all $x, y \in X$, where $\sum_n c_n < \infty$. Then T is a Picard operator.*

However, as shown by Kazimierz Goebel [46] Theorem 3 is in some sense equivalent to the Banach Fixed Point Theorem. Namely, under the above assumptions, there existsts a metric ρ on X which is Lipschitz equivalent to d, i.e., there exist $\alpha, \beta > 0$ such that $\alpha d \le \rho \le \beta d$, and T is a Banach contraction with respect to ρ (see also [47]).

There are various proofs of the Banach Fixed Point Theorem. One proof [17] was based on the Cantor intersection theorem and the observation that the sets $C_m = \left\{x \in X\colon d(x, T(x)) \le \frac{1}{m}\right\}$ $(m = 1, 2, \ldots)$ have a nonempty intersection. Another proof can be found in the paper [6]. Andrei Baranga endowed the Cartesian product of a metric space (X, d) and $[0, \infty)$ with a partial order and then he applied Kleene's fixed point theorem which was formulated for selfmappings of partially ordered sets. Recall that a mapping T is a selfmapping of X if T acts from X into X.

In 2007 Richard S. Palais presented another proof of the Banach Contraction Principle (in [96]). The author proves the Principle without using the formula for the sum of a geometric series but with the use of so called the fundamental contraction inequality (which was obtained from the triangle inequality applied twice): if $T\colon X \to X$ is a contraction mapping with contraction constant α, then for all $x_1, x_2 \in X$,

$$d(x_1, x_2) \le \frac{1}{1-\alpha}\big(d(x_1, T(x_1)) + d(x_2, T(x_2))\big).$$

This inequality immediately implies that T cannot have more than one fixed point. Moreover, using it, one can easily show that for any point x_0, the sequence $(T^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence.

One of disadvantages of the Banach Fixed Point Theorem is that it used rather sharp assumption concerning mapping T. Independently, in 1957 Andrey Kolmogorov and Sergey Fomin, and in 1962 Frank F. Bonsall proved the theorem in which assumptions about T were weakened. It can be formulated in the following way.

Theorem 4 [13, 77] *Let T be a continuous selfmapping of a complete metric space X such that the iterate T^k is a contraction mapping of X for some positive integer k. Then T is a Picard operator.*

In 1968 Victor W. Bryant showed that the assumption about continuity of T is not necessary and one can skip it without changing the thesis of the above theorem.

Theorem 5 [24] *If T is a selfmapping of a complete metric space and if, for some positive integer k, T^k is a contraction, then T has a unique fixed point.*

However, in many applications it happens that Theorem 4 is sufficient. It turns out that in this case there exists a complete metric ρ, which is equivalent to d and such that T is a contraction with respect to ρ.

3 Contractive type mappings

Banach's proof of Theorem 1 was clear and concise. Currently, most math students learn about it during the first year of their studies. As we know every Banach contraction is continuous. Clearly, there are functions which are not continuous but they have fixed points. Thus a natural question arises: do there exist contractive type mappings which need not be continuous but their definition is strong enough to ensure the existence of a fixed point? The first answer was obtained by Rangachary Kannan in 1968 in the following theorem.

Theorem 6 [67] *Let (X, d) be a complete metric space, $T : X \to X$ be a mapping for which there exists constant $r \in (0, \frac{1}{2})$ such that*

$$d(T(x), T(y)) \leq r\big[d(T(x), x) + d(T(y), y)\big] \tag{1}$$

for all $x, y \in X$. Then T has a unique fixed point in X.

An operator T fulfilling condition (1) is called a Kannan contraction. So the above theorem says that if X is complete, then every Kannan mapping has a fixed point. It is interesting that the converse theorem is also true, i.e., if (X, d) is a metric space such that every Kannan contraction for some fixed $r \in (0, \frac{1}{2})$ has a fixed point, then (X, d) is complete. It was proved by Papagudi V. Subrahmanyam [108]). One can find a Kannan contraction which is not continuous, so it cannot be a Banach contraction. On the other hand, if $r = \frac{1}{2}$ then the Kannan contraction may not have a fixed point, even if it is continuous [50].

In 1972 Santi K. Chatterjea replaced condition (1) with

$$d(T(x), T(y)) \leq r\big[d(T(x), y) + d(T(y), x)\big] \text{ for each } x, y \in X \tag{2}$$

and obtained the analogous result [28]. A map T satisfying (2) is said to be a Chatterjea contraction. In 1977 Billy E. Rhoades observed that those three mentioned contractions of Banach, Kannan and Chatterjea are independent [105]. In the meantime, Ljubomir B. Ćirić [33] obtained an extension of theorems of Kannan and Chaterjea: every selfmapping T of a complete metric space (X, d) such that for all $x, y \in X$

$$d(T(x), T(y)) \leq r \max\big\{d(x, y), d(x, T(x)), d(y, T(y)), d(x, T(y)), d(y, T(x))\big\},$$

where $r \in [0, 1)$, has a unique fixed point.

Recently, in [100] it was shown that Banach's, Kannan's and Ćirić's fixed point theorems are equivalent in the following sense: for a complete metric space (X, d) and a Ćirić (in particular, Kannan or Chaterjea) contraction T there exists a metric ρ on X such that (X, ρ) is complete, there exists $\alpha > 0$ such that $d \leq \alpha\rho$, and T is a Banach contraction with respect to ρ. The reverse result is also obtained by redefining the metric.

Several types of contractions and their interrelations have been discussed in the survey due to Rhoades. In the paper [105] he presented 25 definitions of basic contractions (and more than 100 of their modifications) and examined the relations between them. All of the contractions from [105] have the property that if they have a fixed point, then it is unique.

Further generalization of the Banach Contraction Principle was given by Caristi.

Theorem 7 [27] *Let (X, d) be a complete metric space and let T be a selfmapping of X. Assume that there exists a lower semi-continuous function $\psi \colon X \to [0, \infty)$ such that*

$$d(x, T(x)) \leq \psi(x) - \psi(T(x)) \text{ for } x \in X.$$

Then T has a fixed point.

Recall that ψ is lower semi-continuous if

$$\liminf_{x \to x_0} \psi(x) \geq \psi(x_0) \text{ for all } x_0 \in X.$$

It is worth noting that the hypothesis of the Banach Contraction Principle is implied by the Caristi theorem with $\psi(x) = \frac{1}{1-\alpha} d(x, T(x))$, where α is a contraction constant from Theorem 2.

In the next part of the article we will discuss some other contractive type conditions. Let (X, d) be a complete metric space and T be a selfmapping of X. $\mathbb{R}_+$ denotes the set of all non-negative reals.

The Banach theorem can be formulated in the form: every Banach contraction on a complete metric space has a CFP (or every Banach contraction is a PO). There are numerous results in the literature giving sufficient conditions for the existence of a CFP, but the Banach principle is still the most important here for its simplicity and an amazing efficiency in applications.

As far as we know the first significant generalization of Banach's principle was obtained in [101]. E. Rakotch was advised by Haim Hanani to define the family of contractive type mappings which are now called *Rakotch contractions* and then definition is as follows.

Definition 2 [101] We say that T is a Rakotch contraction ($T \in \mathtt{Ra}$) if there is a non-increasing function $\alpha \colon \mathbb{R}_+ \to [0, 1]$ such that $\alpha(t) < 1$ for each $t > 0$ and

$$d(T(x), T(y)) \leq \alpha\big(d(x, y)\big) \cdot d(x, y) \quad \text{for all} \quad x, y \in X. \tag{3}$$

Clearly, a Rakotch contraction with a constant function α is a Banach contraction. Every Rakotch contraction is a PO [101]. It is worth noting that Kannan's definition is independent from that of Rakotch (see [105]).

In 1968 Felix E. Browder introduced more general definition, which now can be formulated in the following way.

Definition 3 [23] If there exists a function $\varphi \colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(t) < t \quad \text{for all} \quad t > 0$$

and

$$d(T(x), T(y)) \leq \varphi\big(d(x, y)\big) \text{ for all } x, y \in X,$$

then we say that T is *φ-contractive*.

A selfmap T is called a *Browder contraction* ($T \in \mathtt{Br}$) if it is φ-contractive for a non-decreasing and right continuous function φ.

Any Browder contraction is a PO. The classical Banach theorem can be obtained for $\varphi(t) = \alpha t$ with a contraction constant $0 < \alpha < 1$. In the paper [23] the proof was done with unnecessary assumption that the space (X, d) is bounded.

The result of Browder was extended by David W. Boyd and James S.W. Wong [18] in 1969. They observed that instead of right continuity (in the definition of Browder contraction) it was enough to assume only the right upper semi-continuity of φ, i.e.

$$\limsup_{s \to t^+} \varphi(s) \leq \varphi(t) \text{ for all } t \in \mathbb{R}_+.$$

Such T is called then a *Boyd–Wong contraction* ($T \in \mathtt{BW}$) and the boundedness of (X, d) is unnecessary. Moreover, in their result the first condition from Definition 3 cannot be dispensed. If the condition $\varphi(t) < t$ is not satisfied even in one point, then T may not have a fixed point or the existing fixed point may not be unique.

The class of Browder contractions is a proper subclass of the Boyd–Wong class. Many equivalent conditions for $T \in \mathtt{BW}$ and also for $T \in \mathtt{Br}$ are given in [60]. In the same paper it was shown that the class of Rakotch contractions is identical with the class of φ-contractive mapping where φ is strictly increasing and concave.

Another generalization of Banach's principle was given in 1972 by Mark A. Krasnosel'skiĭ [78]. We say that T is a *Krasnosel'skiĭ contraction* ($T \in \mathtt{Kr}$) if given $a, b \in \mathbb{R}_+$, with $0 < a < b$ there is an $L(a, b) \in [0, 1)$, such that for all $x, y \in X$,

$$\text{if } a \leq d(x, y) \leq b, \text{ then } d(T(x), T(y)) \leq L(a, b)\, d(x, y).$$

Later it was shown that this definition is equivalent to the one of Browder.

Very soon further two conditions were proposed by Michael A. Geraghty in 1973 and 1974 [43, 44]. A selfmap T is said to be a *Geraghty (I) contraction* ($T \in \mathtt{Ge\ I}$) if T satisfies condition (3) with the function $\alpha\colon \mathbb{R}_+ \to [0, 1]$ having the property that given a sequence $(t_n)_{n\in\mathbb{N}}$

$$\alpha(t_n) \to 1 \text{ implies } t_n \to 0.$$

It turns out that T is a Geraghty (I) contraction if only if it is a Rakotch contraction. The definition of Geraghty (I) class of mappings was modified in [44]. The above property of α was replaced by the following: given a sequence $(t_n)_{n\in\mathbb{N}}$,

$$\text{if } t_n \to 0 \text{ is non-increasing and } \alpha(t_n) \to 1 \text{ then } t_n \to 0$$

and such modification of T was called *Geraghty (II) contraction* ($T \in \mathtt{Ge\ II}$). This class of mappings (as shown in [53]) coincides with the Boyd–Wong class of mappings.

The following variant of the above conditions was also considered: T is a *Geraghty (III) contraction* ($T \in \mathtt{Ge\ III}$) if T satisfies condition (3) with the function $\alpha\colon \mathbb{R}_+ \to [0, 1]$ such that given a sequence $(t_n)_{n\in\mathbb{N}}$

$$\text{if } (t_n)_{n\in\mathbb{N}} \text{ is bounded and } \alpha(t_n) \to 1 \text{ then } t_n \to 0.$$

This definition turned out to be equivalent to the one of Browder, so unfortunately all Geraghty contractions are equivalent to some other known ones.

Another variant of Browder's condition was given by Janusz Matkowski in [87]. We say that T is a *Matkowski contraction* ($T \in \mathtt{Ma}$) if there exists non-decreasing function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{n\to\infty} \varphi^n(t) = 0 \text{ for all } t > 0$$

and T is φ- contractive. The classes of Matkowski and Boyd–Wong are incomparable. This class of mappings is wider than Browder's class, but if T is a Matkowski contraction, then the second iterate T^2 satisfies Browder's condition. In some sense both these conditions are equivalent, more precisely, we can say that the conditions are iteratively equivalent. Moreover, by Browder theorem, since T^2 is a PO, so is T. The class of Browder contractions is a proper subclass of Matkowski's class.

In 1976 James Dugundji established a theorem from which he derived a fixed point theorem for the following class of mappings [36]. We say T is a *Dugundji contraction* ($T \in \mathtt{Du}$) if for given $\varepsilon > 0$ there is $\delta > 0$, such that for all $x, y \in X$,

$$d(x, y) - d(T(x), T(y)) < \delta \text{ implies } d(x, y) < \varepsilon.$$

As shown in the above mentioned paper, Dugundji theorem yields Browder's result under the assumption that (X, d) is bounded. It was shown that without the boundedness assumption, the class of Browder's contractions is essentially wider than the one of Dugundji: There exists a φ-contractive map in Browder's class which is not a Dugundji contraction.

Another contractive definition was given by James Dugundji and Andrzej Granas in [37]. A mapping T is said to be *Dugundji–Granas contraction* ($T \in \mathtt{DG}$) if

$$d(T(x), T(y)) \leq d(x, y) - \Theta(x, y) \quad \text{for all} \quad x, y \in X,$$

where the function $\Theta\colon X \times X \to \mathbb{R}$ is compactly positive on X, i.e., given $a, b \in \mathbb{R}_+$, such that $0 < a < b$

$$\inf\{\Theta(x, y)\colon a \leq d(x, y) \leq b\} > 0.$$

The above definition is equivalent to that of Krasnosel'skiĭ [37], so to the one of Browder as well.

In 1977 Matkowski introduced φ-contractive maps with a non-decreasing and continuous function φ satisfying the following limit condition:

$$\lim_{t\to\infty} (t - \varphi(t)) = \infty.$$

Independently, the same class of functions appeared also in the paper of Wolfgang Walter [114], therefore such φ-contractive maps are said to be *Matkowski–Walter contractions* ($T \in \mathtt{MW}$). There exists a φ-contractive map in Matkowski–Walter class which is not a Rakotch contraction and there is a φ-contractive map in Dugundji's class which is not a Matkowski–Walter contraction.

In 1996 Theodore A. Burton introduced the following definition [25]: we say that T is a large contraction (*Burton contraction* ($T \in \mathtt{Bu}$)) if given $a > 0$ there is $L(a) \in [0, 1)$, such that for all $x, y \in X$,

$$\text{if } d(x, y) \geq a, \text{ then } d(T(x), T(y)) \leq L(a)d(x, y).$$

Unexpectedly, Burton's condition is equivalent to the Rakotch one, so in some sense we come back to the beginning of the theory of contractive type mappings.

In view of the results, the following relations hold

$$\begin{array}{ccccccccccc}
 & & & & \mathtt{Ba} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & \mathtt{Bu} & = & \mathtt{Ra} & = & \mathtt{Ge\ I} & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathtt{MW} & & & & & & \\
 & & & & \downarrow & & & & & & \\
 & & & & \mathtt{Du} & & & & & & \\
 & & & & \downarrow & & & & & & \\
\mathtt{Ta} & = & \mathtt{DG} & = & \mathtt{Br} & = & \mathtt{Kr} & = & \mathtt{Ge\ III} & & \\
 & & & \swarrow & & \searrow & & & & & \\
 & & \mathtt{Ma} & & & & \mathtt{BW} & = & \mathtt{Ge\ II} & = & \mathtt{Mu} \\
 & & \downarrow & & & & \downarrow & & & & \\
 & & \mathtt{Le} & & \longleftarrow & & \mathtt{MK} & & & &
\end{array}$$

In the above diagram [60]:

$\mathtt{MK}$ is the class of *Meir–Keeler contraction* [68]: $T \in \mathtt{MK}$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies } d(T(x), T(y)) < \varepsilon \text{ for all } x, y \in X.$$

Meir and Keeler proved that if $T \in \mathtt{MK}$, then T is a Picard operator.

$\mathtt{Mu}$ is the class of *Mukherjea contractions* [93]: $T \in \mathtt{Mu}$ if T is φ-contractive where φ is right continuous.

$\mathtt{Ta}$ is the class of *Tasković contractions* [112]: $T \in \mathtt{Ta}$ if T is φ-contractive, where φ is such that

$$\limsup_{s \to t} \varphi(s) < t \text{ for } t > 0.$$

$\mathtt{Le}$ is the class of *Leader contractions* [83]: $T \in \mathtt{Le}$ if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \exists r \in \mathbb{N}\ \forall x, y \in X\ (d(x, y) < \varepsilon + \delta \Rightarrow\ d(T^r x, T^r y) < \varepsilon).$$

There are also contractive mappings of integral type which were introduced by Alberto Branciari in 2002 [19]: for a locally Lebesgue integrable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^t f(s)ds > 0$ for any positive t, a selfmapping T satisfying the condition

$$\int_0^{d(T(x),T(y))} f(s)ds \leq \alpha \int_0^{d(x,y)} f(s)ds$$

for some $\alpha \in (0, 1)$ and all $x, y \in X$ is called a Branciari contraction. It was shown in [110] that every Branciari contraction is a Meir–Keeler contraction. Moreover, it is also a Browder contraction.

In 1969 Sam B. Nadler, Jr. extended Banach's Contraction Principle to multivalued contractions (see [95]). By $CB(X)$ he denoted the family of nonempty closed and bounded subsets of a metric space (X, d). The family $CB(X)$ is a metric space with the Hausdorff metric H defined by

$$H(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\right\} \quad \text{for all } A, B \in CB(X).$$

A function $F\colon (X,d)\to (CB(X),H)$ is said to be a multi-valued Lipschitz mapping of X if $H(F(x),F(z))\le \alpha d(x,z)$ for all $x,z\in X$ and some $\alpha\ge 0$. If F has the Lipschitz constant $0<\alpha<1$, then F is called a multi-valued contraction mapping. A point x is called a fixed point of multi-valued mapping F provided $x\in F(x)$. Since the mapping $i : X\to CB(X)$, given by $i(x)=\{x\}$ for each $x\in X$, is an isometry, the fixed point theorems presented by Nadler are generalizations of their single-valued analogues. A corresponding extension of Banach's classical result is known as the Nadler contraction principle.

Theorem 8 [95] *Let (X,d) be a complete metric space. If $F\colon X\to CB(X)$ is a multivalued contraction mapping, then F has a fixed point.*

The above Nadler's fixed point theorem for multi-valued contractive mappings has been extended in many directions. As can be expected, the generalizations concerned both: modifications of a definition of multi-valued contractions and spaces. In many cases, the Hausdorff metric was not involved and some results were proved without using the concept of a Hausdorff metric. Some mutivalued versions of the Banach Contraction Principle with respect to generalized distances also appeared.

It is necessary to emphasize that the information given in this section is far for being comprehensive since there is a huge number of papers dealing with contractive type conditions:

> A complete survey of all that has been written about contraction mappings would appear to be nearly impossible, and perhaps not really useful [74].

4 Other extensions of the Banach Contraction Principle

The contractive condition from the Banach Fixed Point Theorem

$$d(f(x),f(y))\le \lambda d(x,y), \tag{4}$$

where $x,y\in X$ and $\lambda\in[0,1)$, is a global one. In [38] a function fulfilling the above condition was called *a globally contractive mapping*. Michael Edelstein asked whether the Banach Contraction Principle could be modified in such a way that this condition is assumed not for all $x,y\in X$ but for sufficiently close points only. In recent years there has been observed some activity in studying mappings that are only locally (or pointwise) contractive. The name *shrinking mapping* is also used. We may find it among others in [94, p. 182] and [31]. In the latter paper the following global and local conditions are examined.

Definition 4 [31] Let X be a metric space and $f\colon X\to X$. Then

- (C) f is contractive with a contraction constant λ if (4) is fulfilled;
- (S) f is shrinking if $d(f(x),f(y))<d(x,y)$ for all $x,y\in X$ with $x\ne y$ (in [13, 39, 106] they are called contractive);
- (LC) f is locally contractive provided that for every $y\in X$, there exists an open U such that $y\in U$ and $f\restriction U$ is contractive (with a contraction constant λ);
- (LS) f is locally shrinking provided that for every $y\in X$, there exists an open U such that $y\in U$ and $f\restriction U$ is shrinking;
- (ULC) f is uniformly locally contractive provided that there exist $\varepsilon>0$ and $\lambda\in[0,1)$ such that for every $y\in X$, the restriction $f\restriction B(y,\varepsilon)$ is contractive with a contraction constant λ. An abbreviation (ε,λ)-(ULC) is also used;
- (ULS) f is uniformly locally shrinking provided that there exists a number $\varepsilon>0$ such that the restriction $f\restriction B(y,\varepsilon)$ is shrinking for every $y\in X$.

($B(y, \varepsilon)$ is a ball centered at y with radius ε).

Krzysztof Cris Ciesielski and Jakub Jasinski also present some pointwise notions of contractive and shrinking maps and their several uniform versions, which led to twelve classes of mappings. Ten of them turned out to be distinct. We present only six of them. In [31] one may find the complete survey. The authors fully discuss the inclusions among these classes of functions and the fixed and periodic point theorems available for these mapping.

In [38, 39] the following theorem for mappings on compact spaces was proved.

Theorem 9 *Let X be compact and $f\colon X \to X$.*

(1) *If f is (S), then f has a unique fixed point.*
(2) *If f is (LS), then f has a periodic point x (i.e., $f^{(n)}(x) = x$ for some $n \in \mathbb{N}$).*
(3) *If f is (LS) and X is connected, then f has a unique fixed point.*

To prove (1) it suffices to consider the function $X \ni x \mapsto d\big(x, f(x)\big)$, which is continuous, so by compactness of X, it attains its minimum at some x_*. Then $x_* = f(x_*)$.

Part (1) of Theorem 9 has been applied to some integral equations of Abel–Liouville type; see [104].

Observe that in the above theorem part (1) is false for noncompact complete spaces. The map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{2}\big(x + \sqrt{x^2+1}\big)$ is shrinking but it is not a contraction. It has no fixed point (see [94, p. 182] or [31]).

It is worth noting that also a shrinking self-mapping of a complete bounded metric space need not have a fixed point (examples can be found in [47]).

In 1966 Donald Bailey proved that if X is compact and T is continuous and such that for every $x, y \in X$, $x \neq y$, there exists a positive integer $n(x, y)$ such that $d(T^{n(x,y)}x, T^{n(x,y)}y) < d(x, y)$, then T has a unique fixed point in X [2].

Let us also mention that fixed point theorems for mappings satisfying conditions (ULC) or (ULS) from Definition 4 can be proven by the use of remetrization method as done in [31]. We now present their approach.

For $\varepsilon > 0$, we say that X is ε-*chainable*, provided for every $p, q \in X$, there exists a finite sequence $s = \langle x_0, x_1, \ldots, x_n\rangle$, referred to as an ε-*chain from p to q*, such that $x_0 = p$, $x_n = q$, and $d(x_i, x_{i+1}) \le \varepsilon$ for every $i < n$. The length of the ε-*chain* s is defined as $l(s) = \sum_{i<n} d(x_i, x_{i+1})$.

Theorem 10 *Let $\varepsilon > 0$ and assume that (X, d) is connected or, more generally ε-chainable. Then the map $\widehat{D}\colon X^2 \to [0, \infty)$ given as*

$$\widehat{D}(x, y) = \inf\{l(s)\colon s \text{ is an } \varepsilon\text{-chain from } x \text{ to } y\}$$

is a metric on X topologically equivalent to d. If (X, d) is complete, then so is $(X, \widehat{D})$.

Moreover

(i) *If $f\colon (X, d) \to (X, d)$ is (η, λ)-(ULC) for some $\eta > \varepsilon$, then $f\colon (X, \widehat{D}) \to (X, \widehat{D})$ is a contraction with constant λ.*
(ii) *If (X, d) is compact and $f\colon (X, d) \to (X, d)$ is (ULS) with constant $\eta > \varepsilon$, then $f\colon (X, \widehat{D}) \to (X, \widehat{D})$ is shrinking.*

(for abbreviations used in (i) and (ii), see Definition 4).

Another paper in which the assumption of contractiveness is relaxed to a local condition was [56]. The proof of Thakyin Hu and William A. Kirk was corrected in [65].

Definition 5 A map $f : X \to X$ is pointwise contractive, if for every point $x \in X$ there exists a number $\lambda_x \in [0, 1)$ and an open neighborhood $U_x \subset X$ of x such that $d(f(x), f(y)) \leq \lambda_x d(x, y)$ for all $y \in U_x$. If the same $\lambda \in [0, 1)$ works for all $x \in X$ then we say that f is uniformly pointwise contractive.

A fixed point theorem for such functions was obtained under the assumption that a metric space X is rectifiably-path connected, i.e., such that any points $x, y \in X$ can be connected in X by a path $p : [0, 1] \to X$ of finite length.

Theorem 11 [56, 65] *If X is a rectifiably-path connected complete metric space and a map $f : X \to X$ is uniformly pointwise contractive, then f has a unique fixed point.*

A similar theorem was proved for (LC) functions.

Theorem 12 [30] *If X is a rectifiably-path connected complete metric space and a map $f : X \to X$ is locally contractive, then f has a unique fixed point.*

If in condition (4) of Theorem 9 constant λ is equal to 1, then f is called *nonexpansive*. Clearly, any contraction (meaning contractive with a contraction constant λ) is a contractive mapping, any shrinking mapping is nonexpansive, but the converse implications are not true in general. Moreover, all such mappings are continuous. If contractive/shrinking mappings have a fixed point, then this fixed point is obviously unique. For nonexpansive mappings this may not be true. The next theorem was obtained for nonexpansive mappings and it was proved in 1965 independently by three mathematicians: Felix E. Browder, Dietrich Göhde and William A. Kirk.

Theorem 13 [22, 49, 70] *If X is a non-empty closed bounded convex subset of a Hilbert space H and $f : X \to X$ is a nonexpansive map, then f has a fixed point.*

In fact, their results were more general. In particular, it suffices that H is a uniformly convex Banach space. Let us note that the Banach Contraction Principle is used in the proof of Theorem 13 to show that $\inf\{\|x - fx\| : x \in X\} = 0$.

One can find many extensions of Banach's theorem, but according to [75] most of them were just trivial:

> Most of these extensions are fairly routine and many are completely trivial. However, there is one far-reaching extension that appears to be very deep.

This extention, mentioned by Kirk, is the conjecture named *Generalized Banach Contraction Conjecture* (GBCC, in short).

Conjecture 1 [62] Let (X, d) be a complete metric space, $0 < M < 1$, T a selfmapping of X. Let J be a set of positive integers. Assume that T satisfies the condition

$$\inf\{d(T^k x, T^k y) : k \in J\} \leq M d(x, y). \tag{5}$$

Then T has a fixed point.

The authors stated that: if $J = \{1\}$, then we get Banach's original result (see [13]), if $J = \{p\}$ $(p > 0)$, then we obtain Theorem 5 of Bryant, if J is infinite set of positive integers, then GBCC is not true. It remains to check whether GBCC is taking place for $J = \{1, 2, \ldots, N\}$ $(N \in \mathbb{N})$. If T is strongly continuous, then the answer is affirmative. However, the original condition in the conjecture does not demand the continuity neither T nor T^k.

Assume that J is a finite subset of the positive integers. In the above mentioned paper two questions were asked: whether GBCC is true when T satisfies (5) and if GBCC is true when T satisfies (5) and is continuous? The authors gave the affirmative answer to the first question for the case $J = \{1, 2\}$ and the positive answer for the second question when $J = \{1, 2, 3\}$. Moreover, if N is a positive integer, then these results imply the truth of GBCC when: $J = \{N, 2N\}$, $J = \{N, 3N\}$ and $J = \{2N, 3N\}$. In the same 1999 year there was published another paper [63], in which it is shown that GBCC is true for arbitrary J when T is uniformly continuous. This result was improved in 2002 [89] in such a way that the uniform continuity was replaced by continuity. The proof used the Ramsey Colouring Theorem. The full answer to the first question was presented in the same year in the paper [90]. It was shown that GBCC is true for any finite set J and the continuity of T is not needed. Independently, the same result was obtained by Alexander D. Arvanitakis [1].

The Banach Fixed Point Theorem gives us information about the convergence of Picard iterates to the unique fixed point of T. It is natural to ask, if the similar property holds for GBCC. The answer to this question is affirmative which was shown in 2008 by Simeon Reich and Alexander J. Zaslavski. They introduced the notion of *Jachymski–Schröder–Stein contraction* in the following way.

Definition 6 [103] Let (X, d) be any complete metric space, $\phi\colon [0, \infty) \to [0, \infty)$ a function which is right upper semi-continuous and satisfies $\phi(t) < t$ for $t > 0$. A mapping $T\colon X \to X$ is called Jachymski-Schröder-Stein contraction (with respect to ϕ), if for all $x, y \in X$:

$$\min\{d(T^k x, T^k y)\colon k \in \{1, 2, \ldots, N_0\}\} \le \phi(d(x, y)).$$

Using this definition they proved two theorems: in the first they established convergence of iterates to a fixed point, and in the second this conclusion was strengthened to obtain uniform convergence on bounded subsets of X.

5 Mappings on spaces with distance-type functions

In all our considerations we use metric spaces. We can generalize this notion by modifying some of the axioms of metric spaces. Thus, several other types of spaces have been introduced and a lot of results have been extended to the new settings. It was not immediately observed that such spaces may be different than metric spaces: the modified metric need not be continuous in both variables, the topology may not be Hausdorff, a sequence may converge to more than one point, a convergent sequence may fail to be a Cauchy sequence, an open ball cannot be an open set (although the definitions of convergent or Cauchy sequences, of balls and diameter are as usual and the topology is induced by open balls).

Now we want to present some modifications of metric and corresponding versions of the Banach Fixed Point Theorem for them. Recall that a function $d\colon X \times X \to \mathbb{R}_+$ is a metric if the following conditions are satisfied for all $x, y, z \in X$:

- $d(x, y) = 0 \iff x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) \le d(x, z) + d(z, y)$.

If d fulfills only first two conditions, then (X, d) is called a semimetric space. In 1989 another axiom for semimetric spaces (weaker than triangle inequality) was used in [3].

Definition 7 A semimetric space (X, d) is said to be a b-metric space (or quasimetric space) if there exists $s \geq 1$ such that for each $x, y, z \in X$,

$$d(x, y) \leq s\big[d(x, z) + d(z, y)\big].$$

It seems that this notion originates from the monograph [16]. Obviously, every b-metric space is a semimetric space and if $s = 1$ then b-metric space becomes a metric space (examples of b-metric spaces can be found in [75] and in [106]). Ivan A. Bakhtin presented the Contraction Principle for mappings on b-metric spaces that is a generalization of the Banach Fixed Point Theorem.

Theorem 14 [3] *Let (X, d) be a complete b-metric and suppose $f: X \to X$ satisfies for some $\alpha \in [0, 1)$,*

$$d(f(x), f(y)) \leq \alpha d(x, y) \tag{6}$$

for every $x, y \in X$. Then f has a unique fixed point $x_* \in X$ and $\lim_{n\to\infty} f^n(x) = x_*$ for all $x \in X$.

It turns out that if a semimetric space (X, d) is bounded, then we need not any substitute of the triangle inequality to establish a fixed point theorem. The following theorem is a partial extension of the contraction principle.

Theorem 15 [61] *Let (X, d) be a Hausdorff semimetric and d-Cauchy complete space and f be a selfmap on X satisfying the Banach contractive condition:*

$$d(f(x), f(y)) \leq hd(x, y) \text{ for some } h \in [0, 1) \text{ and all } x, y \in X.$$

If (X, d) is bounded, i.e., $M = \sup\{d(x, y) : x, y \in X\} < \infty$, then f has a unique fixed point p and for any $x \in X$, $(f^n(x))_{n\in\mathbb{N}}$ converges to p.

It is shown that the above theorem cannot be extended to unbounded semimetric spaces. Recall that (X, d) is d-Cauchy complete, if every d-Cauchy sequence is convergent. A sequence $(x_n)_{n\in\mathbb{N}}$ is d-Cauchy if given $\varepsilon > 0$ there is a $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq k$.

In [10] the authors extended Bakhtin's theorem to the class of φ-contractions, where φ is a comparison function, i.e., is increasing and such that $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t \geq 0$. Given a semimetric space (X, d) and a comparison function φ, we have the analogous definition of mapping $T: X \to X$ called a φ-contraction as we have in metric spaces. Recall that it fulfills the condition $d(T(x), T(y)) \leq \varphi(d(x, y))$ for $x, y \in X$. The results obtained in [10] for semimetric spaces can be combined in the following theorem.

Theorem 16 [10] *Let (X, d) be a semimetric space and φ be a comparison function.*

1. *If $T: X \to X$ is a φ-contraction, then T has at most one fixed point.*
2. *If (X, d) is d-Cauchy complete and regular, then every φ-contraction has a unique fixed point.*
3. *If (T_n) is a sequence of φ-contractions converging pointwise to a φ-contraction $T_0: X \to X$, then the sequence of the fixed points of (T_n) converges to the unique fixed point of T_0.*

Recall that a semimetric space is regular if the basic triangle function, i.e., function $\Phi_d : \overline{\mathbb{R}_+} \times \overline{\mathbb{R}_+} \to \overline{\mathbb{R}_+}$ defined by the formula

$$\Phi_d(u, v) = \sup\{d(x, y) : \exists_{p\in X}\ d(p, x) \leq u \wedge d(p, y) \leq v\}\ (u, v \in \overline{\mathbb{R}_+})$$

is continuous at the origin. Actually, part 2 of Theorem 16 was established in the paper [61] in which the authors even used somewhat weaker assumption on a space than its regularity. Much more information on the topic of this chapter can be found in a nice book of William A. Kirk and Naseer Shahzad [76].

6 Converses to the Banach Fixed Point Theorem

In 1959 Czesław Bessaga presented his version of the theorem which is a converse to the Banach Fixed Point Theorem. Firstly he observed that if an operator is Lipschitzian with a constant less than 1, so is each iterate of it. So the theorem of Banach can be formulated in the following way: If a selfmapping U of a complete metric space satisfies the Lipschitz condition with a constant less than 1, then each iteration of U has a unique fixed point. This reformulation of Banach theorem admitted the following converse (we quote after Bessaga):

Theorem 17 [9] *Suppose that U is a mapping of an abstract set X into itself such that each iteration U^n $(n = 1, 2, \ldots)$ of U has a unique fixed point. Let K be any number with $0 < K < 1$. Then there exists a complete metric d on X such that U satisfies the Lipschitz condition with constant K.*

Nowadays the converse of the Banach Contraction Principle is formulated in a slightly more general form:

Theorem 18 [34] *Let $X \neq \emptyset$ be an arbitrary set, $F : X \to X$ and $k \in (0, 1)$. Then*

(a) *If F^n has at most one fixed point, for every $n \geq 1$, then there exists a metric d such that $d(Fx, Fy) \leq kd(x, y)$ on $X \times X$.*
(b) *If, in addition, some F^n has a fixed point, then there is a complete metric d such that $d(Fx, Fy) \leq kd(x, y)$ on $X \times X$.*

In fact Bessaga proved the second part of the above theorem. Moreover, he showed that it is equivalent to some form of the Axiom of Choice.

In 1965 James S. Wong published a generalization of Theorem 17 which was a part of his PhD thesis. Firstly, he noticed that if $\{T_1, T_2, \ldots, T_n\}$ is a commuting family of contractions, then every element of the commutative semigroup generated by these mappings is again a contraction. So he could extend the concept of a contraction to the concept of a contractive semigroup. He answered the question whether there exists a metric ρ on X in which mutually commuting mappings $T_1, T_2, \ldots, T_n$ with common unique fixed point are simultaneously contractions with respect to ρ.

Theorem 19 [118] *Let X be an abstract set with n mutually commuting mappings $T_1, T_2, \ldots, T_n$ defined on X into itself such that each composition $T_1^{k_1} \circ T_2^{k_2} \circ \ldots \circ T_n^{k_n}$ (where $k_1, \ldots, k_n$ are non-negative integers not all equal to zero) possesses a unique fixed point which is common to every choice of $k_1, \ldots, k_n$. Then for each $\lambda \in (0, 1)$, there exists a complete metric ρ on X such that $\rho(T_i x, T_i y) \leq \lambda\rho(x, y)$ for $1 \leq i \leq n$ and for all $x, y \in X$.*

The theorem of Bessaga is a special case of the above theorem for $n = 1$. Wong also observed that his theorem cannot be extended to the case of a countable infinite family of mappings.

In 2000 Jachymski published another proof of Bessaga's theorem. He modified the proof presented in [34] to give possibly the simplest proof of Bessaga's theorem. He extended the part (a) of Theorem 18 and showed that for bounded spaces the converse to the contraction principle can be proved without using any form of the axiom of choice (see [58]).

Next version of the converse theorem was proved by Ludvík Janoš in 1963 (published in 1967 in [64]). Janoš assumed that X is a compact metrizable topological space and T is a continuous selfmapping of X such that $\bigcap_{n=1}^{\infty} T^n(X) = \{a\}$ and he asked whether it is possible to find a metric ρ generating the given topology of X such that the mapping T is contractive with respect to ρ. Janoš's answer was affirmative and he gave the method of constructing such a metric.

Independently, the converse to the Banach Fixed Point Theorem was presented by Philip R. Meyers in 1967.

Theorem 20 [91] *Let f be a continuous selfmapping of a metric space (X, d). Assume that f fulfills conditions: f has a fixed point x_*, $(f^n(x))$ converges to x_* for all $x \in X$, and there exists an open neighborhood U of x_* such that $f^n(U) \to \{x_*\}$, i.e., the sequence $(f^n \restriction U)$ is uniformly convergent to the constant function x_*. Then, for any $\lambda \in (0, 1)$, there exists a metric d_λ on X equivalent to d, complete if (X, d) is complete, such that f is a contraction with contraction constant λ.*

The result of Meyers was extended by Solomon Leader in [81]. The first part of his theorem is identical with Meyers' result. He also answered the question when there exists a bounded metric d_λ having the properties as in Theorem 20.

We say that a metric space (X, d) has the Banach Fixed Point Property (BFPP) if every Banach contraction has a fixed point. From the Banach theorem it follows that every complete metric space has the BFPP, but the converse implication does not hold. Indeed, as early as 1959 Edwin H. Connell gave an example of non-closed subset A of $\mathbb{R}^2$ such that every continuous selfmap of A has a fixed point [32]. Clearly, A has the BFPP and is not complete. On the other hand, Hu proved that a metric space (X, d) is complete if and only if every non-empty closed subset of X has the BFPP [55]. An important contribution to this topic was made by Jonathan M. Borwein [15] who showed that a convex subset C of a normed linear space has the BFPP if and only if C is complete. In particular, a normed linear space X has the BFPP if and only if X is a Banach space. In fact, the main result in [15] is yet more general. A surprising result of Tomonari Suzuki goes in a different direction [111]. He considered a selfmapping T of a metric space (X, d) which satisfies the inequality $d(T(x), T(y)) \le \alpha\, d(x, y)$ with some $\alpha \in [0, 1)$ for pairs (x, y) from some subset of $X \times X$. The completeness of (X, d) can be characterized by the fixed point property for such mappings. More precisely, the following result holds.

Theorem 21 [111] *Let (X, d) be a metric space. Define a function Θ on $[0, 1)$ by*

$$\Theta(r) := \begin{cases} 1 & \text{if } 0 \le r \le \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2} & \text{if } \frac{\sqrt{5}-1}{2} < r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r} & \text{if } \frac{1}{\sqrt{2}} < r < 1. \end{cases}$$

Then (X, d) is complete if and only if there exist $r \in (0, 1)$ and $\eta \in (0, \Theta(r)]$ such that every mapping T satisfying the condition

$$\text{for any } x, y \in X,\ \eta d(x, T(x)) \le d(x, y) \text{ implies } d(T(x), T(y)) \le r d(x, y)$$

has a fixed point.

In particular, since $\Theta\colon [0, 1) \to (\frac{1}{2}, 1]$, we may obtain the following

Corollary 1 *A metric space* (X, d) *is complete if and only if there exists* $r \in (0, 1)$ *such that every mapping* T *satisfying the condition*

$$\text{for any } x, y \in X,\ \frac{1}{2}d(x, T(x)) \le d(x, y) \text{ implies } d(T(x), T(y)) \le rd(x, y)$$

has a fixed point.

In 2006 Ehrhard Behrends in a private communication with Marton Elekes asked two questions:

1. Is there an open nonclosed subset of $\mathbb{R}^n$ with the Banach fixed point property for some $n \in \mathbb{N}$?
2. Is there a *simple* nonclosed subset of $\mathbb{R}$ with the Banach fixed point property?

The first question has a negative answer, the second affirmative [40]. Elekes showed that for any $n \in \mathbb{N}$, if X having the BFPP is an open subset of $\mathbb{R}^n$ or X is a subset of $\mathbb{R}$ which is simultaneously F_σ and G_δ, then X is closed and therefore is complete (in fact, it means that X coincides with $\mathbb{R}^n$). The latter result is optimal since the author gives two examples of nonclosed subsets of $\mathbb{R}$ possessing the BFPP in which the first one is an F_σ set and the second one is a G_δ set. Moreover, there is a bounded Borel (even F_σ) subset of $\mathbb{R}$ with the BFPP that is not complete with respect to every equivalent metric. Finally, the author showed that for any positive integer n, there exists a non-measurable subset of $\mathbb{R}^n$ with the BFPP.

7 Selected applications of the Banach Contraction Principle

The Banach Fixed Point Theorem plays an important role in many areas of mathematics. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method of finding those fixed points. Therefore this theorem becomes an essential tool for finding solutions of many problems in pure and applied mathematics.

Below we present some basic applications of the Banach Fixed Point Theorem. However, it is worth emphasizing that in many cases the Contraction Principle was not used in original proofs of various results. In particular, this concerns the Picard–Lindelöf theorem which appeared in 1890, 32 years before the publication of Banach's paper. Nevertheless, nowadays the Picard–Lindelöf theorem is usually presented as an application of Banach's Fixed Point Theorem though it is not a true example of the application. The same can be said about the Weierstrass-Stone theorem or the central limit theorem which were proven via a fixed point argument many years after they appeared. In this section we present these three proofs as well as proofs that in their original versions used the Banach Fixed Point Theorem.

Firstly recall that the iterative sequence $x_{n+1} = T(x_n)$, used in the proof of Theorem 2 with an arbitrary $x_0 \in X$, converges to the unique fixed point x_* of T. The proof of the Banach theorem yields the following useful information about the rate of convergence towards the fixed point.

Corollary 2 *Let f be a contraction mapping on a complete metric space (X, d) with contraction constant α and fixed point x_*. For any $x_0 \in X$, we have the following estimates:*

$$d(x_n, x_*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, f(x_0)), \tag{7}$$

$$d(x_n, x_*) \leq \alpha \cdot d(x_{n-1}, x_*), \tag{8}$$

$$d(x_n, x_*) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n). \tag{9}$$

These inequalities have different purposes. The inequality (7) tells us how many times we need to iterate f starting from x_0 to be certain that we are within a specified distance from the fixed point. This is an upper bound on how long we need to compute. It is called an a priori estimate. The inequality (8) shows that once we find a term by iteration within some desired distance of the fixed point, all further iterates will be within that distance. However, it is not so useful as an error estimate since both sides of (8) involve the unknown fixed point. The inequality (9) tells us, after each computation, how much closer we are to the fixed point in terms of the previous two iterations. This kind of estimate, called an a posteriori estimate, is very important because if two successive iterations x_m and x_{m+1} are nearly equal, then this guarantees that we are very close to the true fixed point x_*.

7.1 Newton's method of finding zeros of functions

A classical application of Banach's Fixed Point Theorem is Newton's method for finding solutions of equations $f(x) = 0$, where $f: (a, b) \to \mathbb{R}$ is differentiable. It begins with taking an initial value x_0 and uses the tangent of f at x_0. The next iterative value x_1 is the zero of the tangent of f at x_0, so assuming that $f'(x_0) \neq 0$ we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Repeating this process we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 0, 1, 2, \ldots$$

provided $f'(x_n) \neq 0$. However, in practice, it is more convenient to try to use the following simplified iterative procedure:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_0)} \quad \text{for } n = 0, 1, 2, \ldots \tag{10}$$

For the latter procedure we have the following theorem on the convergence of sequence $(x_n)_{n\in\mathbb{N}}$ to zero of f.

Theorem 22 *Let $x_0 \in \mathbb{R}$, $r > 0$ and $f: (x_0 - r, x_0 + r) \to \mathbb{R}$ be differentiable with $f'(x_0) \neq 0$. Assume that f' is Lipschitzian with constant $L > 0$, $s := 2\left|\frac{f(x_0)}{f'(x_0)}\right|$ and $4L\frac{|f(x_0)|}{(f'(x_0))^2} \leq 1$. Then there exists a unique $x_* \in [x_0 - s, x_0 + s]$ such that $f(x_*) = 0$ and the sequence $(x_n)_{n\in\mathbb{N}}$ defined by (10) is convergent to x_*.*

A more general version of this result can be found in [97], Theorem 15.1. To prove the theorem, we define the function F on $[x_0 - r, x_0 + r]$ by $F(x) := x - \frac{f(x)}{f'(x_0)}$. It can be shown that F is a Banach contraction with constant not larger than $\frac{1}{2}$ and F is a selfmapping of

$[x_0 - s, x_0 + s]$. By the Banach Contraction Principle, F has a unique fixed point x_*. Since $F(x) = x$ if and only if $f(x) = 0$, we get that x_* is a unique zero of f in $[x_0 - s, x_0 + s]$.

7.2 Systems of linear equations

We present an application of the Banach Contraction Theorem to find the solution of the following system of linear equations with n unknowns:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ & \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{cases} \tag{11}$$

This system can be written in the form

$$\begin{cases} x_1 = (1 - a_{11})x_1 - a_{12}x_2 - \cdots - a_{1n}x_n + b_1 \\ x_2 = -a_{21}x_1 + (1 - a_{22})x_2 - \cdots - a_{2n}x_n + b_2 \\ \cdots \\ x_n = -a_{n1}x_1 - a_{n2}x_2 - \cdots + (1 - a_{nn})x_n + b_n \end{cases} \tag{12}$$

By putting $\alpha_{ij} = -a_{ij} + \delta_{ij}$, where $\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$ the last system is equivalent to

$$x_i = \sum_{j=1}^{n} \alpha_{ij}x_j + b_i, \ i = 1, 2, \ldots, n. \tag{13}$$

If we use the following denotations: $x = [x_1, x_2, \ldots, x_n]^T$, $B = [b_1, b_2, \ldots, b_n]^T$ and $A = [\alpha_{ij}]_{n\times n}$, then system (13) can be rewritten in the matrix form

$$x = Ax + B$$

and finding the solution of it is equivalent to the problem of finding a fixed point of the transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(x) := Ax + B$. If T is a contraction we can use the Banach theorem and the method of successive approximations. In particular, this is the case under the assumptions of the following

Theorem 23 [98] *Let $X = \mathbb{R}^n$ be the metric space with the metric $d_\infty(x, y) = \max_{1\le i\le n} |x_i - y_i|$. If $\sum_{j=1}^{n} |\alpha_{ij}| \le \alpha < 1$ for all $i = 1, 2, \ldots, n$, then the system of linear equations (11) has a unique solution.*

7.3 Differential and integral equations

The most interesting applications of the Banach theorem arise in connection with function spaces. We will start with the results concerning differential equations.

Let us consider the initial value problem (known as the Cauchy problem)

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{14}$$

where $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and x is an unknown function. We suppose that $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R} \times \mathbb{R}^n$ and $(t_0, x_0) \in U$. By integrating both sides of (14) with respect to t, we obtain the equivalent integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{15}$$

Note that $x_0(t) = x_0$ is an approximating solution of (15) for t close enough to t_0. By putting $x_0(\cdot)$ into our integral equation, we get another approximating solution:

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))ds.$$

By iterating this procedure, we get a sequence of approximating solutions

$$x_m(t) = K^m(x_0)(t), \text{ where } K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{16}$$

To find a solution of (14), it suffices to show that K is a contraction and then we may apply the Contraction Principle to infer that K has a fixed point. Since the equation $x = K(x)$ is equivalent to the integral equation (15) (and (14) as well), we obtain then the solution of the initial value problem given at the beginning (for details see [113]). Finally, we can formulate the following famous theorem:

Theorem 24 (Picard–Lindelöf theorem, [113]) *Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, then there exists a unique local solution $\overline{x}(\cdot) \in C^1(I)$ of* (14)*, where I is some interval around x_0.*

Some extensions of the Picard–Lindelöf theorem may be found in [113]. Let us also notice that if a function f is *globally* Lipschitz continuous in the second argument, uniformly with respect to the first one taken from some interval $[0, T]$, then it is possible to prove the existence of a global solution $\overline{x}$ of (14), i.e., $\overline{x}$ acts on $[0, T]$. There are two approaches to show that. The first of them uses Theorem 4: it turns out that some iterate of the integral operator K is then a contraction with respect to the supremum norm on $C([0, T])$. The second approach uses the following Bielecki's norm [11]: for $x \in C([0, T])$ and $\lambda \geq 0$,

$$\|x\|_\lambda := \sup_{t \in [0,T]} e^{-\lambda L t}|x(t)|.$$

(For simplicity, we consider the case, when $n = 1$.) Then it can be shown that for any $\lambda > 1$, K is a contraction with respect to the norm $\|\cdot\|_\lambda$. These two approaches are nicely described, e.g., in [47]. For further examples of the fixed point theorems applied to the differential (and integral) equations, check [97].

Now we will show how the Banach theorem yields the existence and uniqueness results for integral equations. Consider the following linear Fredholm integral equations of the second kind:

$$f(x) = g(x) + \lambda \int_{a}^{b} h(x, y) f(y)dy, \tag{17}$$

where $f : [a, b] \to \mathbb{R}$ is an unknown function, $h : [a, b] \times [a, b] \to \mathbb{R}$ is a given function (called the kernel) and λ is a parameter. This integral equation can be examined in many function spaces. We will show the procedure of solving it in $C([a, b])$ endowed with the supremum norm: $\|f\| = \sup_{x\in[a,b]} |f(x)|$ for $f \in C([a, b])$. Then $(C([a, b]), \|\cdot\|)$ is a Banach space.

The Eq. (17) can be rewritten as $T(f) = f$, where for $f \in C([a, b])$ and $x \in [a, b]$,

$$T(f)(x) = g(x) + \lambda \int_a^b h(x, y) f(y)dy. \tag{18}$$

Since functions g and h are both continuous, operator T is a selfmapping of $C([a, b])$. We should determine for which values of λ the map T is a contraction. Note that since h is continuous, it must also be bounded, so if $|h(x, y)| \le M$, then

$$d_\infty(T(f_1), T(f_2)) \le |\lambda| \cdot M \cdot (b - a) \cdot d_\infty(f_1, f_2).$$

Thus T is a Banach contraction if $|\lambda| < \frac{1}{(b-a)M}$ and we have the theorem:

Theorem 25 *Suppose g is continuous on $[a, b]$, h is continuous on $[a, b] \times [a, b]$ and $|h(x, y)| \le M$ for all $x, y \in [a, b]$. Assume that $|\lambda| < \frac{1}{(b-a)\cdot M}$. Then the integral equation* (17) *has a unique solution $f \in C([a, b])$. The function f is the limit of iterative sequence $(f_0, f_1, \ldots)$, where f_0 is any continuous function on $[a, b]$, and $f_{n+1}(x) = g(x) + \lambda \int_a^b h(x, y) f_n(y)dy$, $n = 0, 1, 2, \ldots$.*

7.4 The Lax–Milgram lemma

Let H be a real Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. The dual of H is denoted by H^*.

Theorem 26 (Lax–Milgram lemma, [80, 97]) *Suppose $B : H \times H \to \mathbb{R}$ is a bilinear form for which there exist constants $\alpha, \beta > 0$ such that*

- *$|B(u, v)| \le \alpha\|u\|\,\|v\|$ for any $u, v \in H$ (boundedness),*
- *$B(u, u) \ge \beta\|u\|^2$ for all $u \in H$ (coercivity).*

Then for each $u^* \in H^*$, there exists a unique element $u \in H$ such that $B(u, v) = u^*v$ for all $v \in H$.

We give a sketch of a proof to show how the Banach Fixed Point Theorem can be used to prove it. We may associate with the form B a linear bounded operator $T : H \to H^*$ defined by the formula: for any $u \in H$,

$$(Tu)(v) := B(u, v) \text{ for all } v \in H.$$

Then $|(Tu)(v)| \le \alpha\|u\|\,\|v\|$, so $\|Tu\| \le \alpha\|u\|$, and hence $\|T\| \le \alpha$, so indeed T is bounded. Moreover, for $u \ne \Theta$,

$$\|Tu\| \ge \left\|(Tu)\left(\frac{u}{\|u\|}\right)\right\| = \frac{1}{\|u\|}B(u, u) \ge \beta\|u\|,$$

so T is injective and $\|T^{-1}\| \le \frac{1}{\beta}$. Consequently, T is an isomorphism of H into H^*. Now, our aim is to show that for every $u^* \in H^*$, there exists a unique $u \in H$ such that $(Tu)(v) = u^*v$ for all $v \in H$, i.e., $Tu = u^*$. That means T is an isomorphism of H onto H^*. Observe that if we consider inner product $\langle\cdot,\cdot\rangle$ as a bilinear form, then the associated operator $\Lambda : H \to H^*$

is an isomorphism of H onto H^* by the Riesz representation theorem. Thus the equation $Tu = u^*$ is equivalent to the equation $\Lambda^{-1}(Tu - u^*) = \Theta$ which in turn is equivalent to the equation $u = u - \rho\Lambda^{-1}(Tu - u^*)$, where $\rho \neq 0$ is an arbitrarily fixed real number. Define the map f by $f(u) := u - \rho\Lambda^{-1}(Tu - u^*)$. It can be shown that f is a Banach contraction if $\rho \in (0, \frac{2\beta}{\alpha^2})$, so by the Banach Fixed Point Theorem, f has a unique fixed point, which is a solution of equation $Tu = u^*$.

The typical application of Lax–Milgram lemma is related to the elliptic partial differential equations. The main task is always to identify the Hilbert space of functions among which we look for solutions to a given partial differential equation, and to check the validity of the assumptions on B.

Let Ω be a bounded region in $\mathbb{R}^n$, $f \in L^2(\Omega)$ be given and $a \geq 0$. Consider the elliptic boundary-value problem

$$\begin{cases} -\Delta u + au = f \text{ in } \Omega \\ u = 0 \text{ on } \delta\Omega, \end{cases} \tag{19}$$

where $u\colon \overline{\Omega} \to \mathbb{R}$ is unknown and Δ is the Laplace operator. We are looking for solutions u in the Sobolev space $H_0^1(\Omega)$ (the completion of $C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is the space of the functions with continuous partial derivatives of any order with compact support). Note that the condition $u \in H_0^1(\Omega)$ implies that u vanishes at the boundary of Ω. By weak solutions of (19) one means the function u which satisfies the condition: $\langle \nabla v, \nabla u\rangle + a\langle v, u\rangle = \langle v, f\rangle$ for any $v \in H_0^1(\Omega)$. As bilinear form B from the Lax-Milgram lemma we put

$$B(u, v) = \langle \nabla v, \nabla u\rangle + a\langle v, u\rangle.$$

Such B is bounded and coercive in $H_0^1(\Omega)$, so by virtue of the Lax–Milgram lemma, there exists exactly one $u \in H_0^1(\Omega)$ such that $B(v, u) = \langle v, f\rangle$ for any $v \in H_0^1(\Omega)$, which means (19) has a unique solution $u \in H_0^1(\Omega)$ [69].

7.5 Theory of monotone operators

The monotonicity methods have been started early in sixties; see, for instance, [79, 92]. The interesting historical remarks concerning monotone operators can be found in [8]. The method of monotone operators have various applications in the study of boundary value problems, optimal control problems, contact mechanics and very many related fields. In [42] one may also find a nice introduction to this topic accessible to non-specialists.

The fundamental result in the theory of monotone operators is the following one due to Eduardo Zarantonello [120].

Theorem 27 *Assume E is a Hilbert space and $A : E \to E$ is Lipschitz continuous, i.e., there is $M > 0$ such that for all $u, v \in E$, we have*

$$\|A(u) - A(v)\| \leq M \|u - v\|$$

and strongly monotone with a constant $m < M$, i.e., for $u, v \in E$, we have

$$(A(u) - A(v), u - v)_E \geq m \|u - v\|^2 .$$

Then for each $h \in E$, the equation

$$A(u) = h \tag{20}$$

has exactly one solution. Moreover, A is invertible and $A^{-1}\colon E \to E$ is Lipschitz continuous.

For the proof take

$$0 < \varepsilon < \frac{2m}{M^2} \tag{21}$$

and define a Lipschitz continuous operator $T_\varepsilon : E \to E$ by

$$T_\varepsilon(u) = u - \varepsilon(A(u) - h)$$

with a Lipschitz constant $L_\varepsilon := 1 + \varepsilon^2 M^2 - 2m\varepsilon$ which by (21) is less than 1. Therefore, by the Banach Contraction Principle, there is exactly one u_ε such that $u_\varepsilon = T_\varepsilon(u_\varepsilon)$ which means that

$$u_\varepsilon = u_\varepsilon - \varepsilon(A(u_\varepsilon) - h).$$

This provides the unique solvability of (20) together with the following iteration method

$$u_{n+1} = u_n - \varepsilon(A(u_n) - h) \text{ for each } n = 0, 1, 2, \ldots$$

which converges to the unique solution u of (20).

7.6 Banach algebras and the Weierstrass–Stone theorem

A Banach algebra is a Banach space B endowed with a multiplication operation "$\cdot$" (assumed to be associative), which is compatible with operations of addition and a scalar multiplication, i.e., for any $x, y, z \in B$ and any $\alpha \in K$ ($K = \mathbb{R}$ or $K = \mathbb{C}$),

$$(x + y) \cdot z = x \cdot z + y \cdot z, \; x \cdot (y + z) = x \cdot y + x \cdot z, \; \alpha(x \cdot y) = (\alpha x) \cdot y = x \cdot (\alpha y).$$

Moreover, the norm is required to satisfy the inequality $\|x \cdot y\| \leq \|x\| \|y\|$ which ensures the continuity of the multiplication. If there exists in B a neutral element e with respect to multiplication such that $\|e\| = 1$, then e is called the unity of B, and B is said to be unital. We mention two important examples of unital Banach algebras here: the algebra $\mathcal{B}(E)$ of all continuous linear operator on a Banach space E (we endow $\mathcal{B}(E)$ with composition as multiplication and the operator norm), and the algebra $\mathcal{B}(\Omega)$ of all bounded real- or complex-valued functions on some set Ω with pointwise multiplication and the supremum norm. In general, $\mathcal{B}(E)$ is non-commutative, i.e., not necessarily $S \circ T = T \circ S$, when $S, T \in \mathcal{B}(E)$, but clearly, $\mathcal{B}(\Omega)$ is always commutative.

In 1972 Frank F. Bonsall and David S.G. Stirling published an elegant paper [14] in which they used the Banach Fixed Point Theorem to show the existence of square roots of certain elements in a Banach algebra B. They began with studying the equation $2x - x^2 = a$, where $a \in B$ is fixed and $\|a\| < 1$, and they proved that this equation has a unique solution $x_* \in B$ such that $\|x_*\| < 1$. Here they applied the Banach Contraction Principle to the operator T defined by

$$T(x) := \frac{1}{2}(a + x^2) \text{ for } x \in B(a) \cap \{x \in B \colon \|x\| < \eta\},$$

where $B(a)$ denotes the least closed subalgebra of B containing a and a real number η is such that $\|a\| < \eta < 1$. As a simple consequence, it can be obtained that for every unital Banach algebra B with unity e and for any $x \in B$ such that $\|x - e\| < 1$, there exists a unique square root y of x (i.e., $x = y^2$) such that $\|y - e\| < 1$.

Using the approach of Bonsall and Stirling, Jaroslav Zemánek [122] presented a beautiful proof of the Weierstrass–Stone theorem.

Theorem 28 (Weierstrass–Stone theorem) *Let Ω be a compact space and $C(\Omega)$ be the Banach algebra of all real-valued continuous functions with the supremum norm. If A is a subalgebra of $C(\Omega)$ containing the unit function and separating Ω, then A is dense in $C(\Omega)$*

Let us recall that A separates Ω if for any $x, y \in \Omega$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$. To prove the Weierstrass–Stone theorem, Zemánek showed first with the help of the Banach Contraction Principle that if B is a complete subalgebra of $B(\Omega)$ and $f \in B$ is non-negative on Ω, then $\sqrt{f} \in B$. This immediately implies that if $f \in B$, then $|f| \in B$, and consequently, if $f, g \in B$ then $\min\{f, g\} \in B$ and $\max\{f, g\} \in B$. The last implication follows from the fact that B is a linear space, $\min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$ and $\max\{f, g\} = -\min\{-f, -g\}$. This ends the algebraic part of the proof. The rest part of the proof of the Weierstrass–Stone theorem is topological and is based on a compactness argument.

7.7 The central limit theorem

In 1984 Gholamhossein G. Hamedani and Gilbert G. Walter [52] gave a proof of a classical theorem of probability—the central limit theorem—with the help of the Banach Fixed Point Theorem. In fact, they developed the suggestion of Julius R. Blum that the central limit theorem could be interpreted and proved as a Fixed Point Theorem. To do that, they introduced first a metric on a space of distribution functions in the following way. For $\lambda \geq 0$, let R_λ denote the set of all random variables X such that the expected value $E(|X|^\lambda)$ is finite and

$$E(X^k) = m_k \text{ for } k = 1, 2, \dots [\lambda],$$

where m_k is the kth moment of the standard normal variable Z and $[\cdot]$ stands for the floor function. Let M_λ denote the set of all distribution functions from R_λ, and for $F, G \in M_\lambda$,

$$d_\lambda(F, G) := \sup_{t \in \mathbb{R} \setminus \{0\}} \left| E\left(\frac{e^{iXt} - e^{iYt}}{|t|^\lambda} \right) \right|.$$

Hamedani and Walter proved that (M_λ, d_λ) is a metric space and, moreover, (M_λ, d_λ) is complete when λ is a positive integer. Next, for $\alpha > 0$, they defined a mapping T_α from M_λ into M_0 using a convolution of distribution functions and they showed that T_α is a Banach contraction if $\alpha^\lambda > 2$ and $T_\alpha(M_\lambda) \subset M_\lambda$. In particular, this is the case if $\alpha = \sqrt{2}$ and $\lambda > 2$. Let Φ denote the standard normal distribution function. It turns out that $\Phi = T_{\sqrt{2}}(\Phi)$ and $\Phi \in M_\lambda$. Thus, by the Banach Fixed Point Theorem, we get that for any $F \in M_\lambda$, $d_\lambda\left(T^n_{\sqrt{2}}F, \Phi\right) \to 0$. Now, Hamedani and Walter observed that if (X_n) is a sequence of independent (or only sub-independent) random variables with the same distribution function F, mean 0, variance 1, and such that $E(|X|^\lambda)| < \infty$ for some $\lambda > 2$ and all $n \in \mathbb{N}$, then $T^n_{\sqrt{2}}F$ is the distribution function of the random variable $2^{-\frac{n}{2}} \sum_{i=1}^{2^n} X_i$. The convergence of $\left(T^n_{\sqrt{2}}F\right)$ to Φ with respect to d_λ implies that the sequence $\left(2^{-\frac{n}{2}} \sum_{i=1}^{2^n} X_i\right)$ converges in distribution to Z. Finally, with the help of a clever lemma, Hamedani and Walter extended the above result to the non-identically distributed case in which the set of distribution functions of random variables X_n is bounded in M_λ. Moreover, they managed to show that $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \to Z$ in distribution as $n \to \infty$ obtaining in this way the central limit theorem.

7.8 Iterated function systems

An iterated function system (abbr. IFS) consists of a metric space (X, d) and a finite family $\{f_1, \ldots, f_N\}$ of continuous selfmappings of X. In order to introduce the concept of an attractor of IFS $\big((X, d), \{f_1, \ldots, f_N\}\big)$, we need to recall first the definition of the Hausdorff metric. For $x \in X$ and nonempty subset A of X, the distance $d(x, A)$ between x and A is defined as follows:

$$d(x, A) := \inf\{d(x, a) \colon a \in A\}.$$

For nonempty subsets A and B of X, we set

$$D(A, B) := \sup\{d(a, B) \colon a \in A\}.$$

Of course, in general it is possible that $D(A, B) = \infty$. However, if A and B are bounded, then both $D(A, B)$ and $D(B, A)$ are finite, so

$$H(A, B) := \max\{D(A, B), D(B, A)\}$$

is also finite. Let $\mathcal{K}(X)$ denote the family of all nonempty compact subsets of X. It can be shown that $(\mathcal{K}(X), H)$ is a metric space and H is then called a Hausdorff (or Pompeiu–Hausdorff) metric on $\mathcal{K}(X)$. Moreover, if (X, d) is complete, so is $(\mathcal{K}(X), H)$.

Now, a set $A_* \in \mathcal{K}(X)$ is called an attractor of IFS $\big((X, d), \{f_1, \ldots, f_N\}\big)$ if

$$A_* = \bigcup_{i=1}^{N} f_i(A_*)$$

and for any set $A \in \mathcal{K}(X)$, $F^n(A) \to A_*$ with respect to the Hausdorff metric, where F is a set-valued operator defined by the formula:

$$F(A) := \bigcup_{i=1}^{N} f_i(A) \text{ for } A \in \mathcal{K}(X).$$

Note that F is a selfmapping of $\mathcal{K}(X)$.

In 1981 John E. Hutchinson applied the Banach Fixed Point Theorem in an impressive way to obtain the following theorem on the existence of attractors, which is a fundamental result in the theory of IFSs.

Theorem 29 [57] *Let (X, d) be a complete metric space and $f_1, \ldots, f_N \colon X \to X$ be Banach contractions. Then the IFS $\big((X, d), \{f_1, \ldots, f_N\}\big)$ has an attractor.*

The idea of Hutchinson's proof is to show that the above defined operator F is a Banach contraction with respect to the Hausdorff metric on $\mathcal{K}(X)$, so the existence of attractor follows immediately from the Banach Fixed Point Theorem. Attractors of IFSs consisting of Banach contractions are often called *fractals in the sense of Hutchinson and Barnsley*. A number of examples of such fractals can be found in the monograph of Michael F. Barnsley [7]. Here we quote two famous examples.

Example 1 (The Cantor set) Let $\mathbb{R}$ be endowed with the Euclidean metric d and for $x \in \mathbb{R}$,

$$f_1(x) := \frac{1}{3}x, \quad f_2(x) := \frac{1}{3}x + \frac{2}{3}.$$

Clearly, f_1 and f_2 are Banach contractions, so by Hutchinson's theorem, the IFS $\big((\mathbb{R}, d), \{f_1, f_2\}\big)$ has an attractor C and in particular, $H(F^n([0,1]), C) \to 0$. Since $F([0,1]) \subset [0,1]$, the sequence $\big(F^n([0,1])\big)_{n\in\mathbb{N}}$ is descending. It is known that in fact any descending sequence of compact sets is convergent with respect to the Hausdorff metric and its limit coincides with the intersection of these sets. Thus we have that $C = \bigcap_{n\in\mathbb{N}} F^n([0,1])$, so we may see that indeed, C is the Cantor set.

Example 2 (The Sierpiński triangle) This time we endow the plane $\mathbb{R}^2$ with the Euclidean metric d and we consider the following three mappings: for $(x, y) \in \mathbb{R}^2$,

$$f_1(x,y) := \frac{1}{2}(x,y), \quad f_2(x,y) := \frac{1}{2}(x,y) + \left(\frac{1}{2}, 0\right), \quad f_3(x) := \frac{1}{2}(x,y) + \left(\frac{1}{4}, \frac{1}{2}\right).$$

The IFS $\big((\mathbb{R}^2, d), \{f_1, f_2, f_3\}\big)$ satisfies the assumptions of Hutchinson's theorem and its attractor is the Sierpiński triangle.

8 Closing comments

In this paper, we have provided only a part of the available information about the Banach Fixed Point Theorem. It is not possible to include all the knowledge on this topic in a single article, because fixed point theory is developing very dynamically, as evidenced by the fact that almost 14,000 articles containing words "fixed point" in their titles have been published since 2009 until 2023, whereas 13000 such articles appeared before 2009. In the former group there are 4100 papers using in their titles the words "metric" and "fixed point". (All these data are taken from the MathSciNet as of December, 2023.) Therefore, our selection had to be subjective and it was certainly not complete. For instance, among others we did not present the results related to:

- common fixed point theorems (according to the MathSciNet there are almost 5000 articles with titles containing words "common fixed point" and probably the first paper on this topic dealing with contractive mappings was [35]);
- cone metric spaces (750 papers; according to the opinion of Petr Zabreĭko [119], the first fixed point theorem for contractions on such spaces was established by Anatolij Perov, [99]); more recent results on this topic are discussed in a survey paper [66].
- fuzzy metric spaces (1300 papers; as far as we know, the research on fixed points of fuzzy contractive mappings was initiated by Stanisław Heilpern, [54]);
- asymptotic contractions (the notion introduced by Kirk, [71]);
- contractions with respect to a partial ordering (we owe this notion to André Ran and Martine Reurings, whose paper [102] has now almost 400 citations according to the MathSciNet);
- contractions with respect to a graph (the class of mappings defined in the paper [59] having now over 150 citations).

We also omitted the discussion on a number of applications of the Contraction Principle, for example, in:

- matrix theory (in particular, the Perron–Frobenius theorem, [20]);
- commutative algebra (the algebraic Weierstrass Preparation Theorem, [45]);
- partial differential equations (for example, the Cauchy–Kowalevsky theorem, [115]);
- actuarial mathematics (approximation of a vector of ruin probabilities in regime-switching models, [41]).

As pointed out in [109]

> the ever growing list of applications of the Banach Fixed Point Theorem would fill volumes.

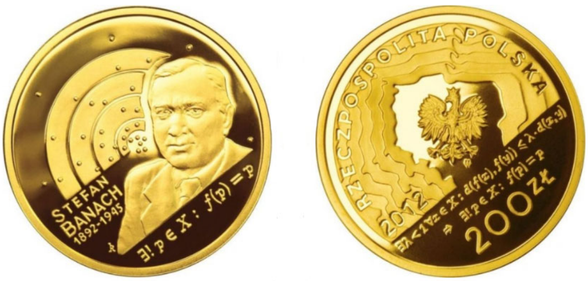
Furthermore, we are not giving a list of open problems in metric fixed point theory, however, we can recommend here to the reader the following recent papers by the leading experts in this field: [48, 72, 73, 107, 121].

Finally, let us quote Stanisław Mazur, PhD student of Stefan Banach from 1932. In 1961, he wrote a text about the importance of Banach's doctoral dissertation (see [88]):

> W ciągu niespełna 40 lat, które upłynęły od czasu ukazania się rozprawy doktorskiej Stefana Banacha, analiza funkcjonalna rozrosła się w potężny dział matematyki, który skupia na sobie uwagę coraz liczniejszych matematyków na świecie [...] Cały dotychczasowy rozwój analizy funkcjonalnej dowodzi, że koncepcje Stefana Banacha posiadają nieprzemijającą wartość w nauce. Analiza funkcjonalna, to wspaniały trwały pomnik jej twórcy.[4]

Even though more than 60 years have passed since then, these sentences have not lost their relevance, and the Banach Fixed Point Theorem is one of the most famous theorems.

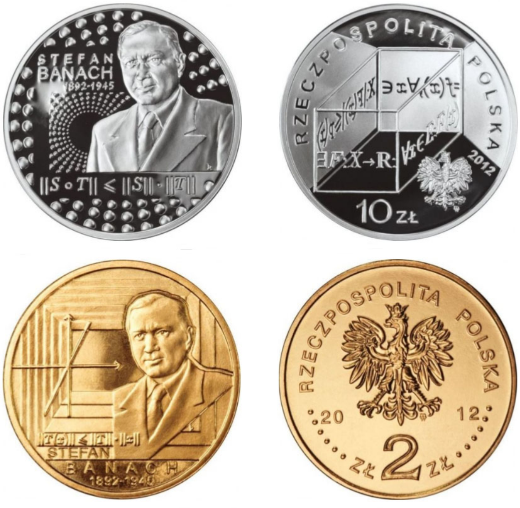
On April 3rd, 2012 r. National Bank of Poland issued three coins (2 PLN, 10 PLN and 200 PLN) commemorating Stefan Banach.[5] The coin 200 PLN was made in gold and was devoted to the Banach Fixed Point Theorem.



10 PLN coin was made in silver and showed the relation between linear mappings on Banach spaces. The third coin (2 PLN) presented inequality characterizing bounded linear mappings on Banach spaces.

[4] In less than 40 years since the publication of Stefan Banach's doctoral dissertation, functional analysis has grown into a powerful branch of mathematics that attracts the attention of more and more mathematicians around the world [...] The entire development of functional analysis so far proves that Stefan Banach's concepts have everlasting value in science. Functional analysis is a wonderful permanent monument to its creator.

[5] Use of the images of coins in accordance with https://www.nbp.pl/home.aspx?f=/banknoty_i_monety/wykorzystanie_wizerunkow.html (in Polish).



Acknowledgements We are grateful to the referees for many valuable remarks which led to an improved version of our paper. Special thanks are due to Professors Krzysztof Cris Ciesielski and Marek Galewski for their helpfull suggestions which made this article much more complete.

Data availability Not applicable.

References

1. Arvanitakis, A.D.: A proof of the generalized Banach contraction conjecture. Proc. Am. Math. Soc. **131**(12), 3647–3656 (2003)
2. Bailey, D.F.: Some theorems on contractive mappings. J. Lond. Math. Soc. **41**, 101–106 (1966)
3. Bakhtin, I.A.: The contraction mapping principle in almost metric spaces. Funct. Anal. Gos. Ped. Inst. Unianowsk **30**, 26–37 (1989)
4. Banach, S.: Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. **3**, 133–181 (1922)
5. Banach, S.: Théorie des Opérations Linéaires. Monogr. Mat., t. I, Warszawa (1932)
6. Baranga, A.: The contraction principle as a particular case of Kleene's fixed point theorem. Discrete Math. **98**(1), 7–79 (1991)
7. Barnsley, M.F.: Fractals Everywhere. Academic Press, San Diego (1993)
8. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham (2017)
9. Bessaga, C.: On the converse of the Banach "fixed-point principle". Colloq. Math. **7**, 41–43 (1959)
10. Bessenyei, M., Páles, Z.: A contraction principle in semimetric spaces. J. Nonlinear. Convex Anal. **18**(3), 515–524 (2017)
11. Bielecki, A.: Une remarque sur la méthode de Banach–Cacciopoli–Tikhonov dans la théorie des équations différentielles ordinaires. Bull. Acad. Polon. Sci. Cl. III. **4**, 261–264 (1956)
12. Birkhoff, G., Kreyszig, E.: The establishment of functional analysis. Hist. Math. **11**, 258–321 (1984)

13. Bonsall, F.F.: Lectures on Some Fixed-point Theorems of Functional Analysis. Tata Inst. Fund. Res. Stud. Math, Mumbai (1962)
14. Bonsall, F.F., Stirling, D.S.G.: Square roots in Banach ∗-algebras. Glasg. Math. J. **13**, 74 (1972)
15. Borwein, J.M.: Completeness and the contraction pronciple. Proc. Am. Math. Soc. **87**(2), 246–250 (1983)
16. Bourbaki, N.: General Topology. Part 2, pp. iv+363. Addison-Wesley Publishing Co., Hermann, Reading-London-Don Mills (1966)
17. Boyd, D.W., Wong, J.S.W.: Another proof of the contraction mapping principle. Can. Math. Bull. **11**, 605–606 (1968)
18. Boyd, D.W., Wong, J.S.W.: On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458–464 (1969)
19. Branciari, A.: A fixed point theorem for mappings satisfying a general contractive condition of integral type. Int. J. Math. Math. Sci. **29**(9), 531–536 (2002)
20. Brooks, R.M., Schmitt, K.: The contraction mapping principle and some applications. Electron. J. Differ. Equ. **Monograph 9**, 90 (2009)
21. Brouwer, L.E.J.: Über Abbildungen von Mannigfaltigkeiten. Math. Ann. **71**, 97–115 (1912)
22. Browder, F.E.: Fixed point theorems for noncompact mappings in Hilbert spaces. Proc. Natl. Acad. Sci. USA **53**, 1272–1276 (1965)
23. Browder, F.E.: On the convergence of successive approximations for nonlinear functional equations. Nederl. Akad. Wetensch. Proc. Ser. A 71 Indag. Math. **30**, 27–35 (1968)
24. Bryant, V.W.: A remark on a fixed-point theorem for iterated mappings. Am. Math. Mon. **75**, 399–400 (1968)
25. Burton, T.A.: Integral equations, implicit functions, and fixed points. Proc. Am. Math. Soc. **124**, 2383–2390 (1996)
26. Caccioppoli, R.: Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale. Atti Accad. Naz. Lincei Rend. **11**, 794–799 (1930)
27. Caristi, J.: Fixed point theorems for mappings satisfying inwardness conditions. Trans. Am. Math. Soc. **215**, 241–251 (1976)
28. Chatterjea, S.K.: Fixed point theorems. C. R. Acad. Bulgare Sci. **25**, 727–730 (1972)
29. Ciesielska, D., Ciesielski, K.: Banach's doctorate: a case of mistaken identity. Math. Intell. **43**(3), 1–7 (2021)
30. Ciesielski, K.C., Jasinski, J.: On fixed points of locally and pointwise contracting maps. Topol. Appl. **204**, 70–78 (2016)
31. Ciesielski, K.C., Jasinski, J.: Fixed point theorems for maps with local and pointwise contraction properties. Can. J. Math. **70**(3), 538–594 (2018)
32. Connell, E.H.: Properties of fixed point spaces. Proc. Am. Math. Soc. **10**, 974–979 (1959)
33. Ćirić, L.B.: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**, 267–273 (1974)
34. Deimling, K.: Nonlinear Functional Analysis. Springer, Berlin (1985)
35. DeMarr, R.: Common fixed points for commuting contraction mappings. Pac. J. Math. **13**, 1139–1141 (1963)
36. Dugundji, J.: Positive definite functions and coincidences. Fund. Math. **90**, 131–142 (1976)
37. Dugundji, J., Granas, A.: Weakly contractive maps and elementary domain invariance theorems. Bull. Greek Math. Soc. **19**, 141–151 (1978)
38. Edelstein, M.: An extension of Banach's contraction principle. Proc. Am. Math. Soc. **12**(1), 7–10 (1961)
39. Edelstein, M.: On fixed and periodic points under contractive mappings. J. Lond. Math. Soc. **37**, 74–79 (1962)
40. Elekes, M.: On a converse to Banach's fixed point theorem. Proc. Am. Math. Soc. **137**, 3139–3146 (2009)
41. Gajek, L., Rudź, M.: Banach contraction principle and ruin probabilities in regime-switching models. Insur. Math. Econ. **80**, 45–53 (2018)
42. Galewski, M.: Basic Monotonicity Methods with Some Applications. Compact Textbooks in Mathematics. Birkhäuser, Basel, Springer Nature, Basingstoke (2021)
43. Geraghty, M.A.: On contractive mappings. Proc. Am. Math. Soc. **40**, 604–608 (1973)
44. Geraghty, M.A.: An improved criterion for fixed points of contraction mappings. J. Math. Anal. Appl. **48**, 811–817 (1974)
45. Gersten, S.M.: A short proof of the algebraic Weierstrass preparation theorem. Proc. Am. Math. Soc. **88**, 751–752 (1983)
46. Goebel, K.: On a property of Lipshitzian transformations. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **16**, 27–28 (1968)

47. Goebel, K., Kirk, W.A.: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990)
48. Goebel, K., Kirk, W.A.: Some problems in metric fixed point theory. J. Fixed Point Theory Appl. **4**(1), 13–25 (2008)
49. Göhde, D.: Zum Prinzip der kontraktiven Abbildung. Math. Nachr. **30**, 251–258 (1965)
50. Górnicki, J.: Various extensions of Kannan's fixed point theorem. J. Fixed Point Theory Appl. **20**(1), 12 (2018). https://doi.org/10.1007/s11784-018-0500-2
51. Granas, A., Dugundi, J.: Fixed Point Theory. Springer, New York (2003)
52. Hamedani, G.G., Walter, G.G.: A fixed point theorem and its application to the central limit theorem. Arch. Math. (Basel) **43**(3), 258–264 (1984)
53. Hegedüs, M., Szilágyi, T.: Equivalent conditions and a new fixed point theorem in the theory of contractive type mappings. Math. Japon. **25**, 147–157 (1980)
54. Heilpern, S.: Fuzzy mappings and fixed point theorem. J. Math. Anal. Appl. **83**(2), 566–569 (1981)
55. Hu, T.K.: On a fixed-point theorem for metric spaces. Am. Math. Mon. **74**(4), 436–437 (1967)
56. Hu, T., Kirk, W.A.: Local contractions in metric spaces. Proc. Am. Math. Soc. **68**, 121–124 (1978)
57. Hutchinson, J.: Fractals and self-similarity. Indiana Univ. Math. J. **30**, 713–747 (1981)
58. Jachymski, J.: A short proof of the converse to the contraction principle and some related results. Dedicated to Juliusz Schauder, 1899–1943. Topol. Methods Nonlinear Anal. **15**(1), 179–186 (2000)
59. Jachymski, J.: The contraction principle for mappings on a metric space with a graph. Proc. Am. Math. Soc. **136**(4), 1359–1373 (2008)
60. Jachymski, J.R., Jóźwik, I.: Nonlinear contractive conditions: a comparison and related problems. Banach Center Publ. **77**, 123–146 (2007)
61. Jachymski, J., Matkowski, J., Świątkowski, T.: Nonlinear contractions on semimetric spaces. J. Appl. Anal. **1**(2), 125–134 (1995)
62. Jachymski, J.R., Schröder, B., Stein, J.D., Jr.: A connection between fixed-point theorems and tiling problems. J. Comb. Theory Ser. A **87**, 273–286 (1999)
63. Jachymski, J.R., Stein, J.D., Jr.: A minimum condition and some related fixed-point theorems. J. Austral. Math. Soc. Ser. A **66**, 224–243 (1999)
64. Janoš, L.: A converse of Banach's contraction theorem. Proc. Am. Math. Soc. **18**(2), 287–289 (1967)
65. Jungck, G.: Local radial contractions—a counter-example. Houst. J. Math. **8**, 501–506 (1982)
66. Janković, S., Kadelburg, Z.; Radenović, S.: On cone metric spaces: a survey. Nonlinear Anal. **74**(7), 2591–2601 (2011)
67. Kannan, R.: Some results on fixed points. Bull. Cal. Math. Soc. **60**, 71–76 (1968)
68. Keeler, E., Meir, A.: A theorem on contraction mappings. J. Math. Anal. Appl. **28**, 326–329 (1969)
69. Kesavan, S.: Topics in Functional Analysis and Applications. New Age International (P) Ltd., Publishers, New Delhi (2007)
70. Kirk, W.A.: A fixed point theorem for mappings which do not increase distances. Am. Math. Mon. **72**, 1004–1006 (1965)
71. Kirk, W.A.: Fixed points of asymptotic contractions. J. Math. Anal. Appl. **277**(2), 645–650 (2003)
72. Kirk, W.A.: Metric fixed point theory: old problems and new directions. Fixed Point Theory **11**(1), 45–58 (2010)
73. Kirk, W.A.: Metric fixed point theory: a brief retrospective. Fixed Point Theory Appl. **215**, 17 (2015)
74. Kirk, W.A., Sims, B.: Handbook of Metric Fixed Point Theory. Kluwer Academic Publishers, Dordrecht (2001)
75. Kirk, W.A.: Some thoughts on polish contributions to fixed point theory. Wiad. Mat. **48**(2), 127–134 (2012)
76. Kirk, W.A., Shahzad, N.: Fixed Point Theory in Distance Spaces. Springer International Publishing Switzerland, Cham (2014)
77. Kolmogorov, A., Fomin, S.V.: Introduction to Functional Analysis, vol. 1. Graylock Press, New York (1957)
78. Krasnosel'skiĭ, M.A., Vaĭnikko, G.M., Zabreĭko, P.P., Rutitskiĭ Ya.B., Stetsenko, V.Ya.: Approximate Solution of Operator Equations, pp. xii+484. Wolters-Noordhoff Publishing, Groningen (1972)
79. Kačurovskiĭ, R.I.: Monotone operators and convex functionals. Uspekhi Mat. Nauk **15**, 213–215 (1960)
80. Lax, P.D., Milgram, A.N.: Parabolic Equations in Contributions to the Theory of Partial Differential Equations. Ann. of Math. Stud., vol. 33, pp. 167–190. Princeton University Press, Princeton (1964)
81. Leader, S.: A topological characterization of Banach contractions. Pac. J. Math. **69**(2), 461–466 (1977)
82. Leader, S.: Uniformly contractive fixed points in compact metric space. Proc. Am. Math. Soc. **86**(1), 153–158 (1982)
83. Leader, S.: Equivalent Cauchy sequences and contractive fixed points in metric spaces. Stud. Math. **76**, 63–67 (1983)

84. Maligranda, L.: “B” jak przestrzeñ Banacha [“B” as Banach space]. Wiad. Mat. **56**(1), 15–27 (2020) **(in Polish)**
85. Maligranda, L.: 100-lecie doktoratu Stefana Banacha [Centenary of Stefan Banach’s PhD]. Wiad. Mat. **56**(2), 301–309 (2020) (**in Polish**)
86. Maligranda, L.: Doktorat i doktoranci Stefana Banacha [Doctorate and PhD students of Stefan Banach]. Antiq. Math. **14**(1), 89–149 (2020) (**in Polish**)
87. Matkowski, J.: Integrable Solutions of Functional Equations. Dissertationes Math. (Rozprawy Mat.), Warszawa, p. 127 (1975)
88. Mazur, S.: Przemówienie wygłoszone na uroczystości ku uczczeniu pamięci Stefana Banacha [Speech delivered at the ceremony in memory of Stefan Banach]. Wiad. Mat. **4**, 249–250 (1961) (**in Polish**)
89. Merryfield, J., Rothschild, B., Stein, J.D., Jr.: An application of Ramsey’s theorem to the Banach contraction principle. Proc. Am. Math. Soc. **130**, 927–933 (2002)
90. Merryfield, J., Stein, J.D., Jr.: A generalization of the Banach contraction principle. J. Math. Anal. Appl. **273**, 112–120 (2002)
91. Meyers, P.R.: A converse to Banach’s contraction theorem. J. Res. Nat. Bur. Stand. Sect. B **71B**, 73–76 (1967)
92. Minty, G.J.: Monotone (nonlinear) operators in Hilbert space. Duke Math. J. **29**, 341–346 (1962)
93. Mukherjea, A.: Contractions and completely continuous mappings. Nonlinear Anal. **1**, 235–247 (1977)
94. Munkres, J.R.: Topology, 2nd edn. Prentice Hall, Saddle River (2000)
95. Nadler, S.B., Jr.: Multi-valued contraction mappings. Pac. J. Math. **30**(2), 475–488 (1969)
96. Palais, R.S.: A simple proof of the Banach contraction principle. J. Fixed Point Theory Appl. **2**, 221–223 (2007)
97. Pata, V.: Fixed Point Theorems and Applications. Springer International Publishing, Cham (2019)
98. Pathak, H.K.: An Introduction to Nonlinear Analysis and Fixed Point Theory. Springer Nature Singapore Pte Ltd, Singapore (2018)
99. Perov, A.I.: On the Cauchy problem for a system of ordinary differential equations. (Russian) Približ. Metod. Rešen. Differencial’. Uravnen., no. 2, pp. 115–134 (1964)
100. Puiwong, J., Saejung, S.: Banach’s fixed point theorem, Ćirić’s and Kannan’s are equivalent via redefining the metric. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **114**(4), 8 (2020). https://doi.org/10.1007/s13398-020-00897-1
101. Rakotch, E.: A note on contractive mappings. Proc. Am. Math. Soc. **13**, 459–465 (1962)
102. Ran, A.C.M., Reurings, M.C.B.: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435–1443 (2004)
103. Reich, S., Zaslavski, A.J.: Two results on Jachymski–Schröder-Stein contractions. Bull. Polish Acad. Sci. Math. **56**, 53–58 (2008)
104. Reinermann, J., Stallbohm, V.: Eine Anwendung des Edelsteinschen Fixpunktsatzes auf Integralgleichungen vom Abel-Liouvilleschen Typ. Arch. Math. **22**, 643–647 (1971)
105. Rhoades, B.E.: A comparison of various definitions of contractive mappings. Trans. Am. Math. Soc. **226**, 257–290 (1977)
106. Rus, I.A., Petruşel, A., Petruşel, G.: Fixed Point Theory, p. 509. Cluj University Press, Cluj-Napoca (2008)
107. Rus, I.A., Şerban, M.-A.: Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem. Carpath. J. Math. **29**(2), 239–258 (2013)
108. Subrahmanyam, P.V.: Completeness and fixed points. Monatsh. Math. **80**, 325–330 (1975)
109. Subrahmanyam, P.V.: Elementary Fixed Point Theorems. Springer, Singapore (2018)
110. Suzuki, T.: Meir-Keeler contractions of integral type are still Meir–Keeler contractions. Int. J. Math. Math. Sci. (2007). https://doi.org/10.1155/2007/39281
111. Suzuki, T.: A generalized Banach contraction principle that characterizes metric completeness. Proc. Am. Math. Soc. **136**(5), 1861–1869 (2008)
112. Tasković, M.R.: A generalization of Banach’s contraction principle. Publ. Inst. Math. (Beograd) (N.S.) **23**(37), 179–191 (1978)
113. Teschl, G.: Ordinary Differential Equations and Dynamical Systems (Graduate Studies in Mathematics). American Mathematical Society, Providence (2012)
114. Walter, W.: Remarks on a paper by F. Browder about contraction. Nonlinear Anal. **5**, 21–25 (1981)
115. Walter, W.: An elementary proof of the Cauchy–Kowalevsky theorem. Am. Math. Mon. **92**, 115–126 (1985)
116. Weissinger, J.: Zur Theorie und Anwendung des Iterationsverfahrens. Math. Nachr., vol. 8, issue 1, pp. 193–212. Wiley-VCH Verlag, Berlin (1952)
117. Wiener, N.: I Am a Mathematician: The Later Life of a Prodigy. MIT Press, Cambridge (1956)

118. Wong, J.S.W.: Generalizations of the converse of contraction mapping principle. Can. J. Math. **18**, 1095–1104 (1966)
119. Zabreĭko, P.P.: K-metric and K-normed linear spaces: survey. Collect. Math. Univ. Barcelona **48**(4), 825–859 (1997)
120. Zarantonello, E.H.: Solving functional equations by contractive averaging. Mathematics Research Center Rep. # 160, Madison (1960)
121. Zeidler, E.: Nonlinear Functional Analysis and Its Applications I: Fixed-Point Theorems. Springer, New York (1986)
122. Zemánek, J.: A simple proof of the Weierstrass–Stone theorem. Comment. Math. Prace Mat. **20**(2), 495–497 (1978)