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ORIGINAL PAPER

Modular differential equations and algebraic systems

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Abstract
This paper investigates the modular differential equation $y'' + sE_4y = 0$ on the upper half-plane $\mathbb{H}$, where E_4 is the weight 4 Eisenstein series and s is a complex parameter. This is equivalent to studying the Schwarz differential equation $\{h, \tau\} = 2sE_4$, where the unknown h is a meromorphic function on $\mathbb{H}$. On the other hand, such a solution h must satisfy $h(\gamma\tau) = \varrho(\gamma)h(\tau)$ for $\tau \in \mathbb{H}$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and ϱ being a 2−dimensional complex representation of $\mathrm{SL}_2(\mathbb{Z})$. Moreover, in order for h to be meromorphic or to have logarithmic singularities at the $\mathrm{SL}_2(\mathbb{Z})$-cusps of $\mathbb{H}$, it is necessary to have $s = \pi^2r^2$ with r being a rational number. For $r = m/n$ in reduced form, it turns out that ϱ is irreducible with finite image if and only if $2 \leq n \leq 5$ and in this case h is a modular function for the genus zero torsion-free principal congruence group $\Gamma(n)$, while ϱ is reducible if and only if $n = 6$. By Solving an explicit algebraic system, we prove that solutions for any $r = m/n$ can be built from a solution corresponding to $r = 1/n$, for $2 \leq n \leq 6$, by integrating certain weight 2 meromorphic modular forms. Together with the earlier work by the authors for r being an integer [20], this provides the solutions to the above-mentioned differential equations for all $r = m/n$ with $1 \leq n \leq 6$.

Keywords Modular differential equations · Schwarz derivative · Modular forms · Eisenstein series · Equivariant functions · Representations of the modular group

Mathematics Subject Classification 11F03 · 11F11 · 34M05

1 Introduction

The theory of modular differential equations, which are linear differential equations with coefficients in the ring of modular forms, have been considered by early automorphic forms experts such as Klein [10], Hurwitz [6] and Van der Pol [26, 27]. There has been a lot of

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interest in these differential equations in recent decades starting with the pioneering work by Kaneko and Zagier [7]. The subject developed into a fertile research area with applications in many areas of mathematics and mathematical physics. A great deal of literature has been produced on the subject, including the works [1, 4, 5, 8, 9, 11, 14, 15]. We shall be concerned with modular differential equations in connection with the Schwarz differential equation and the theory of equivariant functions as we now explain.

Let Γ be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ acting on the upper half-plane $\mathbb{H}$, and denote by $\overline{\Gamma}$ its projection in $\mathrm{PSL}_2(\mathbb{R})$. We consider the following differential equation with an automorphic potential

$$y'' + Q(\tau)\, y = 0, \quad \tau \in \mathbb{H},$$

where $Q(\tau)$ is a weight 4 automorphic form for Γ. If f_1 and f_2 are linearly independent solutions, then $h = f_2/f_1$ satisfy the Schwarz differential equation

$$\{h, \tau\} = 2Q(\tau),$$

where $\{h, \tau\}$ is the Schwarz derivative defined by

$$\{h, \tau\} = \left(\frac{h''(\tau)}{h'(\tau)}\right)' - \frac{1}{2}\left(\frac{h''(\tau)}{h'(\tau)}\right)^2 .$$

The Schwarz derivative has many projective, geometric and analytic properties that can be found in [16, 22]. On the other hand, for a meromorphic function h on $\mathbb{H}$, one can show that $\{h, \tau\}$ is a weight 4 automorphic form for Γ if and only if there exists a 2-dimensional representation ϱ of $\overline{\Gamma}$ such that

$$h(\gamma \cdot \tau) = \varrho(\gamma) \cdot h(\tau), \ \tau \in \mathbb{H}, \ \gamma \in \Gamma.$$

Here, both γ and $\varrho(\gamma)$ act by linear fractional transformation. We call such a function h a $\varrho-$equivariant function for Γ. This class of functions has been studied extensively in [2, 3, 24, 25] with interesting applications in [18–21, 23]. The automorphic functions (of weight zero) are $\varrho-$equivariant with $\varrho = 1$; the constant representation. If $\varrho = Id$, the defining representation, then h is simply called an equivariant function (it commutes with action of Γ). As an example, if f is a weight k automorphic form for Γ, then

$$h_f(\tau) = \tau + k\,\frac{f(\tau)}{f'(\tau)}$$

is an equivariant function for Γ. This also includes the case f being a non-constant automorphic function which leads to the trivial equivariant function $h(\tau) = \tau$.

In this paper, we focus on the case of the modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. A holomorphic weight 4 modular form $Q(\tau)$ is thus a scalar multiple of the weight 4 Eisenstein series $E_4(\tau)$. Therefore, we consider the modular differential equation

$$y'' + s\,E_4(\tau)\, y = 0, \tag{1.1}$$

and the corresponding Schwarz differential equation

$$\{h, \tau\} = 2\,s\,E_4(\tau)\,. \tag{1.2}$$

It should be noted that the modular differential equations studied in [8] and [7] can be reduced to the equation (1.1), [21]. According to [18], any solution h to (1.2) is necessarily locally univalent and leads to solutions $y_1 = 1/\sqrt{h'}$ and $y_2 = h/\sqrt{h'}$ to (1.1). Moreover, for a solution to (1.2) to be meromorphic or to have a logarithmic singularity at ∞, the parameter s must satisfy $s = \pi^2 r^2$ where r is a rational number.

In [18], we investigated solutions to (1.2) that are $\varrho-$equivariant with $\operatorname{Ker}\varrho$ having a finite index in $\mathrm{SL}_2(\mathbb{Z})$, in other words, that are modular functions for a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. It turns out that necessarily ρ is an irreducible representation of $\mathrm{SL}_2(\mathbb{Z})$ and that $s = \pi^2 r^2$ with a rational number $r = n/m$, $2 \le m \le 5$ and $\gcd(m, n) = 1$. Furthermore, the solution h is a modular function for the principal congruence subgroup $\Gamma(m)$. The integers m and n have the following interpretation: We have the two coverings of compact Riemann surfaces

$$\pi : X(\ker \varrho) \longrightarrow X(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{P}_1(\mathbb{C})$$

induced by the natural inclusion $\ker \varrho \subseteq \mathrm{SL}_2(\mathbb{Z})$, and

$$h : X(\ker \varrho) \longrightarrow X(\mathrm{SL}_2(\mathbb{Z})) \cong \mathbb{P}_1(\mathbb{C})$$

induced by the solution h. Here $X(\Gamma)$ is the modular curve attached to the subgroup Γ. Then m and n are the respective ramification indices above ∞ for the two coverings.

When r is an integer ($m = 1$), the situation is completely different. There are always solutions to (1.2) that are simply equivariant, that is, when $\varrho = Id$, while the solutions to (1.1) are constructed from quasi-modular forms [20].

In [19], we investigated the case when solutions to (1.2) correspond to reducible representation ϱ of $\mathrm{SL}_2(\mathbb{Z})$. It turns out that necessarily $r = n/6$ with $\gcd(n, 6) = 1$. The denominator 6 occurs because it is the level of the commutator group of $\mathrm{SL}_2(\mathbb{Z})$ over which the characters of $\mathrm{SL}_2(\mathbb{Z})$ are trivial. In addition, the solutions to (1.2) are integrals of weight 2 meromorphic modular forms with a character. For the case $n = 1$, the weight 2 form in question is η^4. We then constructed solutions for every $n = 1 + 12k$, $k \in \mathbb{N}$, by integrating the modular form

$$f_n = \frac{\eta^4}{\prod_{i=1}^{k}(J - a_i)^2},$$

where the numbers a_i are solution to the algebraic system

$$\frac{4}{x_i} + \frac{3}{x_i - 1} + \sum_{j \neq i} \frac{12}{x_j - x_i} = 0, \quad 1 \le i \le k,$$

which turns out to admit a solution in $(0, 1)^k$. The idea is to adjoin double poles to η^4 in $\mathbb{H}$ with zero residues. In this case, the double poles are not elliptic points.

In this paper, we show that this method extends nicely to the remaining cases of residues modulo 12, namely, for n coprime to 6 such that $n \equiv 5, 7 \text{ or } 11 \mod 12$. More precisely, starting from a fundamental solution f_α to (1.2) with $s = \pi^2(\alpha/6)^2$ for each $\alpha = 5, 7$ or 11, one can construct a solution for each m in the residue class of α modulo 12 by adjoining double poles to f'_α and integrating. In these cases, the double poles are allowed to include one of the elliptic points i or ρ or both. However, it is shown that the whole construction can be carried out by solving the algebraic system

$$\frac{a}{x_i} + \frac{b}{x_i - 1} + \sum_{j \neq i} \frac{c}{x_j - x_i} = 0, \quad 1 \le i \le k,$$

where a, b and c vary with α. These systems are a result of some nice identities involving special values of higher derivatives of classical modular forms.

Furthermore, we revisit the cases where the level $m \in \{2, 3, 4, 5\}$ studied in [18] and we show that our method can be applied to construct the solutions to (1.1) and (1.2). Indeed, starting from a solution t_m corresponding to $r = 1/m$, which turns out to be a Hauptmodul for $\Gamma(m)$, we can construct a solution corresponding to $r = n/m$ with n coprime to m by adjoining to t'_m double zeros and double poles that arise from solutions to the above algebraic system with an appropriate choice of the parameters a, b and c.

It is yet to be fully understood why solutions to a simple algebraic system would lead to solutions to infinite families of modular differential equations.

2 Special values of higher derivatives of modular forms

In this section we recall some classical elliptic modular forms. We also establish some interesting identities involving special values of their higher derivatives at elliptic fixed points.

The Eisenstein series E_2, E_4 and E_6 are defined by their $q-$expansions:

$$E_2(\tau) = 1 - 24 \sum_{n\geq 1} \sigma_1(n)\, q^n\,,$$

$$E_4(\tau) = 1 + 240 \sum_{n\geq 1} \sigma_3(n)\, q^n\,,$$

$$E_6(\tau) = 1 - 504 \sum_{n\geq 1} \sigma_5(n)\, q^n\,.$$

Here τ is a variable in the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} | \mathrm{Im}(\tau) > 0\}$ and $q = \exp(2\pi i \tau)$ is the uniformizer at ∞. The arithmetical function σ_k is defined on positive integers by

$$\sigma_k(n) = \sum_{0<d\,|\,n} d^k.$$

The function E_2 is a quasi-modular form of weight 2 and E_4 and E_6 are modular forms of respective weights 4 and 6 for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

We also define the Dedekind eta-function by

$$\eta(\tau) \;=\; q^{\frac{1}{24}} \prod_{n\geq 1} (1 - q^n),$$

and the weight 12 cusp form Δ (the modular discriminant)

$$\Delta(\tau) \;=\; \eta(\tau)^{24} \;=\; \frac{1}{1728}(E_4(\tau)^3 - E_6(\tau)^2).$$

We also have the elliptic modular function J ($J-$invariant)

$$J(\tau) \;=\; \frac{1}{1728} \frac{E_4(\tau)^3}{\Delta},$$

and the Klein elliptic modular function λ for $\Gamma(2)$

$$\lambda(\tau) \;=\; \left(\frac{\eta(\tau/2)}{\eta(2\tau)}\right)^8.$$

The following relations will be used below [17, Chapter 6]:

$$E_4 \;=\; \frac{J'^2}{(2\pi i)^2 J(J-1)}, \tag{2.1}$$

$$E_6 \;=\; \frac{J'^3}{(2\pi i)^3 J^2(J-1)}, \tag{2.2}$$

$$\Delta \;=\; \frac{-1}{(48\pi^2)^3}\,\frac{J'^6}{J^4(J-1)^3}. \tag{2.3}$$

Let us recall that Δ does not vanish in $\mathbb{H}$, while E_6 (resp. E_4) has a simple zero at i (resp. at $\rho = \exp(2\pi i/3)$) and its $\mathrm{SL}_2(\mathbb{Z})-$ orbit. Meanwhile, $J-1$ has a double zero at i and J has a zero of order 3 at ρ.

The following propositions will be very useful in the next sections.

Proposition 2.1 *We have*

$$12\,\frac{\eta'(i)}{\eta(i)} \;=\; \frac{3}{7}\,\frac{E_6''(i)}{E_6'(i)} \;=\; \frac{J'''(i)}{J''(i)} \;=\; 3i.$$

Proof Taking the logarithmic derivative in (2.3) yields

$$24\,\frac{\eta'}{\eta} \;=\; 6\,\frac{J''}{J'} - 4\,\frac{J'}{J} - 3\,\frac{J'}{J-1}. \tag{2.4}$$

Using the expansion of J near i

$$J(\tau) = 1 + \frac{1}{2}\,J''(i)(\tau-i)^2 + \frac{1}{6}\,J'''(i)(\tau-i)^3 + \mathrm{O}(\tau-i)^4$$

we get

$$\begin{aligned}
\frac{J''(\tau)}{J'(\tau)} &= \frac{1}{\tau-i} + \frac{1}{2}\,\frac{J'''(i)}{J''(i)} + \mathrm{O}(\tau-i),\\
\frac{J'(\tau)}{J(\tau)-1} &= \frac{2}{\tau-i} + \frac{1}{3}\,\frac{J'''(i)}{J''(i)} + \mathrm{O}(\tau-i)
\end{aligned}$$

and

$$\frac{J'(\tau)}{J(\tau)} = \mathrm{O}(\tau-i).$$

Therefore,

$$24\,\frac{\eta'(\tau)}{\eta(\tau)} = 2\,\frac{J'''(i)}{J''(i)} + \mathrm{O}(\tau-i),$$

that is

$$12\,\frac{\eta'(i)}{\eta(i)} = \frac{J'''(i)}{J''(i)}.$$

In the meantime, differentiating $J(-1/\tau) = J(\tau)$ trice yields

$$\frac{6}{\tau^4}\,J'(-1/\tau) - \frac{6}{\tau^5}\,J''(-1/\tau) + \frac{1}{\tau^6}\,J'''(-1/\tau) \;=\; J'''(\tau).$$

Since $J'(i) = 0$, we get

$$\frac{J'''(i)}{J''(i)} = 3i.$$

Finally, using

$$E_6(\tau) = (\tau-i)E_6'(i) + \frac{1}{2}(\tau-i)^2 E_6''(i) + \mathrm{O}(\tau-i)^3$$

and

$$E_6'(\tau) = E_6'(i) + (\tau - i)E_6''(i) + \mathrm{O}(\tau - i)^2,$$

we obtain

$$\frac{E_6'(\tau)}{E_6(\tau)} = \frac{1}{\tau - i} + \frac{1}{2}\frac{E_6''(i)}{E_6'(i)} + \mathrm{O}(\tau - i)^2. \tag{2.5}$$

On the other hand, taking the logarithmic derivative in (2.2) yields

$$\begin{aligned}\frac{E_6'(\tau)}{E_6(\tau)} &= 3\frac{J''(\tau)}{J'(\tau)} - 2\frac{J'(\tau)}{J(\tau)} - \frac{J'(\tau)}{J(\tau) - 1}\\ &= \frac{1}{\tau - i} + \frac{7}{6}\frac{J'''(i)}{J''(i)} + \mathrm{O}(\tau - i)\end{aligned}$$

using the expansions of J''/J', J'/J etc. cited in the beginning of this proof. Now, comparing with (2.5), we get

$$\frac{E_6''(i)}{E_6'(i)} = \frac{7}{3}\frac{J'''(i)}{J''(i)},$$

which concludes the proof. □

Proposition 2.2 *We have*

$$24\frac{\eta'(\rho)}{\eta(\rho)} = \frac{J^{(4)}(\rho)}{J'''(\rho)} = \frac{6}{5}\frac{E_4''(\rho)}{E_4'(\rho)} = 12\,\frac{1 + \rho}{1 - \rho}.$$

Proof Write

$$\begin{aligned}E_4(\tau) &= (\tau - \rho)E_4'(\rho) + \frac{1}{2}(\tau - \rho)^2 E_4''(\rho) + \mathrm{O}(\tau - \rho)^3,\\ E_4'(\tau) &= E_4'(\rho) + (\tau - \rho)E_4''(\rho) + \mathrm{O}(\tau - \rho)^2,\end{aligned}$$

so that

$$\frac{E_4'(\tau)}{E_4(\tau)} = \frac{1}{\tau - \rho} + \frac{1}{2}\frac{E_4''(\rho)}{E_4'(\rho)} + \mathrm{O}(\tau - \rho). \tag{2.6}$$

Now write

$$\begin{aligned}J(\tau) &= \frac{1}{6}J'''(\rho)(\tau - \rho)^3 + \frac{1}{24}J^{(4)}(\rho)(\tau - \rho)^4 + \mathrm{O}(\tau - \rho)^5,\\ J'(\tau) &= \frac{1}{2}J'''(\rho)(\tau - \rho)^2 + \frac{1}{6}J^{(4)}(\rho)(\tau - \rho)^3 + \mathrm{O}(\tau - \rho)^4,\\ J''(\tau) &= J'''(\rho)(\tau - \rho) + \frac{1}{2}J^{(4)}(\rho)(\tau - \rho)^2 + \mathrm{O}(\tau - \rho)^3.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{J''(\tau)}{J'(\tau)} &= \frac{2}{\tau - \rho} + \frac{1}{3}\frac{J^{(4)}(\rho)}{J'''(\rho)} + \mathrm{O}(\tau - \rho),\\ \frac{J'(\tau)}{J(\tau)} &= \frac{3}{\tau - \rho} + \frac{1}{4}\frac{J^{(4)}(\rho)}{J'''(\rho)} + \mathrm{O}(\tau - \rho)\end{aligned}$$

and

$$\frac{J'(\tau)}{J(\tau) - 1} = \mathrm{O}(\tau - \rho)^2.$$

Now, using (2.1), we have

$$\frac{E_4'(\tau)}{E_4(\tau)} = 2\frac{J''(\tau)}{J'(\tau)} - \frac{J'(\tau)}{J(\tau)} - \frac{J'(\tau)}{J(\tau)-1}. \tag{2.7}$$

Therefore,

$$\frac{E_4'(\tau)}{E_4(\tau)} = \frac{1}{\tau-\rho} + \frac{5}{12}\frac{J^{(4)}(\rho)}{J'''(\rho)} + \mathrm{O}(\tau-\rho).$$

Comparing with (2.6), we obtain

$$\frac{E_4''(\rho)}{E_4'(\rho)} = \frac{5}{6}\frac{J^{(4)}(\rho)}{J'''(\rho)}.$$

On the other hand, using (2.4), we get

$$24\frac{\eta'(\tau)}{\eta(\tau)} = \frac{J^{(4)}(\rho)}{J'''(\rho)} + \mathrm{O}(\tau-\rho),$$

which proves that

$$24\frac{\eta'(\rho)}{\eta(\rho)} = \frac{J^{(4)}(\rho)}{J'''(\rho)} = \frac{6}{5}\frac{E_4''(\rho)}{E_4'(\rho)}.$$

Furthermore, differentiating twice the identity

$$E_4\left(\frac{-1}{\tau+1}\right) = (\tau+1)^4 E_4(\tau)$$

and taking $\tau = \rho$ yields the last equality in the proposition. □

3 Level 6 modular differential equations and algebraic systems

Suppose we are given a $\rho-$equivariant function for a finite index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. If ρ is a reducible representation of Γ, then it can be conjugated to an upper triangular representation, i.e. there exists $\sigma \in \mathrm{GL}_2(\mathbb{C})$ such that $\rho_1 = \sigma\rho\sigma^{-1}$ is upper triangular. Moreover $h_1 = \sigma \cdot h$ is ρ_1-equivariant and shares the same Schwarz derivative with h. Thus, if we are looking for a solution to (1.2) corresponding to a reducible representation, we may suppose, without loss of generality, that ρ is upper-triangular. According to Theorem 4.3 in [19], a meromorphic function h is $\rho-$equivariant for a triangular representation ρ of Γ if and only if the derivative h' is a meromorphic weight 2 modular form for Γ with a character. Now, for the Schwarz derivative $\{h, \tau\}$ to be holomorphic, h' must be nonvanishing where h is holomorphic and, elsewhere, h should have only simple poles, which is equivalent to say that h' has only double poles with zero residues. Therefore, if we seek a solution h to (1.2), then we have to integrate nonvanishing weight 2 modular forms for $\mathrm{SL}_2(\mathbb{Z})$ with a character and having double poles (if any) with vanishing residues. The characters in question are trivial on the commutator group of $\mathrm{SL}_2(\mathbb{Z})$ which is a level 6 and index 12 in $\mathrm{SL}_2(\mathbb{Z})$, and therefore these modular forms have a $q-$expansion where $q = \exp(2\pi i\tau/6)$.

According to [19], a holomorphic weight 2 modular form for $\mathrm{SL}_2(\mathbb{Z})$ with a character must be equal to $c\eta^4$ where c is a constant. If we set

$$h(\tau) = \int_i^\tau \eta^4(\tau)\, d\tau,$$

then we have

$$\{h,\tau\} = \frac{2\pi^2}{36}\,E_4(\tau).$$

In other words, h is a solution to (1.2) with $s = \dfrac{\pi^2}{6^2}$. It follows that $y = 1/\sqrt{h'} = \eta^{-2}$ is a solution to $y'' + \dfrac{\pi^2}{36}\,y = 0$; a differential equation that was first mentioned by Klein in [10], and it was Hurwitz who first gave η^{-2} as a solution to this equation [6].

In order to find other solutions, we should look for weight 2 modular forms with double poles and zero residues. To this end, for each triplet of parameters (a, b, c), we introduce the following algebraic system $E^n_{a,b,c}$ of n equations in n variables x_i:

$$\frac{a}{x_i} + \frac{b}{x_i - 1} + \sum_{j\neq i}\frac{c}{x_j - x_i} = 0\,,\quad 1 \le i \le n. \tag{3.1}$$

Notice that for $\alpha \neq 0$, the system $E^n_{a,b,c}$ is equivalent to the system $E^n_{\alpha a,\alpha b,\alpha c}$. According to [19, Theorem 6.2], if a, b and c are positive real numbers, then the system $E^n_{a,b,c}$ has a solution in $(0,1)^n$. Let $(x_i)_{1\le i\le n}$ be a solution to the algebraic system $E^n_{4,3,12}$ and set

$$f_n = \frac{\eta^4}{\prod_{i=1}^n (J - x_i)^2}.$$

Also, write $x_i = J(w_i)$, $w_i \in \mathbb{H}$. Then, as $0 < x_i < 1$, the w_i's are not elliptic fixed points and f_n is a weight 2 modular form with a character and has a double pole at each w_i, $1 \le i \le n$, and is holomorphic elsewhere. Moreover, the fact that the x_i's satisfy the system $E^n_{4,3,12}$ is equivalent to the vanishing of the residues of f_n at each w_i. One of the main results in [19] is that $h_n(\tau) = \displaystyle\int_i^\tau f_n(z)\,dz$ is a solution to (1.2) with $s = \pi^2\left(\dfrac{12n+1}{6}\right)^2$, while the solutions to (1.1) are given by $y_1 = \eta^{-2}\prod_{i=1}^n (J - x_i)$ and $y_2 = h_n y_1$; generalizing Hurwitz's solution when $n = 0$. This solves the modular differential equations (1.1) and (1.2) with $s = \pi^2(m/6)^2$ with $m \equiv 1 \mod 12$. We now focus on finding the solutions for the remaining residues classes modulo 12, that is, when $m \equiv 5, 7$ or $11 \mod 12$. The idea is to allow the nonvanishing weight 2 modular forms to have double poles at elliptic points.

Theorem 3.1 *Let $n \in \mathbb{N}$ and $(x_i)_{1\le i\le n} \in (0,1)^n$ be a solution to the algebraic system $E^n_{4,9,12}$ and let $w_i \in \mathbb{H}$ such that $x_i = J(w_i)$. Then*

$$f_n = \frac{\eta^4}{(J-1)\prod_{i=1}^n (J - x_i)^2} = \frac{\eta^{28}}{E_6^2\prod_{i=1}^n (J - x_i)^2}$$

is a nonvanishing weight 2 modular form with double poles at i and at each w_i with zero residues. Moreover, if $h_n(\tau) = \displaystyle\int_i^\tau f_n(z)dz$, then

$$\{h_n,\tau\} = 2\pi^2\frac{(12n+7)^2}{6^2}\,E_4(\tau).$$

Proof As i is a double zero of $J-1$ and each w_i is not in the $\mathrm{SL}_2(\mathbb{Z})$−orbit of i, it is clear that i is a double pole of f_n. Write $g_n = \eta^4/\prod_{i=1}^n (J - x_i)^2$ so that $f_n = g_n/(J-1)$. Also write

$$J(\tau)-1=\frac{1}{2}J''(i)(\tau-i)^2+\frac{1}{6}J'''(i)(\tau-i)^3+\mathrm{O}(\tau-i)^4.$$

Then

$$\frac{J''(i)}{2}\frac{(\tau-i)^2}{J(\tau)-1}=1-\frac{1}{3}\frac{J'''(i)}{J''(i)}(\tau-i)+\mathrm{O}(\tau-i)^2.$$

It follows that

$$\begin{aligned}\frac{d}{d\tau}((\tau-i)^2 f_n(\tau)) &= \frac{d}{d\tau}\frac{g_n(\tau)(\tau-i)^2}{J(\tau)-1}\\ &= g_n'(\tau)\frac{(\tau-1)^2}{J(\tau)-1}+g_n(\tau)\frac{d}{d\tau}\frac{(\tau-i)^2}{J(\tau)-1}.\end{aligned}$$

Therefore,

$$\mathrm{Res}(f_n,i)=\lim_{\tau\to i}\frac{d}{d\tau}((\tau-i)^2 f_n(\tau))=\frac{2g_n(i)}{J''(i)}\left(\frac{g_n'(i)}{g_n(i)}-\frac{J'''(i)}{3J''(i)}\right).$$

In the meantime, taking the logarithmic derivative of g_n yields

$$\frac{g_n'(i)}{g_n(i)}=4\frac{\eta'(i)}{\eta(i)}-\sum_{i=1}^{n}\frac{2J'(i)}{J(i)-x_i}=4\frac{\eta'(i)}{\eta(i)}.$$

Hence, using Proposition 2.1,

$$\mathrm{Res}(f_n,i)=\frac{2g_n(i)}{J''(i)}\left(4\frac{\eta'(i)}{\eta(i)}-\frac{J'''(i)}{3J''(i)}\right)=0.$$

Now, fix i, $1\le i\le n$, and write $f_n(\tau)=\phi_n(\tau)/(J(\tau)-J(w_i))^2$. A similar calculation as above shows that

$$\mathrm{Res}(f_n,w_i)=\frac{\phi_n'(w_i)}{J'(w_i)^2}-\phi_n(w_i)\frac{J''(w_i)}{J'(w_i)^3}=\frac{\phi_n(w_i)}{J'(w_i)^2}\left(\frac{\phi_n'(w_i)}{\phi_n(w_i)}-\frac{J''(w_i)}{J'(w_i)}\right).$$

Meanwhile,

$$\frac{\phi_n'}{\phi_n}=\frac{4\eta'}{\eta}-\frac{J'}{J-1}-\sum_{j\neq i}\frac{2J'}{J-J(w_j)}.$$

Therefore

$$\mathrm{Res}(f_n,w_i)=\frac{\phi_n(w_i)}{J'(w_i)^2}\left(\frac{4\eta'(w_i)}{\eta(w_i)}-\frac{J'(w_i)}{J(w_i)-1}-\sum_{j\neq i}\frac{2J'(w_i)}{J(w_i)-J(w_j)}-\frac{J''(w_i)}{J'(w_i)}\right).$$

Using (2.4), we have

$$\frac{4\eta'}{\eta}-\frac{J''}{J'}=\frac{J'}{6}\left(\frac{3}{1-J}-\frac{4}{J}\right),$$

and hence

$$\mathrm{Res}(f_n,w_i)=\frac{-\phi_n(w_i)}{6J'(w_i)}\left(\frac{4}{J(w_i)}+\frac{9}{J(w_i)-1}+\sum_{j\neq i}\frac{12}{J(w_i)-J(w_j)}\right)=0$$

because the $x_i = J(w_i)$ were chosen to be a solution to the algebraic system $E^n_{4,9,12}$. Therefore f_n has only double poles with zero residues and is nonvanishing elsewhere. Thus its integral h_n has only simple poles and it is locally univalent elsewhere. It follows that the Schwarz derivative $\{h_n, \tau\}$ is a holomorphic weight 4 modular form for $\mathrm{SL}_2(\mathbb{Z})$ and hence a scalar multiple of E_4. Finally, notice that the leading coefficient of the $q-$expansion of f_n is $q^{\frac{1}{6}+1+2n} = q^{\frac{12n+7}{6}}$ and consequently the leading coefficient of $\{h_n, \tau\} = (f_n'/f_n)' - \frac{1}{2}(f_n'/f_n)^2$ is easily seen to be $2\pi^2 \dfrac{(12n+7)^2}{6^2}$. □

We now seek a similar solution but with a double at the other elliptic fixed points, namely $\tau = \rho$.

Theorem 3.2 *Let $n \in \mathbb{N}$ and $(x_i)_{1 \le i \le n} \in (0,1)^n$ be a solution to the algebraic system $E^n_{8,3,12}$ and let $w_i \in \mathbb{H}$ such that $x_i = J(w_i)$. Then*

$$f_n = \frac{\eta^{20}}{E_4^2 \prod_{i=1}^n (J - x_i)^2}$$

is a nonvanishing weight 2 modular forms with double poles at ρ and at each w_i with zero residues. Moreover, if $h_n(\tau) = \displaystyle\int_i^\tau f_n(z)dz$, then

$$\{h_n, \tau\} = 2\pi^2 \frac{(12n+5)^2}{6^2} E_4(\tau).$$

Proof Write $f_n = \psi_n / E_4^2$ and

$$E_4(\tau) = (\tau - \rho)E_4'(\rho) + \frac{1}{2}(\tau - \rho)^2 E_4''(\rho) + \mathrm{O}(\tau - \rho)^3$$

so that

$$E_4^2(\tau) = (\tau - \rho)^2 E_4'^2(\rho)\left(1 + (\tau - \rho)\frac{E_4''(\rho)}{E_4'(\rho)} + \mathrm{O}(\tau - \rho)^2\right).$$

Hence,

$$\frac{(\tau - \rho)^2 \psi_n(\tau)}{E_4^2(\tau)} = \frac{\psi_n(\tau)}{E_4'^2(\rho)}\left(1 - (\tau - \rho)\frac{E_4''(\rho)}{E_4'(\rho)} + \mathrm{O}(\tau - \rho)^2\right).$$

It follows that

$$\begin{aligned}
\lim_{\tau \to \rho} \frac{d}{d\tau} \frac{(\tau - \rho)^2 \psi_n(\tau)}{E_4^2(\tau)} &= \frac{\psi_n'(\rho)}{E_4'^2(\rho)} - \frac{\psi_n(\rho) E_4''(\rho)}{E_4'^3(\rho)} \\
&= \frac{\psi_n(\rho)}{E_4'^2(\rho)}\left(\frac{\psi_n'(\rho)}{\psi_n(\rho)} - \frac{E_4''(\rho)}{E_4'(\rho)}\right) \\
&= \frac{\psi_n(\rho)}{E_4'^2(\rho)}\left(20\frac{\eta'(\rho)}{\eta(\rho)} - \sum_{i=1}^n \frac{2J'(\rho)}{J(\rho) - x_i} - \frac{E_4''(\rho)}{E_4'(\rho)}\right) \\
&= \frac{\psi_n(\rho)}{E_4'^2(\rho)}\left(20\frac{\eta'(\rho)}{\eta(\rho)} - \frac{E_4''(\rho)}{E_4'(\rho)}\right) \\
&= 0.
\end{aligned}$$

The last equality follows from Proposition 2.2. Therefore, the residue of f_n at the double pole ρ is zero. In a similar manner to the previous theorem and using both (2.4) and (2.7), it is easily seen that the residues of f_n at each w_i is precisely zero because the x_i's satisfy the algebraic system $E^n_{8,3,12}$. Finally, the leading coefficient of the $q-$expansion of f_n is $q^{\frac{12n+5}{6}}$ and thus the leading coefficient of $\{h_n, \tau\}$ is $2\pi^2 \dfrac{(12n+5)^2}{6^2}$. □

Finally, we seek a solution which has both elliptic points i and ρ as double poles.

Theorem 3.3 *Let $n \in \mathbb{N}$ and $(x_i)_{1\le i\le n} \in (0,1)^n$ be a solution to the algebraic system $E^n_{8,9,12}$ and let $w_i \in \mathbb{H}$ such that $x_i = J(w_i)$. Then*

$$f_n = \frac{\eta^{20}}{E_4^2(J-1)\prod_{i=1}^n (J-x_i)^2} = \frac{\eta^{44}}{E_4^2 E_6^2 \prod_{i=1}^n (J-x_i)^2}$$

is a nonvanishing weight 2 modular forms with double poles at i, ρ and at each w_i with zero residues. Moreover, if $h_n(\tau) = \int_i^\tau f_n(z)dz$, then

$$\{h_n, \tau\} = 2\pi^2 \frac{(12n+11)^2}{6^2} E_4(\tau).$$

Proof This can be shown in the same way as the previous two theorems with the use of both Proposition 2.1 and Proposition 2.2. At the same time, the exponent of q in the leading coefficient of f_n is $\dfrac{5}{6} + 1 + 12n = \dfrac{12n+11}{6}$. □

4 Modular solutions and algebraic systems

We have mentioned that according to [18], the Schwarzian equation (1.2) has solutions that are modular functions if and only if $s = \pi^2 n^2/m^2$ with m and n being positive integers such that $2 \le m \le 5$ and $\gcd(m, n) = 1$. For each such pair (m, n), the invariance group for the modular solution h is $\Gamma(m)$ and n is the ramification index above ∞ in the covering $h : X(m) \longrightarrow \mathbb{P}_1(\mathbb{C})$. Here $X(m) = X(\Gamma(m))$. A key fact about the groups $\Gamma(m)$ for $2 \le m \le 5$ is that they are the only principal congruence groups that are genus 0 and torsion-free. In this section, we will establish that these modular solutions are also attached to an algebraic system in the same way the solutions in the previous section were.

Let $m \in \{2, 3, 4, 5\}$ and let t be a Hauptmodul of $\Gamma(m)$. Choose t so that its Fourier expansion has the shape

$$t(\tau) = \frac{1}{q} + \sum_{i\ge 0} a_i q^i, \quad q = e^{\frac{2\pi i\tau}{m}}.$$

Since the Hauptmodul t takes its values only once and $\Gamma(m)$ has no elliptic elements, then according to [12, 13], $\{t, \tau\}$ is a holomorphic weight 4 modular form for the normalizer of $\Gamma(m)$ in $SL_2(\mathbb{Z})$ which is $SL_2(\mathbb{Z})$ itself , and thus it is a scalar multiple of E_4. From the $q-$expansion of t, it is clear that

$$\{t, \tau\} = \frac{\pi^2}{m^2} E_4(\tau).$$

Now let $n \geq 2$ be an integer coprime to m. According to [18], there exists a modular function h for $\Gamma(m)$ which is solution to $\{h, \tau\} = 2\pi^2(n/m)^2 E_4(\tau)$. As n is the ramification index of h at ∞, we can write

$$h(\tau) = q^n + \mathrm{o}(q^n) \quad q = e^{\frac{2\pi i \tau}{m}}.$$

Now, suppose that the poles of h are given by the set $\{w_1, w_2, \dots w_a, s_1, \dots s_b\}$, where, for $1 \leq i \leq a$, $w_i \in \mathbb{H}$ (if any) and the s_j, $1 \leq j \leq b$, are among the cusps of $\Gamma(m)$. Then the degree d of the covering $h : X(m) \longrightarrow \mathbb{P}^1(\mathbb{C})$ satisfies

$$d = a + nb. \tag{4.1}$$

We also consider the modular function $f = t'/h'$ for $\Gamma(m)$. Since h' can have only double poles at the w_i's and it is nonvanishing elsewhere in $\mathbb{H}$, we see that f is holomorphic in $\mathbb{H}$. Therefore, for some polynomials P and Q, we have $f = P(t)/Q(t)$. Moreover, as the Hauptmodul t has a pole at ∞, it is holomorphic on $\mathbb{H}$ and t' does not vanish on $\mathbb{H}$ (because t is a Hauptmodul and $\Gamma(m)$ has no elliptic element, or because the Schwarz derivative of t is holomorphic as we have seen above). It follows that each w_i, $1 \leq i \leq a$, is a zero of order 2 of f. In the meantime, the behaviour of f at the cusps is as follows:

- Near each s_i, $1 \leq i \leq b$: we have, for some constants α, β and γ, $h'(\tau) = \alpha/q^n + \dots$ and $t'(\tau) = \beta q + \dots$ because t has a pole at ∞ and thus it is holomorphic at any other cusp. Therefore,

$$f(\tau) = \gamma q^{n+1} + \dots.$$

- Near ∞:

$$f(\tau) = \frac{\alpha/q + \dots}{\beta q^n + \dots} = \frac{\gamma}{q^{n+1}} + \dots.$$

- Near each cusp $s \notin \{s_1, \dots, s_b, \infty\}$: we have

$$f(\tau) = \frac{\alpha q + \dots}{\beta q^n + \dots} = \frac{\gamma}{q^{n-1}} + \dots.$$

Therefore, we have

$$P(t) = \prod_{i=1}^{a} (t - t(w_i))^2 \prod_{i=1}^{b} (t - t(s_i))^{n+1}$$

and

$$Q(t) = \prod_{s \notin \{s_1, \dots, s_b, \infty\}} (t - t(s))^{n-1}.$$

Furthermore, comparing the order of ∞ in $h'/t' = Q(t)/P(t)$ yields

$$(n+1) = 2a + (n+1)b - (n-1)(\nu_\infty - (b+1)),$$

where ν_∞ is the number of inequivalent cusps for $\Gamma(m)$. Hence, using (4.1), we get

$$2d - 2 = (n-1)\nu_\infty.$$

Notice that this is simply the Riemann-Hurwitz formula for the covering $h : X(m) \longrightarrow \mathbb{P}^1(\mathbb{C})$.

We can have a more precise information on a and b for a given level m.

Proposition 4.1 *With the notation as above, for each positive integer n, we have*

(1) *If $m = 2$, then*

$$(a,b) \in \left\{ \left(\frac{3n-1}{2}, 0 \right), \left(\frac{n-1}{2}, 1 \right) \right\}.$$

(2) *If $m = 3$, then*

$$(a,b) \in \{(2n-1,0), (n-1,1)\}.$$

(3) *If $m = 4$, then*

$$(a,b) \in \{(3n-2,0), (2n-2,1), (n-2,2)\}.$$

(4) *Finally, if $m = 5$, then*

$$(a,b) \in \{(n(6-k)-5,k),\ 0 \le k \le 5\}.$$

Proof If $m = 2$, there are 3 inequivalent cusps and the Riemann-Hurwitz formula reads $2d = 3n-1$ which implies that n is odd. Since $a+nb = d$, we have then $2a+1 = n(3-2b)$. It follows that either $b = 0$ which gives $a = (3n-1)/2$ or $b = 1$ which corresponds to $a = (n-1)/2$. Similarly, for $m = 3$, $\nu_\infty = 4$ and then $a+nb = 2n-1$ which we can rewrite as $a+1 = n(2-b)$. It follows that $b \in \{0,1\}$ and $(a,b) \in \{(2n-1,0), (n-1,1)\}$. The other cases follow similarly knowing that for $n \ge 3$, $\nu_\infty = \frac{1}{2}n^2 \prod_{p|n, p\ prime}(1-1/p^2)$. □

Finally, the solution to $\{h,\tau\} = 2\pi^2(n/m)^2 E_4$ is thus obtained by integrating the weight 2 modular form $h' = t'Q(t)/P(t)$ by choosing the adequate pair (a,b) given in the above proposition. Clearly $t'Q(t)$ does not vanish on $\mathbb{H}$ and the poles in $\mathbb{H}$ are the w_i's which should have a zero residue. Hence, $x_i = t(w_i)$, $1 \le i \le a$, are a solution to a system of type (3.1).

Let us illustrate this construction in the case $m = 2$. The group $\Gamma(2)$ has 3 cusps, namely 0, 1 and ∞. We take $t = 1/\lambda$ which sends the triple $(0,1,\infty)$ to the triple $(1,0,\infty)$.

<u>**Case 1:**</u> If $a = (3n-1)/2$ and $b = 0$, then

$$h' = \frac{t' t^{n-1}(t-1)^{n-1}}{\prod_{j=1}^{a}(t-t(w_j))^2}. \tag{4.2}$$

Fix $i \in \{1\ldots a\}$, and write $h' = g/(t-t(w_i))^2$. Then

$$\mathrm{Res}(h', w_i) = \frac{g(w_i)}{t'(w_i)^2}\left(\frac{g'(w_i)}{g(w_i)} - \frac{t''(w_i)}{t'(w_i)}\right).$$

Meanwhile,

$$\frac{g'}{g} = \frac{t''}{t'} + (n-1)\frac{t'}{t} + (n-1)\frac{t'}{t-1} - \sum_{j\neq i}\frac{2t'}{t-t(w_j)}.$$

It follows that

$$\mathrm{Res}(h', w_i) = \frac{g(w_i)}{t'(w_i)}\left(\frac{n-1}{t(w_i)} - \frac{n-1}{t(w_i)-1} - \sum_{j\neq i}\frac{2}{t(w_i)-t(w_j)}\right).$$

Thus, if we set $x_i = t(w_i)$, then $(x_i)_{1\le i\le a}$ is a solution to the algebraic system $E^a_{n-1,1-n,-2}$.

Case 2: If $a = (n-1)/2$ and $b = 1$, then we have two sub-cases depending on whether we take the cusp 0 or the cusp 1 in the polynomial $P(t)$. We have the two possibilities for h':

$$h_1' = \frac{t' t^{n-1}}{(t-1)^{n+1} \prod_{j=1}^{a} (t - x_j)^2} \quad \text{and} \quad h_2' = \frac{t'(t-1)^{n-1}}{t^{n+1} \prod_{j=1}^{a} (t - x_j)^2}, \tag{4.3}$$

where the $(x_i)_{1 \le i \le a}$ are solution to $E^a_{1-n,n+1,2}$ and $E^a_{n+1,1-n,2}$ respectively. Notice that both functions have q^n as a leading term and their integrals are solution to $\{h, \tau\} = \pi^2 (n/2)^2 E_4(\tau)$. This mean that h_1 and h_2 are linear fraction of one another.

Example 4.2 For $m = 2$ and $n = 3$, we have three solutions for h':

(1) With one pole in $\mathbb{H}$ and one pole at the cusp 0:

$$h_1' = \frac{t' t^2}{(t-x)^2 (t-1)^4}.$$

The residue at the pole in $\mathbb{H}$ is zero lead to $x = -1$. Thus, we have the solution

$$h_1 = \frac{-1}{6} \frac{2t-1}{(t-1)^3 (t+1)}.$$

(2) With one pole in $\mathbb{H}$ and one pole at the cusp 1:

$$h_2' = \frac{t'(t-1)^2}{(t-x)^2 t^4}.$$

The residue at the pole in $\mathbb{H}$ vanishes when $x = 2$. The primitive is given by

$$h_2 = \frac{-1}{6} \frac{2t-1}{t^3 (t-2)}.$$

(3) With four poles in $\mathbb{H}$ and none at the cusps:

$$h_3' = \frac{t' t^2 (t-1)^2}{\prod_{i=1}^{4} (t - x_i)},$$

where $x_1, \ldots, x_4$ are solutions to

$$\frac{1}{x_i} + \frac{1}{x_i - 1} - \sum_{j \ne i} \frac{1}{x_i - x_j}, \quad 1 \le i \le 4.$$

This algebraic system has as solutions (up to a permutation):

$$\frac{1-\sqrt{3}}{2} \pm \left(\frac{3}{4}\right)^{\frac{1}{4}} \quad \frac{1+\sqrt{3}}{2} \pm i \left(\frac{3}{4}\right)^{\frac{1}{4}}.$$

These solutions are the roots of the irreducible polynomial

$$P(x) = x^4 - 2x^3 + 4x - 2.$$

Therefore we can write

$$h_3' = \frac{t' t^2 (t-1)^2}{(t^4 - 2t^3 + 4t - 2)^2} \quad h_3 = \frac{1}{12} \frac{t^3 (t-2)}{t^4 - 2t^3 + 4t - 2}.$$

Remark 4.3 We expect that for each $1 \leq m \leq 5$, every choice of the pair (a, b) gives arise to a solution of the same Schwarz differential equation $\{h, \tau\} = 2\pi^2(n/m)^2 E_4$, and hence these solutions should be linear fractions of each others. This is illustrated in the case of $m = 2$ and $n = 3$, where it can be easily checked that

$$h_2 = \frac{h_1}{6h_1 + 1} \text{ and } h_3 = \frac{6h_1 + 1}{-72h_1 + 12}.$$

Remark 4.4 The lambda function and its derivative can be expressed in terms of Jacobi theta functions. Indeed, according to [17, Chapter 7], we have

$$\lambda = \frac{\theta_2^4}{\theta_3^4} \quad \lambda' = i\pi\theta_4^4\lambda = i\pi\frac{\theta_2^4\theta_4^4}{\theta_3^4}.$$

Hence

$$t = \frac{\theta_3^4}{\theta_2^4} \quad t' = -i\pi\frac{\theta_3^4\theta_4^4}{\theta_2^4}.$$

Therefore, we can see that in the general case for the level 2, the derivatives in (4.2) and in (4.3) are readily squares since n is odd. This allows to easily write down a square root of h' whose reciprocal is a solution to the modular differential equation $y'' + \pi^2(n/2)^2 E_4\, y = 0$.

Data availability Not applicable.

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