



Modes of convergence of random variables and algebraic genericity

G. Araújo¹ · M. Fenoy² · J. Fernández-Sánchez^{3,4} · J. López-Salazar⁵ · J. B. Seoane-Sepúlveda⁶ · J. M. Vecina⁷

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Abstract

Important probabilistic problems require to find the limit of a sequence of random variables. However, this limit can be understood in different ways and various kinds of convergence can be defined. Among the many types of convergence of sequences of random variables, we can highlight, for example, that convergence in L^p -sense implies convergence in probability, which, in turn, implies convergence in distribution, besides that all these implications are strict. In this paper, the relationship between several types of convergence of sequences of random variables will be analyzed from the perspective of lineability theory.

Keywords Algebraability · Lineability · Modes of convergence · Probability theory · Random variable

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✉ J. B. Seoane-Sepúlveda
jseoane@ucm.es

Extended author information available on the last page of the article

1 Preliminaries and background

Since the beginning of the twentyfirst century the study of linearity within non linear settings (known as lineability theory nowadays) has become a sort of a trend within many areas of mathematical research such as real and complex analysis [4, 8, 9, 24, 37], linear dynamics [13, 22, 23], set theory [27], operator theory [14, 37], measure theory [9], and algebraic geometry [12].

Let us recall that if V is a vector space and κ is a (finite or infinite) cardinal number, a subset $A \subset V$ is said to be κ -lineable if there exists a vector subspace $M \subset V$ such that the dimension of M is κ and $M \setminus \{0\} \subset A$. In addition, if A is contained in some commutative algebra, then A is called strongly κ -algebrable if there exists a κ -generated free algebra M with $M \setminus \{0\} \subset A$; that is, there exists a subset B of cardinality κ with the following property: for any positive integer m , any nonzero polynomial P in m variables without constant term and any distinct elements $x_1, \dots, x_m \in B$, we have $P(x_1, \dots, x_m) \neq 0$ and $P(x_1, \dots, x_m) \in A$.

Recently, in [11], it was introduced the notion of convex lineability, which will also be explored in this work. As usual, $\text{conv}(B)$ will denote the convex hull of a subset B of a vector space V , that is,

$$\text{conv}(B) = \left\{ \sum_{i=1}^m a_i x_i : m \in \mathbb{N}, x_i \in B, a_i \in [0, 1], \sum_{i=1}^m a_i = 1 \right\}.$$

We say that a subset A of a vector space V is κ -convex lineable if there exists a linearly independent subset $B \subset V$ such that B has cardinality κ and $\text{conv}(B) \subset A$.

The notion of lineability dates back to 2004 (see [4]). It was originally coined by V. I. Gurariy and it was just recently introduced by the American Mathematical Society in the 2020 Mathematical Subject Classification under the references 15A03 and 46B87. The original motivation for this concept was, probably, the inspiration that came from the famous example of Weierstrass (also known as Weierstrass' monster, a continuous nowhere differentiable function on \mathbb{R} , see [39]). In 1966, V. I. Gurariy showed that the set of Weierstrass' monsters contains (except for the 0 function) an infinite dimensional vector space (see [28, 29]). The current state of the art of this area of research can be consulted, for instance, in [1, 3, 5, 10, 12, 13, 18–20, 25, 33].

In this work, we link the topic of lineability with probability theory. This paper's goal is to continue the ongoing research that was started in [21, 26]. On this occasion we shall analyze the many different types of convergence of sequences of random variables from the perspective of lineability theory. A recent article that also studies this topic is [7].

Throughout the paper, $(\Omega, \mathcal{A}, \mu)$ will always denote a probability space. Any measurable function defined on it will be called a random variable. Two random variables X and Y on $(\Omega, \mathcal{A}, \mu)$ will be said to be equal if $X = Y$ except on a set of probability zero. Given a random variable X and a real number ε , we will frequently write $\mu(X > \varepsilon)$ instead of $\mu(\{\omega \in \Omega : X(\omega) > \varepsilon\})$ (and similarly for other inequalities). We define below some basic kinds of convergence considered in probability theory that will be studied in the article.

- (1) A sequence $\{X_n\}_{n=1}^{\infty}$ of random variables converges almost surely to another random variable X if there exists a subset $B \in \mathcal{A}$ such that $\mu(B) = 1$ and $\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)$ for every $\omega \in B$.
- (2) The sequence $\{X_n\}_{n=1}^{\infty}$ converges almost surely uniformly to X if there exists a set $B \in \mathcal{A}$ such that $\mu(B) = 1$ and $\lim_{n \rightarrow \infty} X_n(\omega) = X(\omega)$ uniformly on B .

(3) It is said that $\{X_n\}_{n=1}^\infty$ converges in probability to X if

$$\lim_{n \rightarrow \infty} \mu (|X_n - X| > \varepsilon) = 0$$

for every $\varepsilon > 0$.

(4) The sequence $\{X_n\}_{n=1}^\infty$ converges completely to X if

$$\sum_{n=1}^\infty \mu (|X_n - X| > \varepsilon) < \infty$$

for every $\varepsilon > 0$.

(5) Let us suppose that X and X_n , for all $n \in \mathbb{N}$, belong to the space $L^p (\Omega, \mathcal{A}, \mu)$ for some $p > 0$. The sequence $\{X_n\}_{n=1}^\infty$ converges to X in L^p -sense or in p -mean if

$$\lim_{n \rightarrow \infty} E (|X_n - X|^p) = 0,$$

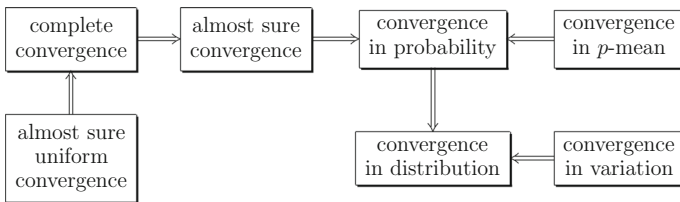
where $E(X)$ denotes the expectation of the random variable X .

(6) Let F and F_n be the distribution functions of X and X_n , respectively. The sequence $\{X_n\}_{n=1}^\infty$ converges in distribution to X if $\lim_{n \rightarrow \infty} F_n(x) = F(x)$ for all $x \in \mathbb{R}$ at which F is continuous. When this property holds, it is also said that $\{F_n\}_{n=1}^\infty$ converges weakly to F .

(7) Let us suppose that F and F_n , for all $n \in \mathbb{N}$, are absolutely continuous distribution functions with densities f and f_n , respectively. The sequence of distribution functions $\{F_n\}_{n=1}^\infty$ converges in variation to F if

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} |f_n(x) - f(x)| dx = 0.$$

The following diagram contains the different implications held by the modes of convergence defined above.



2 Algebraability of sets of sequences of random variables

2.1 Almost sure uniform convergence and complete convergence

Let us suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of random variables on a probability space $(\Omega, \mathcal{A}, \mu)$ and that $X_n \rightarrow 0$ uniformly on a subset B with $\mu(B) = 1$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|X_n(\omega)| \leq \varepsilon$ for every $n > n_0$ and every $\omega \in B$. As a consequence,

$$\sum_{n=1}^\infty \mu (|X_n(\omega)| > \varepsilon) \leq \sum_{n=1}^{n_0} \mu (|X_n(\omega)| > \varepsilon) + \sum_{n=n_0+1}^\infty \mu (\Omega \setminus B) < \infty.$$

That is, $\{X_n\}_{n=1}^\infty$ also converges completely to zero. However, the converse is not true. Based on the example given in [38, Section 14.15], we will prove the strong \mathfrak{c} -algebrability of the collection of sequences that converge completely but do not converge almost surely uniformly. We will consider the following product of sequences of random variables:

$$\{X_n\}_{n=1}^\infty \cdot \{Y_n\}_{n=1}^\infty = \{X_n \cdot Y_n\}_{n=1}^\infty.$$

The symbols \mathcal{B} and λ will denote the Borel σ -algebra in $[0, 1]$ and the Lebesgue measure on $[0, 1]$, respectively, and \mathfrak{c} will denote the cardinality of the continuum.

Theorem 1 *Let \mathcal{S} be the set of all sequences $\{X_n\}_{n=1}^\infty$ of random variables on $([0, 1], \mathcal{B}, \lambda)$ such that $\{X_n\}_{n=1}^\infty$ converges to 0 completely but does not converge almost surely uniformly. Then \mathcal{S} is strongly \mathfrak{c} -algebrable.*

Proof It is based in a general algebrability criteria that appears in [6, Proposition 7]. For each $b > 0$ and each $n \in \mathbb{N}$, the random variable $X_{b,n}$ is defined as follows:

$$X_{b,n} = \begin{cases} e^{b/x} & \text{if } x \in \left(0, \frac{1}{n^2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{H} \subset (0, +\infty)$ be a Hamel basis for \mathbb{R} over the field \mathbb{Q} . It is known that the cardinality of such a basis is \mathfrak{c} (see [35, Theorem 4.2.3]). We will prove that if h_1, \dots, h_m are distinct elements of \mathcal{H} and P is a non-zero polynomial in m variables without constant term, then the sequence

$$\{P(X_{h_1,n}, \dots, X_{h_m,n})\}_{n=1}^\infty$$

is not zero and belongs to \mathcal{S} . The polynomial P can be written as

$$P(x_1, \dots, x_m) = \sum_{i=1}^k a_i x_1^{s_{i1}} \cdots x_m^{s_{im}},$$

where $k \in \mathbb{N}$, $a_i \in \mathbb{R} \setminus \{0\}$, $s_{ij} \in \mathbb{N} \cup \{0\}$ for $1 \leq i \leq k$, $1 \leq j \leq m$, and $s_{i1} + \dots + s_{im} > 0$. We can assume that all the vectors of exponents (s_{i1}, \dots, s_{im}) are different. Let h_1, \dots, h_m be distinct elements of \mathcal{H} and note that

$$X_{h_1,n}(x)^{s_{i1}} \cdots X_{h_m,n}(x)^{s_{im}} = \begin{cases} e^{\frac{s_{i1}h_1 + \dots + s_{im}h_m}{x}} & \text{if } x \in \left(0, \frac{1}{n^2}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$X_{h_1,n}(x)^{s_{i1}} \cdots X_{h_m,n}(x)^{s_{im}} = X_{s_{i1}h_1 + \dots + s_{im}h_m}(x).$$

Setting $b_i = s_{i1}h_1 + \dots + s_{im}h_m$ for each $i \in \{1, \dots, k\}$, we have

$$P(X_{h_1,n}, \dots, X_{h_m,n}) = \sum_{i=1}^k a_i X_{b_i,n}.$$

Since \mathcal{H} is a basis for \mathbb{R} over \mathbb{Q} , it follows that b_1, \dots, b_k are distinct. Moreover, each b_i is strictly positive because h_1, \dots, h_m are also positive and $s_{i1} + \dots + s_{im} > 0$. Without loss of generality, we can suppose that $0 < b_1 < \dots < b_k$.

If $P(X_{h_1,n}, \dots, X_{h_m,n})$ were zero at every point of $[0, 1]$, then $\sum_{i=1}^k a_i e^{b_i/x}$ would be zero for every $x \in (0, 1/n^2]$, which is not possible because

$$\lim_{x \rightarrow 0^+} \frac{\sum_{i=1}^k a_i e^{b_i/x}}{e^{b_k/x}} = a_k + \lim_{x \rightarrow 0^+} \sum_{i=1}^{k-1} \frac{a_i}{e^{(b_k-b_i)/x}} = a_k \neq 0.$$

Furthermore, if $\varepsilon > 0$, then

$$\sum_{n=1}^{\infty} \lambda \left(\left| \sum_{i=1}^k a_i X_{b_i,n} \right| > \varepsilon \right) \leq \sum_{n=1}^{\infty} \lambda((0, 1/n^2]) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

That is, the sequence $\left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^{\infty}$ converges completely to 0.

Let us suppose that $\left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^{\infty}$ also converges almost surely uniformly to 0. Then there is some measurable set B with $\lambda(B) = 1$ and there is $n_0 \in \mathbb{N}$ such that $\left| \sum_{i=1}^k a_i X_{b_i,n}(x) \right| < 1$ for all $n \geq n_0$ and all $x \in B$. Moreover, the fact that $\lambda(B) = 1$ implies that 0 is an accumulation point of B . Hence, fixed $n \geq n_0$ and $A = B \cap (0, 1/n^2]$, we have

$$\begin{aligned} 0 &= \lim_{x \in A, x \rightarrow 0^+} \frac{1}{e^{b_k/x}} \geq \lim_{x \in A, x \rightarrow 0^+} \frac{\left| \sum_{i=1}^k a_i X_{b_i,n}(x) \right|}{e^{b_k/x}} \\ &= \lim_{x \in A, x \rightarrow 0^+} \frac{\left| \sum_{i=1}^k a_i e^{b_i/x} \right|}{e^{b_k/x}} = |a_k| > 0. \end{aligned}$$

This contradiction shows that $\left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^{\infty}$ does not converge almost surely uniformly to 0. That is, the sequence $\{P(X_{h_1,n}, \dots, X_{h_m,n})\}_{n=1}^{\infty} = \left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^{\infty}$ belongs to \mathcal{S} . □

2.2 Complete convergence and almost sure convergence

As a consequence of the Borel–Cantelli lemma, if a sequence of random variables converges completely to 0, then it also converges almost surely (see [34, Theorem 1.27 and Proposition 5.7]). However, the converse does not hold in general (see [38, Section 14.4]). Let us mention that both convergence modes are equivalent if the random variables $\{X_n\}_{n=1}^{\infty}$ are independent (see [31]). Based on these facts, we will study the set of sequences of random variables that converge to zero almost surely but not completely.

Theorem 2 *Let \mathcal{S} be the set of all sequences $\{X_n\}_{n=1}^{\infty}$ of random variables on $([0, 1], \mathcal{B}, \lambda)$ such that $\{X_n\}_{n=1}^{\infty}$ converges to 0 almost surely but does not converge completely. Then \mathcal{S} is strongly \mathfrak{c} -algebraable.*

Proof For each $b \in \mathbb{R}$ and $n \in \mathbb{N}$, the random variable $X_{b,n}$ is defined as follows:

$$X_{b,n}(x) = \begin{cases} e^{b/x} & \text{if } x \in (0, \frac{1}{n}] \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{H} \subset (0, +\infty)$ be a Hamel basis for \mathbb{R} over the field \mathbb{Q} , whose cardinality is \mathfrak{c} (see [35, Theorem 4.2.3]). Let h_1, \dots, h_m be distinct elements of \mathcal{H} and let P be a non-zero

polynomial in m variables without constant term. As it was shown in the proof of Theorem 1, there are positive numbers $b_1 < \dots < b_k$ such that

$$P(X_{h_1,n}, \dots, X_{h_m,n}) = \sum_{i=1}^k a_i X_{b_i,n}.$$

Following the same technique employed in the proof of Theorem 1 it can be shown that $\left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^\infty$ is not the zero sequence. Furthermore, note that $\left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^\infty$ converges to 0 almost surely (in fact, this happens at every point of $[0, 1]$).

Finally, we will prove that the sequence $\left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^\infty$ does not converge completely to zero. Let $\varepsilon > 0$. Since

$$\lim_{x \rightarrow 0^+} \left| \sum_{i=1}^k a_i e^{b_i/x} \right| = \lim_{x \rightarrow 0^+} \left| a_k e^{b_k/x} \right| \cdot \left| 1 + \sum_{i=1}^{k-1} \frac{a_i}{a_k e^{(b_k-b_i)/x}} \right| = +\infty,$$

there exists $n_0 \in \mathbb{N}$ such that $\left| \sum_{i=1}^k a_i e^{b_i/x} \right| > \varepsilon$ for every $x \in (0, 1/n_0]$. Hence, if $n \geq n_0$ and $x \in (0, 1/n]$, then

$$\left| \sum_{i=1}^k a_i X_{b_i,n}(x) \right| = \left| \sum_{i=1}^k a_i e^{b_i/x} \right| > \varepsilon$$

and so

$$\sum_{n=1}^\infty \lambda \left(\left| \sum_{i=1}^k a_i X_{b_i,n} \right| > \varepsilon \right) \geq \sum_{n=n_0}^\infty \lambda \left(\left(0, \frac{1}{n}\right] \right) = \sum_{n=n_0}^\infty \frac{1}{n} = +\infty.$$

This proves that the sequence

$$\left\{ P(X_{h_1,n}, \dots, X_{h_m,n}) \right\}_{n=1}^\infty = \left\{ \sum_{i=1}^k a_i X_{b_i,n} \right\}_{n=1}^\infty$$

belongs to \mathcal{S} and concludes the proof of the strongly c -algebraability of this set. □

Remark 3 Let us recall that a probability space $(\Omega, \mathcal{A}, \mu)$ is said to be *non-atomic* if for every set $B \in \mathcal{A}$ with $\mu(B) > 0$ there exists another set $A \in \mathcal{A}$ such that $A \subset B$ and $0 < \mu(A) < \mu(B)$. By [16, Theorem 9.2.2], Theorems 1 and 2 remain valid if the space $([0, 1], \mathcal{B}, \lambda)$ is replaced by $(\Omega, \mathcal{A}, \mu)$, where Ω is a complete separable metric space, \mathcal{A} is the Borel σ -algebra in Ω , and μ is a non-atomic probability measure.

Remark 4 It is well known that every sequence of random variables that converges almost surely also converges in probability (see [34, Proposition 5.10]). In [2, Theorem 7.1] it was proved the c -lineability of the set of sequences of Lebesgue measurable functions on $[0, 1]$ that converge in measure (that is, in probability) but not almost surely, while the strongly c -algebraability of that family of functions was obtained in [17, Theorem 2.2]. In the case of a non-atomic probability space, the strong c -algebraability of the collection of sequences of random variables that converge in probability but not almost surely was proved in [7, Theorem 11].

3 Lineability of sets of sequences of random variables

Before presenting the next theorems about sequences of random variables, we need to state the following result, whose proof can be seen in [32, Lemma 9.21] and that will be applied in Theorems 6 and 15:

Theorem 5 *There exists a family Δ of subsets of \mathbb{N} with the following properties:*

- *Each $\sigma \in \Delta$ is infinite.*
- *The elements of Δ are almost disjoint. That is, if σ and σ' are two different elements of Δ , then $\sigma \cap \sigma'$ is finite or empty.*
- *The cardinality of Δ is \mathfrak{c} .*

Next we prove a general result about lineability of sets of sequences.

Theorem 6 *Let V be a vector space over \mathbb{R} and let c_1 and c_2 be two types of convergence defined on V such that both c_1 and c_2 have the following properties:*

- (a) *If $\{X_n\}_{n=1}^\infty \subset V$ and $X_n \rightarrow 0$, then also $\lambda X_n \rightarrow 0$ for all $\lambda \in \mathbb{R}$.*
- (b) *The sequence in V constantly equal to 0 converges to 0.*
- (c) *If $\{X_n\}_{n=1}^\infty$ is a sequence in V that can be decomposed into a finite number of subsequences converging to 0, then $\{X_n\}_{n=1}^\infty$ converges to 0.*
- (d) *If $\{X_n\}_{n=1}^\infty$ is a sequence that converges to 0, then all subsequences of $\{X_n\}_{n=1}^\infty$ also converge to 0.*
- (e) *If a finite number of terms from a sequence is deleted, the convergence or lack of convergence does not change.*

Let

$$\mathcal{S} = \left\{ \{X_n\}_{n=1}^\infty \subset V : X_n \xrightarrow{c_1} 0 \text{ and } X_n \not\xrightarrow{c_2} 0 \right\}.$$

If $\mathcal{S} \neq \emptyset$, then \mathcal{S} is \mathfrak{c} -lineable.

Proof Let Δ be a family of \mathfrak{c} almost disjoint infinite subsets of \mathbb{N} (see Theorem 5). If $\sigma \in \Delta$ and $j \in \mathbb{N}$, $\sigma(j)$ denotes the j th element of the set σ . Let us suppose that a sequence $\{X_n\}_{n=1}^\infty$ belongs to \mathcal{S} . Given $\sigma \in \Delta$, we define $\{X_{\sigma,n}\}_{n=1}^\infty$ as follows:

$$X_{\sigma,n} = \begin{cases} X_j & \text{if } n \in \sigma \text{ and } \sigma(j) = n, \\ 0 & \text{if } n \notin \sigma. \end{cases}$$

The set $\mathcal{M} = \{\{X_{\sigma,n}\}_{n=1}^\infty : \sigma \in \Delta\}$ has cardinality \mathfrak{c} and every linear combination of its elements belongs to \mathcal{S} . Indeed, let us suppose that $k \in \mathbb{N}$, $\sigma_1, \dots, \sigma_k$ are distinct elements of Δ , and $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$. Since $\sigma_1, \dots, \sigma_k$ are almost disjoint, there exists $n_0 \in \mathbb{N}$ such that each $n \geq n_0$ belongs at most to one set σ_i , $i \in \{1, \dots, k\}$. Therefore, $\left\{ \sum_{i=1}^k a_i X_{\sigma_i,n} \right\}_{n=n_0}^\infty$ can be decomposed into $k + 1$ subsequences: the zero sequence and $\{a_i X_j : j \in \mathbb{N}, \sigma_i(j) \geq n_0\}$ with $i \in \{1, \dots, k\}$. By the properties (a) and (e) we have that

$$\{a_i X_j : j \in \mathbb{N}, \sigma_i(j) \geq n_0\} \xrightarrow{c_1} 0$$

for every $i \in \{1, \dots, k\}$. Moreover, the zero sequence converges to 0 by the property (b). Then (c) implies that $\left\{ \sum_{i=1}^k a_i X_{\sigma_i,n} \right\}_{n=n_0}^\infty$ converges to zero with respect to c_1 . Using again

(e), it follows that

$$\sum_{i=1}^k a_i X_{\sigma_i, n} \xrightarrow{c_1} 0.$$

Let us see that this linear combination does not converge to zero with respect to c_2 . Since $X_n \not\xrightarrow{c_2} 0$, by (a) we know that $a_1 X_n \not\xrightarrow{c_2} 0$. Then (e) implies

$$\left\{ \sum_{i=1}^k a_i X_{\sigma_i, n} : n \in \sigma_1, n \geq n_0 \right\} = \{a_1 X_j : j \in \mathbb{N}, \sigma_1(j) \geq n_0\} \not\xrightarrow{c_2} 0.$$

By (d) we deduce that

$$\sum_{i=1}^k a_i X_{\sigma_i, n} \not\xrightarrow{c_2} 0.$$

Therefore, $\left\{ \sum_{i=1}^k a_i X_{\sigma_i, n} \right\}_{n=1}^\infty$ belongs to \mathcal{S} .

By (b), the fact that $\sum_{i=1}^k a_i X_{\sigma_i, n} \not\xrightarrow{c_2} 0$ implies that $\left\{ \sum_{i=1}^k a_i X_{\sigma_i, n} \right\}_{n=1}^\infty$ cannot be the zero sequence. Hence, we deduce that the elements of \mathcal{M} are linearly independent and the proof of the c -lineability of \mathcal{S} is complete. \square

Theorem 7 *The following modes of convergence of sequences of random variables satisfy the properties (a), (b), (c), (d), and (e) given in Theorem 6: almost sure convergence, almost sure uniform convergence, complete convergence, convergence in probability, convergence in distribution, and convergence in L^p -sense.*

Proof The unique property that might not be obvious is (a) for convergence in distribution. However, it can be deduced from Slutsky’s theorem (see [34, Theorems 5.22 and 5.23]). \square

3.1 Almost sure uniform convergence, complete convergence and almost sure convergence

The following theorem complements the result given in Theorem 1.

Theorem 8 *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic probability space and let \mathcal{S} be the set of all sequences $\{X_n\}_{n=1}^\infty$ of random variables on $(\Omega, \mathcal{A}, \mu)$ such that $\{X_n\}_{n=1}^\infty$ converges to 0 completely but does not converge almost surely uniformly. Then \mathcal{S} is c -lineable.*

Proof Since $(\Omega, \mathcal{A}, \mu)$ is non-atomic, for every $n \in \mathbb{N}$ there exists a subset $\Omega_n \in \mathcal{A}$ such that $\mu(\Omega_n) = 1/n^2$ (see [16, Corollary 1.12.10]). The random variable X_n is defined as the characteristic function of Ω_n :

$$X_n = \mathbf{1}_{\Omega_n}.$$

Note that if $\varepsilon > 0$, then

$$\sum_{n=1}^\infty \mu(|X_n| > \varepsilon) \leq \sum_{n=1}^\infty \mu(\Omega_n) = \sum_{n=1}^\infty \frac{1}{n^2} < \infty.$$

That is, the sequence $\{X_n\}_{n=1}^\infty$ converges completely to 0.

Let us suppose that $\{X_n\}_{n=1}^\infty$ also converges almost surely uniformly to 0. Then there is some measurable set B with $\mu(B) = 1$ and there is $n_0 \in \mathbb{N}$ such that $|X_n(\omega)| < 1/2$ for all $n \geq n_0$ and all $\omega \in B$. Since $X_{n_0}(\omega) = 1$ if $\omega \in \Omega_{n_0}$, it follows that $B \cap \Omega_{n_0} = \emptyset$ and then

$$0 = \mu(\Omega \setminus B) \geq \mu(\Omega_{n_0}) = \frac{1}{n_0^2} > 0.$$

This contradiction shows that the sequence $\{X_n\}_{n=1}^\infty$ does not converge almost surely uniformly to 0. That is, $\{X_n\}_{n=1}^\infty$ belongs to \mathcal{S} . Now we just need to apply Theorem 6 to get the result. □

The following theorem complements the result given in Theorem 2.

Theorem 9 *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic probability space and let \mathcal{S} be the set of all sequences $\{X_n\}_{n=1}^\infty$ of random variables on $(\Omega, \mathcal{A}, \mu)$ such that $\{X_n\}_{n=1}^\infty$ converges to 0 almost surely but does not converge completely. Then \mathcal{S} is \mathfrak{c} -lineable.*

Proof By [16, Corollary 1.12.10], there exists a sequence $\{\Omega_n : n \in \mathbb{N}\}$ of measurable subsets of Ω such that $\Omega_{n+1} \subset \Omega_n$ and $\mu(\Omega_n) = 1/n$ for each $n \in \mathbb{N}$. Let $X_n = \mathbf{1}_{\Omega_n}$ and note that if $\omega \notin \Omega_{n_0}$ for some $n_0 \in \mathbb{N}$, then $X_n(\omega) = 0$ for all $n \geq n_0$. Therefore,

$$\{\omega \in \Omega : X_n(\omega) \not\rightarrow 0\} \subset \bigcap_{n=1}^\infty \Omega_n.$$

Since $\mu(\bigcap_{n=1}^\infty \Omega_n) = 0$, it follows that $\{X_n\}_{n=1}^\infty$ converges to 0 almost surely. Moreover,

$$\sum_{n=1}^\infty \mu\left(|X_n| > \frac{1}{2}\right) = \sum_{n=1}^\infty \mu(\Omega_n) = \sum_{n=1}^\infty \frac{1}{n} = +\infty.$$

Consequently, the sequence $\{X_n\}_{n=1}^\infty$ does not converge completely to 0. We have proved that $\{X_n\}_{n=1}^\infty$ belongs to \mathcal{S} , so this set is \mathfrak{c} -lineable by Theorem 6. □

3.2 Convergence in L^p -sense and almost sure convergence

Let us recall that $E(X)$ denotes the expectation of a random variable X . By the Lyapunov inequality, it is known that $E(|X|^q)^{1/q} \leq E(|X|^p)^{1/p}$ if $0 < q < p$ (see [15, page 277]). As a consequence, it follows that the convergence in L^p -sense implies the convergence in L^q -sense whenever $0 < q < p$, but the converse is not true (see [38, Section 14.5]). Moreover, it is also known that almost sure convergence and convergence in L^p -sense are independent, that is, none of them implies the other one (see [38, Sections 14.7 and 14.8]). More generally, considering non-atomic probability spaces, Theorems 10, 11, and 12 below bring interesting information related to these modes of convergence.

Theorem 10 *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic probability space and let $p > 0$. Let \mathcal{S} be the set of all sequences $\{X_n\}_{n=1}^\infty$ of random variables on $(\Omega, \mathcal{A}, \mu)$ such that $\{X_n\}_{n=1}^\infty$ converges to 0 in L^q -sense for all $q \in (0, p)$ but does not converge in L^p -sense. Then \mathcal{S} is \mathfrak{c} -lineable.*

Proof Let $n_0 \in \mathbb{N}$ such that $n^{-p+1/n} \leq 1$ for all $n \geq n_0$. For each $n \geq n_0$ there exists $\Omega_n \in \mathcal{A}$ such that $\mu(\Omega_n) = n^{-p+1/n}$ (see [16, Corollary 1.12.10]). Define

$$X_n(\omega) = \begin{cases} n & \text{if } \omega \in \Omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, if $0 < q < p$, then

$$\lim_{n \rightarrow \infty} E(|X_n|^q) = \lim_{n \rightarrow \infty} \int_{\Omega} |X_n|^q d\mu = \lim_{n \rightarrow \infty} n^{q-p+\frac{1}{n}} = 0$$

and so $\{X_n\}_{n=n_0}^{\infty}$ converges to 0 in L^q -sense. On the other hand, we have that

$$\lim_{n \rightarrow \infty} E(|X_n|^p) = \lim_{n \rightarrow \infty} n^{1/n} = 1.$$

Therefore, $\{X_n\}_{n=n_0}^{\infty}$ does not converge to zero in L^p -sense. This proves that $\{X_n\}_{n=1}^{\infty}$ belongs to \mathcal{S} . By Theorem 6, we conclude that \mathcal{S} is c -lineable. \square

Theorem 11 *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic probability space and let \mathcal{S} be the set of all sequences $\{X_n\}_{n=1}^{\infty}$ of random variables on $(\Omega, \mathcal{A}, \mu)$ such that $\{X_n\}_{n=1}^{\infty}$ converges to 0 almost surely but does not converge in L^p -sense for any $p > 0$. Then \mathcal{S} is c -lineable.*

Proof Since the probability space is non-atomic, there exists a collection $\{\Omega_n : n \in \mathbb{N}\}$ of measurable subsets of Ω such that $\mu(\Omega_n) = 1/n$ and $\Omega_{n+1} \subset \Omega_n$ for every $n \in \mathbb{N}$. Set

$$X_n(\omega) = \begin{cases} n^n & \text{if } \omega \in \Omega_n, \\ 0 & \text{otherwise.} \end{cases}$$

On the one hand, if $\omega \notin \bigcap_{n \in \mathbb{N}} \Omega_n$, then $X_n(\omega) = 0$ for n large enough (dependent on ω). That is, the set of points ω for which there is no convergence of $\{X_n(\omega)\}_{n=1}^{\infty}$ is a subset of $\bigcap_{n \in \mathbb{N}} \Omega_n$. Since $\mu(\Omega_n) = 1/n$ for all n , we have that $\mu(\bigcap_{n \in \mathbb{N}} \Omega_n) = 0$. Thus,

$$\mu\left(\left\{\omega \in \Omega : \lim_{n \rightarrow \infty} X_n(\omega) = 0\right\}\right) = 1.$$

On the other hand, if $p > 0$, then

$$\lim_{n \rightarrow \infty} E(|X_n|^p) = \lim_{n \rightarrow \infty} n^{np-1} = +\infty$$

and so $\{X_n\}_{n=1}^{\infty}$ does not converge to 0 in L^p -sense. This proves that $\{X_n\}_{n=1}^{\infty}$ belongs to \mathcal{S} . By Theorem 6 we conclude that \mathcal{S} is c -lineable. \square

Theorem 12 *Let $(\Omega, \mathcal{A}, \mu)$ be a non-atomic probability space and let \mathcal{S} be the set of all sequences $\{X_n\}_{n=n_0}^{\infty}$ of random variables on $(\Omega, \mathcal{A}, \mu)$ such that $\{X_n\}_{n=1}^{\infty}$ converges to 0 in L^p -sense for all $p > 0$ but does not converge almost surely. Then \mathcal{S} is c -lineable.*

Proof Since the probability space is non-atomic, there exists a measurable set Ω_0 such that $\mu(\Omega_0) = 1/2$ (see [16, Corollary 1.12.10]). Defining $\Omega_1 = \Omega \setminus \Omega_0$, we find two measurable sets Ω_0 and Ω_1 such that

$$\Omega = \Omega_0 \cup \Omega_1, \quad \Omega_0 \cap \Omega_1 = \emptyset, \quad \mu(\Omega_0) = \mu(\Omega_1) = \frac{1}{2}.$$

With the same argument, it is possible to find two measurable sets Ω_{00} and Ω_{01} such that

$$\Omega_0 = \Omega_{00} \cup \Omega_{01}, \quad \Omega_{00} \cap \Omega_{01} = \emptyset, \quad \mu(\Omega_{00}) = \mu(\Omega_{01}) = \frac{1}{2^2}.$$

Similarly, there exist two measurable sets Ω_{10} and Ω_{11} such that

$$\Omega_1 = \Omega_{10} \cup \Omega_{11}, \quad \Omega_{10} \cap \Omega_{11} = \emptyset, \quad \mu(\Omega_{10}) = \mu(\Omega_{11}) = \frac{1}{2^2}.$$

If the previous argument is repeated, we obtain a family

$$\{\Omega_{j_1 \dots j_m} : m \in \mathbb{N}, j_1, \dots, j_m \in \{0, 1\}\}$$

of measurable subsets of Ω with the following properties:

- $\Omega_{j_1 \dots j_m} = \Omega_{j_1 \dots j_m 0} \cup \Omega_{j_1 \dots j_m 1}$.
- $\Omega_{j_1 \dots j_m 0} \cap \Omega_{j_1 \dots j_m 1} = \emptyset$.
- $\mu(\Omega_{j_1 \dots j_m 0}) = \mu(\Omega_{j_1 \dots j_m 1}) = 1/2^{m+1}$.
- For each $m \in \mathbb{N}$, $\Omega = \bigcup \{\Omega_{j_1 \dots j_m} : j_1, \dots, j_m \in \{0, 1\}\}$.

Let $X_1 = \mathbf{1}_\Omega$. Moreover, given a natural number $n \geq 2$, we consider its binary expansion:

$$n = 2^m + 2^{m-1}j_1 + 2^{m-2}j_2 + \dots + 2j_{m-1} + j_m,$$

where $m \geq 1$ and $j_1, \dots, j_m \in \{0, 1\}$ are unique for each $n \geq 2$. Hence, we can define

$$X_n = \mathbf{1}_{\Omega_{j_1 \dots j_m}}.$$

Let this m be denoted by n^* . The following table clarifies how each X_n is defined.

$n = 1$		$X_1 = \mathbf{1}_\Omega$
$n = 2 = 2^1 + 0$	$n^* = 1$	$X_2 = \mathbf{1}_{\Omega_0}$
$n = 3 = 2^1 + 1$	$n^* = 1$	$X_3 = \mathbf{1}_{\Omega_1}$
$n = 4 = 2^2 + 2 \cdot 0 + 0$	$n^* = 2$	$X_4 = \mathbf{1}_{\Omega_{00}}$
$n = 5 = 2^2 + 2 \cdot 0 + 1$	$n^* = 2$	$X_5 = \mathbf{1}_{\Omega_{01}}$
$n = 6 = 2^2 + 2 \cdot 1 + 0$	$n^* = 2$	$X_6 = \mathbf{1}_{\Omega_{10}}$
$n = 7 = 2^2 + 2 \cdot 1 + 1$	$n^* = 2$	$X_7 = \mathbf{1}_{\Omega_{11}}$
$n = 8 = 2^3 + 2^2 \cdot 0 + 2 \cdot 0 + 0$	$n^* = 3$	$X_8 = \mathbf{1}_{\Omega_{000}}$
$n = 9 = 2^3 + 2^2 \cdot 0 + 2 \cdot 0 + 1$	$n^* = 3$	$X_9 = \mathbf{1}_{\Omega_{001}}$
\vdots	\vdots	\vdots

Let us see that $\{X_n\}_{n=1}^\infty$ belongs to \mathcal{S} . On the one hand,

$$\lim_{n \rightarrow \infty} E(X_n) = \lim_{n \rightarrow \infty} \int_\Omega |X_n|^p d\mu = \lim_{n \rightarrow \infty} \mu(\Omega_{j_1 \dots j_{n^*}}) = \lim_{n \rightarrow \infty} \frac{1}{2^{n^*}} = 0,$$

which means that $\{X_n\}_{n=1}^\infty$ converges to 0 in L^p for all $p > 0$. On the other hand, for each $\omega \in \Omega$ and each $m \in \mathbb{N}$, there exist unique $j_1, \dots, j_m \in \{0, 1\}$ such that $\omega \in \Omega_{j_1 \dots j_m}$. If $0 \leq \ell < 2^m$, then

$$X_{2^m + \ell}(\omega) = \begin{cases} 1 & \text{if } \ell = 2^{m-1}j_1 + 2^{m-2}j_2 + \dots + j_m, \\ 0 & \text{otherwise.} \end{cases}$$

As a consequence, the sequence $\{X_n(\omega)\}_{n=1}^\infty$ contains infinitely many elements equal to 0 and infinitely many elements equal to 1. We thus conclude that $\{X_n(\omega)\}_{n=1}^\infty$ does not converge for any $\omega \in \Omega$. This proves that $\{X_n(\omega)\}_{n=1}^\infty$ belongs to \mathcal{S} , so this set is \mathfrak{c} -lineable by Theorem 6. □

3.3 Convergence in distribution and convergence in probability

Although every sequence of random variables that converges in probability also converges in distribution, the reciprocal result does not hold in general (see [34, Proposition 5.13] and [38, Section 14.2]). Our next result will show the lineability of the collection of sequences converging in distribution but not in probability.

Theorem 13 *Let $(\Omega, \mathcal{A}, \mu)$ be a probability space on which is possible to define infinitely many independent and identically distributed different random variables. Let \mathcal{S} be the set of all sequences of random variables on $(\Omega, \mathcal{A}, \mu)$ that converge in distribution but do not converge in probability. Then \mathcal{S} is c -lineable.*

Proof Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent and identically distributed different random variables on $(\Omega, \mathcal{A}, \mu)$. Since all random variables X_n have the same distribution function, that we will call F , it follows that the sequence $\{X_n\}_{n=1}^\infty$ converges in distribution to X_1 .

If F only took the values 0 and 1, every random variable X_n would be constant almost surely and equal to the same value ($\min \{x \in \mathbb{R} : F(x) = 1\}$). Therefore, we would have that $X_n = X_m$ for all $n, m \in \mathbb{N}$, which contradicts the hypothesis of the theorem. Then there must exist $x_0 \in \mathbb{R}$ such that $0 < F(x_0) < 1$. Since F is right-continuous, there exists $\delta > 0$ such that if $x_0 \leq x < x_0 + \delta$, then $F(x) < 1$. Choose any a and b so that

$$x_0 < a < b < x_0 + \delta.$$

Since F is non-decreasing, for all m and n we have

$$\mu(X_m < a) \cdot \mu(X_n > b) \geq F(x_0) \cdot (1 - F(b)) > 0. \tag{3.1}$$

These inequalities will be applied later on.

It is known that convergence in probability is metrizable (see [15, Exercise 21.15]). In fact, convergence in probability is equivalent to convergence with respect to the following metric in the space of all random variables on $(\Omega, \mathcal{A}, \mu)$:

$$d(X, Y) = E \left(\frac{|X - Y|}{1 + |X - Y|} \right) = \int_{\Omega} \frac{|X - Y|}{1 + |X - Y|} d\mu.$$

Let us estimate the distance between X_m and X_n for two natural numbers $m \neq n$. To do that, we will use that the function $f(t) = \frac{t}{1+t}$ is increasing on $[0, +\infty)$ and that the variables X_m and X_n are independent:

$$\begin{aligned} d(X_m, X_n) &= \int_{\Omega} f(|X_m - X_n|) d\mu \geq \int_{(X_m < a) \cap (X_n > b)} f(|X_m - X_n|) d\mu \\ &\geq \mu((X_m < a) \cap (X_n > b)) \cdot f(b - a) \\ &= \mu(X_m < a) \cdot \mu(X_n > b) \cdot f(b - a). \end{aligned}$$

By (3.1), if $m \neq n$, then

$$d(X_m, X_n) \geq F(x_0) \cdot (1 - F(b)) \cdot f(b - a) > 0.$$

We conclude that $\{X_n\}_{n=1}^\infty$ is not a Cauchy sequence, which implies that $\{X_n\}_{n=1}^\infty$ cannot converge in probability. Therefore, $\{X_n\}_{n=1}^\infty$ belongs to \mathcal{S} . By Theorem 6, we obtain the c -lineability of the set \mathcal{S} . □

Remark 14 The existence of a probability space and a collection of independent random variables on it with prefixed distributions, as required in Theorems 13 and 15, is guaranteed by [15, Theorem 20.4].

4 Convergence of sums of sequences of random variables

Let $X, Y, X_n,$ and Y_n (with $n \in \mathbb{N}$) be random variables and suppose that $\{X_n\}_{n=1}^\infty$ converges in distribution to X and $\{Y_n\}_{n=1}^\infty$ converges in distribution to Y , which will be denoted by $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$. In general, it is not possible to guarantee the convergence of $\{X_n + Y_n\}_{n=1}^\infty$ to $X + Y$, as it can be seen [38, Section 14.10]. Inspired by the cited counterexample, we will prove a stronger result:

Theorem 15 *Let us suppose that $(\Omega, \mathcal{A}, \mu)$ is a probability space such that there are two subsets $\Omega_1 \in \mathcal{A}$ and $\Omega_2 \in \mathcal{A}$ with $\Omega = \Omega_1 \cup \Omega_2, \Omega_1 \cap \Omega_2 = \emptyset,$ and $\mu(\Omega_1) = \mu(\Omega_2) = 1/2.$ For each $i \in \{1, 2\},$ let \mathcal{A}_i be the elements of \mathcal{A} contained in Ω_i and let $\mu_i(B) = 2\mu(B)$ for all $B \in \mathcal{A}_i.$ In addition, suppose that there is a sequence $\{X_n\}_{n=1}^\infty$ of independent random variables defined on $(\Omega_1, \mathcal{A}_1, \mu_1)$ and there is another sequence $\{Y_n\}_{n=1}^\infty$ of independent random variables defined on $(\Omega_2, \mathcal{A}_2, \mu_2)$ such that the distribution function of every X_n and every Y_n is*

$$F(x) = \begin{cases} 0 & \text{if } x < -1, \\ \frac{1+x}{2} & \text{if } -1 \leq x \leq 1, \\ 1 & \text{if } x \geq 1. \end{cases}$$

Under these conditions, there exist two vector spaces, \mathcal{V}_1 and $\mathcal{V}_2,$ of sequences of random variables on $(\Omega, \mathcal{A}, \mu)$ with the following properties:

- (1) *The dimension of both spaces \mathcal{V}_1 and \mathcal{V}_2 is $c.$*
- (2) *If $\{Z_n\}_{n=1}^\infty \in \mathcal{V}_1 \setminus \{0\}$ and $\{T_n\}_{n=1}^\infty \in \mathcal{V}_2 \setminus \{0\},$ then there are random variables Z and T on $(\Omega, \mathcal{A}, \mu)$ satisfying $Z_n \xrightarrow{d} Z, T_n \xrightarrow{d} T,$ and $Z_n + T_n \not\xrightarrow{d} Z + T.$*

Proof Given $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N},$ let F_a denote the distribution function of $aX_n.$ On the one hand, if $a > 0,$ then

$$F_a(x) = \mu_1(\{\omega \in \Omega_1 : aX_n(\omega) \leq x\}) = F\left(\frac{x}{a}\right) = \begin{cases} 0 & \text{if } x < -a, \\ \frac{1+\frac{x}{a}}{2} & \text{if } -a \leq x < a, \\ 1 & \text{if } x \geq a. \end{cases}$$

On the other hand, if $a < 0,$ then

$$\begin{aligned} F_a(x) &= \mu_1(\{\omega \in \Omega_1 : aX_n(\omega) \leq x\}) = \mu_1\left(\left\{\omega \in \Omega_1 : X_n(\omega) \geq \frac{x}{a}\right\}\right) \\ &= 1 - \mu_1\left(\left\{\omega \in \Omega_1 : X_n(\omega) < \frac{x}{a}\right\}\right) = \begin{cases} 1 & \text{if } x/a \leq -1, \\ 1 - \frac{1+\frac{x}{a}}{2} & \text{if } -1 < x/a \leq 1, \\ 0 & \text{if } x/a > 1, \end{cases} \\ &= \begin{cases} 1 & \text{if } x \geq -a, \\ \frac{1-\frac{x}{a}}{2} & \text{if } a \leq x < -a, \\ 0 & \text{if } x < a. \end{cases} \end{aligned}$$

That is, for every $a \in \mathbb{R} \setminus \{0\}$ and every $n \in \mathbb{N},$ the distribution function of aX_n is

$$F_a(x) = \begin{cases} 0 & \text{if } x < -|a|, \\ \frac{1+\frac{x}{|a|}}{2} & \text{if } -|a| \leq x < |a|, \\ 1 & \text{if } x \geq |a|. \end{cases}$$

As a consequence, the density function of aX_n is

$$f_a(x) = \begin{cases} \frac{1}{2|a|} & \text{if } -|a| < x < |a|, \\ 0 & \text{otherwise.} \end{cases}$$

For each $n \in \mathbb{N}$, we define the following extension of X_n to the whole space $(\Omega, \mathcal{A}, \mu)$:

$$X_n^*(\omega) = \begin{cases} X_n(\omega) & \text{if } \omega \in \Omega_1, \\ 0 & \text{if } \omega \in \Omega_2. \end{cases}$$

If n_1, \dots, n_k are distinct positive integers and a_1, \dots, a_k are non-zero real numbers, then the variables $a_1 X_{n_1}, \dots, a_k X_{n_k}$ are independent, so the density function of $\sum_{i=1}^k a_i X_{n_i}$ is the following convolution:

$$g_{a_1, \dots, a_k} = f_{a_1} * \dots * f_{a_k} \tag{4.1}$$

(see [15, page 267]). Therefore, the distribution function of $\sum_{i=1}^k a_i X_{n_i}$ is

$$G_{a_1, \dots, a_k}(x) = \int_{-\infty}^x g_{a_1, \dots, a_k}(t) dt. \tag{4.2}$$

If S is defined as

$$S(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0, \end{cases}$$

then the distribution function of the extension $\sum_{i=1}^k a_i X_{n_i}^*$ is $(G_{a_1, \dots, a_k} + S) / 2$. Indeed, since $\sum_{i=1}^k a_i X_{n_i}^* = 0$ on Ω_2 , if $x < 0$, then

$$\begin{aligned} \mu \left(\sum_{i=1}^k a_i X_{n_i}^* \leq x \right) &= \mu \left(\left\{ \omega \in \Omega_1 : \sum_{i=1}^k a_i X_{n_i}^*(\omega) \leq x \right\} \right) \\ &= \frac{1}{2} \mu_1 \left(\left\{ \omega \in \Omega_1 : \sum_{i=1}^k a_i X_{n_i}(\omega) \leq x \right\} \right) \\ &= \frac{1}{2} G_{a_1, \dots, a_k}(x) = \frac{G_{a_1, \dots, a_k}(x) + S(x)}{2}. \end{aligned}$$

On the other hand, if $x \geq 0$, then

$$\begin{aligned} \mu \left(\sum_{i=1}^k a_i X_{n_i}^* \leq x \right) &= \mu(\Omega_2) + \mu \left(\left\{ \omega \in \Omega_1 : \sum_{i=1}^k a_i X_{n_i}^*(\omega) \leq x \right\} \right) \\ &= \frac{1}{2} + \frac{1}{2} \mu_1 \left(\left\{ \omega \in \Omega_1 : \sum_{i=1}^k a_i X_{n_i}(\omega) \leq x \right\} \right) \\ &= \frac{1}{2} + \frac{1}{2} G_{a_1, \dots, a_k}(x) = \frac{G_{a_1, \dots, a_k}(x) + S(x)}{2}. \end{aligned}$$

This proves that

$$H_{a_1, \dots, a_k} = \frac{G_{a_1, \dots, a_k} + S}{2} \tag{4.3}$$

is the distribution function of $\sum_{i=1}^k a_i X_{n_i}^*$ whenever n_1, \dots, n_k are distinct positive integers and a_1, \dots, a_k are non-zero real numbers.

Let Δ be a family of c almost disjoint subsets of \mathbb{N} (see Theorem 5). Given $n \in \mathbb{N}$ and $\sigma \in \Delta$, the symbol $\sigma(n)$ denotes the n th element of σ . The space \mathcal{V}_1 is defined as the linear span of the set

$$\left\{ \left\{ X_{\sigma(n)}^* \right\}_{n=1}^\infty : \sigma \in \Delta \right\}.$$

Let us see that any non-zero element of \mathcal{V}_1 is convergent in distribution. Let $\left\{ \sum_{i=1}^k a_i X_{\sigma_i(n)}^* \right\}_{n=1}^\infty \in \mathcal{V}_1$, where $\sigma_1, \dots, \sigma_k$ are distinct elements of Δ and $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$. Since $\sigma_1, \dots, \sigma_k$ are almost disjoint sets, there exists n_0 such that $\sigma_i(n) \neq \sigma_j(n)$ for all $n \geq n_0$ and $i \neq j$. By Eq. (4.3), if $n \geq n_0$, the distribution function of $\sum_{i=1}^k a_i X_{\sigma_i(n)}^*$ is H_{a_1, \dots, a_k} . Moreover, since the distribution function of $\sum_{i=1}^k a_i X_i^*$ is also H_{a_1, \dots, a_k} , we can conclude that

$$\sum_{i=1}^k a_i X_{\sigma_i(n)}^* \xrightarrow{d} \sum_{i=1}^k a_i X_i^*.$$

Note that if $i \in \{1, \dots, k\}$, then

$$\sup \{x \in \mathbb{R} : f_{a_i}(t) = 0 \text{ for all } t \in (-\infty, x)\} = -|a_i|.$$

Using that $g_{a_1, \dots, a_k} = f_{a_1} * \dots * f_{a_k}$ and the Titchmarsh convolution theorem (see [36]), we obtain

$$\sup \{x \in \mathbb{R} : g_{a_1, \dots, a_k}(t) = 0 \text{ for all } t \in (-\infty, x)\} = \sum_{i=1}^k (-|a_i|) < 0.$$

Since $G_{a_1, \dots, a_k}(x) = \int_{-\infty}^x g_{a_1, \dots, a_k}(t) dt$, it follows that

$$\sup \{x \in \mathbb{R} : G_{a_1, \dots, a_k} = 0 \text{ on } (-\infty, x)\} < 0.$$

Hence,

$$\begin{aligned} \sup \{x : H_{a_1, \dots, a_k} = 0 \text{ on } (-\infty, x)\} &= \sup \{x : G_{a_1, \dots, a_k} = 0 \text{ on } (-\infty, x)\} \\ &< 0 = \sup \{x : S = 0 \text{ on } (-\infty, x)\}. \end{aligned}$$

Therefore, $H_{a_1, \dots, a_k} \neq S$. Since S is the distribution function of the random variable which is equal to zero everywhere, we deduce that $\sum_{i=1}^k a_i X_{\sigma_i(n)}^* \neq 0$ if $n \geq n_0$, so $\left\{ \sum_{i=1}^k a_i X_{\sigma_i(n)}^* \right\}_{n=1}^\infty$ is not the zero sequence. That is, the set

$$\left\{ \left\{ X_{\sigma(n)}^* \right\}_{n=1}^\infty : \sigma \in \Delta \right\}$$

is linearly independent and the dimension of \mathcal{V}_1 is c .

For each $n \in \mathbb{N}$, the random variable Y_n defined on Ω_2 is extended to the whole space Ω as follows:

$$Y_n^*(\omega) = \begin{cases} Y_n(\omega) & \text{if } \omega \in \Omega_2, \\ 0 & \text{if } \omega \in \Omega_1. \end{cases}$$

The space \mathcal{V}_2 is defined as the linear span of

$$\left\{ \left\{ Y_{\sigma(n)}^* \right\}_{n=1}^{\infty} : \sigma \in \Delta \right\}.$$

Similar arguments to those used previously imply that the dimension of \mathcal{V}_2 is \mathfrak{c} and that any non-zero element of \mathcal{V}_2 is convergent in distribution. In particular, if $\varphi_1, \dots, \varphi_\ell$ are distinct elements of Δ and $b_1, \dots, b_\ell \in \mathbb{R} \setminus \{0\}$, then there is $n_1 \in \mathbb{N}$ such that if $n \geq n_1$, the distribution functions of $\sum_{i=1}^\ell b_i Y_{\varphi_i(n)}$ and $\sum_{i=1}^\ell b_i Y_{\varphi_i(n)}^*$ are G_{b_1, \dots, b_ℓ} and H_{b_1, \dots, b_ℓ} , respectively. By Eq. (4.3), the distribution function of $\sum_{i=1}^\ell b_i X_{k+i}^*$ is also H_{b_1, \dots, b_ℓ} , so we deduce that

$$\sum_{i=1}^\ell b_i Y_{\varphi_i(n)}^* \xrightarrow{d} \sum_{i=1}^\ell b_i X_{k+i}^*.$$

To conclude the proof, we have to prove that the sum of both sequences does not converge in distribution to the sum of the limits. Setting

$$Z_n = \sum_{i=1}^k a_i X_{\sigma_i(n)}^*, \quad T_n = \sum_{i=1}^\ell b_i Y_{\varphi_i(n)}^*, \quad L = \sum_{i=1}^k a_i X_i^* + \sum_{i=1}^\ell b_i X_{k+i}^*,$$

we have to prove that $Z_n + T_n \not\xrightarrow{d} L$. Let $n \geq \max\{n_0, n_1\}$. Since $Z_n = 0$ on Ω_2 and $T_n = 0$ on Ω_1 , if $x \in \mathbb{R}$, then

$$\begin{aligned} \mu(Z_n + T_n \leq x) &= \mu(\{\omega \in \Omega_1 : Z_n(\omega) \leq x\}) + \mu(\{\omega \in \Omega_2 : T_n(\omega) \leq x\}) \\ &= \frac{1}{2} \mu_1 \left(\left\{ \omega \in \Omega_1 : \sum_{i=1}^k a_i X_{\sigma_i(n)}(\omega) \leq x \right\} \right) \\ &\quad + \frac{1}{2} \mu_2 \left(\left\{ \omega \in \Omega_2 : \sum_{i=1}^\ell b_i Y_{\varphi_i(n)}(\omega) \leq x \right\} \right) \\ &= \frac{1}{2} G_{a_1, \dots, a_k}(x) + \frac{1}{2} G_{b_1, \dots, b_\ell}(x), \end{aligned}$$

where we have applied Eq. (4.2) (and a similar one for $\sum_{i=1}^\ell b_i Y_{\varphi_i(n)}$). This proves that if $n \geq \max\{n_0, n_1\}$, then the distribution function of $Z_n + T_n$ is

$$\frac{G_{a_1, \dots, a_k} + G_{b_1, \dots, b_\ell}}{2}.$$

Moreover, by Eq. (4.3), the distribution function L is

$$H_{a_1, \dots, a_k, b_1, \dots, b_\ell} = \frac{G_{a_1, \dots, a_k, b_1, \dots, b_\ell} + S}{2}.$$

Let us see that these two distribution functions are not equal. On the one hand, by the Titchmarsh convolution theorem (see [36]), we have

$$\sup \{x \in \mathbb{R} : g_{a_1, \dots, a_k} = 0 \text{ on } (-\infty, x)\} = \sum_{i=1}^k (-|a_i|).$$

Since $G_{a_1, \dots, a_k}(x) = \int_{-\infty}^x g_{a_1, \dots, a_k}(t) dt$, we deduce that

$$\sup \{x \in \mathbb{R} : G_{a_1, \dots, a_k} = 0 \text{ on } (-\infty, x)\} = \sum_{i=1}^k (-|a_i|).$$

Similarly,

$$\sup \left\{ x \in \mathbb{R} : G_{b_1, \dots, b_\ell} = 0 \text{ on } (-\infty, x) \right\} = \sum_{i=1}^{\ell} (-|b_i|).$$

Since G_{a_1, \dots, a_k} and G_{b_1, \dots, b_ℓ} are non-negative at every point, it follows that

$$\sup \left\{ x : \frac{G_{a_1, \dots, a_k} + G_{b_1, \dots, b_\ell}}{2} = 0 \text{ on } (-\infty, x) \right\} = \min \left\{ \sum_{i=1}^k (-|a_i|), \sum_{i=1}^{\ell} (-|b_i|) \right\}.$$

On the other hand, applying again the Titchmarsh convolution theorem, we obtain

$$\sup \left\{ x \in \mathbb{R} : g_{a_1, \dots, a_k, b_1, \dots, b_\ell} = 0 \text{ on } (-\infty, x) \right\} = - \sum_{i=1}^k |a_i| - \sum_{i=1}^{\ell} |b_i|,$$

which implies

$$\sup \left\{ x \in \mathbb{R} : G_{a_1, \dots, a_k, b_1, \dots, b_\ell} = 0 \text{ on } (-\infty, x) \right\} = - \sum_{i=1}^k |a_i| - \sum_{i=1}^{\ell} |b_i|$$

and then

$$\begin{aligned} &\sup \left\{ x \in \mathbb{R} : H_{a_1, \dots, a_k, b_1, \dots, b_\ell} = 0 \text{ on } (-\infty, x) \right\} \\ &= \sup \left\{ x \in \mathbb{R} : G_{a_1, \dots, a_k, b_1, \dots, b_\ell} = 0 \text{ on } (-\infty, x) \right\} \\ &= - \sum_{i=1}^k |a_i| - \sum_{i=1}^{\ell} |b_i| < \min \left\{ \sum_{i=1}^k (-|a_i|), \sum_{i=1}^{\ell} (-|b_i|) \right\}. \end{aligned}$$

Therefore,

$$\frac{G_{a_1, \dots, a_k} + G_{b_1, \dots, b_\ell}}{2} \neq H_{a_1, \dots, a_k, b_1, \dots, b_\ell},$$

which proves that $\{Z_n + T_n\}_{n=1}^\infty$ does not converge in distribution to L . □

5 Convex lineability of sets of sequences of random variables

5.1 Convergence of distribution functions and convergence of densities

Let us suppose that F and F_n (with $n \in \mathbb{N}$) are absolutely continuous distribution functions with densities f and f_n , respectively. By the Scheffé theorem, if $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for almost all $x \in \mathbb{R}$, then $\{F_n\}_{n=1}^\infty$ converges to F at every point of \mathbb{R} (see [15, Theorem 16.12]). However, the converse result does not hold in general (see [38, Section 14.9]). In this sense, we have the following result that relates both modes of convergence from the point of view of lineability:

Theorem 16 *Let S be the set of all sequences $\{F_n\}_{n=1}^\infty$ of absolutely continuous distribution functions such that $\{F_n\}_{n=1}^\infty$ converges at every point of \mathbb{R} , but the sequence of their densities $\{f_n\}_{n=1}^\infty$ does not converge at almost any point of $[0, 1]$. Then S is \mathfrak{c} -convex lineable.*

Proof For each $b \in (0, +\infty)$ and $n \in \mathbb{N}$, define

$$F_{b,n}(x) = \begin{cases} 0 & \text{if } x < 0, \\ \left(x - \frac{\sin(2\pi nx)}{2\pi n}\right) e^{b(x-1)} & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1. \end{cases}$$

In order to prove that $F_{b,n}$ is a distribution functions, we only need to show that $F_{b,n}$ is nondecreasing. To do that, we will study its derivative on $(0, 1)$, which will be denoted by $f_{b,n}$. Thus, if $0 < x < 1$, then

$$f_{b,n}(x) = \left(1 - \cos(2\pi nx) + b \left(x - \frac{\sin 2\pi nx}{2\pi n}\right)\right) e^{b(x-1)}.$$

Let $g_n(x) = x - \frac{\sin(2\pi nx)}{2\pi n}$ and observe that $g'_n(x) = 1 - \cos(2\pi nx) > 0$ for all $x \in (0, 1)$, which means that g_n is strictly increasing on $(0, 1)$. Since $g_n(0) = 0$, it follows that $g_n(x) > 0$ for all $x \in (0, 1)$. Therefore, $f_{b,n}(x) > 0$ for all $x \in (0, 1)$, which implies that $F_{b,n}$ is increasing on $(0, 1)$ and, consequently, it is a distribution function. We also consider another distribution function:

$$G_b(x) = \begin{cases} 0 & \text{if } x < 0, \\ x e^{b(x-1)} & \text{if } x \in [0, 1], \\ 1 & \text{if } x > 1. \end{cases}$$

The set $\mathcal{M} = \{ \{F_{b_i,n}\}_{n=1}^\infty : b \in (0, +\infty) \}$ is linearly independent. Indeed, let us suppose that $0 < b_1 < \dots < b_k$ and $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$. If for some $n \in \mathbb{N}$ we had

$$\sum_{i=1}^k a_i F_{b_i,n}(x) = \sum_{i=1}^k a_i \left(x - \frac{\sin(2\pi nx)}{2\pi n}\right) e^{b_i(x-1)} = 0$$

for every $x \in [0, 1]$, the Identity Theorem for holomorphic functions would imply that

$$\sum_{i=1}^k a_i \left(x - \frac{\sin(2\pi nx)}{2\pi n}\right) e^{b_i(x-1)} = 0$$

at every point x of the complex plane. As a consequence, it would follow that

$$\begin{aligned} 0 &= \lim_{x \rightarrow +\infty} \frac{\sum_{i=1}^k a_i \left(x - \frac{\sin(2\pi nx)}{2\pi n}\right) e^{b_i(x-1)}}{a_k e^{b_k(x-1)}} \\ &= \lim_{x \rightarrow +\infty} \left(x - \frac{\sin(2\pi nx)}{2\pi n}\right) + \lim_{x \rightarrow +\infty} \sum_{i=1}^{k-1} \frac{a_i \left(x - \frac{\sin(2\pi nx)}{2\pi n}\right)}{a_k e^{(b_k-b_i)(x-1)}} = +\infty. \end{aligned}$$

This contradiction shows that, for every $n \in \mathbb{N}$, $\sum_{i=1}^k a_i F_{b_i,n}$ is not identically zero on $[0, 1]$. Therefore, $\left\{ \sum_{i=1}^k a_i F_{b_i,n} \right\}_{n=1}^\infty$ is not the zero sequence and then the elements of \mathcal{M} are linearly independent.

Next we will prove that every convex combination of elements of \mathcal{M} belongs to \mathcal{S} . Let us suppose again that $0 < b_1 < \dots < b_k$, $a_1, \dots, a_k \in [0, 1]$, and $\sum_{i=1}^k a_i = 1$. The sum

$\sum_{i=1}^k a_i F_{b_i,n}$ is a new distribution function such that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^k a_i F_{b_i,n}(x) = \sum_{i=1}^k a_i G_{b_i}(x)$$

for all $x \in \mathbb{R}$.

Now let us see that the sequence of densities $\left\{ \sum_{i=1}^k a_i f_{b_i,n} \right\}_{n=1}^\infty$ does not converge at any point of $(0, 1)$. If $x \in (0, 1)$, then

$$\sum_{i=1}^k a_i f_{b_i,n}(x) = \sum_{i=1}^k a_i \left(1 + b_i \left(x - \frac{\sin(2\pi nx)}{2\pi n} \right) \right) e^{b_i(x-1)} - \cos(2\pi nx) \sum_{i=1}^k a_i e^{b_i(x-1)}.$$

Note that

$$\lim_{n \rightarrow \infty} \sum_{i=1}^k a_i \left(1 + b_i \left(x - \frac{\sin(2\pi nx)}{2\pi n} \right) \right) e^{b_i(x-1)} = \sum_{i=1}^k a_i (1 + b_i x) e^{b_i(x-1)}.$$

Moreover, $\sum_{i=1}^k a_i e^{b_i(x-1)}$ is strictly positive and does not depend on n . Hence, if $\left\{ \sum_{i=1}^k a_i f_{b_i,n}(x) \right\}_{n=1}^\infty$ were convergent, the sequence $\{\cos(2\pi nx)\}_{n=1}^\infty$ should be convergent as well, which is not possible. On the one hand, if $x \in \mathbb{Q}$, then that sequence is periodic. On the other hand, if $x \notin \mathbb{Q}$, then $\{(\cos(2\pi nx), \sin(2\pi nx))\}_{n=1}^\infty$ is dense in the unit circle and so $\{\cos(2\pi nx)\}_{n=1}^\infty$ is dense in $[-1, 1]$ (see [30, Proposition 4.1.1]). Therefore, $\left\{ \sum_{i=1}^k a_i f_{b_i,n} \right\}_{n=1}^\infty$ does not converge at any point of $(0, 1)$. The densities are unique almost everywhere, which implies that any other sequence of densities will not converge at almost any point of $(0, 1)$. □

5.2 Convergence in distribution and convergence in variation

Let us suppose again that F and F_n (with $n \in \mathbb{N}$) are absolutely continuous distribution functions with density functions f and f_n , respectively. If $\{F_n\}_{n=1}^\infty$ converges in variation to F , for each $x \in \mathbb{R}$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} |F_n(x) - F(x)| &= \lim_{n \rightarrow \infty} \left| \int_{-\infty}^x f_n(t) dt - \int_{-\infty}^x f(t) dt \right| \\ &\leq \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} |f_n(t) - f(t)| dt = 0. \end{aligned}$$

Therefore, $\{F_n\}_{n=1}^\infty$ converges weakly to F . However, the converse is not true in general (see [38, Section 14.12]). The next result shows that the set of sequences of absolutely continuous distribution functions converging weakly but not in variation contains a large convex space in the lineability sense.

Theorem 17 *Let F denote the function on \mathbb{R} defined as follows:*

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x < 1, \\ 1 & \text{if } x \geq 0. \end{cases}$$

Let S denote the set of all sequences of absolutely continuous distribution functions that converge to F at every point of \mathbb{R} but do not converge to F in variation. Then S is ϵ -convex lineable.

Proof We will work restricted to $[0, 1]$, since all the distribution functions considered in this proof will take the value 0 on $(-\infty, 0]$ and the value 1 on $[1, +\infty)$. Given $b \in (0, 1)$ and $n \in \mathbb{N}$, the function $g_{b,n}$ is defined as follows:

$$g_{b,n}(t) = \begin{cases} \frac{1}{b} & \text{if } t \in \left[\frac{j}{n}, \frac{j+b}{n}\right] \text{ for some } j \in \{0, \dots, n-1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\int_0^1 g_{b,n}(t)dt = 1$. Let $G_{b,n}$ be the distribution function associated to the density $g_{b,n}$. For each $x \in [0, 1]$ there exists $j \in \{0, \dots, n-1\}$ such that either $x \in \left[\frac{j}{n}, \frac{j+b}{n}\right]$ or $x \in \left[\frac{j+b}{n}, \frac{j+1}{n}\right]$. On the one hand, if $x \in \left[\frac{j}{n}, \frac{j+b}{n}\right]$ for some $j \in \{0, \dots, n-1\}$, then

$$G_{b,n}(x) = \int_0^x g_{b,n}(t)dt = \sum_{i=0}^{j-1} \frac{1}{b} \cdot \frac{b}{n} + \int_{j/n}^x \frac{1}{b} dt = \frac{j}{n} - \frac{j}{nb} + \frac{x}{b}. \tag{5.1}$$

Hence,

$$G_{b,n}(x) - x = \frac{j}{n} - \frac{j}{nb} + \frac{1-b}{b}x \leq \frac{j}{n} - \frac{j}{nb} + \frac{1-b}{b} \cdot \frac{j+b}{n} = \frac{1-b}{n}$$

and

$$G_{b,n}(x) - x = \frac{j}{n} - \frac{j}{nb} + \frac{1-b}{b}x \geq \frac{j}{n} - \frac{j}{nb} + \frac{1-b}{b} \cdot \frac{j}{n} = 0.$$

On the other hand, if $x \in \left[\frac{j+b}{n}, \frac{j+1}{n}\right]$ for some $j \in \{0, \dots, n-1\}$, then

$$G_{b,n}(x) = \int_0^x g_{b,n}(t)dt = \int_0^{\frac{j+b}{n}} g_{b,n}(t)dt = \sum_{i=0}^j \frac{1}{b} \cdot \frac{b}{n} = \frac{j+1}{n}. \tag{5.2}$$

Hence,

$$G_{b,n}(x) - x = \frac{j+1}{n} - x \leq \frac{j+1}{n} - \frac{j+b}{n} = \frac{1-b}{n}$$

and

$$G_{b,n}(x) - x = \frac{j+1}{n} - x \geq \frac{j+1}{n} - \frac{j+1}{n} = 0.$$

Therefore, we obtain that

$$0 \leq G_{b,n}(x) - x \leq \frac{1-b}{n} < \frac{1}{n} \tag{5.3}$$

for every $x \in [0, 1]$.

Let us prove that the set $\{G_{b,1} : b \in (0, 1)\}$ is linearly independent. Let $k \in \mathbb{N}$, $0 < b_1 < \dots < b_k < 1$, and $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$. If $n = 1$ and $x \in [b_{k-1}, b_k]$, then the equalities in

(5.2) imply that $G_{b_i,1}(x) = 1$ for every $i \in \{1, \dots, k - 1\}$, while $G_{b_k,1}(x) = \frac{x}{b_k}$ by (5.1). Therefore, for $x \in [b_{k-1}, b_k]$, we have

$$\sum_{i=1}^k a_i G_{b_i,1}(x) = a_1 + \dots + a_{k-1} + \frac{x}{b_k}.$$

Consequently, it cannot happen that $\sum_{i=1}^k a_i G_{b_i,1}(x) = 0$ for all $x \in [b_{k-1}, b_k]$. This proves the set of functions $\{G_{b,1} : b \in (0, 1)\}$ is linearly independent, which in turn implies that the set of sequences

$$\mathcal{M} = \{\{G_{b,n}\}_{n=1}^\infty : b \in (0, 1)\}$$

is linearly independent as well.

Next we will prove that every convex combination of elements of the set \mathcal{M} belongs to \mathcal{S} . Let $k \in \mathbb{N}$, $0 < b_1 < \dots < b_k < 1$, and $a_1, \dots, a_k \in \mathbb{R} \setminus \{0\}$ such that $\sum_{i=1}^k a_i = 1$. By the inequalities in (5.3), if $x \in [0, 1]$, then

$$\begin{aligned} \left| \sum_{i=1}^k a_i G_{b_i,n}(x) - x \right| &= \left| \sum_{i=1}^k a_i G_{b_i,n}(x) - \sum_{i=1}^k a_i x \right| \\ &\leq \sum_{i=1}^k a_i |G_{b_i,n}(x) - x| < \frac{1}{n}. \end{aligned}$$

Therefore, $\{\sum_{i=1}^k a_i G_{b_i,n}\}_{n=1}^\infty$ converges to F at every point.

Finally, the variation between F and $\sum_{i=1}^k a_i G_{b_i,n}$ is equal to

$$\int_0^1 \left| 1 - \sum_{i=1}^k a_i g_{b_i,n}(x) \right| dx.$$

Since $b_1 < \dots < b_k$, we have that

$$\left[\frac{j + b_k}{n}, \frac{j + 1}{n} \right] \subset \left[\frac{j + b_i}{n}, \frac{j + 1}{n} \right]$$

for every $i \in \{1, \dots, k\}$ and every $j \in \{0, \dots, n - 1\}$. Therefore, if $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, n - 1\}$, then $g_{b_i,n}(x) = 0$ for all $x \in \left[\frac{j+b_k}{n}, \frac{j+1}{n} \right]$. Hence,

$$\begin{aligned} \int_0^1 \left| 1 - \sum_{i=1}^k a_i g_{b_i,n}(x) \right| dx &\geq \sum_{j=0}^{n-1} \int_{\frac{j+b_k}{n}}^{\frac{j+1}{n}} \left| 1 - \sum_{i=1}^k a_i g_{b_i,n}(x) \right| dx \\ &= \sum_{j=0}^{n-1} \int_{\frac{j+b_k}{n}}^{\frac{j+1}{n}} dx = 1 - b_k > 0. \end{aligned}$$

Consequently, $\{\sum_{i=1}^k a_i G_{b_i,n}\}_{n=1}^\infty$ does not converge to F in variation and the proof is complete. □

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Authors and Affiliations

G. Araújo¹ · M. Fenoy² · J. Fernández-Sánchez^{3,4} · J. López-Salazar⁵ ·
J. B. Seoane-Sepúlveda⁶ · J. M. Vecina⁷

G. Araújo
gustavoaraujo@servidor.uepb.edu.br

M. Fenoy
mmfenoy@ucm.es

J. Fernández-Sánchez
juanfernandez@ual.es

J. López-Salazar
jeronimo.lopezsalazar@upm.es

J. M. Vecina
josemariavecina@gmail.com

¹ Departamento de Matemática, Universidade Estadual da Paraíba, Campina Grande 58.429-500, Brazil

- ² Instituto de Matemática Interdisciplinar (IMI), Departamento de Estadística e Investigación Operativa, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain
- ³ Instituto de Matemática Interdisciplinar (IMI), Universidad Complutense de Madrid, Madrid, Spain
- ⁴ Grupo de investigación de Teoría de Cópulas y Aplicaciones, Universidad de Almería, Carretera de Sacramento s/n, 04120 Almería, Spain
- ⁵ Departamento de Matemática Aplicada a las Tecnologías de la Información y las Comunicaciones, Universidad Politécnica de Madrid, Calle Nikola Tesla, s/n, 28031 Madrid, Spain
- ⁶ Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Instituto de Matemática Interdisciplinar (IMI), Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain
- ⁷ Departamento de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain