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ORIGINAL PAPER

On complete Kählerian manifolds endowed with closed conformal vector fields

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Abstract
Let $\overline{M}^{2n}$, $n > 1$, be a complete, noncompact Kählerian manifold, endowed with a nontrivial closed conformal vector field ξ having at least one singular point. Under a reasonable set of conditions, we show that ξ has just one singular point p and that $\overline{M}\backslash\{p\}$ is isometric to a one dimensional cone over a simply connected Sasakian manifold N diffeomorphic to $\mathbb{S}^{2n-1}$.As a straightforward consequence, we conclude that if the addition of a single point to the Kählerian cone of a $(2n-1)$-dimensional Sasakian manifold N has the structure of a complete, noncompact, $2n$-dimensional Kählerian manifold whose metric extends that of the cone, and such that the canonical vector field of the cone extends to it having a singularity at the extra point, then N is isometric to $\mathbb{S}^{2n-1}$, endowed with an appropriate Sasakian structure.

Keywords Kählerian manifold · Sasakian manifold · Conformal vector field · Maximum principle at infinity

Mathematics Subject Classification 53B35 · 53C21 · 53C25 · 53C24

1 Introduction and preliminaries

Given an m-dimensional Riemannian manifold (M^m, g) with Levi–Civita connection ∇, we recall that a conformal vector field ξ on M is said to be *closed* if the 1-form $\xi^\flat$ is closed. This is easily seen to be equivalent to the existence of a smooth function $\psi : M \to \mathbb{R}$ (called the

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conformal factor of ξ) such that

$$\nabla_X \xi = \psi X,$$

for all $X \in \mathfrak{X}(M)$. In turn, this readily yields

$$\psi = \frac{1}{m} \mathrm{div}_M \xi. \tag{1}$$

Also in this setting, item 1 of Lemma 1 of [9] shows that, for every nontrivial closed and conformal vector field ξ,

$$|\xi|^2 \nabla \psi = -\,\mathrm{Ric}_M(\xi)\xi, \tag{2}$$

where $\nabla \psi$ stands for the gradient of ψ and

$$\mathrm{Ric}_M(\xi) = \frac{1}{m-1} \mathrm{Ric}_M(\xi, \xi)$$

for the normalized Ricci curvature of (M^m, g, ∇) in the direction of ξ. Moreover, if $m > 2$, then item 3 of that result shows that $\xi^{-1}(0)$, the set of singular points of ξ, is a set of isolated points and $\psi(p) \neq 0$ for every $p \in \xi^{-1}(0)$. See also Lemma 1 in [5], which summarizes some of the known results about Riemannian manifolds which admit closed and conformal vector fields. We also refer the reader to [6] and [10] for some recent results on the structure and geometric properties of Riemannian manifolds endowed with closed conformal vector fields.

The standard class of examples of Riemannian manifolds equipped with closed conformal vector fields is that of Riemannian warped products with one dimensional fibers, as we now recall. Let $I \subset \mathbb{R}$ be an open interval with its standard metric dt^2 and N^{m-1} be an $(m-1)$-dimensional Riemannian manifold with metric g_N. We set $M^m = I \times N^{m-1}$ and let $\pi_I : M \to I$ and $\pi_N : M \to N$ denote the projections. If $h : I \to (0, +\infty)$ is a smooth function and $\tilde{h} = h \circ \pi_I : M \to (0, +\infty)$, then

$$\langle \cdot, \cdot \rangle = \pi_I^* dt^2 + \tilde{h}^2 \pi_N^* g_N$$

is a metric tensor on M, with respect to which M is said to be the warped product of I and N, with warping function h. We summarize this by writing

$$M^m = I \times_h N^{m-1}.$$

If $\tilde{h} = h \circ \pi_I$, ∂_t denotes the canonical vector field on I and $\tilde{\partial}_t$ its horizontal lift to M, then it is a standard fact that the vector field $\tilde{h}\tilde{\partial}_t$ is a closed conformal vector field on M with no singular points, with conformal factor $\tilde{h'} = h' \circ \pi_I$, where h' is the derivative of h.

In particular, letting $I = (0, +\infty)$ and $h(t) = t$ for every $t > 0$, we obtain the one dimensional cone $M^m = (0, +\infty) \times_t N^{m-1}$, with closed conformal vector field $t\tilde{\partial}_t$ of conformal factor 1.

A Sasakian manifold is a Riemannian manifold (N, g_N) with Levi–Civita connection D such that the one dimensional cone

$$M = (0, +\infty) \times_t N$$

is a Kählerian manifold; in particular, N is odd dimensional. In such a case, if we let J denote the complex structure of M and $\xi(t, p) = t\tilde{\partial}_t$ the closed conformal vector field, then it can be proved (cf. [11], for instance) that:

(a) $Z := J\xi$ is a unit Killing vector field on $N \approx \{1\} \times N$.

(b) For $X \in \mathfrak{X}(N)$, one has

$$JX = -\langle X, Z\rangle\xi + D_X Z. \tag{3}$$

(c) If $\Phi \in \mathrm{End}(TN)$ is given by $\Phi(X) = D_X Z$, then

$$\big(D_X\Phi\big)(Y) = \langle Y, Z\rangle X - \langle X, Y\rangle Z, \tag{4}$$

for all $X, Y \in \mathfrak{X}(N)$.

Conversely, let (N, g_N) be an odd dimensional Riemannian manifold with Levi–Civita connection D and $M = (0, +\infty)\times_t N$ be the one dimensional cone over N. Assume that there exists a unit Killing vector field Z on N for which the field of endomorphisms $\Phi \in \mathrm{End}(TN)$, given by $\Phi(X) = D_X Z$, satisfies (4). Then (see also [11]), N is a Sasakian manifold and the restriction of the complex structure J of M to TN satisfies (3).

With notations as in the previous paragraph, warped product geometry (cf. Corollary 7.43 of [7]) readily shows that

$$\mathrm{Ric}_M(\xi) = 0. \tag{5}$$

Actually, if the Sasakian manifold N is $(2n-1)$-dimensional, then, computing as suggested above with the aid of (4), it can be shown that (N, g_N) is Einstein if, and only if, $M = (0, +\infty)\times_t N$ is Ricci flat.

In this paper, we aim at proving the following

Theorem 1.1 *Let $(\overline{M}^{2n}, g = \langle\cdot,\cdot\rangle, J)$, $n > 1$, be a complete, noncompact Kählerian manifold, endowed with a nontrivial closed conformal vector field ξ, of conformal factor ψ and having at least one singular point. Assume that $\psi \geq 1$ on $\overline{M}$ and $\psi \to 1$ at infinity. If the Ricci curvature of $\overline{M}$ in the direction of ξ is nonpositive, then:*

(a) $\psi \equiv 1$ on $\overline{M}$ and ξ has just one singular point, say p.
(b) $\overline{M}\backslash\{p\}$ is isometric to a one dimensional cone over a Sasakian manifold diffeomorphic to $\mathbb{S}^{2n-1}$.

The fact that the conformal factor $\psi \equiv 1$ on $\overline{M}$ implies that the Lie derivative of the metric tensor g with respect to ξ satisfies $\mathcal{L}_\xi g = 2g$. Geometrically, this means that the flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of the vector field ξ consists of hometheties of positive coefficient, since $(\varphi_t^* g)_p = e^{2t} g_p$ for all $p \in \overline{M}$ and $t \in \mathbb{R}$. For that reason ξ is also said to be a homothetic vector field (see, for instance, Chapter 5 in [8]).

It is worth pointing out that this result lies in the complementary setting of the one dealt with by the second author in [4].

For the coming corollary, given a $(2n-1)$-dimensional Sasakian manifold N, we say that the Kählerian cone $M := (0, +\infty)\times_t N$ *has a removable singularity* if the following condition is satisfied: for some symbol p not in M, the space $\overline{M} := M \cup \{p\}$ has the structure of a complete, noncompact, $2n$-dimensional Kählerian manifold whose metric and complex structure extend that of M, and such that the closed conformal vector field $t\partial_t$ of M likewise extends to $\xi \in \mathfrak{X}(\overline{M})$.

Corollary 1.2 *Let N be a $(2n-1)$-dimensional Sasakian manifold whose Kählerian cone $M := (0, +\infty)\times_t N$ has a removable singularity. With notations as above, if p is a singular point of ξ, then N is isometric to $\mathbb{S}^{2n-1}$, endowed with an appropriate Sasakian structure.*

2 Proof of Theorem 1.1

First of all, as observed in the second paragraph of Sect. 1, since $2n > 2$ and ξ is nontrivial, it has isolated zeros. Recall also that we are assuming $\mathrm{Ric}_{\overline{M}}(\xi) \leq 0$ on $\overline{M}$. We divide the subsequent analysis in several steps.

Claim 1. $\psi = 1$ on $\overline{M}$ and $\mathrm{Ric}_{\overline{M}}(\xi) = 0$ on $\overline{M}\backslash\xi^{-1}(0)$.

We shall need the following result.

Theorem 2.1 (Theorem 2.2 of [1]) *Let* $(\overline{M}, \langle\cdot,\cdot\rangle)$ *be a connected, oriented, complete non-compact Riemannian manifold, and let* $\eta \in \mathfrak{X}(\overline{M})$ *be a vector field on* $\overline{M}$. *Assume that there exists a nonnegative, non identically vanishing function* $f \in \mathcal{C}^\infty(\overline{M})$, *converging to zero at infinity and such that* $\langle\nabla f, \eta\rangle \geq 0$ *on* $\overline{M}$. *If* $\mathrm{div}_{\overline{M}}\eta \geq 0$ *on* $\overline{M}$, *then:*

(a) $\langle\nabla f, \eta\rangle \equiv 0$ *on* $\overline{M}$.
(b) $\mathrm{div}_{\overline{M}}\eta \equiv 0$ *on* $\overline{M}\backslash f^{-1}(0)$.

Back to the proof of Claim 1, let $\eta = \psi\xi$. It follows from (1) and (2) that, at every nonsingular point of ξ, we have

$$\mathrm{div}_{\overline{M}}\eta = \langle\nabla\psi, \xi\rangle + \psi\mathrm{div}_{\overline{M}}\xi = -\mathrm{Ric}_{\overline{M}}(\xi) + 2n\psi^2 \geq 0. \tag{6}$$

By continuity, $\mathrm{div}_{\overline{M}}\eta \geq 0$ on $\overline{M}$.

Assume, for the sake of contradiction, that ψ is not identically 1 on $\overline{M}$. Setting $f = \psi - 1$, we have $f \geq 0$ and $f \not\equiv 0$ on $\overline{M}$. Also from (2), we get, at the nonsingular points of ξ,

$$\langle\nabla f, \eta\rangle = \langle\nabla\psi, \psi\xi\rangle = -\mathrm{Ric}_{\overline{M}}(\xi)\psi \geq 0.$$

Again, by continuity, $\langle\nabla f, \eta\rangle \geq 0$ on $\overline{M}$.

Item (b) of Theorem 2.1 gives $\mathrm{div}_{\overline{M}}\eta \equiv 0$ on $\overline{M}\backslash f^{-1}(0) = \overline{M}\backslash\psi^{-1}(1)$. Back to (6), this shows that $\psi \equiv 0$ on $\overline{M}\backslash\psi^{-1}(1)$, thus contradicting the fact that $\psi \geq 1$ on $\overline{M}$. Therefore, $\psi \equiv 1$ on $\overline{M}$, which means that

$$\nabla_X\xi = X \tag{7}$$

for all $X \in \mathfrak{X}(\overline{M})$.

Once we know that $\psi \equiv 1$, Eq. (2) shows that $\mathrm{Ric}_{\overline{M}}(\xi) \equiv 0$ on $\overline{M}\backslash\xi^{-1}(0)$, and, trivially, on $\xi^{-1}(0)$. Anyway, as we have already noticed, this will also follow once we show that $\overline{M}\backslash\{p\}$ is isometric to a one dimensional cone over a simply connected Sasakian manifold.

Claim 2. ξ has exactly one singular point.

Arguing once more by contradiction, assume that $\xi(p) = 0$ and $\xi(q) = 0$, for some distinct points $p, q \in \overline{M}$. Thanks to the completeness of $\overline{M}$, we can take a normalized geodesic $\gamma : [0, \ell] \to \overline{M}$ from p to q. Letting $\xi(t)$ denote the restriction of ξ to γ and $\varphi(t) := \langle\xi(t), \gamma'(t)\rangle$, we get $\varphi(0) = \varphi(\ell) = 0$ and

$$\varphi'(t) = \langle\frac{D\xi}{dt}(t), \gamma'(t)\rangle = \langle\nabla_{\gamma'(t)}\xi, \gamma'(t)\rangle = \langle\gamma'(t), \gamma'(t)\rangle = 1,$$

whence $\varphi(t) = t + \varphi(0) = t$. However, this contradicts the fact that $\varphi(\ell) = 0$.

It thus follows from the previous claim and our hypotheses that ξ has exactly one singular point.

Claim 3. If p is the singular point of ξ, then, for each $q \neq p$, there is just one normalized geodesic γ from p to q. Moreover, letting $\xi(t)$ denote the restriction of ξ to γ, we have $\xi(t) = t\gamma'(t)$.

Let $\gamma_1, \gamma_2 : [0, \ell_i] \to \overline{M}$ be two normalized geodesics from p to q. Since $\xi(p) = 0$, reasoning as in the proof of Claim 2, we get $\langle \xi(\gamma_i(t)), \gamma_i'(t)\rangle = t$ for every $t \in [0, \ell_i]$ and $i = 1, 2$. On the other hand,

$$\frac{d}{dt}|\xi(\gamma_i(t))|^2 = 2\Big\langle \nabla_{\gamma_i'(t)}\xi, \xi(\gamma_i(t))\Big\rangle = 2\Big\langle \gamma_i'(t), \xi(\gamma_i(t))\Big\rangle = 2t,$$

so that $|\xi(\gamma_i(t))|^2 = t^2 + |\xi(0)|^2 = t^2$ for every $t \in [0, \ell_i]$ and $i = 1, 2$. Finally, Cauchy-Schwarz inequality gives

$$t = \Big\langle \xi(\gamma_i(t)), \gamma_i'(t)\Big\rangle \leq |\xi(\gamma_i(t))||\gamma_i'(t)| = t,$$

so that $\xi(\gamma_i(t)) = t\gamma_i'(t)$ for every $t \in [0, \ell_i]$ and $i = 1, 2$. In particular, since $\gamma_1(\ell_1) = \gamma_2(\ell_2) = q$ and $\ell_1\gamma_1'(\ell_1) = \xi(q) = \ell_2\gamma_2'(\ell_2)$ and they are both normalized geodesics, it must be $\ell_1 = \ell_2 = |\xi(q)| = \ell > 0$. Therefore, $\gamma_1(\ell) = \gamma_2(\ell)$ with $\gamma_1'(\ell) = \gamma_2'(\ell)$, which implies that $\gamma_1(t) = \gamma_2(t)$ for every $t \in [0, \ell]$.

From now on, we let p denote the unique singular point of ξ, and

$$M := \overline{M}\backslash\{p\}.$$

Claim 4. The exponential map $\exp_p : T_p\overline{M} \to \overline{M}$ is a diffeomorphism.

The previous claim accounts for the injectivity of $\exp_p$, and the completeness of $\overline{M}$ for its surjectivity. On the other hand, if $v \in T_p\overline{M}$ is a unit vector and $\gamma_v : [0, +\infty) \to \overline{M}$ is the geodesic ray issuing from p with $\gamma_v'(0) = v$, then, also from the previous claim, $\gamma_{v|[0,t]}$ is minimizing, for every $t > 0$. Therefore, $\exp_p$ has no conjugate points along γ_v. Since this happens for every unit vector $v \in T_p\overline{M}$, it assures that $\exp_p$ is a local diffeomorphism. Being bijective, it is actually a global diffeomorphism.

Claim 5. If $\mathcal{E} = \{Y \in \mathfrak{X}(M);\ \langle Y, \xi\rangle = 0\}$, then $\mathcal{E}$ is integrable.

For $Y, Z \in \mathcal{E}$, we have

$$\begin{aligned}\Big\langle [Y, Z], \xi\Big\rangle &= \Big\langle \nabla_Y Z - \nabla_Z Y, \xi\Big\rangle \\ &= Y\Big\langle Z, \xi\Big\rangle - \Big\langle Z, \nabla_Y\xi\Big\rangle - Z\langle Y, \xi\Big\rangle + \langle Y, \nabla_Z\xi\Big\rangle \\ &= -\Big\langle Z, Y\Big\rangle + \Big\langle Y, Z\Big\rangle = 0.\end{aligned}$$

Therefore, $[Y, Z] \in \mathcal{E}$, as we wished to show.

Claim 6. If $N = \exp_p(\mathbb{S}^{2n-1})$, then $\xi_{|N}$ is a unit normal vector field along N.

Let $v \in \mathbb{S}^{2n-1}$, $\gamma(t) = \exp_p(tv)$, $t \geq 0$, and $q = \gamma(1) \in N$. Claim 3 gives $|\xi_q| = |\gamma'(1)| = |\gamma'(0)| = |v| = 1$; together with Gauss' lemma, it shows that $\xi_q \in T_qN^{\perp}$. The rest is immediate from the previous claim.

It follows in particular from the previous claim that N is a leaf of the distribution $\mathcal{E}$ and N is diffeomorphic to $\mathbb{S}^{2n-1}$.

Claim 7. If $M = \overline{M}\backslash\{p\}$, then M is isometric to $(0, +\infty) \times_t N$.

We already know, from Claim 4, that $\exp_p : T_p\overline{M}\backslash\{0\} \to M$ is a diffeomorphism. Since $(0, +\infty) \times N$ is diffeomorphic to $T_p\overline{M}\backslash\{0\}$ via $(t, q) \mapsto t(\exp_p)^{-1}(q)$, it follows that the map

$$\begin{aligned} \Psi : (0, +\infty) \times N &\longrightarrow \quad M \\ (t, q) \quad &\longmapsto \exp_p(t(\exp_p)^{-1}(q)) \end{aligned} \tag{8}$$

is also a diffeomorphism.

Thus, it suffices to show that the metric g_0, induced on $(0, +\infty) \times N$ by such a diffeomorphism, is the warping metric of $(0, +\infty) \times_t N$.

To this end, first note that, from Claim 3, the integral curve of ξ through any point $q \in M$ is a pregeodesic of M. Actually, a simple computation shows that the flow of $\xi_{|M}$ is

$$\begin{aligned} \Phi : \mathbb{R} \times M &\longrightarrow \quad M \\ (u, q) &\longmapsto \exp_p\left(e^u (\exp_p)^{-1}(q)\right) \end{aligned}. \tag{9}$$

Actually, since

$$\Phi(u, q) = \Psi(e^u, q) = \gamma_{(\exp_p)^{-1}(q)}(e^u)$$

we have, by Claim 3,

$$\frac{\partial \Phi}{\partial u}(u, q) = e^u \gamma'_{(\exp_p)^{-1}(q)}(e^u) = \xi(\Phi(u, q)). \tag{10}$$

Let $\alpha : (-\varepsilon, \varepsilon) \to N$ be an arc length parametrized curve with $\alpha(0) = q$. We shall consider the parametrized surface in M given by the map

$$\begin{aligned} \varphi : \mathbb{R} \times (-\varepsilon, \varepsilon) &\longrightarrow \mathbb{R} \times N \longrightarrow \quad M \\ (u, s) \quad &\longmapsto (u, \alpha(s)) \longmapsto \Phi(t, \alpha(s)) \end{aligned}.$$

Since $\xi_{|N} \in TN^{\perp}$, we have

$$\left\langle \frac{\partial \varphi}{\partial u}(u, s), \frac{\partial \varphi}{\partial s}(u, s) \right\rangle_{\varphi(u,s)} = \left\langle \xi(\varphi(u, s)), \frac{\partial \varphi}{\partial s}(u, s) \right\rangle_{\varphi(u,s)} = 0.$$

On the other hand, (10) gives

$$\left\langle \frac{\partial \varphi}{\partial u}(u, s), \frac{\partial \varphi}{\partial u}(u, s) \right\rangle_{\varphi(u,s)} = e^{2u}.$$

Now, since ξ is a closed conformal vector field with conformal factor $\psi \equiv 1$ (see Eq. (7)), we conclude that

$$\frac{D}{\partial s}\xi(\varphi(u, s)) = \nabla_{\frac{\partial \varphi}{\partial s}} \xi = \frac{\partial \varphi}{\partial s}(u, s).$$

Set

$$f(u, s) : \left\langle \frac{\partial \varphi}{\partial s}(u, s), \frac{\partial \varphi}{\partial s}(u, s) \right\rangle_{\varphi(u,s)}.$$

Computing pretty much as in the proof of item (b) of Theorem 3.4 of [3] (and omitting the point $\varphi(u, s)$ from the computations, for the sake of clarity), we get

$$\begin{aligned} \frac{\partial f}{\partial u}(u, s) &= \frac{d}{du} \left\langle \frac{\partial \varphi}{\partial s}(u, s), \frac{\partial \varphi}{\partial s}(u, s) \right\rangle \\ &= 2 \left\langle \frac{D}{\partial s} \frac{\partial \varphi}{\partial u}(u, s), \frac{\partial \varphi}{\partial s}(u, s) \right\rangle \\ &= 2 \left\langle \frac{D}{\partial s}\xi(\varphi(u, s)), \frac{\partial \varphi}{\partial s}(u, s) \right\rangle \end{aligned}$$

$$
= 2\left\langle \frac{\partial \varphi}{\partial s}(u,s), \frac{\partial \varphi}{\partial s}(u,s) \right\rangle
$$
$$
= 2f(u,s).
$$

Hence, $f(u,s) = e^{2u} f(0,s)$, which, by $\varphi(0,s) = \alpha(s)$ and $|\alpha'(s)| = 1$, is the same as

$$
\left\langle \frac{\partial \varphi}{\partial s}(u,s), \frac{\partial \varphi}{\partial s}(u,s) \right\rangle_{\varphi(u,s)} = e^{2u} \left\langle \frac{\partial \varphi}{\partial s}(0,s), \frac{\partial \varphi}{\partial s}(0,s) \right\rangle_{\varphi(0,s)} = e^{2u}.
$$

Finally, consider the parametrized surface

$$
\begin{array}{cccccc} \psi : (0,+\infty) \times (-\varepsilon, \varepsilon) & \longrightarrow & (0,+\infty) \times N & \longrightarrow & M \\ (t,s) & \longmapsto & (t, \alpha(s)) & \longmapsto & \Psi(t, \alpha(s)) \end{array}.
$$

Since $\psi(t,s) = \varphi(\log t, s)$, the above computations translate into

$$
\left\langle \frac{\partial \psi}{\partial t}(t,s), \frac{\partial \psi}{\partial s}(t,s) \right\rangle_{\psi(t,s)} = 0, \quad \left\langle \frac{\partial \psi}{\partial t}(t,s), \frac{\partial \psi}{\partial t}(t,s) \right\rangle_{\psi(t,s)} = 1
$$

and

$$
\left\langle \frac{\partial \psi}{\partial s}(t,s), \frac{\partial \psi}{\partial s}(t,s) \right\rangle_{\psi(t,s)} = t^2.
$$

Therefore, $((0,+\infty) \times N, g_0)$ is isometric to $(0,+\infty) \times_t N$, as we wished to show.

Claim 8. (N, g_N, D) is a Sasaki manifold, where g_N is the induced metric on N, which we denote also by $\langle \cdot, \cdot \rangle$, and D its corresponding Levi–Civita connection.

Since $\Psi : (0,+\infty) \times_t N \to M$ is an isometry from Claim 7, it follows that $(0,+\infty) \times_t N$ is naturally a Kählerian manifold: one just has to use the isometry to import, to $(0,+\infty) \times_t N$, the complex atlas and the complex structure of M. Therefore, according to the definition of Sasaki manifold and the discussion about that given in Sect. 1, N is a Sasaki manifold.

Remark 2.2 $\mathbb{S}^{2n-1}$, endowed with the canonical round metric, is the simplest example of a Sasakian manifold. Nevertheless, a sphere may, at first, be endowed with several distinct Sasakian structures. For instance, as observed at page 353 of [2], there are 63 distinct Sasakian structures on $\mathbb{S}^5$. Therefore, the conclusion of item (b) in Theorem 1.1 is, under our hypotheses, the best possible one. We would like to thank professor Vicente Muñoz for calling our attention to these examples of Sasakian structures on $\mathbb{S}^5$.

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References

1. Alías, L.J., Caminha, A., do Nascimento, F.Y.: A maximum principle at infinity with applications. J. Math. Anal. Appl. **474**, 242–247 (2019)
2. Boyer, C.P., Galicki, K.: Sasakian Geometry. Oxford Mathematical Monographs. Oxford University Press, Oxford (2008)
3. Caminha, A.: The geometry of closed conformal vector fields on Riemannian spaces. Bull. Braz. Math. Soc. **42**, 277–300 (2011)
4. Caminha, A.: A rigidity result for Kählerian manifolds endowed with closed conformal vector fields. Rev. UMA **60**, 469–484 (2019)
5. Castro, I., Montealegre, C.R., Urbano, F.: Closed conformal vector fields and Lagrangian submanifolds in complex space forms. Pac. J. Math. **199**, 269–302 (2001)
6. Diógenes, R., Ribeiro, E., Jr., da Silva Filho, J.F.: Gradient Ricci solitons admitting a closed conformal vector field. J. Math. Anal. Appl. **455**, 1975–1983 (2017)
7. O'Neill, B.: Semi-Riemannian geometry with applications to Relativity. Academic Press, New York (1983)
8. Poor, W.A.: Differential geometric structures. McGraw-Hill Book Co., New York (1981)
9. Ros, A., Urbano, F.: Lagrangian submanifolds of $\mathbb{C}^n$ with conformal Maslov form and the Whitney sphere. J. Math. Soc. Japan **50**, 203–226 (1998)
10. da Silva Filho, J.F.: Critical point equation and closed conformal vector fields. Math. Nachr. **293**, 2299–2305 (2020)
11. Sparks, J.: Sasaki–Einstein manifolds. Surveys Diff. Geom. **16**, 265–324 (2011)