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ORIGINAL PAPER

Uniqueness and asymptotics of singularly perturbed equations involving implicit boundary conditions

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Abstract
A class of one-dimensional convection–diffusion equations with a singularly perturbed parameter in a bounded domain is presented, where the boundary condition is nonlocal type with an implicit form related to the unknown solutions. In general, the validity of the maximum principle of this type equation is unassurable. Employing a singular perturbations method as a natural tool, we establish the uniqueness and maximum principle as the singularly perturbed parameter is sufficiently small. Such an argument is different from the standard fixed point approaches. Moreover, as this parameter tends to zero, the boundary and interior asymptotics of solutions is obtained.

Keywords Convection–diffusion equations · Singular perturbation · Nonlocality · Uniqueness · Asymptotic analysis

Mathematics Subject Classification 34B10 · 34D15 · 34E05 · 34K26 · 35J25

1 Introduction

1.1 Position of the problem

Considerable attention has been directed recently toward singularly perturbed convection–diffusion equations with nonlocal reaction terms which are appeared either in the equations or on the boundary conditions [3, 8, 11, 14, 17, 25, 30, 33, 35]. Due to the practical applications and various numerical developments, this category is recently one of the central objects of investigation in nonlocal type differential equations. However, the refined asymptotic analysis of such singularly perturbed equations seems not to be received attention in the vast related literature.

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In this note, we shall investigate the singularly perturbed convection–diffusion equation [5, 6, 26]:

$$-\varepsilon^2 u_\varepsilon''(x) + \varepsilon a(x) u_\varepsilon'(x) + b(x) f(u_\varepsilon(x)) = 0 \quad \text{in} \quad (0, \ell), \tag{1.1}$$

where u_ε is imposed by the integral/implicit type boundary condition

$$u_\varepsilon(0) = 0 \quad \text{and} \quad u_\varepsilon(\ell) = \mu + \int_{\ell_0}^{\ell_1} p(y) g(u_\varepsilon(y), u_\varepsilon'(y))\, \mathrm{d}y, \tag{1.2}$$

which means that the *unknown* boundary data $u_\varepsilon(\ell)$ satisfies an *implicit form* with a non-local dependence on the unknown u_ε in a sub-domain $(\ell_0, \ell_1) \Subset (0, \ell)$, i.e., $0 < \ell_0 < \ell_1 < \ell$, where (1.2) is called the implicit boundary condition since it is not an explicit form for $u_\varepsilon(\ell)$. Here $\varepsilon > 0$ is a singularly perturbed parameter tending to zero. Besides, μ is a constant independent of ε, the variable coefficients $a, b \in \mathrm{C}^1([0, \ell]; \mathbb{R})$, $p \in \mathrm{C}([0, \ell]; \mathbb{R})$ and the source terms $f \in \mathrm{C}^2(\mathbb{R}; \mathbb{R})$ and $g \in \mathrm{C}^1(\mathbb{R}^2; \mathbb{R})$ satisfy certain conditions which will be specified later.

The differential equations in bounded domains involve the boundary conditions determined by the unknown solution at an interior region are known as *nonlocal boundary value problems*. The boundary condition (1.2) is quite different from the standard Dirichlet boundary condition since it may rely implicitly on the unknown solution u_ε. Accordingly, the exact solutions of (1.1)–(1.2) are in general not specified, and the validity of the maximum principle is unassurable. Note also that the asymptotics of solution u_ε with respect to $0 < \varepsilon \ll 1$ is influenced by properties of these variable coefficients and source terms, resulting in that the asymptotic behaviors of u_ε and the non-local term $\int_{\ell_0}^{\ell_1} pg(u_\varepsilon, u_\varepsilon')\, \mathrm{d}y$ are influenced by each other. However, the rigorous asymptotic analysis is definitely of great challenge since the nonlocal terms depend on the unknown solutions.

1.2 Scientific background

The theory of singularly perturbed equations with implicit boundary conditions such as integral-type boundary conditions or multi-point boundary conditions are recently an active area of researches due to the important applications in the various fields. Such boundary conditions are known to make a much better description of practical models than the standard boundary conditions. At the best of our knowledge, Byszewski in [4] provided the first study involving the related physical significance. It further generates increased interest in a wide variety of nonlocal models. To formulate our study in a more concrete fashion, let us review the related background of (1.1)–(1.2) as follows.

Let us start from the importance of both convection and diffusion which are a combination of two dissimilar phenomena and play a significant role in heat transfer and fluid flow. Such a model can be regarded as a simplified model to the governing equations of the fluid flow [18, 27]. For the linear case, i.e., $f(u_\varepsilon) = u_\varepsilon$ and $g(u_\varepsilon, u_\varepsilon') = u_\varepsilon$, (1.1)–(1.2) has also been applied to some optimal control problems [7, 10, 20, 21, 29, 36, 37]. There are a lot of papers studying multiple solutions of elliptic type or parabolic type equations with nonlocal boundary conditions. We refer the reader to [3, 16, 38, 39].

Basic outcomes on equations related to the implicit type boundary conditions can be found in [1, 11, 14, 22] and references therein for commentary. By far the majority of related works, only some special cases of (1.1)–(1.2) have been investigated numerically (cf. [5, 6, 17, 35]). To the best of our knowledge, the standard method of matched asymptotic expansions seems not to accurately deal with the singularly perturbed equation with such implicit boundary conditions, and the rigorous asymptotic analysis is of great challenge [12, 13, 28, 34]. On the other hand, it is known that the numerical simulation for dealing with such singularly perturbed models are usually unstable and do not give satisfactory results for sufficiently small $\varepsilon > 0$ (cf. [30, 31]).

Thus, the focus of the current work is to investigate the non-local effect on the asymptotics of u_ε as ε goes to zero. Later on we will make precise assumptions on functions a, b and f to capture the more refined asymptotics of u_ε with respect to $0 < \varepsilon \ll 1$. We will first show that the equation (1.1) with the boundary condition (1.2) has a unique solution u_ε provided that $\varepsilon > 0$ is sufficiently small. Our argument is based on the asymptotic method which is different from [9] where the author used the method of difference scheme in the framework of numerical simulation under suitable parameters to prove the uniqueness. On the other hand, we shall stress that the uniqueness of (1.1)–(1.2) may not hold when $\varepsilon > 0$ is not small. More precisely, we provide an example for (1.1)–(1.2) with $f(u_\varepsilon) = u_\varepsilon$ and $g(u_\varepsilon, u'_\varepsilon) = u_\varepsilon$ to explain that there exists $\varepsilon_* > 0$ and μ_* such that when $\varepsilon = \varepsilon_*$ and $\mu = \mu_*$, (1.1)–(1.2) has infinitely many solutions u_{ε_*}; when $\varepsilon = \varepsilon_*$ and $\mu \neq \mu_*$, (1.1)–(1.2) has no solution. However, when $0 < \varepsilon < \varepsilon_*$, for any $\mu \in \mathbb{R}$, (1.1)–(1.2) has a unique solution (see Remark 1.3).

Finally, it is worth mentioning that the integral-type boundary condition (1.2) at $x = \ell$ is formally approximated to the multi-point boundary condition

$$u_\varepsilon(\ell) = \mu + \sum_{j=1}^{m} p(\xi_j)u_\varepsilon(\xi_j) + \sum_{k=1}^{\widetilde{m}} p(\widetilde{\xi}_k)u'_\varepsilon(\widetilde{\xi}_k) \tag{1.3}$$

when $g(u_\varepsilon, u'_\varepsilon) \sim \sum_{j=1}^{m} \delta_{\xi_j}(y)u_\varepsilon(y) + \sum_{k=1}^{\widetilde{m}} \delta_{\widetilde{\xi}_k}(y)u'_\varepsilon(y)$ in a weak sense in $\mathrm{C}([0, \ell]; \mathbb{R})$, where $m, \widetilde{m} \in \mathbb{N}$, ξ_j, $\widetilde{\xi}_k \in (\ell_0, \ell_1)$ are given points, and δ_{ξ_j} (resp., $\delta_{\widetilde{\xi}_k}$) is a Dirac delta function concentrated at ξ_j (resp., $\widetilde{\xi}_k$). In general, (1.2) and (1.3) can be regarded as implicit type boundary conditions since they depend on unknown solutions u_ε. Here we point out a close relation between (1.2) and (1.3), but the rigorous study of (1.1) with this two type boundary conditions are quite different. This work focuses mainly on the analysis of (1.1)–(1.2).

1.3 Significant ideas and the main contribution

We now make assumptions on functions a, b and f. In what follows, f is a strictly increasing and concave–convex function on $\mathbb{R}$,

$$f(0) = 0, \quad f' > 0 \quad \text{on } \mathbb{R} \quad \text{and} \quad f''(x_1) \geq 0 \geq f''(x_2) \quad \text{for } x_1 > 0 > x_2, \tag{1.4}$$

and we focus on the case that the convection-diffusion equation (1.1) has a *weak convection effect* under a mathematical setting

$$a^2(x) < 4f'(0)b(x) \quad \text{on} \quad [0, \ell]. \tag{1.5}$$

Equation (1.1) with (1.4) and (1.5) fulfills many well-known convection–diffusion equations. Typical examples of (1.4) is $f(u_\varepsilon) = \sinh u_\varepsilon$ appearing in Poisson–Boltzmann equations and size-modified Poisson–Boltzmann equations of electro-chemistry and sinh–Gordon equations of plasma physics (cf. [19, 23, 25]), and can also be found in the monotonic kinetic systems (see, e.g., [2]). (1.5) includes the case $a(x) \equiv 0$ and $b(x) > 0$ which arising in many applications such as the homogeneous chemical reactions. In particular, (1.5) can also be regarded as the convection–diffusion equation (1.1) with a large reaction $b(x)$. For example, $\min_{[0,\ell]} b$ is sufficiently large so that $\max_{[0,\ell]} |a| \leq 2\sqrt{f'(0)\min_{[0,\ell]} b}$. It should also be stressed that most of the related literature assume $a \geq 0$ which is exactly a simple situation for investigating the asymptotics of (1.1)–(1.2) with $0 < \varepsilon \ll 1$. This study with assumption (1.5) includes the case $\min_{[0,\ell]} a < 0$.

It is known that when $g(u_\varepsilon, u'_\varepsilon) = u_\varepsilon$ and $p \geq 0$ satisfies $\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y < 1$, for each $\varepsilon > 0$, under (1.4) and $b > 0$, the maximum principle is valid for (1.1)–(1.2); see, e.g., [32, 33]. However, for the general $g(u_\varepsilon, u'_\varepsilon)$ and p, the maximum principle of (1.1)–(1.2) seems not to have known.

Let us notice that (1.1)–(1.2) has a close relation with

$$\begin{cases} -\varepsilon^2 v''_{\varepsilon,\lambda}(x) + \varepsilon a(x) v'_{\varepsilon,\lambda}(x) + b(x) f(v_{\varepsilon,\lambda}(x)) = 0 \quad \text{in} \quad (0,\ell), \\ \qquad v_{\varepsilon,\lambda}(0) = 0, \quad v_{\varepsilon,\lambda}(\ell) = \lambda. \end{cases} \tag{1.6}$$

For more specific details, (1.1)–(1.2) has a solution $u_\varepsilon \equiv v_{\varepsilon,\lambda}$ if there exists $\lambda = \lambda_\varepsilon$ depending on ε such that the solution $v_{\varepsilon,\lambda}$ of (1.6) satisfies $\lambda = \mu + \int_{\ell_0}^{\ell_1} p(y) g(v_{\varepsilon,\lambda}(y), v'_{\varepsilon,\lambda}(y))\,\mathrm{d}y$. In Sect. 3.1, we will prove that equation (1.6) has a unique solution. To deal with this algebraic equation of λ, it suffices to consider the mapping $\mathcal{T}_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\mathcal{T}_\varepsilon(\lambda) := -\lambda + \mu + \int_{\ell_0}^{\ell_1} p(y) g(v_{\varepsilon,\lambda}(y), v'_{\varepsilon,\lambda}(y))\,\mathrm{d}y. \tag{1.7}$$

Since we focus on $v_{\varepsilon,\lambda}$ with $\varepsilon > 0$ approaching zero, the main difficulty in analysis of $\mathcal{T}_\varepsilon$ comes from the refined estimate of the nonlocal term $\int_{\ell_0}^{\ell_1} p(y) g(v_{\varepsilon,\lambda}(y), v'_{\varepsilon,\lambda}(y))\,\mathrm{d}y$ with respect to ε and λ. To the best of our knowledge, some well-known fixed point theorems such as Krasnoselskii's fixed point, Leray–Schauder nonlinear alternative Schauder fixed point and weakly contractive mapping theorems cannot apply to the investigation of $\mathcal{T}_\varepsilon$ unless ε is far away from zero. However, this is not our case study. Accordingly, to deal with the property of $\mathcal{T}_\varepsilon$ with respect to $0 < \varepsilon \ll 1$, in Section 2 we first introduce the asymptotic behavior of $v_{\varepsilon,\lambda}$. Next, we will establish a uniqueness property of $\mathcal{T}_\varepsilon(\lambda) = 0$ with respect to sufficiently small $\varepsilon > 0$.

In this note, under (1.4)–(1.5), we show that as $\varepsilon > 0$ is sufficiently small, (1.1)–(1.2) has a unique classical solution which is uniformly bounded in $[0, \ell]$ as $\varepsilon > 0$ approaches zero. Moreover, we obtain the boundary asymptotics of $u_\varepsilon(\ell)$ and interior asymptotics as ε approaches zero. The main result is stated as follows.

Theorem 1.1 *Assume that* $a, b \in \mathrm{C}^1([0,\ell];\mathbb{R})$, $p \in \mathrm{C}([0,\ell];\mathbb{R})$, $f \in \mathrm{C}^2(\mathbb{R};\mathbb{R})$ *and* $g \in \mathrm{C}^1(\mathbb{R}^2;\mathbb{R})$ *satisfying* (1.4) *and* (1.5). *Let*

$$\mathfrak{m} := \mu + g(0,0) \int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y. \tag{1.8}$$

Then for any constant $\mathfrak{M} > |\mathfrak{m}|$, *there exists a positive constant* $\eta := \eta(\mu, \mathfrak{M})$ *depending mainly on* μ *and* $\mathfrak{M}$ *such that for each* $\varepsilon \in (0, \eta)$, *equation* (1.1) *with the boundary condition* (1.2) *has a unique solution* $u_\varepsilon \in \mathrm{C}^1([0,\ell];\mathbb{R}) \cap \mathrm{C}^3((0,\ell);\mathbb{R})$ *with* $\sup_{\varepsilon\in(0,\eta)} \max_{[0,\ell]} |u_\varepsilon| \leq \mathfrak{M}$, *and*

$$u_\varepsilon(\ell) \xrightarrow{\varepsilon\downarrow 0} \mathfrak{m}. \tag{1.9}$$

Moreover, there hold the following properties:

(i) *If* $\mathfrak{m} > 0$ (< 0, *respectively), then for* $\varepsilon \in (0, \eta)$, *we have* $u'_\varepsilon > 0$ ($u'_\varepsilon < 0$, *respectively) on* $[0, \ell]$.

(ii) *For each interior point* $x \in [0, \ell)$, $|u_\varepsilon(x)| + |u'_\varepsilon(x)| \xrightarrow{\varepsilon \downarrow 0} 0$ *exponentially.*

(iii) u_ε *with* $\varepsilon \downarrow 0$ *develops boundary layers near* $x = \ell$ *in the sense that, for* $\tau \in (0, 1)$ *independent of* ε,

$$\lim_{\varepsilon \downarrow 0} \max_{[\ell - \varepsilon^\tau, \ell]} \left| u_\varepsilon(x) - U\left(\frac{\ell - x}{\varepsilon}\right) \right| = 0, \tag{1.10}$$

where U *is the unique classical solution of*

$$\begin{cases} -U''(t) - a(\ell)U'(t) + b(\ell) f(U(t)) = 0, & t > 0, \\ U(0) = \mathfrak{m}, \quad \lim_{t \to \infty} U(t) = 0. \end{cases} \tag{1.11}$$

Remark 1.2 Under (1.4)–(1.5), when $\mathfrak{m} = 0$ and $0 < \varepsilon < \eta$, (1.1)–(1.2) only has trivial solution $u_\varepsilon \equiv 0$. When $\mathfrak{m} \neq 0$, Theorem 1.1(i) shows that $|u_\varepsilon|$ attains its maximum value at boundary point $x = \ell$ and $|u_\varepsilon(\ell) - \mathfrak{m}| \leq \kappa_\varepsilon$, where κ_ε is a positive quantity tending to zero as $\varepsilon \downarrow 0$.

(ii) and (iii) of Theorem 1.1 mean that the asymptotic profiles of u_ε with $\varepsilon \downarrow 0$ exhibit boundary layers in the region $[\ell - \varepsilon^\tau, \ell]$ and become flat in $[0, \ell - \varepsilon^\tau]$, where $U(\frac{\ell - x}{\varepsilon})$ is the so-called *zeroth-order approximate boundary layer solution* to (1.1)–(1.2). A difference from other related works is that, for the boundary asymptotics of u_ε, we present a rigorous analysis (1.10)–(1.11) based on the maximum principle (see Section 4 for the proof).

When f satisfying (1.4) is linear, i.e., $f(s) = f'(0)s$, (1.11) has a unique solution $U(t) = \mathfrak{m}\mathrm{e}^{-\boldsymbol{q}t}$ with $\boldsymbol{q} = \frac{a(\ell)}{2} + \sqrt{\frac{a^2(\ell)}{4} + b(\ell) f'(0)} > 0$. This along with (1.10) implies $\max_{[\ell - \varepsilon^\tau, \ell]} \left| u_\varepsilon(x) - \mathfrak{m}\mathrm{e}^{-\frac{\boldsymbol{q}}{\varepsilon}(\ell - x)} \right| \xrightarrow{\varepsilon \downarrow 0} 0$. However, for the general nonlinear term f, the exact solution U cannot be obtained specifically. To understand basically the behavior of boundary layers of u_ε near $x = \ell$, it suffices to consider the linearized equation of (1.11) around $t = 0$ for an approximation scheme of u_ε near $x = \ell$:

$$-\widetilde{U}''(t) - a(\ell)\widetilde{U}'(t) + b(\ell) f'(\mathfrak{m}) \left(\widetilde{U}(t) - \mathfrak{m} + \frac{f(\mathfrak{m})}{f'(\mathfrak{m})} \right) = 0, \quad t > 0, \tag{1.12}$$

where $\widetilde{U}(0) = \mathfrak{m}$ and $\widetilde{U}'(t) \xrightarrow{t \to \infty} 0$ (in accordance with Theorem 1.1(ii)). It cannot be expected $\widetilde{U}(t) \xrightarrow{t \to \infty} 0$ since the linearization (1.12) is obtained from (1.11) around $t = 0$. More precisely, if f is nonlinear, by (1.4) and (1.12), there holds $\widetilde{U}(t) \xrightarrow{t \to \infty} \mathfrak{m} - \frac{f(\mathfrak{m})}{f'(\mathfrak{m})} > 0$ (resp., < 0) as $\mathfrak{m} > 0$ (resp., < 0). Consequently, when $\mathfrak{m} \neq 0$, a formal approximation of u_ε near $x = \ell$ is depicted as

$$u_\varepsilon(x) \approx \widetilde{U}(\frac{\ell - x}{\varepsilon}) = \mathfrak{m} - \frac{f(\mathfrak{m})}{f'(\mathfrak{m})} \left(1 - \mathrm{e}^{-\frac{\boldsymbol{P}}{\varepsilon}(\ell - x)} \right)$$

$$\text{with } \boldsymbol{P} = \frac{a(\ell)}{2} + \sqrt{\frac{a^2(\ell)}{4} + b(\ell) f'(\mathfrak{m})} > 0. \tag{1.13}$$

Rather, we shall emphasize again that such an approximation is not valid if x is far away from the boundary point ℓ.

Before closing this section, we return to Theorem 1.1, which verifies the uniqueness of equation (1.1)–(1.2) as $\varepsilon > 0$ is sufficiently small. To point out the importance of Theorem 1.1, we shall explain that equation (1.1)–(1.2) may have either infinitely many solution or no solution as $\varepsilon > 0$ is not small. A practical example is provided as follows.

Example 1.3 Consider the following equation:

$$\begin{cases} -\varepsilon^2 u_\varepsilon''(x) + u_\varepsilon(x) = 0 \quad \text{in} \quad (0, 1), \\ u_\varepsilon(0) = 0 \quad \text{and} \quad u_\varepsilon(1) = \mu + (\log 8) \int_0^1 u_\varepsilon(y)\, dy. \end{cases} \tag{1.14}$$

Then for $\varepsilon_* = \frac{1}{\log 2}$, we have that:

(i) When $0 < \varepsilon < \varepsilon_*$, (1.14) has a unique solution $u_\varepsilon(x) = c(\varepsilon, \mu)\left(e^{\frac{x}{\varepsilon}} - e^{-\frac{x}{\varepsilon}}\right)$, where

$$c(\varepsilon, \mu) = \frac{\mu}{2\left(1 - (\log 8)\varepsilon \tanh \frac{1}{2\varepsilon}\right) \sinh \frac{1}{\varepsilon}} \overset{0<\varepsilon\ll 1}{\approx} \mu e^{-\frac{1}{\varepsilon}}.$$

(ii) if $\varepsilon = \varepsilon_*$ and $\mu = 0$, then for any constant c_*, all $u_{\varepsilon_*}(x) = c_*\left(e^{\frac{x}{\varepsilon_*}} - e^{-\frac{x}{\varepsilon_*}}\right)$ are solutions of (1.14);
(iii) if $\varepsilon = \varepsilon_*$ and $\mu \neq 0$, then (1.14) has no solution.

This result can be obtained via simple calculations. We omit the detail here.

Outline and notation

The rest of this paper is structured as follows. In Section 2, we introduce the properties of $v_{\varepsilon,\lambda}$ (cf. (1.6)) and the uniqueness of $\mathcal{T}_\varepsilon(\lambda) = 0$ (cf. (1.7)) with respect to small $\varepsilon > 0$ (cf. Propositions 2.1–2.2) and point some difficulties for the uniqueness and asympotitcs of solutions to (1.1)–(1.2). We will give the proofs of Propositions 2.1–2.2 in Sections 3.1–3.2. In Section 4 we state the proof of Theorem 1.1. In our proofs, we will frequently abbreviate "$\leq C$" to "$\lesssim$", where $C > 0$ is a generic constant independent of parameters ε and λ. Finally, we make concluding remarks in Section 5.

2 Preliminaries and the main difficulty

To get useful properties of $\mathcal{T}_\varepsilon$ (defined by (1.7)) with respect to $0 < \varepsilon \ll 1$, we shall establish some required estimates of $v_{\varepsilon,\lambda}$.

Proposition 2.1 *Assume that $a, b \in C^1([0, \ell]; \mathbb{R})$, $p \in C([0, \ell]; \mathbb{R})$, $f \in C^2(\mathbb{R}; \mathbb{R})$ and $g \in C^1(\mathbb{R}^2; \mathbb{R})$ satisfying (1.4) and (1.5). Then for $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, (1.6) has a unique solution $v_{\varepsilon,\lambda} \in C^1([0, \ell]; \mathbb{R}) \cap C^3((0, \ell); \mathbb{R})$. Moreover, there hold the following properties:*

(i) *$\lambda = 0$ implies that $v_{\varepsilon,\lambda} \equiv 0$ is a trivial solution. Besides, if $\lambda > 0$ (< 0, respectively), then $0 \leq v_{\varepsilon,\lambda} \leq \lambda$ ($\lambda \leq v_{\varepsilon,\lambda} \leq 0$, respectively) and $v'_{\varepsilon,\lambda} > 0$ ($v'_{\varepsilon,\lambda} < 0$, respectively) on $[0, \ell]$.*
(ii) *There exist positive constants $\widetilde{M}$ and C_0 independent of ε and λ (cf. (3.5) and (3.19)) such that*

$$\begin{cases} |v_{\varepsilon,\lambda}(x)| \leq \sqrt{2}|\lambda| e^{-\frac{\widetilde{M}}{4\varepsilon}(\ell - x)}, \\ \varepsilon |v'_{\varepsilon,\lambda}(x)| \leq C_0 |\lambda| \left(1 + f'(|\lambda|)\right) e^{-\frac{\widetilde{M}}{8\varepsilon}(\ell - x)}, \quad \forall x \in [0, \ell]. \end{cases} \tag{2.1}$$

(iii) *For $\Lambda > 0$, there exists a positive constant L_Λ depending mainly on Λ (and independent of ε) such that for arbitrary $\varepsilon > 0$ and $\lambda_1, \lambda_2 \in [-\Lambda, \Lambda]$,*

$$\begin{cases} |(v_{\varepsilon,\lambda_1} - v_{\varepsilon,\lambda_2})(x)| \le \sqrt{2}|\lambda_1 - \lambda_2| \mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell - x)}, \\ \varepsilon|(v_{\varepsilon,\lambda_1} - v_{\varepsilon,\lambda_2})'(x)| \le L_\Lambda |\lambda_1 - \lambda_2| \mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell - x)}, \ \forall x \in [0, \ell], \end{cases} \tag{2.2}$$

where $v_{\varepsilon,\lambda_j}$ is the unique solution of (1.6) corresponding to $\lambda = \lambda_j$.

For $\eta > 0$ defined in Theorem 1.1, we will use (2.2) to verify $\sup_{\varepsilon\in(0,\eta)} |\mathcal{T}_\varepsilon(\lambda_1) - \mathcal{T}_\varepsilon(\lambda_2)| \le \widetilde{L}_{\Lambda,\eta}|\lambda_1 - \lambda_2|$ for some positive constant $\widetilde{L}_{\Lambda,\eta}$ depending on Λ and η (cf. (3.24)), which particularly implies the uniform continuity of $\mathcal{T}_\varepsilon$ in $[-\Lambda, \Lambda]$. The other important point is the uniqueness property of $\mathcal{T}_\varepsilon(\lambda) = 0$ with respect to sufficiently small $\varepsilon > 0$. This is established in the next result.

Proposition 2.2 *Under the same assumptions as in Proposition 2.1, for each $\mu \in \mathbb{R}$, there exists a positive constant $\eta^* := \eta^*(\mu)$ depending mainly on μ such that for each $\varepsilon \in (0, \eta^*)$, $\mathcal{T}_\varepsilon$ defined by (1.7) has a unique root $\lambda := \lambda_\varepsilon$ in $\mathbb{R}$, which satisfies*

$$\lambda_\varepsilon \xrightarrow{\varepsilon\downarrow 0} \mathfrak{m}, \tag{2.3}$$

where $\mathfrak{m}$ is defined by (1.8).

Let us point out some difficulties in the proof of Propositions 2.1–2.2. Since $b(x)$ is positive and (1.4)–(1.5) hold, we can apply directly the comparison theorem of second order elliptic equations to establishing the estimate of $v_{\varepsilon,\lambda}$ in (2.1). However, following the similar argument, we can only establish a second order differential inequality of $v'^2_{\varepsilon,\lambda}$ (see (3.10)):

$$\varepsilon^2(v'^2_{\varepsilon,\lambda})''(x) \ge \mathrm{C}^*_{1,\varepsilon} v'^2_{\varepsilon,\lambda}(x) - \mathrm{C}^*_{2,\lambda} v^2_{\varepsilon,\lambda}(x) \quad \text{in } (0, \ell),$$

where $\mathrm{C}^*_{1,\varepsilon}$ is a positive constant depending mainly on ε and $\mathrm{C}^*_{2,\lambda}$ is a positive constant depending mainly on λ. Even if we have a good estimate of $v_{\varepsilon,\lambda}$ in (2.1), we do not have enough information to verify whether $v'^2_{\varepsilon,\lambda}$ can be bounded by $\mathrm{C}^*_{1,\varepsilon} v'^2_{\varepsilon,\lambda}(x) - \mathrm{C}^*_{2,\lambda} v^2_{\varepsilon,\lambda}(x)$ or not. Hence, the standard comparison theorem of elliptic equation may not be applied to dealing with this differential inequality. To establish the desired estimate of $v'_{\varepsilon,\lambda}$, we consider a linear combination of $v'^2_{\varepsilon,\lambda}$ and $v^2_{\varepsilon,\lambda}$. More precisely, we prove that there exist positive constants ξ and $\widetilde{M}$ independent of ε such that $\widetilde{v}_{\varepsilon,\lambda,\xi} := v'^2_{\varepsilon,\lambda} + \xi v^2_{\varepsilon,\lambda}$ satisfies $\max_{[0,\ell]} \widetilde{v}_{\varepsilon,\lambda,\xi} = \max\{\widetilde{v}_{\varepsilon,\lambda,\xi}(0), \widetilde{v}_{\varepsilon,\lambda,\xi}(\ell)\}$ and $\frac{\widetilde{v}_{\varepsilon,\lambda,\xi}(x)}{\max_{[0,\ell]} \widetilde{v}_{\varepsilon,\lambda,\xi}} \le \mathrm{e}^{-\frac{\widetilde{M}}{2\varepsilon}x} + \mathrm{e}^{-\frac{\widetilde{M}}{2\varepsilon}(\ell - x)}$ pointwise in $[0, \ell]$ (cf. (3.13)–(3.14)). As a consequence, we can arrive at the estimate of $v'_{\varepsilon,\lambda}$ in (2.1). To the best of our knowledge, such an approach for the interior estimate (2.1) seems not to appear in the related works.

To analyze the uniqueness of the equation $\mathcal{T}_\varepsilon(\lambda) = 0$ (see (1.7)) as $\varepsilon > 0$ is small, we sufficiently apply the interior estimate (2.1) to investigating the property of $\mathcal{T}_\varepsilon$. The main step for this technique is to establish an estimate $|\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}| \le \widetilde{C}|\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}| \mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}\min\{\ell_0, \ell - \ell_1\}}$ for $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ satisfying $\mathcal{T}_\varepsilon(\lambda_{1,\varepsilon}) = \mathcal{T}_\varepsilon(\lambda_{2,\varepsilon}) = 0$ (cf. (3.30)), where $\widetilde{C}$ and $\widetilde{M}$ are positive constants independent of ε and $\lambda_{j,\varepsilon}$'s.

These ideas will be presented in detail in subsequent proofs.

3 Proof of Propositions 2.1 and 2.2

3.1 Proof of Proposition 2.1

The proof for the existence of $v_{\varepsilon,\lambda}$ to the equation (1.6) is standard. Indeed, by (1.4) we note that $\underline{v}(x) := \min\{0, \lambda\}$ is a sub-solution of (1.6), while $\overline{v}(x) := \max\{0, \lambda\}$ admits a super-solution. Furthermore, by the condition $f \in \mathrm{C}^2(\mathbb{R}; \mathbb{R})$ we know that f' is uniformly bounded in any bounded domain. Hence, for each $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, following directly the argument of sub-solutions and super-solutions (see, e.g., Evans's textbook [15, Section 9.3]), we verify the existence of classical solution $v_{\varepsilon,\lambda}$ of (1.6) and

$$\min\{0, \lambda\} \le v_{\varepsilon,\lambda}(x) \le \max\{0, \lambda\}, \quad \forall x \in [0, \ell]. \tag{3.1}$$

It is worth mentioning that the existence of classical solutions to (1.6) can also be obtained via the direct method in the calculus of variations and the regularity theory for elliptic equations[1].

We next claim the uniqueness of (1.6) for each $\varepsilon > 0$. Note first that by (1.4)–(1.5), there exists a positive constant β such that

$$b(x) \ge \beta, \quad \forall x \in [0, \ell]. \tag{3.2}$$

Note also that $a,\ b \in \mathrm{C}^1([0, \ell]; \mathbb{R})$. Thus, for $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, by (1.4) and (3.2), applying the standard monotone iteration scheme one immediately obtains the existence of classical solutions $v_{\varepsilon,\lambda} \in \mathrm{C}^1([0, \ell]; \mathbb{R}) \cap \mathrm{C}^3((0, \ell); \mathbb{R})$ to (1.6).

Suppose by contradiction that (1.6) with a fixed $\varepsilon > 0$ has at least two solutions v_1 and v_2 satisfying $V := v_1 - v_2 \not\equiv 0$ in $[0, \ell]$. One obtains $-\varepsilon^2 V''(x) + \varepsilon a(x) V'(x) + b(x)[f(v_1(x)) - f(v_2(x))] = 0$ in $(0, \ell)$, and $V(0) = V(l) = 0$. This implies that V attains its positive maximum or negative minimum values at interior point. Without loss of generality, we assume that V attains its positive maximum value at $x_M \in (0, \ell)$. Then $V''(x_M) \le 0 = V'(x_M)$ and $v_1(x_M) > v_2(x_M)$. Along with (1.4), we have $-\varepsilon^2 V''(x_M) + \varepsilon a(x_M) V'(x_M) + b(x_M)[f(v_1(x_M)) - f(v_2(x_M))] > 0$ in $(0, \ell)$, a contradiction. Hence, the maximum value of V occurs at boundary points. Similarly, we can prove that the minimum value of V occurs at boundary points. As a consequence, from boundary data $V(0) = V(l) = 0$ we obtain $V \equiv 0$ on $[0, \ell]$, which also leads to a contradiction. This proves the uniqueness of (1.6). In particular, we re-obtain (3.1) and $v_{\varepsilon,\lambda}(0) \le 0 \le v_{\varepsilon,\lambda}(\ell)$ if $\lambda \ge 0$; $v_{\varepsilon,\lambda}(0) \ge 0 \ge v_{\varepsilon,\lambda}(\ell)$ if $\lambda < 0$.

To prove (i), it suffices to focus on the case $\lambda > 0$ since $\lambda = 0$ indicates $v_{\varepsilon,\lambda} \equiv 0$ on $[0, \ell]$. By (1.4), (3.2) and (3.1) with $\lambda > 0$ we have $b(x) f(v_{\varepsilon,\lambda}(x)) \ge 0$. Hence, one may check from (1.6) that

$$\varepsilon^2 \left(v'_{\varepsilon,\lambda}(x) \mathrm{e}^{\int_x^\ell \frac{a(y)}{\varepsilon}\,\mathrm{d}y} \right)' = b(x) f(v_{\varepsilon,\lambda}(x)) \mathrm{e}^{\int_x^\ell \frac{a(y)}{\varepsilon}\,\mathrm{d}y} \ge 0, \ \ x \in [0, \ell]. \tag{3.3}$$

Thus, $v'_{\varepsilon,\lambda}(x) \mathrm{e}^{\int_x^\ell \frac{a(y)}{\varepsilon}\,\mathrm{d}y}$ is increasing on $(0, \ell)$ and attains the minimum value at $x = 0$. We shall stress that $v'_{\varepsilon,\lambda}(0) > 0$ because if $v'_{\varepsilon,\lambda}(0) = 0$, we can replace the boundary

[1] Note that the equation of $v_{\varepsilon,\lambda}$ is equivalent to the equation $-\varepsilon^2 (D_\varepsilon(x) v'_{\varepsilon,\lambda}(x))' + D_\varepsilon(x) b(x) f(v_{\varepsilon,\lambda}) = 0$, $x \in (0, \ell)$, with the corresponding energy functional $E_\varepsilon[v_{\varepsilon,\lambda}] = \int_0^\ell D_\varepsilon \left(\frac{\varepsilon^2}{2} v'^2_{\varepsilon,\lambda} + bF(v_{\varepsilon,\lambda}) \right) \mathrm{d}x$, where $D_\varepsilon(x) := \exp\left(-\int_0^x \frac{a(y)}{\varepsilon}\,\mathrm{d}y \right)$ is positive on $[0, \ell]$, and $F(t) = \int_0^t f(s)\,\mathrm{d}s \ge 0$ is convex (by (1.4)). Since $b > 0$, for each fixed $\varepsilon > 0$ and $\lambda \in \mathbb{R}$, by the direct method in the calculus of variations one obtains that $E_\varepsilon[v_{\varepsilon,\lambda}]$ is a convex functional and has a minimizer over the space $\mathcal{H} = \{v_{\varepsilon,\lambda} \in \mathrm{H}^1((0, \ell)) : v_{\varepsilon,\lambda}(0) = 0,\ v_{\varepsilon,\lambda}(\ell) = \lambda\}$. As a consequence, the regularity theory for elliptic equations implies that this minimizer is a classical solution of (1.6).

condition $v_{\varepsilon,\lambda}(\ell) = \lambda$ with $v'_{\varepsilon,\lambda}(0) = 0$ in (1.6). Along with the uniqueness of (1.6), we obtain $v_{\varepsilon,\lambda} \equiv 0$ on $[0, \ell]$, a contradiction. Consequently, by (3.3) we have $v'_{\varepsilon,\lambda}(x)\mathrm{e}^{\int_x^\ell \frac{a(y)}{\varepsilon}\,\mathrm{d}y} \geq v'_{\varepsilon,\lambda}(0)\mathrm{e}^{\int_0^\ell \frac{a(y)}{\varepsilon}\,\mathrm{d}y} > 0$. Hence, $0 \leq v_{\varepsilon,\lambda}(x) \leq \lambda$ and $v'_{\varepsilon,\lambda}(x) > 0$ for the case $\lambda > 0$. Similarly, we can prove $\lambda \leq v_{\varepsilon,\lambda}(x) \leq 0$ and $v'_{\varepsilon,\lambda}(x) < 0$ for the case $\lambda < 0$. This completes the proof of (i).

We will prove (2.1) focusing on the case $\lambda > 0$ since the same argument can be applied to the case $\lambda < 0$. Multiplying the equation (1.6) by $v_{\varepsilon,\lambda}$ and using (1.5), one may check that

$$\begin{aligned} \varepsilon^2(v^2_{\varepsilon,\lambda})''(x) &= 2\varepsilon^2\left(v'^2_{\varepsilon,\lambda}(x) + v_{\varepsilon,\lambda}(x)v''_{\varepsilon,\lambda}(x)\right) \\ &= 2\varepsilon^2 v'^2_{\varepsilon,\lambda}(x) + 2\varepsilon a(x)v_{\varepsilon,\lambda}(x)v'_{\varepsilon,\lambda}(x) + 2b(x)v_{\varepsilon,\lambda}(x)f(v_{\varepsilon,\lambda}(x)) \\ &\geq \left(-\frac{a^2(x)}{2} + 2b(x)f'(0)\right)v^2_{\varepsilon,\lambda}(x) \\ &\geq \widetilde{M}^2 v^2_{\varepsilon,\lambda}(x), \end{aligned} \tag{3.4}$$

where

$$\widetilde{M} = \min_{[0,\ell]}\left(-\frac{a^2(x)}{2} + 2b(x)f'(0)\right)^{\frac{1}{2}} > 0 \quad (\text{cf. } (1.5)). \tag{3.5}$$

Here we have used (1.4) to obtain $v_{\varepsilon,\lambda}f(v_{\varepsilon,\lambda}) \geq f'(0)v^2_{\varepsilon,\lambda}$. The differential inequality (3.4) is an important form so that we can use the comparison theorem of the second order ordinary differential equations. Our goal is to verify

$$0 \leq v_{\varepsilon,\lambda}(x) \leq \sqrt{2}\lambda\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell-x)}, \quad \forall x \in [0, \ell]. \tag{3.6}$$

The proof is stated as follows.

Proof of (3.6) Firstly, applying the comparison theorem to (3.4) and using the boundary conditions of $v_{\varepsilon,\lambda}$, we have

$$\begin{aligned} v^2_{\varepsilon,\lambda}(x) &\leq \max\{v^2_{\varepsilon,\lambda}(0), v^2_{\varepsilon,\lambda}(\ell)\}\left(\mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}x} + \mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}(\ell-x)}\right) \\ &= \lambda^2\left(\mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}x} + \mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}(\ell-x)}\right). \end{aligned} \tag{3.7}$$

We shall further improve this estimate as follows. Recall $v'_{\varepsilon,\lambda} > 0$ on $(0, \ell)$. We have $v_{\varepsilon,\lambda}(x) \leq v_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right)$, $\forall x \in [0, \frac{\ell}{2}]$. This along with (3.7) implies

$$v^2_{\varepsilon,\lambda}(x) \leq 2\lambda^2\mathrm{e}^{-\frac{\widetilde{M}\ell}{2\varepsilon}} \leq 2\lambda^2\mathrm{e}^{-\frac{\widetilde{M}}{2\varepsilon}(\ell-x)}, \quad \forall x \in \left[0, \frac{\ell}{2}\right]. \tag{3.8}$$

On the other hand, by (3.7), we have

$$v^2_{\varepsilon,\lambda}(x) \leq \lambda^2\left(\mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}x} + \mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}(\ell-x)}\right) \leq 2\lambda^2\mathrm{e}^{-\frac{\widetilde{M}}{\varepsilon}(\ell-x)}, \quad \forall x \in \left[\frac{\ell}{2}, \ell\right]. \tag{3.9}$$

Combining (3.8) with (3.9), we arrive at (3.6) for the case $\lambda > 0$.

Next we deal with the gradient estimate of $v_{\varepsilon,\lambda}$. Differentiating (1.6) to x and multiplying the expansion by $v'_{\varepsilon,\lambda}$, we have

$$\begin{aligned} \varepsilon^2 v'''_{\varepsilon,\lambda}(x)v'_{\varepsilon,\lambda}(x) =& \varepsilon a(x)v'_{\varepsilon,\lambda}(x)v''_{\varepsilon,\lambda}(x) \\ &+ \left(\varepsilon a'(x) + b(x)f'(v_{\varepsilon,\lambda}(x))v'^2_{\varepsilon,\lambda}(x) + b'(x)f(v_{\varepsilon,\lambda}(x))\right)v'_{\varepsilon,\lambda}(x), \end{aligned}$$

together with the identity $\left(v_{\varepsilon,\lambda}'^2(x)\right)'' = 2\left(v_{\varepsilon,\lambda}''^2(x) + v_{\varepsilon,\lambda}'''(x)v_{\varepsilon,\lambda}'(x)\right)$, we obtain

$$\begin{aligned}\frac{\varepsilon^2}{2}\left(v_{\varepsilon,\lambda}'^2(x)\right)'' &= \varepsilon^2 v_{\varepsilon,\lambda}''^2(x) + \varepsilon a(x)v_{\varepsilon,\lambda}'(x)v_{\varepsilon,\lambda}''(x)\\ &\quad + \left(\varepsilon a'(x) + b(x)f'(v_{\varepsilon,\lambda}(x))v_{\varepsilon,\lambda}'^2(x) + b'(x)f(v_{\varepsilon,\lambda}(x))\right)v_{\varepsilon,\lambda}'(x).\end{aligned}$$

As for (3.6), we shall establish a second-order differential inequality for $v_{\varepsilon,\lambda}'$. Firstly, by the elementary inequality $\varepsilon^2 v_{\varepsilon,\lambda}''^2(x) + \varepsilon a(x)v_{\varepsilon,\lambda}'(x)v_{\varepsilon,\lambda}''(x) \geq -\frac{a^2(x)}{4}v_{\varepsilon,\lambda}'^2(x)$ one may check that

$$\begin{aligned}\frac{\varepsilon^2}{2}\left(v_{\varepsilon,\lambda}'^2(x)\right)'' &\geq \left(\varepsilon a'(x) - \frac{a^2(x)}{4} + b(x)f'(v_{\varepsilon,\lambda}(x))\right)v_{\varepsilon,\lambda}'^2(x)\\ &\quad + b'(x)f(v_{\varepsilon,\lambda}(x))v_{\varepsilon,\lambda}'(x)\\ &\geq \left(\varepsilon a'(x) + \frac{\widetilde{M}^2}{2} - \gamma\right)v_{\varepsilon,\lambda}'^2(x) - \frac{1}{4\gamma}\left(f'(\lambda)\max_{[0,\ell]}|b'|\right)^2 v_{\varepsilon,\lambda}^2(x), \quad (3.10)\end{aligned}$$

where $\widetilde{M} > 0$ has been defined in (3.4) and $\gamma > 0$ will be determined later on. The last estimate of (3.10) is obtained since (1.4) and (3.1) with $\lambda > 0$ imply $f'(0) \leq f'(v_{\varepsilon,\lambda}(x)) \leq f'(\lambda)$ and

$$\begin{aligned}|b'(x)f(v_{\varepsilon,\lambda}(x))v_{\varepsilon,\lambda}'(x)| &\leq \left(f'(\lambda)\max_{[0,\ell]}|b'|\right)v_{\varepsilon,\lambda}(x)v_{\varepsilon,\lambda}'(x)\\ &\leq \frac{1}{4\gamma}\left(f'(\lambda)\max_{[0,\ell]}|b'|\right)^2 v_{\varepsilon,\lambda}^2(x) + \gamma v_{\varepsilon,\lambda}'^2(x).\end{aligned}$$

Taking a closer looking at (3.10), we find that the last term in the last line would give a difficulty in estimating $v_{\varepsilon,\lambda}'$, which is not the case of (3.4). An important idea is to establish a new differential inequality for a linear combination of $(v_{\varepsilon,\lambda}'^2)''$ and $(v_{\varepsilon,\lambda}^2)''$ with a suitable coefficient γ. More precisely, since a' is uniformly bounded on $[0,\ell]$, for any $\xi > 0$, by (3.4) and (3.10), as $\varepsilon > 0$ is sufficiently small, we have

$$\begin{aligned}&\varepsilon^2\left(v_{\varepsilon,\lambda}'^2(x) + \xi v_{\varepsilon,\lambda}^2(x)\right)''\\ &\geq \left(2\varepsilon a'(x) + \widetilde{M}^2 - 2\gamma\right)v_{\varepsilon,\lambda}'^2(x) + \left[\xi\widetilde{M}^2 - \frac{1}{2\gamma}\left(f'(\lambda)\max_{[0,\ell]}|b'|\right)^2\right]v_{\varepsilon,\lambda}^2(x)\\ &\geq \left(\frac{\widetilde{M}^2}{2} - 2\gamma\right)v_{\varepsilon,\lambda}'^2(x) + \left[\xi\widetilde{M}^2 - \frac{1}{2\gamma}\left(f'(\lambda)\max_{[0,\ell]}|b'|\right)^2\right]v_{\varepsilon,\lambda}^2(x).\end{aligned}$$

Since $\gamma > 0$ is arbitrary, in order to ensure the coefficients of its right-hand side are positive constants independent of ε, we can set $\frac{\widetilde{M}^2}{2} - 2\gamma = \frac{\widetilde{M}^2}{4}$ and $\xi\widetilde{M}^2 - \frac{1}{2\gamma}\left(f'(\lambda)\max_{[0,\ell]}|b'|\right)^2 = \frac{\xi\widetilde{M}^2}{4}$ implying

$$\gamma = \frac{\widetilde{M}^2}{8} \quad \text{and} \quad \xi = \frac{16}{3\widetilde{M}^4}\left(f'(\lambda)\max_{[0,\ell]}|b'|\right)^2. \quad (3.11)$$

Then we arrive at the following second-order differential inequality for $v_{\varepsilon,\lambda}'^2 + \xi v_{\varepsilon,\lambda}^2$ with sufficiently small $\varepsilon > 0$:

$$\varepsilon^2\left(v_{\varepsilon,\lambda}'^2(x) + \xi v_{\varepsilon,\lambda}^2(x)\right)'' \geq \frac{\widetilde{M}^2}{4}\left(v_{\varepsilon,\lambda}'^2(x) + \xi v_{\varepsilon,\lambda}^2(x)\right), \quad \forall x \in (0,\ell). \quad (3.12)$$

After applying the comparison theorem to (3.12), one yields

$$v_{\varepsilon,\lambda}'^2(x) + \xi v_{\varepsilon,\lambda}^2(x) \le A_{v_{\varepsilon,\lambda}} \left(e^{-\frac{\widetilde{M}}{2\varepsilon}x} + e^{-\frac{\widetilde{M}}{2\varepsilon}(\ell - x)} \right), \quad \forall x \in [0, \ell], \tag{3.13}$$

where

$$A_{v_{\varepsilon,\lambda}} = \max_{[0,\ell]} \left(v_{\varepsilon,\lambda}'^2 + \xi v_{\varepsilon,\lambda}^2 \right) \le \max_{[0,\ell]} v_{\varepsilon,\lambda}'^2 + \frac{16}{3\widetilde{M}^4} \left(\lambda f'(\lambda) \max_{[0,\ell]} |b'| \right)^2. \tag{3.14}$$

Next we shall estimate $A_{v_{\varepsilon,\lambda}}$ with respect to ε and λ. By (1.6), (1.4)–(1.5) and (3.1) with $\lambda > 0$, we have

$$-\varepsilon v_{\varepsilon,\lambda}'(x) \max_{[0,\ell]} |a| \le \varepsilon^2 v_{\varepsilon,\lambda}''(x) \le \varepsilon v_{\varepsilon,\lambda}'(x) \max_{[0,\ell]} |a| + v_{\varepsilon,\lambda}(x) f'(\lambda) \max_{[0,\ell]} b, \quad x \in (0, \ell). \tag{3.15}$$

Here we have used the fact $v_{\varepsilon,\lambda}' > 0$ on $[0, \ell]$. On the other hand, by the mean value theorem, there exists an interior point $x_\varepsilon \in (0, \ell)$ such that $v_{\varepsilon,\lambda}'(x_\varepsilon) = \frac{\lambda}{\ell}$. For $0 \le x_- < x_\varepsilon < x_+ \le \ell$, by integrating (3.15) over (x_-, x_ε) and (x_ε, x_+) and using (3.6), we arrive at

$$0 < \varepsilon v_{\varepsilon,\lambda}'(x_+) \le \lambda \left(\max_{[0,\ell]} |a| + \frac{4\sqrt{2}}{\widetilde{M}} f'(\lambda) \max_{[0,\ell]} b + \frac{\varepsilon}{\ell} \right), \tag{3.16}$$

$$0 < \varepsilon v_{\varepsilon,\lambda}'(x_-) \le \lambda \left(\max_{[0,\ell]} |a| + \frac{\varepsilon}{\ell} \right). \tag{3.17}$$

By (3.11), (3.14) and (3.16)–(3.17) we have the estimate

$$\sqrt{A_{v_{\varepsilon,\lambda}}} \le \frac{\lambda}{\varepsilon} \left(\max_{[0,\ell]} |a| + \frac{4\sqrt{2}}{\widetilde{M}} f'(\lambda) \max_{[0,\ell]} b \right) + \lambda \left(\frac{1}{\ell} + \frac{4}{\sqrt{3}\widetilde{M}^2} f'(\lambda) \max_{[0,\ell]} |b'| \right).$$

This along with (3.13) immediately implies that, as $0 < \varepsilon \ll 1$,

$$\begin{aligned} 0 < v_{\varepsilon,\lambda}'(x) &\le \sqrt{A_{v_{\varepsilon,\lambda}}} \left(e^{-\frac{\widetilde{M}}{4\varepsilon}x} + e^{-\frac{\widetilde{M}}{4\varepsilon}(\ell - x)} \right) \\ &\le \frac{\lambda}{\varepsilon} \left(C_1 + C_2 f'(\lambda) \right) \left(e^{-\frac{\widetilde{M}}{4\varepsilon}x} + e^{-\frac{\widetilde{M}}{4\varepsilon}(\ell - x)} \right), \quad \forall x \in [0, \ell], \end{aligned} \tag{3.18}$$

where C_i's can be chosen by

$$C_1 = 2 \max_{[0,\ell]} |a|, \quad C_2 = \frac{8}{\widetilde{M}} \max_{[0,\ell]} b. \tag{3.19}$$

In particular, (3.18) implies

$$0 < v_{\varepsilon,\lambda}'(x) \le \frac{2\lambda}{\varepsilon} \left(C_1 + C_2 f'(\lambda) \right) e^{-\frac{\widetilde{M}}{4\varepsilon}(\ell - x)}, \quad \forall x \in \left[\frac{\ell}{2}, \ell \right]. \tag{3.20}$$

On the other hand, for $z \in [0, \frac{\ell}{2}]$, by integrating the equation in (1.6) over the interval $[z, \frac{\ell}{2}]$ and using (1.4) and property $v_{\varepsilon,\lambda},\ v_{\varepsilon,\lambda}' \ge 0$ (since we assume $\lambda \ge 0$), one gets an estimate for $v_{\varepsilon,\lambda}'(z)$:

$$0 \le v'_{\varepsilon,\lambda}(z) = v'_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right) - \frac{1}{\varepsilon^2}\int_z^{\frac{\ell}{2}} \left(\varepsilon a(x)v'_{\varepsilon,\lambda}(x) + b(x)f(v_{\varepsilon,\lambda}(x))\right) \mathrm{d}x$$
$$\le v'_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right) + \frac{1}{\varepsilon}v_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right)\max_{\left[0,\frac{\ell}{2}\right]}|a| \quad \text{(since } v_{\varepsilon,\lambda}(z) \ge 0 \text{ and } f(v_{\varepsilon,\lambda}(x)) \ge 0). \tag{3.21}$$

Recall (3.6) and (3.18). Hence, we have the estimates $0 \le v_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right) \le \sqrt{2}\lambda \mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}\ell}$ and $0 \le v'_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right) \le \frac{2\lambda}{\varepsilon}\left(C_1 + C_2 f'(\lambda)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}\ell}$, together with (3.21) we arrive at

$$0 \le v'_{\varepsilon,\lambda}(z) \le v'_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right) + \frac{1}{\varepsilon}v_{\varepsilon,\lambda}\left(\frac{\ell}{2}\right)\max_{[0,\frac{\ell}{2}]}|a| \le \frac{\lambda}{\varepsilon}\left(\widetilde{C}_1 + \widetilde{C}_2 f'(\lambda)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}\ell}$$
$$\le \frac{\lambda}{\varepsilon}\left(\widetilde{C}_1 + \widetilde{C}_2 f'(\lambda)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell - z)}, \qquad \forall z \in \left[0, \frac{\ell}{2}\right]. \tag{3.22}$$

By (3.6), (3.20) and (3.22), we thus set $C_0 = \max\{C_1, C_2, \widetilde{C}_1, \widetilde{C}_2\}$ and arrive at (2.1) of the case $\lambda \ge 0$. When $\lambda < 0$, we can follow the same argument to obtain (2.1).

It remains to prove (2.2). Without loss of generality, we may assume $\lambda_1, \lambda_2 \in [-\Lambda, \Lambda]$ satisfying $\lambda_1 > \lambda_2$. Then by (1.4), we can apply the comparison theorem to (1.6) and obtain

$$v_{\varepsilon,\lambda_1} \ge v_{\varepsilon,\lambda_2} \quad \text{on } [0.\ell].$$

Setting $\Psi_\varepsilon := v_{\varepsilon,\lambda_1} - v_{\varepsilon,\lambda_2}$, we have

$$\begin{cases} -\varepsilon^2\Psi''_\varepsilon(x) + \varepsilon a(x)\Psi'_\varepsilon(x) + b(x)f'(\Theta_\varepsilon(x)))\Psi_\varepsilon(x) = 0 \quad \text{in} \quad (0, \ell), \\ \Psi_\varepsilon(0) = 0, \quad \Psi_\varepsilon(\ell) = \lambda_1 - \lambda_2 > 0, \end{cases} \tag{3.23}$$

where $\Theta_\varepsilon(x)$ lies between $v_{\varepsilon,\lambda_1}(x)$ and $v_{\varepsilon,\lambda_2}(x)$ so we have $\max_{[0,\ell]}|\Theta_\varepsilon| \le \max\{|\lambda_1|, |\lambda_2|\}$. Furthermore, both (1.4) and (1.5) imply that $f'(\Theta_\varepsilon(x))$ is positive on $[0, \ell]$, and

$$a^2(x) < 4b(x)f'(0) \le 4b(x)f'(\Theta_\varepsilon(x)).$$

For (3.23), we rewrite the equation as the form $-\varepsilon^2\Phi''_\varepsilon(x) + \varepsilon a(x)\Phi'_\varepsilon(x) + \widetilde{b}(x)\widetilde{f}(\Phi_\varepsilon) = 0$, where $\widetilde{b}(x) := b(x)f'(\Theta_\varepsilon(x))$ and $\widetilde{f}(t) = t$. As a consequence, although (3.23) is linear, $a(x)$ and $\widetilde{b}(x)$ still satisfy assumption (1.5) corresponding to $b = \widetilde{b}$ since $\widetilde{f}'(0) = 1$.

Hence, an argument similar to (2.1) gives $|\Psi_\varepsilon(x)| \le \sqrt{2}|\lambda_1 - \lambda_2|\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell - x)}$ and $|\Psi'_\varepsilon(x)| \le \frac{C_0}{\varepsilon}|\lambda_1 - \lambda_2|\left(1 + f'(2\Lambda)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell - x)}$, $\forall x \in [0, \ell]$. Here we have used (1.4) to verify the relation $f'(2\Lambda) \ge f'(\lambda_1 - \lambda_2) \ge f'(0) > 0$. Along with $\Psi_\varepsilon = v_{\varepsilon,\lambda_1} - v_{\varepsilon,\lambda_2}$, we arrive at (2.2) with $L_\Lambda = C_0(1 + f'(2\Lambda))$ and complete the proof of Proposition 2.1.

3.2 Proof of Proposition 2.2

Let $\widetilde{\eta} > 0$ and $\Lambda > 0$ be fixed. Then for $\varepsilon \in (0, \widetilde{\eta})$, and $\lambda_1, \lambda_2 \in [-\Lambda, \Lambda]$, by (2.2) we have

$$|\mathcal{T}_\varepsilon(\lambda_1) - \mathcal{T}_\varepsilon(\lambda_2)| \le |\lambda_1 - \lambda_2| + \max_{[0,\ell]}|p| \max_{[0,\delta(\widetilde{\eta},\Lambda)]^2}|\nabla g|$$
$$\times \int_{\ell_0}^{\ell_1}\left(|v_{\varepsilon,\lambda_1}(y) - v_{\varepsilon,\lambda_2}(y)| + |v'_{\varepsilon,\lambda_1}(y)) - v'_{\varepsilon,\lambda_2}(y))|\right)\mathrm{d}y$$
$$\le |\lambda_1 - \lambda_2|\left[1 + \delta(\widetilde{\eta}, \Lambda)(\ell_1 - \ell_0)\max_{[0,\ell]}|p| \max_{[]0,\delta(\widetilde{\eta})]}|\nabla g|\right],$$

where $\delta(\widetilde{\eta}, \Lambda) = \sup_{\varepsilon\in(0,\widetilde{\eta})} \left(\sqrt{2}\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell-\ell_1)} + \frac{L_\Lambda}{\varepsilon}\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}\right) < \infty$ since $\ell_1 < \ell$. As a consequence, there is a positive constant $\widetilde{L}(\widetilde{\eta}, \Lambda)$ depending mainly on $\widetilde{\eta}$ and Λ such that

$$\sup_{\varepsilon(0,\widetilde{\eta})} |\mathcal{T}_\varepsilon(\lambda_1) - \mathcal{T}_\varepsilon(\lambda_2)| \le \widetilde{L}(\widetilde{\eta}, \Lambda)|\lambda_1 - \lambda_2|, \quad \forall\, \lambda_1, \lambda_2 \in [-\Lambda, \Lambda]. \tag{3.24}$$

Next we prove that there exists $\eta_0 \in (0, \widetilde{\eta})$ such that for each $\varepsilon \in (0, \eta_0)$, $\mathcal{T}_\varepsilon(\lambda) = 0$ has at least one root. Since $[\ell_0, \ell_1] \Subset (0, \ell)$, by (2.1) we have

$$|v_{\varepsilon,\lambda}(x)| \le \sqrt{2}|\lambda|\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell-\ell_1)},$$
$$|v'_{\varepsilon,\lambda}(x)| \le \frac{C_0|\lambda|}{\varepsilon}\left(1 + f'(|\lambda|)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}, \quad \forall\, x \in [\ell_0, \ell_1].$$

Hence, for each $\delta^* > 0$, as $0 < \varepsilon \ll 1$, there holds

$$\begin{aligned}
&\left|\int_{\ell_0}^{\ell_1} p(y)g(v_{\varepsilon,\lambda}(y), v'_{\varepsilon,\lambda}(y))\,\mathrm{d}y - g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y\right| \\
&\qquad \le \max_{[\ell_0,\ell_1]} |p| \max_{[0,\delta^*]^2} |\nabla g| \int_{\ell_0}^{\ell_1} \left(|v_{\varepsilon,\lambda}(y)| + |v'_{\varepsilon,\lambda}(y)|\right)\mathrm{d}y \\
&\qquad \lesssim |\lambda|\left(1 + f'(|\lambda|)\right)\max\left\{\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell-\ell_1)}, \frac{1}{\varepsilon}\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}\right\} \\
&\qquad \lesssim \frac{|\lambda|}{\varepsilon}\left(1 + f'(|\lambda|)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}
\end{aligned} \tag{3.25}$$

since $\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell-\ell_1)} \ll \frac{1}{\varepsilon}\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}$ as $0 < \varepsilon \ll 1$. Thus, by (1.7) and (3.25), one obtains

$$\left|\mathcal{T}_\varepsilon(\lambda) + \lambda - \mu - g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y\right| \lesssim \frac{|\lambda|}{\varepsilon}\left(1 + f'(|\lambda|)\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}. \tag{3.26}$$

Note that, for any finite number λ, the right-hand side of (3.26) approaches zero as $\varepsilon \downarrow 0$. In particular, choosing a number $\Lambda > \max\{|\underline{r}|, |\overline{r}|\}$ with

$$\underline{r} = -1 + \mu + g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y, \quad \overline{r} = 1 + \mu + g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y,$$

by (3.24) and (3.26) we can pick up a number $\eta_0 \in (0, \widetilde{\eta})$ such that for each $\varepsilon \in (0, \eta_0)$, $\mathcal{T}_\varepsilon$ is continuous on $[-\Lambda, \Lambda]$, and $\mathcal{T}_\varepsilon(\overline{r}) < 0 < \mathcal{T}_\varepsilon(\underline{r})$. Hence, there exists $\lambda_\varepsilon \in (\underline{r}, \overline{r})$ such that $\mathcal{T}_\varepsilon(\lambda_\varepsilon) = 0$. Furthermore, we have

$$\lim_{\varepsilon\downarrow 0} \mathcal{T}_\varepsilon(\mu + g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y) = 0 \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \lambda_\varepsilon = \mu + g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y. \tag{3.27}$$

(3.27) shows the existence of λ_ε satisfying (2.3) and $|\lambda_\varepsilon| \le \mathfrak{M}$ as $0 < \varepsilon \ll 1$, where $\mathfrak{M}$ is defined in Theorem 1.1.

We now claim that there exists $\eta^* := \eta^*(\mu) \in (0, \eta_0)$ depending on μ such that for each $\varepsilon \in (0, \eta^*)$, λ_ε is the unique root of $\mathcal{T}_\varepsilon$. Suppose by contradiction that for arbitrary small $\varepsilon > 0$, $\mathcal{T}_\varepsilon$ always has at least two distinct roots $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ satisfying (2.3) and $|\lambda_{1,\varepsilon}|, |\lambda_{2,\varepsilon}| \le \mathfrak{M}$. Let $v_{j,\varepsilon}$ be the solution of (1.6) corresponding to $\lambda = \lambda_{j,\varepsilon}$. Without loss of generality, we focus on the case

$$\mathfrak{m} = \mu + g(0,0)\int_{\ell_0}^{\ell_1} p(y)\,\mathrm{d}y > 0 \tag{3.28}$$

and it suffices to assume that $\lambda_{1,\varepsilon}$ and $\lambda_{2,\varepsilon}$ are positive. Then $V_\varepsilon := v_{1,\varepsilon} - v_{2,\varepsilon}$ is the solution of

$$\begin{cases} -\varepsilon^2 V_\varepsilon''(x) + \varepsilon a(x) V_\varepsilon'(x) + b(x) f'(\theta_\varepsilon(x)) V_\varepsilon(x) = 0 \quad \text{in} \quad (0, \ell), \\ V_\varepsilon(0) = 0, \quad V_\varepsilon(\ell) = \lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}, \end{cases} \tag{3.29}$$

where $\theta_\varepsilon(x)$ lies between $v_{1,\varepsilon}(x)$ and $v_{2,\varepsilon}(x)$, and hence, $0 \leq \theta_\varepsilon(x) \leq \max\{\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}\}$. Therefore, $b(x) f'(\theta_\varepsilon(x)) \geq \beta f'(0)$ (by (1.4) and (3.2)) is positive on $[0, \ell]$, and we can apply the same argument of (3.23) to (3.29). Thus, we verify that V_ε satisfies the same estimate of (2.1) with $\lambda = \lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}$. As a consequence, by (1.7) and $\mathcal{T}_\varepsilon(\lambda_{1,\varepsilon}) = \mathcal{T}_\varepsilon(\lambda_{2,\varepsilon}) = 0$, we can follow the same argument of (3.25)–(3.26) to obtain

$$\begin{aligned} |\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}| &= \left| \int_{\ell_0}^{\ell_1} p(y) g(v_{1,\varepsilon}(y), v_{1,\varepsilon}'(y))\, \mathrm{d}y - \int_{\ell_0}^{\ell_1} p(y) g(v_{2,\varepsilon}(y), v_{2,\varepsilon}'(y))\, \mathrm{d}y \right| \\ &\leq \max_{[\ell_0,\ell_1]} |p| \max_{[0,\delta^*]^2} |\nabla g| \int_{\ell_0}^{\ell_1} (|V_\varepsilon(y)| + |V_\varepsilon'(y)|)\, \mathrm{d}y \\ &\leq \frac{L}{\varepsilon} \mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)} \max_{[\ell_0,\ell_1]} |p| \max_{[0,\delta^*]^2} |\nabla g| |\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}|, \end{aligned} \tag{3.30}$$

where L is a positive constant independent of ε and $\lambda_{j,\varepsilon}$'s since $\sup_{\varepsilon \in (0,\eta^*)} |\lambda_{j,\varepsilon}| \leq \mathfrak{M}$. Here we stress again that in the last term of (3.30), there is no f' term because (3.29) is linear. Note that $\ell_1 \in (0, \ell)$. As a consequence, we can choose an $\eta^{**} \in (0, \eta^*)$ satisfying

$$\frac{\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-\ell_1)}}{\varepsilon} \leq \frac{1}{2L \max_{[\ell_0,\ell_1]} |p| \max_{[0,\delta^*]^2} |\nabla g| + 1}, \quad \text{as } \varepsilon \in (0, \eta^{**}).$$

Along with (3.30), one can arrive at $|\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}| \leq \frac{1}{2} |\lambda_{1,\varepsilon} - \lambda_{2,\varepsilon}|$ as $\varepsilon \in (0, \eta^{**})$. This indicates $\lambda_{1,\varepsilon} = \lambda_{2,\varepsilon}$, a contradiction. Therefore, for $\varepsilon \in (0, \eta^*)$ we prove the uniqueness of $\mathcal{T}_\varepsilon(\lambda) = 0$ and complete the proof of Proposition 2.2.

4 Proof of Theorem 1.1

By Proposition 2.2 and (1.7), for each $\varepsilon \in (0, \eta^*)$, there exists λ_ε depending on ε such that

$$\lambda_\varepsilon = \mu + \int_{\ell_0}^{\ell_1} p(y) g(v_{\varepsilon,\lambda_\varepsilon}(y), v_{\varepsilon,\lambda_\varepsilon}'(y))\, \mathrm{d}y. \tag{4.1}$$

Hence, by the existence of solution $v_{\varepsilon,\lambda}$ of (1.6) and (4.1), we obtain that $u_\varepsilon = v_{\varepsilon,\lambda_\varepsilon}$ is a solution of (1.1)–(1.2). This proves the existence.

We now prove the uniqueness of (1.1)–(1.2) with $\varepsilon \in (0, \eta^*)$, where η^* has been defined by Proposition 2.2. Suppose on the contrary that there exists $\varepsilon^* \in (0, \eta^*)$ such that (1.1)–(1.2) with $\varepsilon = \varepsilon^*$ has at least two distinct solutions u_{1,ε^*} and u_{2,ε^*}. Then by (1.6) and (1.7), there exist λ_1^* and λ_2^* such that $u_{j,\varepsilon^*} = v_{\varepsilon^*,\lambda_j^*}$ and

$$\mathcal{T}_{\varepsilon^*}(\lambda_j^*) = -\lambda_j^* + \mu + \int_{\ell_0}^{\ell_1} p(y) g(v_{\varepsilon^*,\lambda_j^*}(y), v_{\varepsilon^*,\lambda_j^*}'(y))\, \mathrm{d}y = 0, \quad j = 1, 2.$$

However, since $\varepsilon^* \in (0, \eta^*)$, by Proposition 2.2 we have $\lambda_1^* = \lambda_2^*$. This implies $u_{1,\varepsilon^*} \equiv u_{2,\varepsilon^*}$, a contradiction. Hence, for each $\varepsilon \in (0, \eta^*)$, (1.1)–(1.2) has a unique solution satisfying $u_\varepsilon = v_{\varepsilon,\lambda_\varepsilon}$. By (1.6) and (2.3), we obtain $u_\varepsilon(\ell) = v_{\varepsilon,\lambda_\varepsilon}(\ell) = \lambda_\varepsilon \xrightarrow{\varepsilon \downarrow 0} \mathfrak{m}$. This proves (1.8) and (1.9). Moreover, for $\mathfrak{M} > |\mathfrak{m}|$, there exists $\eta \in (0, \eta^*)$ such that for $\varepsilon \in (0, \eta)$, we have $|\lambda_\varepsilon| \le \mathfrak{M}$, i.e., $\sup_{\varepsilon\in(0,\eta)} \max_{[0,\ell]} |u_\varepsilon| \le \mathfrak{M}$.

We notice further that for $\varepsilon \in (0, \eta)$, (1.1)–(1.2) merely has trivial solution $u_\varepsilon \equiv 0$ if and only if $\mathfrak{m} = 0$. Hence, for the case $\mathfrak{m} > 0$, i.e., (3.28), there holds $u_\varepsilon(\ell) > 0$ as $\varepsilon \in (0, \eta)$, and by Proposition 2.1(i) with $\lambda := u_\varepsilon(\ell) > 0$, we have $u_\varepsilon' > 0$ on $[0, \ell]$. Similarly, for the case $\mathfrak{m} < 0$, we have $u_\varepsilon' < 0$ on $[0, \ell]$. Therefore, we prove (i).

For $l^* \in [0, \ell)$, by (1.4) and (2.1) with $u_\varepsilon = v_{\varepsilon,\lambda_\varepsilon}$ and $|\lambda_\varepsilon| \le \mathfrak{M}$, we obtain

$$|u_\varepsilon(l^*)| \le \sqrt{2}\mathfrak{M}\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon}(\ell-l^*)} \xrightarrow{\varepsilon\downarrow 0} 0,$$

and

$$|u_\varepsilon'(l^*)| \le \frac{\mathfrak{M}C_0}{\varepsilon}\left(1 + f'(\mathfrak{M})\right)\mathrm{e}^{-\frac{\widetilde{M}}{8\varepsilon}(\ell-l^*)} \xrightarrow{\varepsilon\downarrow 0} 0.$$

The argument can be directly applied to the case $\mathfrak{m} < 0$ and we complete the proof of (ii).

It remains to prove (iii). First we notice

$$|u_\varepsilon(\ell - \varepsilon^\tau)| \le \sqrt{2}\mathfrak{M}\mathrm{e}^{-\frac{\widetilde{M}}{4\varepsilon^{1-\tau}}} \quad \text{as } \varepsilon \in (0, \eta) \text{ and } \tau \in (0, 1). \tag{4.2}$$

Here, for the simplicity of notation, we assume $\eta < \min\{1, \ell\}^{\frac{1}{\tau}}$ since we mainly perform the asymptotics of u_ε with respect to $0 < \varepsilon \ll 1$. Setting

$$\widetilde{U}_\varepsilon(x) = u_\varepsilon(x) - U\left(\frac{\ell - x}{\varepsilon}\right), \quad x \in \left[\ell - \varepsilon^\tau, \ell\right], \tag{4.3}$$

we will estimate $\max_{[\ell-\varepsilon^\tau,\ell]} \widetilde{U}_\varepsilon^2$. By (1.1)–(1.2), (1.9), (1.11) and (4.2), we have

$$\widetilde{U}_\varepsilon^2(\ell - \varepsilon^\tau) + \widetilde{U}_\varepsilon^2(\ell) \xrightarrow{\varepsilon\downarrow 0} 0, \tag{4.4}$$

and

$$\begin{aligned} & -\varepsilon^2\widetilde{U}_\varepsilon''(x) + \varepsilon a(\ell)\widetilde{U}_\varepsilon'(x) \\ &= \varepsilon\left(a(\ell) - a(x)\right)u_\varepsilon'(x) + \left(b(\ell) - b(x)\right)f(u_\varepsilon(x)) \\ &\quad + b(\ell)\left(-f(u_\varepsilon(x)) + f(u_\varepsilon(x) - \widetilde{U}_\varepsilon(x))\right) \\ &= \varepsilon\left(a(\ell) - a(x)\right)u_\varepsilon'(x) + \left(b(\ell) - b(x)\right)f(u_\varepsilon(x)) \\ &\quad - b(\ell)f'(\Theta_\varepsilon^*(x))\widetilde{U}_\varepsilon(x), \qquad x \in (\ell - \varepsilon^\tau, \ell). \end{aligned} \tag{4.5}$$

Here we have used (1.4) to verify $-f(u_\varepsilon(x)) + f(u_\varepsilon(x) - \widetilde{U}_\varepsilon(x)) = -f'(\Theta_\varepsilon^*(x))\widetilde{U}_\varepsilon(x)$, where Θ_ε^* lies between u_ε and $u_\varepsilon - \widetilde{U}_\varepsilon$. Thus, we have $\max_{[0,\ell]} |\Theta_\varepsilon^*| \le 2\mathfrak{M}$.

We next prove

$$\max_{[\ell-\varepsilon^\tau,\ell]} \widetilde{U}_\varepsilon^2 \le \max\left\{\widetilde{U}_\varepsilon^2(\ell - \varepsilon^\tau), \widetilde{U}_\varepsilon^2(\ell), M^*\varepsilon^{2\tau}\right\}, \tag{4.6}$$

where M^* is a positive constant independent of ε.

Proof of (4.6) We have $\max_{[\ell-\varepsilon^\tau,\ell]} |a(\ell) - a(x)| \le \varepsilon^\tau \max_{[0,\ell]} |a'|$ and $\max_{[\ell-\varepsilon^\tau,\ell]} |b(\ell) - b(x)| \le \varepsilon^\tau \max_{[0,\ell]} |b'|$. For $\varepsilon \in (0, \eta)$, it is known that $\max_{[0,\ell]} |\varepsilon u_\varepsilon'|$ and $\max_{[0,\ell]} |u_\varepsilon|$ are uniformly bounded. Thus, for the convenience of calculations, we rewrite (4.5) as a new estimate

$$\left|-\varepsilon^2 \widetilde{U}_\varepsilon''(x) + \varepsilon a(\ell)\widetilde{U}_\varepsilon'(x) + b(\ell) f'(\Theta_\varepsilon^*(x))\widetilde{U}_\varepsilon(x)\right| \leq C_{**}\varepsilon^\tau, \quad x \in (\ell - \varepsilon^\tau, \ell), \tag{4.7}$$

where $C_{**} > 0$ is independent of ε. Multiplying both sides of (4.7) by $|\widetilde{U}_\varepsilon|$, following the argument of (3.4) and employing the Cauchy–Schwarz's inequality, we can perform appropriate manipulations to obtain

$$\begin{aligned}
\varepsilon^2 (\widetilde{U}_\varepsilon^2)''(x) &\geq 2\left(b(\ell) f'(\Theta_\varepsilon^*(x)) - \frac{a^2(\ell)}{4}\right)\widetilde{U}_\varepsilon^2(x) - C_{**}\varepsilon^\tau |\widetilde{U}_\varepsilon(x)| \\
&\geq \left(b(\ell) f'(0) - \frac{a^2(\ell)}{4}\right)\widetilde{U}_\varepsilon^2(x) - \frac{C_{**}^2 \varepsilon^{2\tau}}{4b(\ell) f'(0) - a^2(\ell)}, \qquad x \in (\ell - \varepsilon^\tau, \ell).
\end{aligned} \tag{4.8}$$

Here we have used $f'(\Theta_\varepsilon^*(x)) \geq f'(0)$ (cf. (1.4)) and $b(\ell) f'(0) - \frac{a^2(\ell)}{4} > 0$ (cf. (1.5)) to obtain the last inequality. Apply the maximum principle to (4.8), one arrives at (4.6) with $M^* = \frac{1}{4}\left(\frac{C_{**}}{b(\ell) f'(0) - \frac{a^2(\ell)}{4}}\right)^2$, and the proof is done accordingly.

By (4.4) and (4.6), we obtain $\max_{[\ell - \varepsilon^\tau, \ell]} \widetilde{U}_\varepsilon^2 \xrightarrow{\varepsilon \downarrow 0} 0$. Along with (4.3), we therefore establish (1.10) and complete the proof of Theorem 1.1.

5 Concluding remark

In this note we study a class of stationary nonlinear convection–diffusion equations with a singularly perturbed parameter and implicit boundary conditions in a one-dimensional bounded domain, where the boundary conditions including the standard integral-type boundary conditions are more generalized than that in the related research works. More precisely, by employing an argument of singular perturbations (cf. [24]), we establish the uniqueness and maximum principle as the singularly perturbed parameter is sufficiently small. We further establish the refined boundary asymptotics of solutions. As a consequence, the asymptotic analysis is expected to provide a parallel reference for the numerical studies, since the numerical simulation for such singularly perturbed equations are usually unstable and do not give satisfactory results as the singularly perturbed parameter is sufficiently close to zero. Such an argument is different from the standard fixed point approaches. We further provide Example 1.3 to support the non-triviality of Theorem 1.1. To the best our knowledge, the problem about the refined asymptotics of such nonlocal type singularly perturbed equations remains to be open. In further works, we intend to systematically investigate the refined asymptotic behavior of singularly perturbed nonlocal models which are more generalized than (1.1)–(1.2).

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