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ORIGINAL PAPER

Higher analogues of discrete topological complexity

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Abstract
In this paper, we introduce the nth discrete topological complexity and study its properties such as its relation with simplicial Lusternik–Schnirelmann category and how the higher dimensions of discrete topological complexity relate with each other. Moreover, we find a lower bound of n-th discrete topological complexity which is given by the nth usual topological complexity of the geometric realisation of that complex. Furthermore, we give an example for the strict case of that lower bound.

Keywords Discrete topological complexity · Simplicial Lusternik Schnirelmann category · Simplicial complex

Mathematics Subject Classification 55M30 · 55U05

1 Introduction

Topological complexity is a topological invariant that was introduced by Farber [3]. It is intended to solve problems such as motion planning in robotics. To catch this aim, a computable algorithm is needed, for each pair of points of the configuration space of a mechanical

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or physical device, a path connecting them in a continuous way. Farber used a well-known map in algebraic topology that is a section of the path-fibration and he interpreted that algorithm in terms of this map. In 2010, Rudyak introduced a notion of higher topological complexity in [8] which is later improved by Basabe, Gonzalez, Rudyak and Tamaki in [2]. After that a discrete version of topological complexity is established by Fernandez-Ternero et al. [4].

The importance of discretisation is based on the fact that many motion planning methods transform a continuous problem into a discrete one. So the aim of the present paper is to extend the Rudyak's approach to discrete version. Namely, we define the nth discrete topological complexity.

Before we introduce the nth discrete topological complexity, let us recall some known definitions and theorems.

Definition 1.1 Let K be a simplicial complex. An edge path in K is a finite or infinite sequence of vertices such that any two consecutive vertices span an edge. We say that K is edge path connected if any two vertices can be joined by a finite edge path.

Definition 1.2 Two simplicial maps $\varphi, \psi : K \to L$ are said to be contiguous (denoted by $\varphi \sim_c \psi$) if for every simplex $\{v_0, \ldots, v_k\}$ in K, $\{\varphi(v_0), \ldots, \varphi(v_k), \psi(v_0), \ldots, \psi(v_k)\}$ constitutes a simplex in L.

Definition 1.3 Two simplicial maps $\varphi, \psi : K \to L$ are said to be in the same contiguity class (denoted by $\varphi \sim \psi$) if there exists a finite sequence of simplicial maps $\varphi_i : K \to L$ for $i = 0, 1, \ldots m$ such that $\varphi = \varphi_1 \sim_c \varphi_2 \sim_c \cdots \sim_c \varphi_m = \psi$.

For two simplicial complexes K and L, if there are simplicial maps $\varphi : K \to L$ and $\psi : L \to K$ such that $\varphi \circ \psi \sim 1_K$ and $\psi \circ \varphi \sim 1_L$, then K and L are said to have the same strong homotopy type, and is denoted by $K \sim L$. If $K \sim \{v_0\}$ where v_0 be a vertex in K, then K is said to be strongly collapsible.

The cartesian product of simplicial complexes is not necessarily a simplicial complex. So a "new" product on simplicial complexes is defined to make their product into a simplicial complex. It is the categorical product of simplicial complexes and for two simplicial complexes K_1 and K_2, it is defined as follows.

(i) The set of vertices is defined by $V(K_1 \prod K_2) := V(K_1) \times V(K_2)$.
(ii) If $p_i : V(K_1 \prod K_2) \to V(K_i)$ is the projection map for $i = 1, 2$, then a simplex σ is said to be in $K_1 \prod K_2$ if $p_1(\sigma) \in K_1$ and $p_2(\sigma) \in K_2$.

(for more details, see [7]).

Definition 1.4 [6] Let K be a simplicial complex. A subcomplex $\Omega \subset K$ is said to be categorical if the inclusion $i : \Omega \hookrightarrow K$ and a constant map $c_{v_0} : \Omega \to K$, where $v_0 \in K$ is some fixed vertex, are in the same contiguity class.

Definition 1.5 [6] The simplicial Lusternik-Schnirelmann category $\mathrm{scat}(K)$ is the least integer $k \geq 0$ such that there exist categorical subcomplexes $\Omega_0, \Omega_1, \ldots, \Omega_k$ of K covering K.

Definition 1.6 [4] We say that $\Omega \subset K^2$ is a Farber subcomplex if there is a simplicial map $\sigma : \Omega \to K$ such that $\Delta \circ \sigma \sim \iota_\Omega$ where $\Delta : K \to K^2$, $\Delta(v) = (v, v)$ is the diagonal map and $\iota_\Omega : \Omega \hookrightarrow K^2$ is the inclusion map.

Definition 1.7 [4] The discrete topological complexity $\mathrm{TC}(K)$ of a simplicial complex K is the least non-negative integer k such that K^2 can be covered by $k + 1$ Farber subcomplexes. More precisely, $K^2 = \Omega_0 \cup \cdots \cup \Omega_k$ and there exist simplicial maps $\sigma_j : \Omega_j \to K$ satisfying $\Delta \circ \sigma_j \sim \iota_j$ where $\iota_j : \Omega_j \hookrightarrow K^2$ are inclusions for each $j = 0, \ldots, k$.

2 Higher analogues of discrete topological complexity

Proposition 2.1 *If $\varphi, \psi : K \to L$ are in the same contiguity class, then so are φ^n and ψ^n.*

Proof Without loss of generality we will focus on being contiguous and show that if $\varphi \sim_c \psi$, so is $\varphi^n \sim_c \psi^n$.

Suppose that $\varphi \sim_c \psi$. If

$$\sigma = \{(a_{11}, a_{12}, \dots, a_{1m}), (a_{21}, a_{22}, \dots, a_{2m}), \dots, (a_{n1}, a_{n2}, \dots, a_{nm})\}$$

is a simplex in K^m, then one can say that $\pi_1(\sigma) = \{a_{11}, a_{21}, \dots, a_{n1}\}$, $\pi_2(\sigma) = \{a_{12}, a_{22}, \dots, a_{n2}\}, \dots, \pi_n(\sigma) = \{a_{1m}, a_{2m}, \dots, a_{nm}\}$ are simplices of K. Hence, for each $i = 1, 2, \dots, m$,

$$\varphi(\pi_i(\sigma)) \cup \psi(\pi_i(\sigma)) = (\varphi(a_{1i}), \cdots, \varphi(a_{ni}), \psi(a_{1i}), \cdots, \psi(a_{ni}))$$

belongs to L, so that $\varphi^n(\sigma) \cup \psi^n(\sigma)$ belongs to L^m. □

Definition 2.1 We say that $\Omega \subset K^n$ is an $n-$Farber subcomplex if there is a simplicial map $\sigma : \Omega \to K$ such that $\Delta \circ \sigma \sim \iota_\Omega$ where $\Delta : K \to K^n$, $\Delta(v) = (v, v, \dots, v)$ is the diagonal map and $\iota_\Omega : \Omega \hookrightarrow K^n$ is the inclusion map.

Definition 2.2 The nth discrete topological complexity $\mathrm{TC}_n(K)$ of a simplicial complex K is the least non-negative integer k such that K^n can be covered by $k+1$ n-Farber subcomplexes. More precisely, $K^n = \Omega_0 \cup \cdots \cup \Omega_k$ and there exist simplicial maps $\sigma_j : \Omega_j \to K$ satisfying $\Delta \circ \sigma_j \sim \iota_j$ where $\iota_j : \Omega_j \hookrightarrow K^n$ are inclusions for each $j = 0, \dots, k$.

Theorem 2.1 *For a subcomplex $\Omega \subset K^n$, the followings are equivalent.*

(1) *Ω is an n-Farber subcomplex.*
(2) *$(\pi_i)_{|\Omega} \sim (\pi_j)_{|\Omega}$ for all $i, j \in \{1, 2, \dots, n\}$.*
(3) *One of the restrictions $(\pi_1)_{|\Omega}, (\pi_2)_{|\Omega}, \dots, (\pi_n)_{|\Omega}$ is a section (up to contiguity) of the diagonal map $\Delta : K \to K^n$.*

Proof (1) $\Rightarrow$ (2): Suppose that $\Omega \subset K^n$ is an $n-$Farber subcomplex. Then there exists a simplicial map $\sigma : \Omega \to K$ such that $\Delta \circ \sigma \sim \iota_\Omega$.

Observe that $\Delta \circ \sigma$ is an n-tuple of σ's, that is, $\Delta \circ \sigma = (\sigma, \dots, \sigma) : \Omega \to K^n$ by $\Delta \circ \sigma(\omega) = (\sigma(\omega), \dots, \sigma(\omega))$.

On the other hand, the inclusion map $\iota_\Omega : \Omega \hookrightarrow K^n$ can be written as

$$\iota_\Omega = (\pi_1|_\Omega, \dots, \pi_n|_\Omega).$$

So we have

$$(\sigma, \dots, \sigma) = \Delta \circ \sigma \sim (\pi_1|_\Omega, \dots, \pi_n|_\Omega).$$

Hence, $\sigma \sim \pi_i|_\Omega$ for each i.

(2) $\Rightarrow$ (3): Fix $i_0 \in \{1, 2, \dots, n\}$ and suppose that $\pi_i|_\Omega \sim \pi_{i_0}|_\Omega$ for all $i \in \{1, 2, \dots, n\}$. Then

$$\iota_\Omega = \big(\pi_1|_\Omega, \dots, \pi_n|_\Omega\big) \sim \big(\pi_{i_0}|_\Omega, \dots, \pi_{i_0}|_\Omega\big) = \Delta \circ \pi_{i_0}|_\Omega.$$

which means that one of the $\pi_{i_0}|_\Omega$'s is a section of Δ.

(3) $\Rightarrow$ (1): Suppose that $(\pi_i)_{|\Omega}$ is a section (up to contiguity) of the diagonal map Δ, for some $i \in \{1, 2, \dots, n\}$ and choose the simplicial map $\sigma := (\pi_i)_{|\Omega} : \Omega \to K$. Then Ω is an $n-$Farber subcomplex. □

Theorem 2.2 $\mathrm{TC}_m(K) \leq \mathrm{TC}_{m+1}(K)$.

Proof Let $\mathrm{TC}_{m+1}(K) = \ell$. Then there exist $\Omega_0, \ldots, \Omega_\ell$ $(m+1)-$Farber subcomplexes of K^{m+1} covering K^{m+1}. By Theorem 2.1,

$$(\pi_1)_{|\Omega_j} \sim (\pi_2)_{|\Omega_j} \sim \ldots \sim (\pi_{m+1})_{|\Omega_j}$$

for each $j \in \{0, 1, \ldots, \ell\}$, where $\pi_i : K^{m+1} \to K$ is the projection to the ith factor.

If we show that there are $\Lambda_0, \ldots, \Lambda_\ell$ subcomplexes of K^m covering K^m such that $(\pi^1)_{|\Lambda_j} \sim (\pi^2)_{|\Lambda_j} \sim \ldots \sim (\pi^m)_{|\Lambda_j}$ for each $j \in \{0, 1, \ldots, \ell\}$, where $\pi^i : K^m \to K$ is the projection to the ith factor, then we are done.

For a fixed vertex $\omega \in K$, define the simplicial map

$$g : K^m \to K^{m+1}, \quad \text{by} \quad g(v^1, \ldots, v^m) = (v^1, \ldots, v^m, \omega)$$

and for each $j \in \{0, 1, \ldots, \ell\}$, define the subcomplex $\Lambda_j := g^{-1}(\Omega_j)$.

We have $\pi^i|_{\Lambda_j} = \pi_i|_{\Omega_j} \circ g : \Lambda_j \to \Omega_j \to K$, for $i \in \{1, \ldots, m\}$ and $j \in \{0, 1, \ldots, \ell\}$.

For every $j \in \{0, 1, \ldots, \ell\}$, since $(\pi_i)_{|\Omega_j} \sim (\pi_k)_{|\Omega_j}$ for each $i, k \in \{1, \ldots, m+1\}$, we have $(\pi^i)_{|\Lambda_j} \sim (\pi^k)_{|\Lambda_j}$ for each $i, k \in \{1, \ldots, m\}$. □

Theorem 2.3 *If* $K \sim L$*, then* $\mathrm{TC}_n(K) = \mathrm{TC}_n(L)$*, i.e., the* n*th discrete topological complexity is an invariant of the strong homotopy type.*

Proof We proceed in three steps.

Step 1. Let us show that if $K \sim L$, then $K^n \sim L^n$.

Suppose that K and L are of the same strong homotopy type, then by definition, there are simplicial maps φ, ψ satisfying $\varphi \circ \psi \sim 1_L$ and $\psi \circ \varphi \sim 1_K$. First consider the case $\varphi \circ \psi \sim 1_L$. By Proposition 2.1, we obtain

$$\varphi^n \circ \psi^n = (\varphi \circ \psi)^n \sim (1_L)^n = 1_{L^n}.$$

Using same arguments, we show that $\psi^n \circ \varphi^n \sim 1_{K^n}$. Hence, we have $K^n \sim L^n$.

Step 2. Next we show that if $\Omega \subset K^n$ is an nth Farber subcomplex, then so is $(\psi^n)^{-1}(\Omega) \subset L^n$.

There exists a simplicial map $\sigma : \Omega \to K$ so that $\Delta_K \circ \sigma \sim \iota_\Omega$ is satisfied, by definition, provided that $\Omega \subset K^n$ is an nth Farber subcomplex.

On the other hand, from the following commutative diagram, it follows that $\Delta_L \circ \varphi = \varphi^n \circ \Delta_K$ and $\Delta_K \circ \psi = \psi^n \circ \Delta_L$.

$$\begin{array}{ccc} K & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & L \\ \downarrow{\scriptstyle \Delta_K} & & \downarrow{\scriptstyle \Delta_L} \\ K^n & \underset{\psi^n}{\overset{\varphi^n}{\rightleftarrows}} & L^n \end{array}$$

Combining these two facts, we have the diagram

$$\begin{array}{ccccccc} & \overset{\sigma}{\nearrow} & K & \underset{\psi}{\overset{\varphi}{\rightleftarrows}} & L & \nwarrow & \\ & & \downarrow{\scriptstyle \Delta_K} & & \downarrow{\scriptstyle \Delta_L} & & \\ \Omega & \overset{\iota_\Omega}{\longrightarrow} & K^n & \underset{\psi^n}{\overset{\varphi^n}{\rightleftarrows}} & L^n & \underset{\iota_{\widetilde{\Omega}}}{\longleftarrow} & \widetilde{\Omega} \end{array}$$

and we find out that $\widetilde{\Omega} := (\psi^n)^{-1}(\Omega) \subset L^n$ is an $n-$Farber subcomplex as there exists a simplicial map

$$\widetilde{\sigma} := \varphi \circ \sigma \circ (\varphi^n)|_{\widetilde{\Omega}} : \widetilde{\Omega} \to L$$

satisfing the following

$$\begin{aligned}
\Delta_L \circ \widetilde{\sigma} &= \Delta_L \circ \varphi \circ \sigma \circ \psi^n|_{\widetilde{\Omega}} \\
&= \Delta_L \circ \varphi \circ \sigma \circ \psi^n \circ \iota_{\widetilde{\Omega}} \\
&= \varphi^n \circ \Delta_K \circ \sigma \circ \psi^n \circ \iota_{\widetilde{\Omega}} \\
&\sim \varphi^n \circ \iota_{\Omega} \circ \psi^n \circ \iota_{\widetilde{\Omega}} \\
&= (\varphi^n \circ \psi^n)_{|\widetilde{\Omega}} \\
&\sim 1_{L^n} \circ \iota_{\widetilde{\Omega}} \\
&= \iota_{\widetilde{\Omega}}\,.
\end{aligned}$$

Step 3. In the last step, we show that $\mathrm{TC}_n(K) \leq \mathrm{TC}_n(L)$. Similarly, $\mathrm{TC}_n(K) \geq \mathrm{TC}_n(L)$ can be showed and the result follows.

Let say $\mathrm{TC}_n(K) = k$. So there is a covering $K^n = \Omega_0 \cup \cdots \cup \Omega_k$ such that each Ω_j is an $n-$Farber subcomplex. From Step 2, each $\Lambda_j = (\psi^n)^{-1}(\Omega_j)$ is an $n-$Farber subcomplex for $j \in \{0, \ldots, k\}$ and they cover L^n. Hence, $\mathrm{TC}_n(K) \leq k$. □

Proposition 2.2 [4, Lemma 4.4] *K is path-edge connected if and only if any two constant simplicial maps to K are in the same contiguity class.*

Theorem 2.4 $\mathrm{scat}(K^{n-1}) \leq \mathrm{TC}_n(K)$ *provided that K is an edge-path connected simplicial complex K.*

Proof Suppose $\mathrm{TC}_n(K) = k$. Then there exist subcomplexes $\Omega_0, \Omega_1, \ldots, \Omega_k$ of K^n covering K^n such that each Ω_i is $n-$Farber subcomplex, i.e., there exists $\sigma_i : \Omega_i \to K$ satisfying $\Delta_K \circ \sigma_i \sim \iota_{\Omega_i}$, where $\iota_{\Omega_i} : \Omega_i \hookrightarrow K^n$ is the inclusion and $\Delta_K : K \to K^n$ is the diagonal map.

For a fixed vertex v_1^0, define $\iota_0 : K^{n-1} \to K^n$ by $\iota_0(\omega) = (v_1^0, \omega)$. Set $V_i := \iota_0^{-1}(\Omega_i) \subset K^{n-1}$ subcomplex. Notice that $V_0, V_1, \ldots, V_k$ cover K^{n-1}. If we prove that each V_i a categorical subcomplex of K^{n-1}, then we can conclude that $\mathrm{scat}(K^{n-1}) \leq k$, and the result follows.

For simplicity, we ignore the subscripts i. From the assumption, $\Delta_K \circ \sigma \sim \iota_\Omega$, that is, there exist $\varphi_{\bar{j}} : \Omega \to K^n$ simplicial maps, for $\bar{j} \in \{1, 2, \ldots, m\}$, such that $\varphi_1 = \Delta_K \circ \sigma$, $\varphi_m = \iota_\Omega$ and $\varphi_{\bar{j}}$ is contiguous to $\varphi_{\bar{j}+1}$:

$$\Delta_K \circ \sigma = \varphi_1 \sim_c \varphi_2 \sim_c \ldots \sim_c \varphi_m = \iota_\Omega.$$

Denoting by $\mathrm{pr}_j : K^n \to K$ the projection to the jth factor, take the compositions

$$\begin{aligned}
\mathrm{pr}_j \circ (\Delta_K \circ \sigma) \circ \iota_0 \circ \iota_V \;&=\; \mathrm{pr}_j \circ \varphi_1 \circ \iota_0 \circ \iota_V \\
&\sim_c \mathrm{pr}_j \circ \varphi_2 \circ \iota_0 \circ \iota_V \\
&\sim_c \cdots \\
&\sim_c \mathrm{pr}_j \circ \varphi_m \circ \iota_0 \circ \iota_V \\
&= \mathrm{pr}_j \circ \iota_\Omega \circ \iota_0 \circ \iota_V
\end{aligned}$$

for each $j = 1, \ldots, n$.

Notice that $\mathrm{pr}_j \circ \Delta_K \circ \sigma \circ \iota_0 \circ \iota_V(\omega) = \sigma(v_1^0, \omega) \in K$ for each $j \in \{1, \ldots, n\}$.

On the other hand

$$\mathrm{pr}_1 \circ (\Delta_K \circ \sigma) \circ \iota_0 \circ \iota_V = v_1^0$$
$$\mathrm{pr}_j \circ (\Delta_K \circ \sigma) \circ \iota_0 \circ \iota_V = \pi_{j-1}$$

for $j \in \{2, \ldots, n\}$, where $\pi_j : K^{n-1} \to K$ is the projection to the jth factor. Here, we can write ι_V in terms of π_j's. Thus,

$$\iota_V = \big(\pi_1, \pi_2, \ldots, \pi_{n-1}\big) \sim \big(\mathrm{pr}_2 \circ (\Delta_K \circ \sigma) \circ \iota_0 \circ \iota_V, \ldots, \mathrm{pr}_n \circ (\Delta_K \circ \sigma) \circ \iota_0 \circ \iota_V\big).$$

Since K is path-edge connected, by Lemma 2.2, the constant map $c : V \to K^{n-1}, c(\omega) = (v_1^0, \ldots, v_{n-1}^0)$ can be realised as another constant map $\bar{c} : V \to K^{n-1}, \bar{c}(\omega) = (v_1^0, \ldots, v_1^0)$. Therefore, $\iota_V \sim c$. □

Theorem 2.5 $\mathrm{TC}_n(K) \leq \mathrm{scat}(K^n)$ *provided that* K *is an edge-path connected simplicial complex* K.

Proof Suppose that $\mathrm{scat}(K^n) = k$. Then there is a categorical covering $\{U_0, U_1, \ldots, U_k\}$ of K^n. If we show that U_i is an $n-$Farber subcomplex, for each $i \in \{0, 1, \ldots, k\}$, then the proof is concluded.

Since we have $\iota \sim c$ where $c : U \to K^n$ is a constant map and $\iota : U \to K^n$ is the inclusion map, there exists a sequence of simplicial maps $h_t : V \to K^n$ for $t \in \{1, \ldots, m\}$ such that $h_0 = \iota$, $h_m = c$ and (h_t, h_{t+1}) are contiguous for all $t \in \{1, 2, \ldots, m-1\}$.

Now let $\pi_j : K^n \to K$ denote the projection map to the jth factor. Hence, $\pi_j \circ h_t$ and $\pi_j \circ h_{t+1}$ are contiguous for all $t \in \{1, 2, \ldots, m-1\}$. From the fact that

$$\pi_j \circ \iota = \pi_j \circ h_0 \sim \pi_j \circ c,$$

it follows that $\pi_j \circ \iota$'s are all in the same contiguity class for all j. By Theorem 2.1, U is an $n-$Farber subcomplex. □

The following lemma, which can be proved by induction, is a generalisation of Theorem 5.5 in [5] and it will be used to prove the later corollary.

Lemma 2.1 *For finite simplicial complexes* $K_1, K_2, \ldots, K_m$*, we have*

$$\mathrm{scat}(K_1 \times K_2 \times \cdots \times K_m) + 1 \leq (\mathrm{scat}K_1 + 1)(\mathrm{scat}K_2 + 1)\cdots(\mathrm{scat}K_m + 1).$$

Corollary 2.1 *Let* K *be an abstract simplicial complex. Then* K *is strongly collapsible if and only if* $\mathrm{TC}_n(K) = 0$.

Proof We have $\mathrm{scat}K = 0$, since K is strongly collapsible. On the other hand, by Lemma 2.1, we obtain $\mathrm{scat}(K^n) + 1 \leq (\mathrm{scat}K + 1)^n$. Combining these two, we have $\mathrm{scat}(K^n) + 1 \leq 1$. Hence, $\mathrm{scat}(K^n) = 0$. It follows from Theorem 2.5 that $\mathrm{TC}_n(K) = 0$.

Conversely, if $\mathrm{TC}_n(K) = 0$, then by Theorem 2.2 $\mathrm{TC}_2(K) \leq \mathrm{TC}_n(K) = 0$. Hence the result follows from Corollary 4.7 in [4]. □

2.1 Geometric realisation

It is proved that $|K^2| \approx |K|^2$ and $|K \times K| \sim |K|^2$, see Theorem 10.21 and Proposition 15.23 in [7]. Moreover, the higher dimensional versions are also valid, as mentioned in Remark 5.2 in [5]. Combining these facts with Lemma 5.1 in [4], we get the following lemma.

Lemma 2.2 *There is a homotopy equivalence $u : |K|^n \to |K^n|$ such that the following diagram is commutative for each $i \in \{1, \dots, n\}$*

$$\begin{array}{ccc} |K|^n & \xrightarrow{u} & |K^n| \\ & \underset{p_i}{\searrow} & \downarrow |\pi_i| \\ & & |K| \end{array}$$

where $p_i : |K|^n \to |K|$ and $\pi_i : K^n \to K$ are projections.

Theorem 2.6 $\mathrm{TC}_n(|K|) \leqslant \mathrm{TC}_n(K)$.

Proof Let $\mathrm{TC}_n(K) = k$. So there exist $n-$Farber subcomplexes $\Omega_0, \dots \Omega_k$ of K^n covering K^n. By Theorem 2.1, $\pi_i \circ \iota_{\Omega_\ell}$ and $\pi_j \circ \iota_{\Omega_\ell}$ are in the same contiguity class for each pair $i, j \in \{1, 2, \dots, n\}$ and each $\ell \in \{0, 1, \dots, k\}$. Considering the geometric realisations of these simplicial maps, we have

$$|\pi_i \circ \iota_{\Omega_\ell}| \backsimeq |\pi_j \circ \iota_{\Omega_\ell}|$$

since being in the same contiguity class can be realised as being homotopic in the continuous realm (see for details [9]).

On the other hand, $|\pi_i \circ \iota_{\Omega_\ell}| = |\pi_i| \circ |\iota_{\Omega_\ell}|$ and $|\iota_{\Omega_\ell}| = \iota_{|\Omega_\ell|}$. Hence, we get

$$|\pi_i| \circ \iota_{|\Omega_\ell|} \backsimeq |\pi_j| \circ \iota_{|\Omega_\ell|}.$$

Now consider the preimage $F_\ell = u^{-1}(\Omega_\ell) \subset |K|^n$ for each $\ell = 0, 1, \dots, k$ where u is a homotopy equivalence as given in Lemma 2.2. Here, all F_ℓ's are closed and the subsets $F_0, F_1, \dots, F_k$ cover $|K|^n$. Moreover,

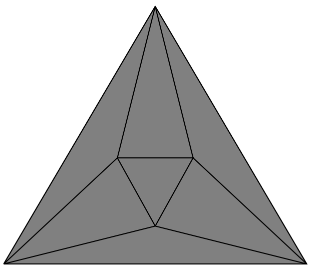
$$p_i \circ \iota_{F_\ell} = |\pi_i| \circ u \circ \iota_{F_\ell} = |\pi_i| \circ \iota_{|\Omega_\ell|} \backsimeq |\pi_j| \circ \iota_{|\Omega_\ell|} = |\pi_j| \circ u \circ \iota_{F_\ell} = p_j \circ \iota_{F_\ell}.$$

Here, the first and the last equalities follow from Lemma 2.2. This completes the proof. □

The following is an example for the strict case of the inequality in Theorem 2.6.

Example 2.1 Consider the simplicial complex in Fig. 1.

Fig. 1 $\mathrm{TC}_n(|K|) < \mathrm{TC}_n(K)$



As mentioned in [1], K is not strongly collapsible. By Example 3.3 in [6], $\mathrm{scat}(K) = 1$. Theorem 2.4 yields that $\mathrm{scat}(K) \leq \mathrm{TC}_2(K)$. From Theorem 2.2, it follows that $1 \leq \mathrm{TC}_n(K)$ for all $n \geq 2$. On the other hand, the geometric realisation of the simplicial complex in Fig. 1 is homeomorphic to a disc. Thus, $\mathrm{TC}_n(|K|) = 0$ for any $n \geq 2$.

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