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ORIGINAL PAPER

Some Korovkin type approximation applications of power series methods

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Abstract
Korovkin type approximation via summability methods is one of the recent interests of the mathematical analysis. In this paper, we prove some Korovkin type approximation theorems in $L_q[a, b]$, the space of all measurable real valued qth power Lebesgue integrable functions defined on $[a, b]$ for $q \geq 1$, and $C[a, b]$, the space of all continuous real valued functions defined on $[a, b]$, via statistical convergence with respect to power series (summability) methods, integral summability methods and μ-statistical convergence of the power series transforms of positive linear operators. We also show with examples that the results obtained in the present paper are stronger than some existing approximation theorems in the literature.

Keywords Power series method · P-Statistical convergence · Integral summability · Korovkin type approximation theorem

Mathematics Subject Classification 40C10 · 40G15 · 41A36

1 Introduction

The classical Korovkin type approximation theory deals with the convergence of sequences of positive linear operators [2, 25]. Korovkin [25] presented a simple criterion in order to decide for a sequence of positive linear operators $\left(L_j\right)$ on $C[a, b]$, the space of all continuous real functions defined on $[a, b]$, whether $(L_j f)$ converges uniformly to f for all

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$f \in C[a, b]$. Besides many researchers have extended Korovkin's theorem by considering other function spaces or considering summability methods whenever the sequence of positive linear operators does not converge in the ordinary sense with respect to the structure of the space (see, e.g., [1, 3, 5–7, 9, 13, 16, 18, 21–23, 27–29, 31–34, 36, 39, 40, 43]). Actually, the main motivation of the summability theory is to make a non-convergent sequence or series converge in some more general senses [10].

We denote the space of all bounded real functions defined on $[a, b]$ by $B[a, b]$. It is well known that the spaces $C[a, b]$ and $B[a, b]$ are Banach spaces with the norm $\|\cdot\|_\infty$ defined by

$$\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

Let $1 \leq q < \infty$ and let $L_q[a, b]$ denote the Banach space of all measurable real valued qth power Lebesgue integrable functions defined on $[a, b]$ with the norm $\|\cdot\|_q$ defined by

$$\|f\|_q := \left(\int_a^b |f(t)|^q \, dt \right)^{1/q}.$$

In this paper we give some Korovkin type approximation theorems via power series methods, P-statistical convergence and integral summability methods in $L_q[a, b]$ and $C[a, b]$.

A power series method is a function theoretical type method and methods of function theoretical type are exceptionally appropriate for applications connected with analytic continuation and numerical solutions of systems of linear equations (see, [10], Sects. 5.2 and 5.3). Power series methods are also very useful in Korovkin type approximation theory. First Korovkin type approximation theorem via Abel convergence, a particular power series method, was given by Unver [40]. Following this study many authors have given Korovkin type approximation results with power series methods (see, e.g., [4, 6, 8, 12, 15, 35, 37–39, 41]).

Definition 1 Let (p_j) be a non-negative real sequence such that $p_0 > 0$ and assume that corresponding power series $p(t) := \sum_{j=0}^{\infty} p_j t^j$ has radius of convergence R with $0 < R \leq \infty$. Let

$$C_p := \left\{ f : (-R, R) \to \mathbb{R} \,\middle|\, \lim_{0<t\to R^-} \frac{1}{p(t)} f(t) \text{ exists} \right\}$$

and

$$C_{P_p} := \left\{ x = (x_j) \,\middle|\, p_x(t) := \sum_{j=0}^{\infty} p_j x_j t^j \text{ has radius of convergence } \geq R \text{ and } p_x \in C_P \right\}.$$

The functional $P - \lim : C_{P_p} \to \mathbb{R}$ (for short P) defined by

$$P - \lim x = \lim_{0<t\to R^-} \frac{1}{p(t)} \sum_{j=0}^{\infty} p_j t^j x_j$$

is called a power series method and x is said to be P-convergent if the limit in the right hand side exists (see, e.g., [10]).

Let P be a power series method. P is said to be regular if $P - \lim x = L$ provided that $\lim x = L$. A power series method is regular if and only if for any non-negative integer j, $\lim_{0<t\to R^-} \dfrac{p_j t^j}{p(t)} = 0$ (see, e.g., [10]). Throughout this paper we assume that $0 \le t < R$. P is said to be Borel type if $R = \infty$ and it is said to be non-polynomial if p is not a polynomial, that is, $p_k \neq 0$ for infinitely many non-negative integer k. A Borel type power series method is regular if and only if it is non-polynomial [10].

If we take $p_j = \dfrac{1}{j!}$ for any non-negative integer j in Definition 1, then we have $p(t) = e^t$ with radius of convergence $R = \infty$. In this case, the corresponding regular power series method is called the *Borel method* B. In other words, a real sequence $x = (x_j)$ is said to be *Borel convergent* to a real number, L, if the series $\sum_{j=0}^{\infty} \dfrac{1}{j!} t^j x_j$ is convergent for any $t \ge 0$, and

$$\lim_{0<t\to\infty} \frac{1}{e^t} \sum_{j=0}^{\infty} \frac{1}{j!} t^j x_j = L.$$

If we take $p_j = 1$ for any non-negative integer j in Definition 1, then we have $p(t) = \dfrac{1}{1-t}$ with radius of convergence $R = 1$. In this case, the corresponding regular power series method is called the *Abel method* , i.e., a real sequence $x = (x_j)$ is said to be *Abel convergent* to a real number, L, if the series $\sum_{j=0}^{\infty} t^j x_j$ is convergent for any $0 \le t < 1$, and

$$\lim_{0<t\to 1^-} (1-t) \sum_{j=0}^{\infty} t^j x_j = L.$$

The concept of statistical convergence that is introduced by Fast [19] and its generalization A-statistical convergence where A is an infinite matrix are interesting concepts of the summability theory and they have many applications in Korovkin type approximation theory. A real sequence $x = (x_j)$ is said to be statistically convergent to a real number L if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n+1} \left|\left\{ j \le n : \left|x_j - L\right| \ge \varepsilon \right\}\right| = 0$$

where vertical bars denote the cardinality (see, e.g., [20, 24, 26, 30]) and we write $st - \lim x = L$.

In Sect. 2, we prove a Korovkin type approximation theorem in $L_q[a, b]$ via applying P-statistical convergence. The concept of P-statistical convergence has been defined in [42] where P stands for a regular power series method. Now, we recall this concept.

Definition 2 [42] Let P be a regular power series method. A real sequence $x = (x_j)$ is said to P-statistically convergent to a real number L if for any $\varepsilon > 0$

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{j:\left|x_j - L\right| \ge \varepsilon} p_j t^j = 0. \tag{1.1}$$

In this case, we write $st_P - \lim x = L$.

In Sect. 3, we prove a Korovkin type approximation theorem by applying integral summability methods to Borel type power series transforms of positive linear operators on $C[a, b]$.

Let $K : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a Lebesgue measurable function such that $K(s, \bullet)$ is Lebesgue integrable for any $s \in [0, \infty)$. If for a Lebesgue measurable function f and a real number L

$$\lim_{s \to \infty} \int_{0}^{\infty} K(s, t) f(t) dt = L$$

whenever $\lim_{t \to \infty} f(t) = L$, then K is called a regular integral summability method [11].

In Sect. 4, we prove a Korovkin type approximation theorem by applying μ-statistical convergence to power series transforms of positive linear operators in $L_q[a, b]$. For this purpose, we need the following definitions.

Definition 3 [11] An f-measure μ is a monotone non-negative finitely additive set function defined on a collection of subsets Γ of $[0, \infty)$ which has the following properties:
(i) For any bounded $B \in \Gamma$, $\mu(B) = 0$,
(ii) $\mu([0, \infty)) = 1$,
(iii) If A is Lebesgue measurable, $A \subset B$ and $\mu(B) = 0$, then $A \in \Gamma$ and $\mu(A) = 0$.

Definition 4 [11] Let f be a real function defined on $[0, \infty)$. If for any $\varepsilon > o$

$$\mu\left(\{t : |f(t) - L| \geq \varepsilon\}\right) = 0,$$

then f is said to be μ-statistically convergent to real number L. In this case, we write $st_{\mu} - \lim f(t) = L$.

2 Approximation via *P*-statistical convergence on $L_q[a, b]$

In this section, we prove a Korovkin type approximation theorem in $L_q[a, b]$ by using P-statistical convergence. A Korovkin type theorem in $L_q[a, b]$ given by Gadjiev and Orhan [21] via statistical convergence. Further Korovkin type approximation results in $L_q[a, b]$ may be found in [17, 28, 38]. First of all, we recall the classical case of these theorems [17].

Theorem 1 [17] *Let (T_j) be a sequence of positive linear operators from $L_q[a, b]$ into $L_q[a, b]$ such that the sequence $\|T_j\| := \|T_j\|_{L_q[a,b] \to L_q[a,b]}$ is uniformly bounded. Then for any $f \in L_q[a, b]$*

$$\lim ||T_j f - f||_q = 0$$

if and only if for $i = 0, 1, 2$

$$\lim ||T_j e_i - e_i||_q = 0$$

where $e_i(t) = t^i$ for $i = 0, 1, 2$.

Now , we are ready to present the following Korovkin type approximation theorem in $L_q[a, b]$ via P-statistical convergence. A version of this theorem in $C[a, b]$ can be found in [42].

Theorem 2 *Let P be a regular power series method and let (T_j) be a sequence of positive linear operators from $L_q[a, b]$ into $L_q[a, b]$ such that the sequence $\|T_j\|$ is uniformly bounded. Then for any $f \in L_q[a, b]$ we have*

$$st_P - \lim ||T_j f - f||_q = 0$$

if and only if for $i = 0, 1, 2$

$$st_P - \lim ||T_j e_i - e_i||_q = 0.$$

Proof The necessity is trivial. To prove the sufficiency let $f \in L_q[a, b]$. Given $\varepsilon > 0$, from the Lusin's theorem, there exists a continuous function φ on $[a, b]$ such that $||f - \varphi||_q < \varepsilon$. Since the function φ is continuous there exist $\delta > 0$ such that

$$|\varphi(t) - \varphi(x)| < \varepsilon$$

for any $t, x \in [a, b]$ whenever $|t - x| < \delta$. Moreover, from the hypothesis there exists $M > 0$ such that $\sup_j ||T_j|| \le M$. If we follow the technique in [17], then we get for any non-negative integer j that

$$||T_j f - f||_q \le \varepsilon(2 + M) + \left(\varepsilon + C + \frac{2C}{\delta^2} d^2\right) ||T_j e_0 - e_0||_q + \frac{4C}{\delta^2} d ||T_j e_1 - e_1||_q + \frac{2C}{\delta^2} ||T_j e_2 - e_2||_q$$

where $d := \max\{|a|, |b|\}$ and C is a uniform bound of φ. Now since $\varepsilon > 0$ is arbitrary, by the assumptions we have

$$st_P - \lim ||T_j f - f||_q = 0.$$

□

In Example 1 we show that Theorem 2 is stronger than Theorem 1.

Example 1 Let $\alpha = (\alpha_j)$ be a non-negative divergent sequence that is B-statistically convergent to 1. Here B stands for Borel power series method. Consider the sequence of Bernstein operators (L_j) where for any positive integer j and $f \in L_1[0, 1]$

$$L_j(f; x) = \sum_{k=0}^{j} f\left(\frac{k}{j}\right) \binom{j}{k} x^k (1 - x)^{j-k}.$$

It is known that for any positive integer j

$$L_j(e_0; x) = 1, \tag{2.1}$$
$$L_j(e_1; x) = x, \tag{2.2}$$

and

$$L_j(e_2; x) = x^2 + \frac{x - x^2}{j}. \tag{2.3}$$

Let us define a sequence of positive linear operators (T_j) by $T_j := \alpha_j L_j$ for any non-negative integer j where $L_0 = 0$. It is obvious that (T_j) does not satisfy the conditions of Theorem 1. On the other hand, we obtain from (2.1), (2.2) and (2.3) that

$$\left\| T_j e_0 - e_0 \right\|_1 = \left| \alpha_j - 1 \right| \|e_0\|_1, \tag{2.4}$$
$$\left\| T_j e_1 - e_1 \right\|_1 = \left| \alpha_j - 1 \right| \|e_1\|_1 \tag{2.5}$$

and

$$\left\| T_j e_2 - e_2 \right\|_1 \le \left| \alpha_j - 1 \right| \|e_2\|_1 + \frac{\alpha_j}{j} \|e_1 - e_2\|_1 \tag{2.6}$$

for $j = 1, 2, ...$ It is obvious from the hypothesis, (2.4) and (2.5) that

$$st_B - \lim \left\| T_j e_0 - e_0 \right\|_1 = 0$$

and

$$st_B - \lim \left\| T_j e_1 - e_1 \right\|_1 = 0.$$

Moreover regularity of B-statistical convergence yields that $st_B - \lim \frac{1}{j} = 0$ which implies $st_B - \lim \frac{\alpha_j}{j} = 0$. Hence, taking into account $st_B - \lim \alpha_j = 1$ in (2.6) we obtain

$$st_B - \lim \left\| T_j e_2 - e_2 \right\|_1 = 0.$$

So from Theorem 2 we conclude that

$$st_B - \lim \left\| T_j f - f \right\|_1 = 0$$

for any $f \in L_1[0, 1]$.

3 Approximation of Borel type power series transforms in $C[a, b]$ via integral summability

In this section, we deal with the sequences of positive linear operators that are not convergent with a Borel-type power series method. If a divergent sequence of positive linear operators is still not convergent with a Borel-type power series method, it can be made convergent by considering integral summability method which is a function-to-function method.

Let P be a non-polynomial Borel-type power series method and let $T = \left(T_j\right)$ be a sequence of positive linear operators from $C[a, b]$ to $B[a, b]$ such that

$$H := \sup_{t>0} \frac{1}{p(t)} \sum_{j=0}^{\infty} \left\| T_j e_0 \right\|_\infty p_j t^j < \infty. \tag{3.1}$$

Then for any $t > 0$ the operator $\mathcal{V}^t_{P,T} : C[a, b] \to B[a, b]$ defined by

$$\left(\mathcal{V}^t_{P,T} f\right)(x) = \frac{1}{p(t)} \sum_{j=0}^{\infty} T_j(f; x) p_j t^j \tag{3.2}$$

is a positive linear operator. Note that $\mathcal{V}^t_{P,T}$ is well-defined from (3.1) for each $t > 0$. Since point-wise (indeed, here the series converges uniformly) limit of a sequence of measurable functions is measurable we obtain that for any $x \in [a, b]$ the function $\left(\mathcal{V}^t_{P,T} f\right)(x)$ is a measurable function of the variable t. Now we can define a new positive linear operator by using integral summability. Let K is a non-negative regular integral summability method. For any $s \in [0, \infty)$ the operator $K^s_{p,T} : C[a, b] \to B[a, b]$ defined by

$$\left(K^s_{p,T} f\right)(x) := \int_0^\infty K(s, t) \left(\mathcal{V}^t_{P,T} f\right)(x)\, dt \tag{3.3}$$

is again a positive linear operator. As K is non-negative and regular we can write from [11] that

$$\lim_{s\to\infty}\int_0^\infty K(s,t)dt = 1. \tag{3.4}$$

On the other hand, we have for any $s\in[0,\infty)$ that

$$\begin{aligned}\left\|K_{p,T}^{s}\right\| &:= \left\|K_{p,T}^{s}\right\|_{C[a,b]\to B[a,b]}\\ &= \sup_{\|f\|=1}\sup_{x\in[a,b]}\left|\int_0^\infty K(s,t)\left(\mathcal{V}_{P,T}^{t}f\right)(x)\,dt\right|\\ &\le H\int_0^\infty K(s,t)dt<\infty \end{aligned}\tag{3.5}$$

which implies that the operator $K_{p,T}^{s}$ is well-defined and bounded. Besides, from (3.4) and (3.5) it is easy to see that

$$\limsup_{s\to\infty}\left\|K_{p,T}^{s}\right\|\le H.$$

Hence, if a sequence of positive linear operator does not convergent with a Borel type power series method then we can use integral summability.

Theorem 3 *Let P be a non-polynomial Borel-type power series method and let $\left(T_j\right)$ be a sequence of positive linear operators from $C[a,b]$ to $B[a,b]$ that satisfies (3.1). Then for any $f\in C[a,b]$ we have*

$$\lim_{s\to\infty}\left\|K_{p,T}^{s}f-f\right\|_\infty=0 \tag{3.6}$$

if and only if for any $i=0,1,2$

$$\lim_{s\to\infty}\left\|K_{p,T}^{s}e_i-e_i\right\|_\infty=0. \tag{3.7}$$

Proof We trivially have (3.6) implies (3.7). Now let $\left\{T_j\right\}$ be a sequence of positive linear operators from $C[a,b]$ into $B[a,b]$ that satisfies (3.1) and let $f\in C[a,b]$. From the continuity of f, for every $\varepsilon>0$ there exists a real number $\delta>0$ such that $|f(t)-f(x)|<\varepsilon$ whenever $t,x\in[a,b]$ satisfying $|x-t|<\delta$. On the other hand we obtain

$$\begin{aligned}|f(t)-f(x)| &\le |f(t)|+|f(x)|\\ &< \frac{2\|f\|_\infty}{\delta^2}(t-x)^2\end{aligned}$$

for all $t,x\in[a,b]$ satisfying $|x-t|\ge\delta$. Hence, for any $t,x\in[a,b]$ we have

$$|f(t)-f(x)|<\varepsilon+\frac{2\|f\|_\infty}{\delta^2}(t-x)^2. \tag{3.8}$$

Using (3.8), as in the classical case [25] we finally have for any $x \in [a, b]$ that

$$\left|\left(K^{s}_{p,T} f\right)(x) - f(x)\right| = \left(\varepsilon + H + \frac{2\,\|f\|_{\infty}}{\delta^2} c^2\right) \left|\int_0^{\infty} K(s,y)\left(\mathcal{V}^{t}_{P,T} e_0\right)(x)\,dy - e_0(x)\right|$$

$$\leq \frac{4c\,\|f\|_{\infty}}{\delta^2} \left|\int_0^{\infty} K(s,y)\left(\mathcal{V}^{t}_{P,T} e_1\right)(x)\,dy - e_1(x)\right|$$

$$+ \frac{2\,\|f\|_{\infty}}{\delta^2} \left|\int_0^{\infty} K(s,y)\left(\mathcal{V}^{t}_{P,T} e_2\right)(x)\,dy - e_2(x)\right| \tag{3.9}$$

where $c = \max\{|a|, |b|\}$. Now, (3.8), (3.9) and hypothesis prove (3.6). □

The regularities of P and K imply that if the conditions of classical Korovkin theorem hold, then (3.7) is satisfied. Example 2 shows that the converse of this fact is not valid in general. Hence, Theorem 3 is stronger than classical Korovkin theorem. It also proves that Theorem 3 is stronger than Korovkin type approximation theorem of [38].

Example 2 Let $\alpha = (\alpha_j)$ be a sequence which is Abel convergent to 1 and which is not Borel convergent and consider the sequence of Bernstein operators (L_j) recalled in Example 1. Define the sequence of positive linear operators (T_j) to be $T_j = \alpha_j L_j$. As α is not convergent, (2.1) implies that $(T_j e_0)$ is not uniformly convergent. Therefore, classical Korovkin theorem is not applicable to the sequence (T_j). On the other hand, since α is not Borel convergent, again (2.1) implies that $(T_j e_0)$ is not convergent in Borel sense. Hence, theorem of [38] fails. Now, If we consider the Abel integral summability method [11] K which is defined by

$$K(s,t) = \frac{1}{s} e^{-t/s}$$

and the Borel power series method B, then for any $s \in (0, \infty)$ the operator in (3.3) turns the operator

$$(K^{s}_{B,T} f)(x) = \frac{1}{s} \int_0^{\infty} e^{-t/s} e^{-t} \left(\sum_{j=0}^{\infty} T_j(f;x) \frac{t^j}{j!}\right) dt.$$

Using the Lebesgue Monotone Convergence Theorem and the Gamma function we have

$$(K^{s}_{B,T} f)(x) = \frac{1}{s+1} \sum_{j=0}^{\infty} \left(\frac{s}{s+1}\right)^{j} T_j(f;x). \tag{3.10}$$

Now, if we consider the substitution of $t = \dfrac{s}{s+1}$, then we obtain $0 < t < 1$, $t \to 1^{-}$ as $s \to \infty$ and

$$\frac{1}{s+1} \sum_{j=0}^{\infty} \left(\frac{s}{s+1}\right)^{j} T_j(f;x) = (1-t) \sum_{j=0}^{\infty} t^j T_j(f;x).$$

Thus, from (2.1), (2.2), (2.3) and Abel convergence of α, condition (3.7) is satisfied. Hence, we have from Theorem 1 that

$$\lim_{s\to\infty} \left\| K^{s}_{B,T} f - f \right\|_{\infty} = 0$$

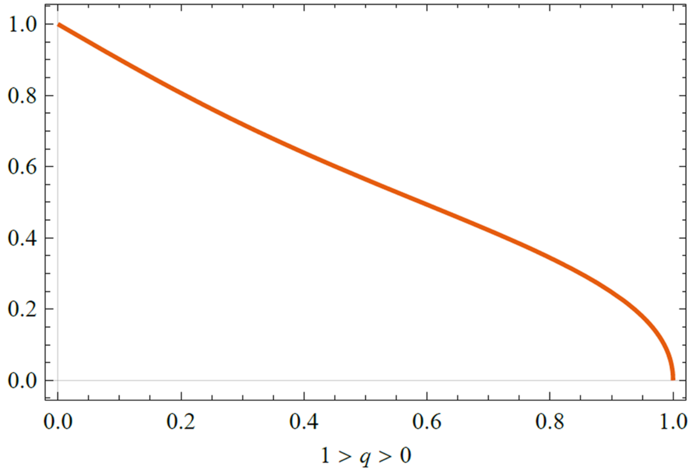

Fig. 1 Limit in (3.13)

for any $f \in C[0, 1]$.

In Example 3 we illustrate the uniform convergence of $K^s_{B,T} f$ to $f \in C[0, 1]$ as $s \to \infty$ for particular choice of sequence of positive linear operators. Note that the sequence of positive linear operators given in the example does not satisfy the conditions of classical Korovkin theorem.

Example 3 Consider again the Abel integral summability method K, the Borel power series method B and the sequence of Bernstein operators (L_j). Define the sequence of positive linear operators (T_j) to be

$$T_j(f; x) = \begin{cases} 0, & \text{if } j \text{ is perfect square or } j = 0 \\ L_j(f; x), & \text{otherwise.} \end{cases} \tag{3.11}$$

From (2.1) and (3.10) we have

$$(K^s_{B,T} e_0)(x) = \frac{1}{s+1} \left(\sum_{j=0}^{\infty} \left(\frac{s}{s+1} \right)^j - \sum_{k=0}^{\infty} \left(\frac{s}{s+1} \right)^{k^2} \right). \tag{3.12}$$

It is easy to see that the first series on the right hand side converges $(s+1)$. On the other hand, we know that Abel convergence is stronger than and consistent with the Cesàro summability. Therefore, the Cesàro summability of the characteristic sequence of the perfect square integers implies (see also Fig. 1)

$$\begin{aligned} \lim_{s\to\infty} \frac{1}{s+1} \sum_{k=0}^{\infty} \left(\frac{s}{s+1} \right)^{k^2} &= \lim_{q\to 1^-} (1-q) \sum_{k=0}^{\infty} q^{k^2} \\ &= 0. \end{aligned} \tag{3.13}$$

So, from (3.12) and (3.13) we have

$$\lim_{s\to\infty} \left\| K^s_{B,T} e_0 - e_0 \right\|_\infty = 0.$$

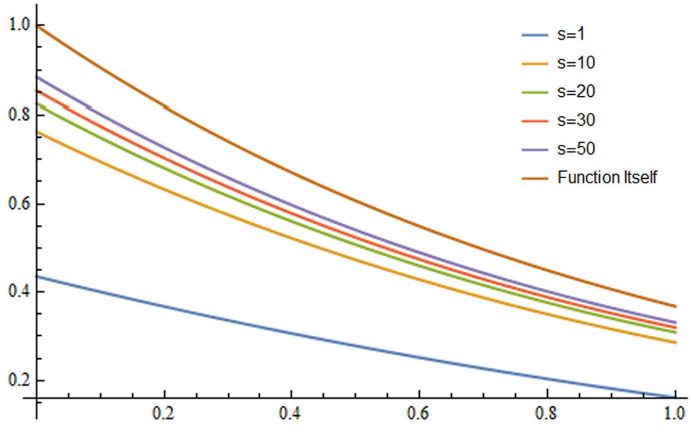

Fig. 2 Approximation of $f(t) = e^{-t}$

Similarly, from (2.2) and (3.10) we obtain

$$(K^s_{B,T}e_1)(x) = \frac{x}{s+1}\left(\sum_{j=0}^{\infty}\left(\frac{s}{s+1}\right)^j - \sum_{k=0}^{\infty}\left(\frac{s}{s+1}\right)^{k^2}\right),$$

which implies that

$$\lim_{s\to\infty}\left\|K^s_{B,T}e_1 - e_1\right\|_\infty = 0.$$

Finally, from (2.3) and (3.10) we have

$$(K^s_{B,T}e_2)(x) = \frac{x^2}{s+1}\left(\sum_{j=1}^{\infty}\left(\frac{s}{s+1}\right)^j - \sum_{k=1}^{\infty}\left(\frac{s}{s+1}\right)^{k^2}\right) + \frac{x-x^2}{s+1}\left(\sum_{j=1}^{\infty}\left(\frac{s}{s+1}\right)^j\frac{1}{j} - \sum_{k=1}^{\infty}\left(\frac{s}{s+1}\right)^{k^2}\frac{1}{k^2}\right). \tag{3.14}$$

On the other hand the sequence $a = (a_k)$ defined by

$$a_k = \begin{cases} \frac{1}{k}, & k \text{ is perfect square} \\ 0, & \text{otherwise} \end{cases}$$

is convergent to zero which implies its Abel convergence to zero, i.e.,

$$\lim_{q\to 1^-}(1-q)\sum_{k=1}^{\infty} q^{k^2}\frac{1}{k^2} = 0.$$

Hence, we have

$$\lim_{s\to\infty}\frac{1}{s+1}\sum_{k=1}^{\infty}\left(\frac{s}{s+1}\right)^{k^2}\frac{1}{k^2} = \lim_{q\to 1^-}(1-q)\sum_{k=1}^{\infty} q^{k^2}\frac{1}{k^2} = 0$$

and from the Abel convergence of the sequence $\left(\frac{1}{j}\right)$ to zero we obtain

$$\lim_{s\to\infty}\frac{1}{s+1}\sum_{j=1}^{\infty}\left(\frac{s}{s+1}\right)^{j}\frac{1}{j}=0.$$

Therefore, we have from (3.14) that

$$\lim_{s\to\infty}\left\|K_{B,T}^{s}e_2-e_2\right\|_{\infty}=0.$$

So, the hypotheses of Theorem 3 are satisfied. Hence, we can say that

$$\lim_{s\to\infty}\left\|K_{B,T}^{s}f-f\right\|_{\infty}=0$$

for any $f\in C[0,1]$ by Theorem 1. Figure 2 illustrates the approximation of the function $f\in C[0,1]$ defined by $f(t)=e^{-t}$ with the net $\left(K_{B,T}^{s}\right)$.

4 Approximation of Borel type power series transforms in $L_q[a,b]$ via μ-statistical convergence

In this section, applying μ-statistical convergence to Borel type power series transforms of positive linear operators we get a Korovkin type approximation theorem in $L_q[a,b]$. Throughout this section we assume that μ is an f-measure on a collection of subsets Γ of $[0,\infty)$.

Let $\left(T_j\right)$ be a sequence of positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that

$$H:=\sup_{t>0}\sum_{j=0}^{\infty}p_j\|T_j\|_{L_q\to L_q}t^{j}<\infty. \tag{4.1}$$

Now we consider the operators $\mathcal{V}_{P,T}^{t}$ defined by (3.2). Observe that $\mathcal{V}_{P,T}^{t}$ is also linear positive operator acting from $L_q[a,b]$ into $L_q[a,b]$ (see, [38]).

Theorem 4 *Let $\left(T_j\right)$ be a sequence of positive linear operators from $L_q[a,b]$ into $L_q[a,b]$ such that (4.1) is satisfied. Then for any $f\in L_q[a,b]$ we have*

$$st_{\mu}-\lim||\mathcal{V}_{P,T}^{t}f-f||_q=0$$

if and only if

$$st_{\mu}-\lim||\mathcal{V}_{P,T}^{t}e_i-e_i||_q=0$$

for $i=0,1,2$.

Proof The necessity is trivial. Let $f\in L_q[a,b]$. Given $\varepsilon>0$, from the Lusin's Theorem, there exists a continuous function φ on $[a,b]$ such that $||f-\varphi||_q<\epsilon$. Since the function φ is continuous there exist $\delta>0$ such that

$$|\varphi(y)-\varphi(x)|<\varepsilon$$

for any $x, y \in [a, b]$ whenever $|y - x| < \delta$. If we follow the technical in [38], then we can write

$$||\mathcal{V}^t_{P,T} f - f||_q \leq \varepsilon \left(2 + \frac{H}{p_0}\right) + \left(\varepsilon + M + \frac{2M}{\delta^2} d^2\right) ||\mathcal{V}^t_{P,T} e_0 - e_0||_q + \frac{4M}{\delta^2} d ||\mathcal{V}^t_{P,T} e_1 - e_1||_q + \frac{2M}{\delta^2} ||\mathcal{V}^t_{P,T} e_2 - e_2||_q$$

where $d := \max\{|a|, |b|\}$ and M is a uniform bound of φ. So from the hypothesis we immediately conclude that

$$st_\mu - \lim ||\mathcal{V}^t_{P,T} f - f||_q = 0.$$

□

5 Conclusion

Function theoretical type summability methods have various applications. One of these applications is making a Korovkin type approximation with a sequence of positive linear operators whenever the ordinary convergence of the space fails. In this paper, we prove some Korovkin type approximation theorems by using P-statistical convergence, integral summability and μ-statistical convergence in the spaces of $C[a, b]$ and $L_q[a, b]$ where P is a power series method and μ is an f measure. We also give some examples that show that the result obtained in this paper is stronger than some previous results.

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