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ORIGINAL PAPER

Schatten class Hankel operators on exponential Bergman spaces

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Abstract
We characterize Schatten p-class Hankel operators with general symbols acting on Bergman spaces with exponential weights when $0 < p < \infty$.

Keywords Bergman spaces with exponential weights · Hankel operators · Schatten classes

Mathematics Subject Classification 47B35 · 30H20

1 Introduction

Let $\mathbb{D}$ be the unit disc on the complex plane. For a subharmonic function φ on $\mathbb{D}$, the weighted Lebesgue space L^2_φ is the set of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^2_\varphi} = \left\{ \int_{\mathbb{D}} \left| f(z) e^{-\varphi(z)} \right|^2 dA(z) \right\}^{\frac{1}{2}} < \infty,$$

where dA is the usual Lebesgue area measure on $\mathbb{D}$. Let $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. The weighted Bergman space A^2_φ is defined as

$$A^2_\varphi = L^2_\varphi \cap H(\mathbb{D}).$$

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In this paper, we are interested in A^2_φ with the weight function $\varphi \in \mathcal{W}$ which was first introduced in [15]. To describe $\mathcal{W}$ precisely, let $\mathcal{C}_0$ be the space of all continuous functions $\rho : \mathbb{D} \to (0, \infty)$ satisfying $\lim_{|z|\to 1} \rho(z) = 0$. The set $\mathcal{L}$ is defined as

$$\mathcal{L} = \left\{ \rho : \|\rho\|_{\mathcal{L}} = \sup_{z,w\in\mathbb{D}, z\neq w} \frac{|\rho(z) - \rho(w)|}{|z - w|} < \infty,\ \rho \in \mathcal{C}_0 \right\}.$$

Let $\mathcal{L}_0$ be the set of those $\rho \in \mathcal{L}$ with the property that for each $\varepsilon > 0$, there exists a compact subset $E \subseteq \mathbb{D}$ such that $|\rho(z) - \rho(w)| \le \varepsilon|z - w|$, whenever $z, w \in \mathbb{D}\backslash E$. Then the weight class $\mathcal{W}$ is defined as

$$\mathcal{W} = \left\{ \varphi \in \mathcal{C}^2 : \Delta\varphi > 0,\ \text{and} \ \exists\ \rho \in \mathcal{L}_0 \ \text{such that} \frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \right\},$$

where Δ denotes the standard Laplace operator. Here and afterward, the notation $a \simeq b$ indicates that there exists some positive constant C such that $C^{-1}b \le a \le Cb$.

The weight class $\mathcal{W}$ covers a large class of weights. There are two weight classes that are closely related to $\mathcal{W}$. One is introduced by Borichev et al. [4]. The other is introduced by Oleinik [24] and Perel'man [25]. For simplicity, we set $\mathcal{BDK}$ to be the weight class considered by Borichev et al., and set $\mathcal{OP}$ to be the weight class considered by Oleinik and Perel'man. As stated in [27], there is a gap in the theory of non-radial weighted Bergman spaces in which properties differ greatly from the radial ones. The weight class $\mathcal{BDK}$ is radial. However, both the collection $\mathcal{W}$ and $\mathcal{OP}$ contain non-radial weights. For example, a non-radial weight in $\mathcal{W}$ is given by

$$\varphi(z) = \frac{1 + \left(\frac{z+\bar{z}}{2}\right)^2}{1 - |z|^2}.$$

As mentioned in [15], the weight class $\mathcal{W}$ covers $\mathcal{BDK}$, but there is no inclusion relation between the weight class $\mathcal{W}$ and $\mathcal{OP}$.

From [15], we know that A^2_φ is a Hilbert space, and there exists a reproducing kernel $K_z(\cdot) = K(\cdot, z)$. Furthermore, one can see that $K(w, z) = \overline{K(z, w)}$. Hence, the Bergman projection P can be represented as

$$P(g)(z) = \int_{\mathbb{D}} g(w)K(z, w)e^{-2\varphi(w)}dA(w), \qquad \forall g \in L^2_\varphi \text{ and } z \in \mathbb{D}.$$

Given some symbol function f, one defines the Hankel operator H_f as

$$H_f(g) = fg - P(fg).$$

Let $\mathbb{C}$ be the set of complex number and $\mathbb{N}^+$ be the set of positive integers. From [15], we know that

$$\Gamma = \left\{ \sum_{j=1}^{N} a_j K(\cdot, z_j) : N \in \mathbb{N}^+, a_j \in \mathbb{C}, z_j \in \mathbb{D},\ \text{for } 1 \le j \le N \right\}$$

is dense in A^2_φ. We consider those f in the symbol class $\mathcal{S}$ defined as

$$\mathcal{S} = \left\{ f \text{ is measurable on } \mathbb{D} : fg \in L^1_\varphi \text{ for } g \in \Gamma \right\}.$$

Since $||K(\cdot, z)||_{L^\infty_\varphi} < \infty$ (see [15]), $P(fg)(z)$ is well defined for $f \in \mathcal{S}$.

For a bounded linear operator $T : H_1 \to H_2$ between two Hilbert spaces, the singular values $s_j(T)$ of T are defined by

$$s_j(T) = \inf \{||T - K|| : K : H_1 \to H_2, \text{ rank } K \le j\},$$

where rank K denotes the rank of K. The operator T is compact if and only if $s_j(T) \to 0$ as $j \to \infty$. For $0 < p < \infty$, we say that T is in the Schatten class S_p and write $T \in S_p(H_1, H_2)$ if

$$||T||_{S_p}^p = \sum_{j=1}^{\infty} s_j(T)^p < \infty.$$

This defines a norm when $1 \le p < \infty$ and a quasi-norm otherwise.

In the past 50 years, a great deal of mathematical effort has been devoted to the study of Schatten class membership of Hankel operators. Problems about this issue fall into the following three broad categories.

For an analytic function f, one can see that $H_f = 0$. So a natural problem is to characterize Schatten class membership of $H_{\bar{f}}$. In 1988, Arazy et al. [1] proved that $H_{\bar{f}}$ belongs to the Schatten class S_p if and only if f belongs to the Besov class when $1 < p < \infty$. In the case of $0 < p \le 1$, they showed that $H_{\bar{f}} \in S_p$ if and only if f is a constant. In 1993, Li [18] extended this result to the case of strongly pseudoconvex domains. In [9], the problem of the smallest value of p such that Hankel operators are in the Schatten p-class was studied. In the setting of Fock space, Constantin and Ortega-Cerdà [5] obtained a characterization of Schatten p-class Hankel operators $H_{\bar{f}}$ acting on $L^2(e^{2\phi})$ where ϕ is a subharmonic function such that $\Delta\phi$ is a doubling measure. In 2013, Seip and Youssfi [30] characterized the Schatten class membership of Hankel operator $H_{\bar{f}}$ on weighted Fock space $L^2(\Psi)$ where Ψ belongs to a class of certain radial logarithmic growth functions. Bommier-Hato and Constantin [3] generalized the result of [30] to the setting of vector-valued Hankel operators.

For a general symbol function f, the problem of simultaneous membership in S_p of H_f and $H_{\bar{f}}$ has been studied by many mathematicians. In 1991, Zhu [34] studied Schatten class Hankel operators on the Bergman space of the unit ball and proved that H_f and $H_{\bar{f}}$ are both in the Schatten p-class if and only if the mean oscillation of f satisfies the L^p condition when $2 \le p < \infty$. Xia [31] showed that the same result holds for $1 < p < 2$ in the setting of the unit disc. In the case of weighted Bergman spaces of the unit ball, Pau [26] completely solved this problem for all $0 < p < \infty$ in terms of the behaviour of a local mean oscillation function in 2015. Notice that Lv and Xu [22] gave a characterization using the method of global mean oscillation for $2 \le p < \infty$ in 1994. Miao [23] studied the same problem in the setting of harmonic Hankel operators and obtained analogous result. In 2008, Raimondo [29] got a characterization of joint membership of H_f and $H_{\bar{f}}$ in S_p on the Bergman space of planar domains. In the setting of Fock space, Xia and Zheng [32] characterized the simultaneous membership in S_p of H_f and $H_{\bar{f}}$ in terms of the standard deviation for $1 \le p < \infty$. In 2011, Isralowitz [16] showed that the same result in [32] also holds for $0 < p < 1$.

The final question is characterizing the Schatten class membership of single Hankel operators. In 1992, Luecking [21] characterized the Schatten p-class of a single Hankel operator H_f for the first time on the Bergman spaces of the unit disc when $1 \le p < \infty$. In 1996, Lin and Rochberg [19] gave a description of Schatten class Hankel operators acting on exponential Bergman spaces A^2_{φ} where the weight φ belongs to $\mathcal{OP}$ when $p \ge 1$. Galanopoulos [8] solved the open case when $0 < p < 1$. In the setting of Fock space, recently, Hu and Virtanen [13] considered the Schatten p-class membership for single Hankel operators H_f

on the Fock space in $\mathbb{C}^n$ when $0 < p < \infty$ in terms of their recently introduced notion of integral distance to analytic functions [14].

The study of Schatten class Hankel operators plays an important role in the spectral theory of Toeplitz operators (see, e.g., [28] and the references therein) which have wide applications in mathematical physics. It is worth mentioning that the relevant question of Schatten class Teoplitz operators has also been extensively studied in Fock spaces and Bergman spaces. See, for example, [2, 17–20, 30, 33].

For a weight $\varphi \in \mathcal{W}$, Hu and Pau [12] completely described the boundedness and compactness of Hankel operators with general symbols. However, characterizing Schatten class membership of Hankel operators with such weights remains open. In this paper, we are concerned with the Schatten p-class of Hankel operators acting on weighted Bergman spaces with $\varphi \in \mathcal{W}$ when $0 < p < \infty$. $\bar{\partial}$-techniques are important for our analysis. As the canonical solution to $\bar{\partial} u = g\bar{\partial} f$, $H_f g$ is naturally connected with the $\bar{\partial}$-theory. We will use Hörmander's theory to obtain the L^2-estimate. Another useful tool is the space of bounded distance to analytic functions which was initiated by Luecking [21]. A number of techniques in this paper are inspired by [8, 13, 14, 19, 21]. A crucial step in the proof of our main theorem for $0 < p < 1$ is that decomposing a lattice into its diagonal part and off-diagonal part. Such method has been previously applied in [8, 13].

Throughout the paper, C stands for some positive constant which may change from line to line, but does not depend on functions being considered. We also give an expression $A \lesssim B$, which means that there is some constant C such that $A \le CB$.

2 Preliminaries

In this section, we are going to present some lemmas that we need in the proof of our main result. For $z \in \mathbb{D}$ and $r > 0$, set $D(z, r) = \{w : |w - z| < r\}$ to be the Euclidean disc with center z and radius r. For a function ρ, we will simply write

$$D^r(z) = D(z, r\rho(z)).$$

Lemma 2.1 *Let $\rho \in \mathcal{L}$ be positive. Then there exists a constant $\alpha_\rho > 0$ such that*

$$\frac{2}{3}\rho(w) < \rho(z) < 2\rho(w) \tag{2.1}$$

for every $z \in \mathbb{D}$ and $w \in D^{\alpha_\rho}(z)$.

Proof Set $\alpha_\rho = \frac{1}{2\|\rho\|_{\mathcal{L}}}$. By the definition of $\mathcal{L}$, we know that

$$|\rho(w) - \rho(z)| \le \|\rho\|_{\mathcal{L}} \cdot |w - z|,$$

for every $w, z \in \mathbb{D}$. Clearly, $|w - z| < \alpha_\rho \rho(z)$ for $w \in D^{\alpha_\rho}(z)$. It follows that, for every $z \in \mathbb{D}$ and $w \in D^{\alpha_\rho}(z)$,

$$|\rho(w) - \rho(z)| < \frac{1}{2}\rho(z).$$

Therefore, we have

$$\frac{2}{3}\rho(w) < \rho(z) < 2\rho(w).$$

The proof is complete. □

In what follows, we always let $\rho \in \mathcal{L}$ be fixed. So we can simply write $\alpha = \alpha_\rho$. We always suppose that α is from Lemma 2.1. For our analysis, we need a covering lemma which is almost identical to Lemma 3.1 of [6].

Lemma 2.2 *Let $\rho \in \mathcal{L}$ be positive. There are positive constants α and s, depending only on $||\rho||_{\mathcal{L}}$, such that for $0 < r \le \alpha$, there exists a sequence $\{a_j\}_{j=1}^{\infty} \subseteq \mathbb{D}$ satisfying*

(A) $\mathbb{D} = \cup_{j=1}^{\infty} D^r(a_j)$;
(B) $D^{sr}(a_j) \cap D^{sr}(a_m) = \emptyset$ *for* $j \neq m$;
(C) $\{D^{2\alpha}(a_j)\}_{j=1}^{\infty}$ *is a covering of $\mathbb{D}$ of finite multiplicity.*

A sequence $\{a_j\}_{j=1}^{\infty}$ satisfying (A)–(C) of Lemma 2.2 will be called a (ρ, r) lattice. The set of (ρ, r) lattices will be denoted by $L(\rho, r)$. The statement (C) of Lemma 2.2 says that for $\{a_j\}_{j=1}^{\infty} \in L(\rho, r)$, there exists an integer N_0 such that

$$1 \le \sum_{j=1}^{\infty} \chi_{D^{2\alpha}(a_j)}(z) \le N_0, \qquad \forall z \in \mathbb{D}. \tag{2.2}$$

For $z \in \mathbb{D}$, let $k_z = K_z/||K_z||_{A^2_\varphi}$ be the normalized reproducing kernel of A^2_φ. The following lemma is an easy consequence of Lemma 2.1, Theorem 3.2 in [15], and Theorem 3.3 in [15].

Lemma 2.3 *Let $\varphi \in \mathcal{W}$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$. For any $w \in \mathbb{D}$, the following statement holds*

(A) *Suppose $r \in (0, \alpha]$. Then we have*

$$|k_w(z)|e^{-\varphi(z)} \simeq \frac{1}{\rho(w)} \simeq \frac{1}{\rho(z)}, \qquad \forall z \in D^r(w); \tag{2.3}$$

(B) *For each positive number N, there exists a constant C such that*

$$|k_w(z)|e^{-\varphi(z)} \le C\frac{1}{\rho(z)}\left(\frac{\min\{\rho(z), \rho(w)\}}{|z-w|}\right)^N, \ z \in \mathbb{D}. \tag{2.4}$$

Lemma 2.4 (Lemma 2.4 [12]) *Let $\varphi \in \mathcal{W}$ and $\{a_j\}_{j=1}^{\infty}$ be some (ρ, r)-lattice with $0 < r \le \alpha$. Then for $\{\lambda_j\}_{j=1}^{\infty} \in l^2$, we have $\sum_{j=1}^{\infty} \lambda_j k_{a_j} \in A^2_\varphi$ with the norm estimation*

$$\left\|\sum_{j=1}^{\infty} \lambda_j k_{a_j}\right\|_{L^2_\varphi} \le C \left\|\{\lambda_j\}_{j=1}^{\infty}\right\|_{l^2}.$$

Lemma 2.5 (Lemma 5.4 [33]) *Let $\rho \in \mathcal{L}_0$, $r \in (0, \alpha]$ and $k \in \mathbb{N}^+$. For any (ρ, r)-lattice $\{a_j\}_{j=1}^{\infty}$ on $\mathbb{D}$, we can divide it into M subsequences which satisfies that if a_i and a_j are two different points in the same subsequence, then $|a_i - a_j| \ge 2^k r \min\big(\rho(a_i), \rho(a_j)\big)$.*

Given a measurable function f on $\mathbb{D}$, we define the Toeplitz operator with symbol f as

$$T_f g(z) = \int_{\mathbb{D}} g(w)K(z, w)e^{-2\varphi(w)} f(w)dA(w), \qquad \forall g \in A^2_\varphi \text{ and } z \in \mathbb{D}.$$

Clearly, the Teoplitz operator T_f is well defined when $f \in \Gamma$. The averaging function $\widehat{f}_r$ with respect to f is defined to be

$$\widehat{f}_r(z) = \frac{1}{|D^r(z)|} \int_{D^r(z)} f(w)dA(w).$$

where $|D^r(z)|$ is the Lebesgue measure of $D^r(z)$. The following result on Schatten class Toeplitz operators is useful in our analysis. One can find its proof in [33].

Theorem 2.1 *Let $\varphi \in \mathcal{W}$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, $0 < p < \infty$, $r \in (0, \alpha]$, and f be a positive borel measurable function such that the Toeplitz operator T_f is well defined. Then T_f belongs to S_p if and only if the function $\widehat{f_r}$ is in $L^p(\mathbb{D}, \rho^{-2}dA)$.*

Lemma 2.6 *Let $0 < p < \infty$, $r \in (0, \alpha]$ and f be a measurable function. Then we have*

$$\int_{\mathbb{D}} |g(z)e^{-\varphi(z)}|^p |f(z)| dA(z) \lesssim \int_{\mathbb{D}} |g(z)e^{-\varphi(z)}|^p \widehat{|f|}_r(z) dA(z) \tag{2.5}$$

for $g \in H(\mathbb{D})$.

Proof Similar to that of Lemma 2.4 in [11]. □

In our study, we need some results for $0 < p < 1$.

Lemma 2.7 ([7]) *Let A and B be two bounded operators. Then*

$$||AB||_{S_p}^p \le ||B||^p ||A||_{S_p}^p \quad \text{and} \quad ||AB||_{S_p}^p \le ||A||^p ||B||_{S_p}^p \tag{2.6}$$

for every $p \in (0, 1)$.

Lemma 2.8 ([7]) *Let $A = A_1 + A_2$ be the sum of two finite-rank operators on a Hilbert space. Then for every $p \in (0, 1)$, we have*

$$||A||_{S_p}^p \le C\left(||A_1||_{S_p}^p + ||A_2||_{S_p}^p\right), \tag{2.7}$$

where the constant C depends only on p.

3 Schatten class membership of Hankel operators

In this section, we are going to give our characterization of Schatten class Hankel operators.

Let $L^2_{loc}(\mathbb{D})$ be the collection of square locally Lebesgue integrable functions on $\mathbb{D}$. For $f \in L^2_{loc}(\mathbb{D})$ and $z \in \mathbb{D}$, we define $G_r(f)(z)$ as

$$G_r(f)(z) = \inf\left\{\left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f-h|^2 dA\right)^{1/2} : h \in H(D^r(z))\right\},$$

where $H(D^r(z))$ is the set of all analytic functions on $D^r(z)$. For $z \in \mathbb{D}$, $f \in L^2(D^r(z), dA)$ and $r > 0$, we define the square mean of $|f|$ over $D^r(z)$ by setting

$$M_r(f)(z) = \left(\frac{1}{|D^r(z)|}\int_{D^r(z)} |f|^2 dA\right)^{1/2}.$$

Lemma 3.1 *For $z \in \mathbb{D}$, $f \in L^2(D^r(z), dA)$ and $r > 0$, there exists some $h \in H(D^r(z))$ such that*

$$M_r(f-h)(z) = G_r(f)(z). \tag{3.1}$$

Proof The approach is similar to Lemma 3.3 of [14] and we omit the proof. □

For $z \in \mathbb{D}$ and $r > 0$, let

$$A^2(D^r(z), dA) = L^2(D^r(z), dA) \cap H(D^r(z))$$

be the Bergman space over $D^r(z)$. Denote by $B_{z,r}$ the corresponding Bergman projection induced by the Bergman kernel of $A^2(D^r(z), dA)$. It is well known that $B_{z,r}$ is bounded and $B_{z,r}h = h$ for $h \in A^2(D^r(z), dA)$. The following lemma is a special case of Lemma 3.4 in [14] (where it is proved for all $1 \le q < \infty$ while our case is just $q = 2$). We prove it using a different method that is both easier and more natural for operators on the Hilbert space.

Lemma 3.2 *For $z \in \mathbb{D}$, $f \in L^2(D^r(z), dA)$ and $r > 0$, there holds*

$$M_r(f - B_{z,r}(f))(z) \simeq G_r(f)(z). \tag{3.2}$$

Proof Choose h as in Lemma 3.1. Then $h \in A^2(D^r(z), dA)$ since $f \in L^2_{loc}(\mathbb{D})$. Thus $B_{z,r}h = h$. By the triangle inequality and Lemma 3.1, we have

$$\begin{aligned} M_r(f - B_{z,r}(f))(z) &\le M_r(f-h)(z) + M_r(h - B_{z,r}(f))(z) \\ &= M_r(f-h)(z) + M_r(B_{z,r}(h-f))(z) \\ &\lesssim M_r(f-h)(z) = G_r(f)(z). \end{aligned}$$

Obviously, $G_r(f)(z) \le M_r(f - B_{z,r}(f))(z)$. □

Given $r > 0$, let $\{a_j\}_{j=1}^{\infty}$ be a $(\rho, r/2)$-lattice, set $J_z = \{j : z \in D^r(a_j)\}$ and denote by $|J_z|$ the cardinal number of J_z. By (2.2), $1 \le |J_z| \le N$. If $\{\psi_j\}_{j=1}^{\infty}$ is a partition of unity subordinate to $\{D^{r/2}(a_j)\}_{j=1}^{\infty}$. Precisely,

$$\psi_j \in \mathcal{C}^\infty(\mathbb{D}), \qquad \text{supp}\psi_j \subseteq D^{r/2}(a_j), \qquad |\bar\partial \psi_j| \le C\rho(a_j)^{-1}, \qquad \sum_{j=1}^{\infty} \psi_j = 1, \qquad \psi_j \ge 0.$$

Clearly, by (2.1)

$$\rho(z)|\bar\partial \psi_j(z)| \le C, \qquad \forall j = 1, 2, \ldots \text{ and } z \in \mathbb{D}.$$

Given $f \in L^2_{loc}(\mathbb{D})$, for $j = 1, 2, \ldots$, pick $h_j \in H(D^r(a_j))$ as in Lemma 3.1 such that

$$M_r(f - h_j) = G_r(f)(a_j).$$

Define

$$f_1 = \sum_{j=1}^{\infty} h_j \psi_j \text{ and } f_2 = f - f_1. \tag{3.3}$$

Notice that $f_1(z)$ is a finite sum for every $z \in \mathbb{D}$ and hence well defined since $\text{supp}\psi_j \subseteq D^{r/2}(a_j) \subseteq D^r(a_j)$.

Lemma 3.3 *For $f \in L^2_{loc}(\mathbb{D})$ and $r > 0$, decomposing $f = f_1 + f_2$ as in (3.3), we have $f_1 \in \mathcal{C}^1(\mathbb{D})$ and*

$$|\rho(z)\bar\partial f_1(z)| + M_{r/4}(\bar\partial f_1)(z) + M_{r/4}(f_2)(z) \le CG_{7r}(f)(z) \tag{3.4}$$

for $z \in \mathbb{D}$, where the constant C is independent of f.

Proof Since $h_j \in H(D^r(a_j))$ and $\psi_j \in \mathcal{C}^\infty(\mathbb{D})$, we have $f_1 \in \mathcal{C}^1(\mathbb{D})$. Given $z \in \mathbb{D}$, without loss of generality, we may assume $z \in D^{r/2}(a_1)$. For $z \in D^{r/2}(a_j)$, we have $D^{r/4}(z) \subseteq D^r(a_j)$. Since $|h_j - h_1|^2$ is subharmonic on $D^{r/4}(z) \subseteq D^r(a_j)$ and $\sum_{j=1}^\infty \bar\partial\psi_j(z) = \bar\partial\sum_{j=1}^\infty \psi_j(z) = \bar\partial 1 = 0$, we obtain

$$\begin{aligned}
|\rho(z)\bar\partial f_1(z)| &= \left|\sum_{j=1}^\infty (h_j(z) - h_1(z))\rho(z)\bar\partial\psi_j(z)\right| \\
&\leq \sum_{j=1}^\infty |h_j(z) - h_1(z)||\rho(z)\bar\partial\psi_j(z)| \\
&\leq C \sum_{\{j: z\in D^{r/2}(a_j)\}} M_{r/4}(h_j - h_1)(z) \\
&\leq C \sum_{\{j: z\in D^{r/2}(a_j)\}} [M_{r/4}(f - h_j)(z) + M_{r/4}(f - h_1)(z)] \\
&\lesssim \sum_{\{j: z\in D^{r/2}(a_j)\}} G_r(f)(a_j).
\end{aligned}$$

For $z \in D^{r/2}(a_j)$, we have $D^r(a_j) \subseteq D^{3r}(z)$. Therefore,

$$G_r(f)(a_j) \leq CG_{3r}(f)(z).$$

Consequently,

$$|\rho(z)\bar\partial f_1(z)| \leq CG_{3r}(f)(z). \tag{3.5}$$

For $w \in D^{r/4}(z)$, we have $D^{3r}(w) \subseteq D^{7r}(z)$. Thus, integrating on both sides of (3.5)

$$\begin{aligned}
M_{r/4}(\rho\bar\partial f_1)(z)^2 &\leq C\rho(z)^{-2} \int_{D^{r/4}(z)} G_{3r}(f)(w)^2 dA(w) \\
&\leq CG_{7r}(f)(z)^2.
\end{aligned} \tag{3.6}$$

Next, we prove the part with regard to f_2. By the Cauchy–Schwarz inequality, we get

$$|f_2(z)|^2 \leq \sum_{j=1}^\infty |f(z) - h_j(z)|^2\psi_j(z).$$

Hence,

$$\begin{aligned}
M_{r/4}(f_2)(z)^2 &\leq \sum_{j=1}^\infty \frac{1}{|D^{r/4}(z)|} \int_{D^{r/4}(z)} |f - h_j|^2\psi_j dA \\
&\leq C \sum_{\{j: z\in D^{r/2}(a_j)\}} \frac{1}{|D^r(a_j)|} \int_{D^r(a_j)} |f - h_j|^2 dA \\
&= C \sum_{\{j: z\in D^r(a_j)\}} G_r(f)(a_j)^2 \\
&\leq CG_{3r}(f)(z)^2.
\end{aligned}$$

Since $G_{3r}(f)(z) \leq CG_{7r}(f)(z)$, the result follows immediately. □

Lemma 3.4 *Let* $0 < p < \infty$ *and* $f \in L^2_{loc}(\mathbb{D})$. *Then the following statements are equivalent:*

(A) $M_r(f)(z) \in L^p(\mathbb{D}, \rho^{-2}dA)$ *for some (or any)* $r \leq \alpha$;
(B) *The sequence* $\{M_\delta(f)(a_j)\}_{j=1}^\infty \in l^p$ *for some (or any)* (ρ, δ)*-lattice* $\{a_j\}_{j=1}^\infty$ *with* $\delta \leq \alpha$.

Moreover, we have

$$||M_r(f)||_{L^p(\mathbb{D},\rho^{-2}dA)} \simeq ||\{M_\delta(f)(a_j)\}_{j=1}^\infty||_{l^p}. \tag{3.7}$$

Proof This lemma is essentially proved in Proposition 2.4 of [33] when the L^p condition is given by $L^p(\mathbb{D}, dA)$. The proof is easy to modify. □

Let $z \in \mathbb{D}$, $r > 0$. Consider the space $L^2(D^r(z), e^{-2\varphi}dA) := L^2_\varphi(D^r(z))$ and the closed subspace of analytic functions $A^2_\varphi(D^r(z))$. Let $P_{z,r}$ be the projection of $L^2_\varphi(D^r(z))$ onto $A^2_\varphi(D^r(z))$. Given a function $f \in L^2_\varphi(D^r(z))$, we extend $P_{z,r}(f)$ to $\mathbb{D}$ by setting $P_{z,r}(f)(w) = 0$ for $w \in \mathbb{D} \setminus D^r(z)$. For $f, g \in L^2_\varphi$, we have $f, g \in L^2_\varphi(D^r(z))$. So there holds $P^2_{z,r}(f) = P_{z,r}(f)$ and $\langle f, P_{z,r}(g)\rangle = \langle P_{z,r}(f), g\rangle$ for $f, g \in L^2_\varphi$. Moreover, if $h \in A^2_\varphi$, then $P_{z,r}(h) = \chi_{D^r(z)}h$. Hence, for $h \in A^2_\varphi$ and $g \in L^2_\varphi$, we get

$$\langle h, \chi_{D^r(z)}g\rangle = \langle \chi_{D^r(z)}h, g\rangle = \langle P_{z,r}(h), g\rangle = \langle h, P_{z,r}(g)\rangle.$$

or equivalently,

$$\langle h, \chi_{D^r(z)}g - P_{z,r}(g)\rangle = 0. \tag{3.8}$$

Lemma 3.5 *Let* $f, g \in L^2_\varphi$. *Then*

$$\langle f - P(f), \chi_{D^r(z)}g - P_{z,r}(g)\rangle = \langle \chi_{D^r(z)}f - P_{z,r}(f), \chi_{D^r(z)}g - P_{z,r}(g)\rangle.$$

Proof The approach is similar to Lemma 5.1 of [13] and we omit the detail here. □

From [12], we know that $H_f : A^2_\varphi \to L^2_\varphi$ is bounded if and only if $G_r(f) \in L^\infty$. As a corollary, $G_r(f) \in L^\infty$ is independent of the choice of r. Furthermore, $||G_r(f)||_{L^\infty} \simeq ||G_\delta(f)||_{L^\infty}$. If $G_r(f) \in L^\infty$, by lemma (3.3), one can get

$$||M_r(f_2)||_{L^\infty} \lesssim ||G_r(f)||_{L^\infty}. \tag{3.9}$$

In the study of Schatten class Hankel operators, it is natural to consider the symbol function to be $f \in \mathcal{S}$ such that $G_r(f) \in L^\infty$.

Lemma 3.6 *Let* $\varphi \in \mathcal{W}$ *with* $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, $0 < p < \infty$, $r \in (0, \alpha]$, *and* $f \in \mathcal{S}$ *such that* $G_r(f) \in L^\infty$, *and decompose* $f = f_1 + f_2$ *as in Lemma* 3.3. *Then* H_{f_1} *and* H_{f_2} *are well defined on* Γ. *Furthermore,*

$$||H_{f_1}(g)||_{L^2_\varphi} \lesssim ||g\rho\bar{\partial} f_1||_{L^2_\varphi} \text{ and } ||H_{f_2}(g)||_{L^2_\varphi} \lesssim ||f_2 g||_{L^2_\varphi}.$$

Proof Recall that

$$\widehat{f_r}(z) = \frac{1}{|D^r(z)|}\int_{D^r(z)} f(w)dA(w).$$

Obviously, by definition,

$$M_r(f)(z)^2 = \widehat{|f|^2}_r(z). \tag{3.10}$$

Applying Hölder's inequality, we have

$$\widehat{|f|}_r(z) \le M_r(f)(z). \tag{3.11}$$

Applying Lemma 2.6 with $p = 1$, (3.9) and (3.11), we get

$$\begin{aligned}\int_{\mathbb{D}} |gK_z|e^{-2\varphi}|f_2|dA &\lesssim \int_{\mathbb{D}} |gK_z|e^{-2\varphi}\widehat{|f_2|}_r dA \\ &\lesssim \int_{\mathbb{D}} |gK_z|e^{-2\varphi}M_r(f_2)dA \\ &\lesssim ||G_r(f)||_{L^\infty}||g||_{L^2_\varphi}||K_z||_{L^2_\varphi} < \infty.\end{aligned}$$

This implies that H_{f_2} is well defined on Γ. Since $H_{f_1} = H_f - H_{f_2}$, H_{f_1} is also well defined on Γ.

By the boundedness of P, we get

$$||H_{f_2}(g)||^2_{L^2_\varphi} \le ||I - P||^2 \cdot ||f_2 g||^2_{L^2_\varphi}$$

Since $\bar\partial(H_{f_1}(g)) = g\bar\partial f_1$ and $H_{f_1}(g) \perp A^2_\varphi$, $H_{f_1}(g)$ is the canonical solution of the equation $\bar\partial u = g\bar\partial f_1$. By Hörmander's L^2 estimation (see Lemma 4.4.1 of [10]), there holds

$$||H_{f_1}(g)||_{L^2_\varphi} \le ||u||_{L^2_\varphi} \lesssim ||g\rho\bar\partial f_1||_{L^2_\varphi}$$

The proof is complete. □

Theorem 3.1 *Let $\varphi \in \mathcal{W}$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, $0 < p < \infty$, $0 < r \le \alpha$, and $f \in \mathcal{S}$ such that $G_r(f) \in L^\infty$. Then the following statements are equivalent:*

(A) *The Hankel operator H_f belongs to S_p;*
(B) *For some (or any) (ρ,r)-lattice $\{a_j\}_{j=1}^\infty$, $\{G_r(f)(a_j)\}_{j=1}^\infty \in l^p$;*
(C) *For some (or any) r, $G_r(f) \in L^p(\mathbb{D}, \rho^{-2}dA)$;*
(D) *For some (or any) r, f admits a decomposition $f = f_1 + f_2$ such that $f_1 \in \mathcal{C}^1(\mathbb{D})$, $M_r(\rho\bar\partial f_1) \in L^p(\mathbb{D}, \rho^{-2}dA)$, and $M_r(f_2) \in L^p(\mathbb{D}, \rho^{-2}dA)$;*
(E) *For some (or any) (ρ,r)-lattice $\{a_j\}_{j=1}^\infty$, f admits a decomposition $f = f_1 + f_2$ such that $f_1 \in \mathcal{C}^1(\mathbb{D})$, $\{M_r(\rho\bar\partial f_1)(a_j)\}_{j=1}^\infty \in l^p$ and $\{M_r(f_2)(a_j)\}_{j=1}^\infty \in l^p$.*

Proof (A) ⇒ (B): We deal with the case $0 < p < 1$ first. Let $\{a_j\}_{j=1}^\infty$ be a (ρ, r)-lattice as in Lemma 2.2. Let $k \in \mathbb{N}^+$ and $k > 5$. According to Lemma 2.5, we can partition the sequence into M subsequences which satisfies that if a_i and a_j are two different points in the same subsequence, then

$$|a_i - a_j| \ge 2^k r \min(\rho(a_i), \rho(a_j)). \tag{3.12}$$

It is enough to work with one of these sequences. Without loss of generality, we assume the sequence to be $\{a_j\}_{j=1}^\infty$. Let J be any finite subcollection of $\mathbb{N}^+$. Suppose $\{e_j\}_{j=1}^\infty$ to be an orthonormal set of A^2_φ. Set

$$A(g) = \sum_{j\in J}\langle g, e_j\rangle k_{a_j}, \qquad g \in A^2_\varphi.$$

By Parseval's equality,

$$\sum_{j\in J}|\langle g, e_j\rangle|^2 \le \sum_{j=1}^{\infty}|\langle g, e_j\rangle|^2 = ||g||_{\varphi}^2.$$

According to Lemma 2.4, A is bounded on A_{φ}^2.

If $||\chi_{D^r(a_j)} g k_{a_j} - P_{a_j,r}(g k_{a_j})||_{L_{\varphi}^2} \neq 0$, we can set

$$h_j = \frac{\chi_{D^r(a_j)} f k_{a_j} - P_{a_j,r}(f k_{a_j})}{||\chi_{D^r(a_j)} f k_{a_j} - P_{a_j,r}(f k_{a_j})||_{L_{\varphi}^2}},$$

otherwise we set $h_j = 0$. Clearly $||h_j||_{\varphi}^2 \le 1$. If $D^r(a_i) \cap D^r(a_j) \neq \emptyset$, then we have $|a_i - a_j| \le 3r\min\{\rho(a_i), \rho(a_j)\}$. If we choose k to be sufficiently large, then there must holds $D^r(a_i) \cap D^r(a_j) = \emptyset$ for $i \neq j$. Therefore $\langle h_i, h_j\rangle = 0$ for $i \neq j$.
For a series of nonnegative numbers $\{c_j\}_{j\in J}$, we define an operator B as

$$B(g) = \sum_{j\in J} c_j \langle g, h_j\rangle e_j.$$

It is easy to see that B is bounded on A_{φ}^2 and $||B|| \le \sup_{j\in J}\{c_j\}$. By the definition, we get

$$\begin{aligned} BH_f A(g) &= \sum_{j\in J} c_j \langle H_f A(g), h_j\rangle e_j \\ &= \sum_{j\in J}\sum_{i\in J} c_j \langle H_f k_{a_i}, h_j\rangle \langle g, e_i\rangle e_j. \end{aligned}$$

Applying Lemma 2.7, we have

$$||BH_f A||_{S_p}^p \le ||B||^p ||H_f||_{S_p}^p ||A||^p \le C \sup_{j\in J} c_j^p.$$

We decompose the operator $BH_f A$ to the diagonal part defined by

$$Y(g) = \sum_{j\in J} c_j \langle H_f k_{a_j}, h_j\rangle \langle g, e_j\rangle e_j$$

and the non-diagonal part defined by

$$Z(g) = \sum_{j,i\in J: i\neq j} c_j \langle H_f k_{a_i}, h_j\rangle \langle g, e_i\rangle e_j.$$

By Lemma 2.8, there holds

$$||Y||_{S_p}^p \lesssim ||BH_f A||_{S_p}^p + ||Z||_{S_p}^p.$$

Lemma 2.3 tells us that there is some constant $C > 0$ such that

$$|k_{a_j}(z)| \ge C e^{\varphi(z)} \rho(a_j)^{-1} > 0,$$

for $z \in D^r(a_j)$. Therefore, $k_{a_j}^{-1} \in H(D^r(a_j))$. By Lemma 3.5 and (2.3), we get

$$\begin{aligned} ||Y||_{S_p}^p &= \sum_{j\in J} c_j^p |\langle H_f k_{a_j}, h_j\rangle|^p = \sum_{j\in J} c_j^p |\langle f k_{a_j} - P(f k_{a_j}), h_j\rangle|^p \\ &= \sum_{j\in J} c_j^p |\langle \chi_{D^r(a_j)} f k_{a_j} - P_{a_j,r}(f k_{a_j}), h_j\rangle|^p \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in J} c_j^p ||\chi_{D^r(a_j)} f k_{a_j} - P_{a_j,r}(f k_{a_j})||^p_{L^2_\varphi} \\
&= \sum_{j\in J} c_j^p \left\{ \int\limits_{D^r(a_j)} |f k_{a_j} - P_{a_j,r}(f k_{a_j})|^2 e^{-2\varphi} dA \right\}^{p/2} \\
&= \sum_{j\in J} c_j^p \left\{ \int\limits_{D^r(a_j)} |k_{a_j}|^2 e^{-2\varphi} |f - k_{a_j}^{-1} P_{a_j,r}(f k_{a_j})|^2 dA \right\}^{p/2} \\
&\simeq \sum_{j\in J} c_j^p \left\{ \frac{1}{|D^r(a_j)|} \int\limits_{D^r(a_j)} |f - k_{a_j}^{-1} P_{a_j,r}(f k_{a_j})|^2 dA \right\}^{p/2} \\
&\geq \sum_{j\in J} c_j^p G_r(f)(a_j)^p.
\end{aligned}
$$

By Proposition 1.29 of [35], Lemma 3.5, and Cauchy–Schwarz inequality, we have

$$
\begin{aligned}
||Z||^p_{S_p} &\leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\langle Ze_n, e_m\rangle|^p = \sum_{j,i\in J: i\neq j} c_j^p |\langle H_f k_{a_i}, h_j\rangle|^p \\
&= \sum_{j,i\in J: i\neq j} c_j^p |\langle \chi_{D^r(a_j)} f k_{a_i} - P_{a_j,r} f k_{a_i}, h_j\rangle|^p \\
&\leq \sum_{j,i\in J: i\neq j} c_j^p ||\chi_{D^r(a_j)} f k_{a_i} - P_{a_j,r} f k_{a_i}||^p_{L^2_\varphi} \\
&= \sum_{j,i\in J: i\neq j} c_j^p \left\{ \int\limits_{D^r(a_j)} |f k_{a_i} - P_{a_j,r}(f k_{a_i})|^2 e^{-2\varphi} dA \right\}^{p/2} \\
&\leq \sum_{j,i\in J: i\neq j} c_j^p \left\{ \int\limits_{D^r(a_j)} |f k_{a_i} - k_{a_i} B_{a_j,r}(f)|^2 e^{-2\varphi} dA \right\}^{p/2}.
\end{aligned}
$$

where $B_{z,r}$ is the projection of $L^2(D^r(z))$ onto $A^2(D^r(z))$. Therefore, applying Lemma 3.2, we obtain

$$
\begin{aligned}
||Z||^p_{S_p} &\leq \sum_{j\in J} c_j^p \sum_{i\in J: i\neq j} \left\{ \int\limits_{D^r(a_j)} \left(|k_{a_i}(z)|^2 e^{-2\varphi(z)}\right) |f(z) - B_{a_j,r}(f)(z)|^2 dA(z) \right\}^{p/2} \\
&\lesssim \sum_{j\in J} c_j^p \sum_{i\in J: i\neq j} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|^2 e^{-2\varphi(z)} \right)^{p/2} \rho(a_j)^p \\
&\quad \left\{ \frac{1}{|D^r(a_j)|} \int\limits_{D^r(a_j)} |f(z) - B_{a_j,r}(f)(z)|^2 dA(z) \right\}^{p/2}
\end{aligned}
$$

$$\simeq \sum_{j\in J} c_j^p G_r(f)(a_j)^p \rho(a_j)^p \sum_{i\in J: i\neq j} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|^2 e^{-2\varphi(z)} \right)^{p/2}. \tag{3.13}$$

For each $i, j \in J$ with $i \neq j$, there is some $w_{j,i} \in \overline{D^r(a_j)}$ such that

$$|w_{j,i} - a_i| = \inf_{z\in D^r(a_j)} |z - a_i|$$

By (2.4) and (2.1), for $z \in D^r(a_j)$

$$|k_{a_i}(z)|e^{-\varphi(z)} \leq C\frac{1}{\rho(z)}\left(\frac{\min(\rho(z), \rho(a_i))}{|z-a_i|}\right)^N \lesssim \frac{1}{\rho(a_j)}\left(\frac{\min(\rho(w_{j,i}), \rho(a_i))}{|w_{j,i}-a_i|}\right)^N.$$

We assert that $|w_{j,i} - a_i| \geq 2^{k-2} r \min(\rho(w_{j,i}), \rho(a_i))$. Otherwise, if $|w_{j,i} - a_i| \leq 2^{k-2} r \min(\rho(w_{j,i}), \rho(a_i))$, by (2.1) and the triangle inequality, one can get

$$|a_j - a_i| \leq |a_j - w_{j,i}| + |w_{j,i} - a_i| \leq r\rho(a_j) + 2^{k-2} r\rho(w_{j,i}) < 2^k r\rho(a_j),$$

and

$$|a_j - a_i| \leq r\rho(a_j) + 2^{k-2} r\rho(w_{j,i}) \leq 2r\rho(w_{j,i}) + 2^{k-2} r\rho(w_{j,i}).$$

For each $k > 5$, we can choose r such that $2^{k-2}r < \alpha$. Then by (2.1) again,

$$|a_j - a_i| \leq 4r\rho(a_i) + 2^{k-1} r\rho(a_i) < 2^k r\rho(a_i).$$

It follows that $|a_j - a_i| < 2^k r \min(\rho(a_j), \rho(a_i))$ which is in contradiction with (3.12). Therefore, for $z \in D^r(a_j)$

$$|k_{a_i}(z)|e^{-\varphi(z)} \lesssim \frac{1}{\rho(a_j)}\left(\frac{1}{2^{k-2}r}\right)^N \lesssim \frac{1}{\rho(a_j)}\left(\frac{1}{2}\right)^{Nk}. \tag{3.14}$$

Combining (3.13) and (3.14), it turns out that

$$||Z||_{S_p}^p \lesssim \left(\frac{1}{2}\right)^{\frac{Npk}{2}} \sum_{j\in J} c_j^p G_r(f)(a_j)^p \rho(a_j)^{\frac{p}{2}} \sum_{i\in J: i\neq j} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2}. \tag{3.15}$$

Set $r_0 = 3r$. Let $j \in J$ be fixed.

$$\sum_{i\in J: i\neq j} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2} \leq \sum_{\{i: |a_j - a_i| \leq r_0\rho(a_j)\}} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2}$$
$$+ \sum_{n=0}^{\infty} \sum_{\{i: 2^n r_0\rho(a_j) < |a_j - a_i| \leq 2^{n+1} r_0\rho(a_j)\}} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2}. \tag{3.16}$$

By (2.1) and (2.3), the first part of (3.16) can be estimated as

$$\sum_{\{i: |a_j - a_i| \leq r_0\rho(a_j)\}} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2} \lesssim \sum_{\{i: |a_j - a_i| \leq r_0\rho(a_j)\}} \rho(a_j)^{-p/2}.$$

By (2.1), we know that $a_j \in D^{2r_0}(a_i)$ when $|a_j - a_i| \le r_0\rho(a_j)$. From the finite multiplicity property (2.2), we deduce that

$$\sum_{\{i:|a_j-a_i|\le r_0\rho(a_j)\}} \rho(a_j)^{-p/2} \le \rho(a_j)^{-p/2} \sum_{i=1}^{\infty} \chi_{D^{2r_0}(a_i)}(a_j) \lesssim \rho(a_j)^{-p/2}.$$

It follows that

$$\sum_{\{i:|a_j-a_i|\le r_0\rho(a_j)\}} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2} \lesssim \rho(a_j)^{-p/2}. \tag{3.17}$$

For each i with $2^n r_0\rho(a_j) < |a_j - a_i| \le 2^{n+1} r_0\rho(a_j)$, we get

$$\begin{aligned} |w_{j,i} - a_i| &\ge |a_j - a_i| - |a_j - w_{j,i}| \\ &> 2^n r_0\rho(a_j) - r\rho(a_j) \\ &= \left(2^n - \frac{1}{3}\right) r_0\rho(a_j) \\ &\ge \frac{1}{2} \cdot 2^n r_0\rho(a_j). \end{aligned}$$

By (2.4), the second part of (3.16) is estimated as

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{\{i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \left(\sup_{z\in D^r(a_j)} |k_{a_i}(z)|e^{-\varphi(z)} \right)^{p/2} \\ &\lesssim \sum_{n=0}^{\infty} \sum_{\{i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \left[\sup_{z\in D^r(a_j)} \left(\frac{1}{\rho(z)} \frac{\rho(z)\rho(a_i)^{N-1}}{|w_{j,i}-a_i|^N} \right) \right]^{p/2} \\ &\lesssim \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{nNp/2} \rho(a_j)^{-Np/2} \sum_{\{i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \rho(a_i)^{(N-1)p/2}. \end{aligned}$$

By the definition of $\mathcal{L}$ and the triangle inequality, it is easy to see that, for $a_i \in D^{r_0 2^{n+1}}(a_j)$,

$$D^{r_0}(a_i) \subseteq D^{[1+(1+\|\rho\|_{\mathcal{L}} r_0)2^{n+1}]r_0}(a_j).$$

We can choose N satisfying $(N-1)p/4 \ge 1$. Therefore, by the finite multiplicity property (2.2), we have

$$\begin{aligned} &\sum_{\{i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \rho(a_i)^{(N-1)p/2} \\ &\le \left(\sum_{\{i:2^n r_0\rho(a_j)<|a_j-a_i|\le 2^{n+1}r_0\rho(a_j)\}} \rho(a_i)^2 \right)^{(N-1)p/4} \\ &\simeq \left(\sum_{i=1}^{\infty} \int_{D^{[1+(1+\|\rho\|_{\mathcal{L}} r_0)2^{n+1}]r_0}(a_j)} \chi_{D^{r_0}(a_i)}(w)\,dA(w) \right)^{(N-1)p/4} \end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{D^{[1+(1+\|\rho\|_{\mathcal{L}} r_0)2^{n+1}]r_0}(a_j)} \sum_{i=1}^{\infty} \chi_{D^{r_0}(a_i)}(w) dA(w) \right)^{(N-1)p/4} \\
&\lesssim \left| D^{[1+(1+\|\rho\|_{\mathcal{L}} r_0)2^{n+1}]r_0}(a_j) \right|^{(N-1)p/4} \\
&\simeq 2^{n(N-1)p/2} \rho(a_j)^{(N-1)p/2}.
\end{aligned}
$$

On the other hand, we know that

$$
\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{np/2} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{p/2}}\right)^{n} \leq C.
$$

Hence,

$$
\sum_{n=0}^{\infty} \sum_{\{i: 2^n r_0 \rho(a_j) < |a_j - a_i| \leq 2^{n+1} r_0 \rho(a_j)\}} \left(\sup_{z \in D^r(a_j)} |k_{a_i}(z)| e^{-\varphi(z)} \right)^{p/2} \lesssim \rho(a_j)^{-p/2}. \tag{3.18}
$$

From (3.17), and (3.18), we deduce that

$$
\sum_{i \in J: i \neq j} \left(\sup_{z \in D^r(a_j)} |k_{a_i}(z)| e^{-\varphi(z)} \right)^{p/2} \lesssim \rho(a_j)^{-p/2},
$$

which, together with (3.15), further implies that

$$
||Z||_{S_p}^p \leq C_1 \left(\frac{1}{2}\right)^{\frac{Npk}{2}} \sum_{j \in J} c_j^p G_r(f)(a_j)^p. \tag{3.19}
$$

Notice that

$$
\|BH_f A\|_{S_p}^p \leq C_2 \sup_{j \in J} c_j^p \quad \text{and} \quad C_3 \sum_{j \in J} c_j^p G_r(f)(a_j)^p \leq \|Y\|_{S_p}^p \leq C_4 \left(\|BH_f A\|_{S_p}^p + \|Z\|_{S_p}^p \right).
$$

One can choose k to be sufficiently large such that

$$
C_5 := C_3 - C_4 C_1 \left(\frac{1}{2}\right)^{\frac{Npk}{2}} > 0.
$$

Finally, we obtain

$$
\sum_{j \in J} c_j^p G_r(f)(a_j)^p \leq C_5^{-1} C_4 C_2 \sup_{j \in J} c_j^p
$$

for every collection J. Applying the duality between l^1 and l^∞, we get the desired result.

Now we treat the case $1 \leq p < \infty$. Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis of A^2_φ. We define an operator T on A^2_φ such that

$$
Te_j = \frac{\chi_{D^r(a_j)} H_f(k_{a_j})}{\left(\int_{D^r(a_j)} |H_f(k_{a_j})|^2 e^{-2\varphi} dA \right)^{\frac{1}{2}}} := s_j \chi_{D^r(a_j)} H_f(k_{a_j}).
$$

Then T is bounded. In fact, for $g \in A^2_\varphi$, by Cauchy–Schwarz inequality and the finite multiplicity property, we have

$$\begin{aligned}|Tg(z)|^2 &= \left|\sum_{j=1}^{\infty}\langle g, e_j\rangle s_j \chi_{D^r(a_j)}(z) H_f(k_{a_j})(z)\right|^2 \\ &\le \left(\sum_{j=1}^{\infty}\chi_{D^r(a_j)}(z)\right)\left(\sum_{j=1}^{\infty}|\langle g, e_j\rangle|^2 s_j^2 \chi_{D^r(a_j)}(z)|H_f(k_{a_j})(z)|^2\right) \\ &\lesssim \sum_{j=1}^{\infty}|\langle g, e_j\rangle|^2 s_j^2 \chi_{D^r(a_j)}(z)|H_f(k_{a_j})(z)|^2.\end{aligned}$$

Hence,

$$\begin{aligned}||Tg||^2_\varphi &\lesssim \int_{\mathbb{D}}\sum_{j=1}^{\infty}|\langle g, e_j\rangle|^2 s_j^2 \chi_{D^r(a_j)}(z)|H_f(k_{a_j})(z)|^2 e^{-2\varphi(z)}dA(z) \\ &= \sum_{j=1}^{\infty}|\langle g, e_j\rangle|^2 s_j^2 \int_{D^r(a_j)}|H_f(k_{a_j})(z)|^2 e^{-2\varphi(z)}dA(z) \\ &= \sum_{j=1}^{\infty}|\langle g, e_j\rangle|^2 = ||g||^2_\varphi.\end{aligned}$$

By Theorem 1.27 of [35], we get

$$\begin{aligned}||H_f||^p_{S_p} &\gtrsim ||T^* H_f A||^p_{S_p} \gtrsim \sum_{j=1}^{\infty}|\langle T^* H_f A e_j, e_j\rangle|^p \\ &= \sum_{j=1}^{\infty}|s_j\langle H_f k_{a_j}, \chi_{D^r(a_j)} H_f k_{a_j}\rangle|^p \\ &= \sum_{j=1}^{\infty}\left(\int_{D^r(a_j)}|H_f(k_{a_j})|^2 e^{-2\varphi}dA\right)^{p/2} \\ &= \sum_{j=1}^{\infty}\left(\int_{D^r(a_j)}\left|f - \frac{1}{k_{a_j}}P(fk_{a_j})\right|^2 |k_{a_j}|^2 e^{-2\varphi}dA\right)^{p/2} \\ &\simeq \sum_{j=1}^{\infty}\left(\frac{1}{|D^r(a_j)|}\int_{D^r(a_j)}\left|f - \frac{1}{k_{a_j}}P(fk_{a_j})\right|^2 dA\right)^{p/2} \\ &\ge \sum_{j=1}^{\infty}G_r(f)(a_j)^p.\end{aligned}$$

(B) $\Rightarrow$ (C): Let $\{a_j\}_{j=1}^{\infty}$ be a (ρ, r)-lattice. Then $\{a_j\}_{j=1}^{\infty}$ is also a $(\rho, 3r)$-lattice (just replace s with a smaller one, for example $s/3$). Suppose $\sum_{j=1}^{\infty} G_{3r}(f)(a_j)^p < \infty$. Since $D^r(z) \subseteq D^{3r}(a_j)$ for $z \in D^r(a_j)$, there holds

$$
\begin{aligned}
\int_{\mathbb{D}} G_r(f)(z)^p \rho(z)^{-2} dA(z) &\leq \sum_{j=1}^{\infty} \int_{D^r(a_j)} G_r(f)(z)^p \rho(z)^{-2} dA(z) \\
&\lesssim \sum_{j=1}^{\infty} \sup_{z \in D^r(a_j)} G_r(f)(z)^p \\
&\lesssim \sum_{j=1}^{\infty} G_{3r}(f)(a_j)^p < \infty.
\end{aligned}
$$

(C) $\Rightarrow$ (D): Suppose $G_r(f)(z) \in L^p(\mathbb{D}, \rho^{-2}dA)$. Decompose $f = f_1 + f_2$ as in Lemma 3.3. Then $f_1 \in \mathcal{C}^1(\mathbb{D})$ and

$$
|\rho(z)\bar{\partial} f_1(z)| + M_{r/28}(\bar{\partial} f)(z) + M_{r/28}(f_2)(z) \leq C G_r(f)(z).
$$

By Lemma 3.4, we have

$$
||M_r(\rho\bar{\partial} f_1)||_{L^p(\mathbb{D},\rho^{-2}dA)} \simeq ||M_{r/28}(\rho\bar{\partial} f_1)||_{L^p(\mathbb{D},\rho^{-2}dA)} \leq C||G_r(f)||_{L^p(\mathbb{D},\rho^{-2}dA)} < \infty
$$

and

$$
||M_r(f_2)||_{L^p(\mathbb{D},\rho^{-2}dA)} \simeq ||M_{r/28}(f_2)||_{L^p(\mathbb{D},\rho^{-2}dA)} \leq C||G_r(f)||_{L^p(\mathbb{D},\rho^{-2}dA)} < \infty.
$$

(D) $\Leftrightarrow$ (E) is just Lemma 3.4.

To prove that (D) implies (A), we need to consider the multiplication operators M_{f_2} and $M_{\rho\bar{\partial} f_1}$. Let ϕ to be f_2 or $\rho\bar{\partial} f_1$. With the assumption $G_r(f)(z) \in L^\infty$ and Lemma 3.3, we have $M_r(\phi)(z) \in L^\infty$. We assert that M_ϕ is bounded from A^2_φ to L^2_φ. In fact, by Lemma 2.6 with $p = 2$, for $g \in A^2_\varphi$,

$$
\begin{aligned}
||M_\phi g||^2_{L^2_\varphi} &= \int_{\mathbb{D}} |g|^2 e^{-2\varphi} |\phi|^2 dA \\
&\lesssim \int_{\mathbb{D}} |g(z)|^2 e^{-2\varphi(z)} \widehat{|\phi|^2}_r(z) dA(z) \\
&= \int_{\mathbb{D}} |g(z)|^2 e^{-2\varphi(z)} M_r(\phi)(z)^2 dA(z) \\
&\leq ||M_r(\phi)||^2_{L^\infty} ||g||^2_{L^2_\varphi}
\end{aligned}
$$

Since for any $g, h \in A^2_\varphi$, there holds

$$
\langle M^*_\phi M_\phi g, h\rangle = \langle M_\phi g, M_\phi h\rangle = \langle T_{|\phi|^2} g, h\rangle.
$$

It follows that $M^*_\phi M_\phi = T_{|\phi|^2}$ on A^2_φ. By Theorem 1.26 of [35], we know that $M_\phi \in S_p$ if and only if $M^*_\phi M_\phi = T_{|\phi|^2} \in S_{p/2}$. According to Theorem 2.1, $T_{|\phi|^2} \in S_{p/2}$ if and only if $\widehat{|\phi|^2}_r(z) \in L^{p/2}(\mathbb{D}, \rho^{-2}dA)$, or equivalently, $M_r(\phi)(z) \in L^p(\mathbb{D}, \rho^{-2}dA)$. Hence, $M_\phi \in S_p$. By Lemma 3.6, we know that $||H_{f_1}(g)||_{L^2_\varphi} \lesssim ||g\rho\bar{\partial} f_1||_{L^2_\varphi}$ and $||H_{f_2}(g)||_{L^2_\varphi} \lesssim ||f_2 g||_{L^2_\varphi}$. It follows that H_{f_1} and H_{f_2} belong to S_p which leads to $H_f \in S_p$. The proof is complete. □

4 Simultaneous membership of H_f and $H_{\bar{f}}$ in S_p

As an application of our result, we provide a characterization of simultaneous membership of H_f and $H_{\bar{f}}$ in S_p. Previously known characterizations for those f such that both H_f and $H_{\bar{f}}$ are in S_p were given in terms of the mean oscillation of f by Zhu [35] and Xia [31] for the Bergman space and by Xu [31] and Pau [26] in the context of the weighted Bergman spaces. We show that an analogous statement of [26] remains true with $\varphi \in \mathcal{W}$ for all $0 < p < \infty$. To this end, we give the notion of mean oscillation. Let $f \in L^2_{loc}(\mathbb{D})$ and $r > 0$. The mean oscillation of f at $z \in \mathbb{D}$ is defined by

$$MO_r(f)(z) = \left(\frac{1}{D^r(z)} \int_{D^r(z)} |f - \widehat{f}_r(z)|^2 \, dA \right)^{1/2}.$$

The following lemma shows the connection between $MO_r(f)(z)$ and $G_r(f)(z)$.

Lemma 4.1 *Suppose $\rho \in \mathcal{L}$, $0 < r < \infty$, $0 < p \leq \infty$ and $f \in L^2_{loc}(\mathbb{D})$. Then $G_r(f) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$ and $G_r(\bar{f}) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$ if and only if $MO_r(f) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$. Moreover,*

$$\|G_r(f)\|_{L^p(\mathbb{D},\rho^{-2}\,dA)} + \|G_r(\bar{f})\|_{L^p(\mathbb{D},\rho^{-2}\,dA)} \simeq \|MO_r(f)\|_{L^p(\mathbb{D},\rho^{-2}\,dA)}.$$

Proof It is trivial that

$$G_r(f)(z) \leq MO_r(f)(z) \quad \text{and} \quad G_r(\bar{f})(z) \leq MO_r(f)(z).$$

As shown in the proof of Lemma 6.2 of [13], the reverse inequality

$$MO_r(f)(z) \lesssim G_r(f)(z) + G_r(\bar{f})(z),$$

is easy to modify. We leave the detail to interested readers. □

Combining the previous lemma with Theorem 3.1, we obtain the characterization of simultaneous membership of H_f and $H_{\bar{f}}$ in S_p.

Theorem 4.1 *Let $\varphi \in \mathcal{W}$ with $\frac{1}{\sqrt{\Delta\varphi}} \simeq \rho \in \mathcal{L}_0$, $0 < p < \infty$, $0 < r \leq \alpha$, and $f \in \mathcal{S}$ such that $G_r(f) \in L^\infty$. Then both H_f and $H_{\bar{f}}$ in S_p if and only if $MO_r(f) \in L^p(\mathbb{D}, \rho^{-2}\, dA)$.*

References

1. Arazy, J., Fisher, S.D., Peetre, J.: Hankel operators on weighted Bergman spaces. Am. J. Math. **110**, 989–1053 (1988)
2. Arroussi, H., Park, I., Pau, J.: Schatten class Toeplitz operators acting on large weighted Bergman spaces. Stud. Math. **229**, 203–221 (2015)
3. Bommier-Hato, H., Constantin, O.: Big Hankel operators on vector-valued Fock spaces in $\mathbb{C}^d$. Integr. Equ. Oper. Theory **90** (2018)
4. Borichev, A., Dhuez, R., Kellay, K.: Sampling and interpolation in large Bergman and Fock spaces. J. Funct. Anal. **242**, 563–606 (2007)
5. Constantin, O., Ortega-Cerdà, J.: Some spectral properties of the canonical solution operator to $\bar{\partial}$ on weighted Fock spaces. J. Math. Anal. Appl. **377**, 353–361 (2011)
6. El-Fallah, O., Mahzouli, H., Marrhich, I., Naqos, H.: Asymptotic behavior of eigenvalues of Toeplitz operators on the weighted analytic spaces. J. Funct. Anal. **270**, 4614–4630 (2016)
7. Fang, Q., Xia, J.: Hankel operators on weighted Bergman spaces and norm ideals. Complex Anal. Oper. Theory **12**, 629–668 (2018)
8. Galanopoulos, P.: Schatten Class Hankel Operators on Large Bergman spaces, arXiv:2106.05898v1

9. Göğüş, N.G., Şahutoğlu, S.: Schatten class Hankel and $\overline{\partial}$-Neumann operators on pseudoconvex domains in $\mathbb{C}^n$. Monatsh. Math. **187**, 237–245 (2018)
10. Hörmander, L.: An Introduction to Complex Analysis in Several Variables, 3rd edn revised. North Holland, Amsterdam (1990)
11. Hu, Z., Lv, X.: Positive Toeplitz operators between different doubling Fock spaces. Taiwan. J. Math. **21**, 467–487 (2017)
12. Hu, Z., Pau, J.: Hankel operators on exponential Bergman spaces. Sci. China Math. **65**(2), 421–442 (2022)
13. Hu, Z., Virtanen, J.A.: Schatten class Hankel operators on the Segal–Bargmann space and the Berger–Coburn phenomenon. Trans. Am. Math. Soc. **375**(5), 3733–3753 (2022)
14. Hu, Z., Virtanen, J.A.: IDA and Hankel operators on Fock spaces. Anal. PDE (to appear). arXiv:2111.04821
15. Hu, Z., Lv, X., Schuster, A.P.: Bergman spaces with exponential weights. J. Funct. Anal. **276**, 1402–1429 (2019)
16. Isralowitz, J.: Schatten p-class Hankel operators on the Segal–Bargmann space $H^2(\mathbb{C}^n, d\mu)$ for $0 < p < 1$. J. Oper. Theory **66**, 145–160 (2011)
17. Isralowitz, J., Virtanen, J.A., Wolf, L.: Schatten class Toeplitz operators on generalized Fock spaces. J. Math. Anal. Appl. **421**, 329–337 (2015)
18. Li, H.: Schatten class of Hankel and Toeplitz operators on the Bergman space of strongly pseudoconvex domains. Proc. Am. Math. Soc. **119**, 1211–1221 (1993)
19. Lin, P., Rochberg, R.: Trace ideal criteria for Toeplitz and Hankel operators on the weighted Bergman spaces with exponential type weights. Pac. J. Math. **173**, 127–146 (1996)
20. Luecking, D.H.: Trace ideal criteria for Toeplitz operators. J. Funct. Anal. **73**, 345–368 (1987)
21. Luecking, D.H.: Characterizations of certain classes of Hankel operators on the Bergman spaces of the unit disk. J. Funct. Anal. **110**, 247–271 (1992)
22. Lv, S., Xu, X.: Schatten Class Hankel operators on the weighted Bergman space of the unit ball. J. Math. Res. Expos. (1995)
23. Miao, J.: Schatten class Hankel operators on the harmonic Bergman space of the unit ball. Integr. Equ. Oper. Theory **59**, 53–65 (2007)
24. Oleinik, V.L.: Embedding theorems for weighted classes of harmonic and analytic functions. J. Sov. Math. **9**, 228–243 (1978)
25. Oleinik, V.L., Perel'man, G.S.: Carleson's embedding theorems for a weighted Bergman spaces. Math. Notes **7**, 577–581 (1990)
26. Pau, J.: Characterization of Schatten-class Hankel operators on weighted Bergman spaces. Duke Math. J. **165**, 2771–2791 (2016)
27. Pelaez, J.A., Rattya, J.: On the boundedness of Bergman projection. In: Advanced Courses of Mathematical Analysis VI, pp. 113–132. World Sci. Publ., Hackensack, NJ (2017)
28. Perälä, A., Virtanen, J.: A note on the Fredholm properties of Toeplitz operators on weighted Bergman spaces with matrix-valued symbols. Oper. Matrices **5**, 97–106 (2011)
29. Raimondo, R.C.: Schatten–von Neumann Hankel operators on the Bergman space of planar domains. Integr. Equ. Oper. Theory **62**, 219–232 (2008)
30. Seip, K., Youssfi, E.H.: Hankel operators on Fock spaces and related Bergman kernel estimates. J. Geom. Anal. **23**, 170–201 (2013)
31. Xia, J.: Hankel operators in the Bergman space and Schatten p-classes: the case $1 < p < 2$. Proc. Am. Math. Soc. **129**, 3559–3567 (2001)
32. Xia, J., Zheng, D.: Standard deviation and Schatten class Hankel operators on the Segal–Bargmann space. Indiana Univ. Math. J. **53**, 1381–1399 (2004)
33. Zhang, Y., Wang, X., Hu, Z.: Toeplitz operators on Bergman spaces with exponential weights. Complex Variables Elliptic Equ. **2**, 1–34 (2022)
34. Zhu, K.: Schatten class Hankel operators on the Bergman space of the unit ball. Am. J. Math. **113**, 147–167 (1991)
35. Zhu, K.: Operator Theory in Function Spaces, Mathematical Surveys and Monographs, 2nd edn. American Mathematical Society, Providence (2007)