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ORIGINAL PAPER

# On a new type of boundary condition

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**Abstract**
Pushed by inverse problems in conductivity in the 3-dimensional setting, we introduce new types of boundary conditions for variational and PDE problems, that in some sense cover the middle space between the classical Dirichlet and Neumann conditions, meant in a essentially different way with respect to mixed boundary conditions. These new boundary conditions are associated with special subspaces of Sobolev spaces between $H^1_0(\Omega)$ and the full space $H^1(\Omega)$. Though problems can be considered in $W^{1,p}(\Omega)$ for $p \neq 2$, in this initial contribution we just examine existence and optimality for regular variational problems under typical assumptions within the scope of $H^1(\Omega)$. In addition to the existence of minimizers, we would like to stress the intriguing form of optimality at the boundary $\partial\Omega$. We especially treat the case $N = 3$, which is the most interesting case, and describe similar conditions in any dimension $N \geq 2$. The numerical approximation definitely requires new ideas.

**Keywords** Sobolev subspaces · Variational problems · Optimality conditions

**Mathematics Subject Classification** 35J20 · 35J25 · 49K20

## 1 Introduction

We would like to examine some new types of boundary conditions for variational problems and PDEs that are motivated by the application of some variational methods to inverse problems in conductivity in the 3-dimensional situation. The 2-dimensional situation has been analyzed recently in [7]. Check also [2] for different ideas for the 3-dimensional case. Though one can deal with the general case $N \geq 2$, for the sake of definiteness, we will restrict attention most of the time to the case $N = 3$ to better understand the ideas. For dimension $N = 1$, the situation is meaningless essentially because the boundary of an interval is a disconnected set consisting of two isolated points. We will point out to this difficulty later. In some sense, this new type of boundary condition is a middle point between the classical Dirichlet and Neumann conditions as they are associated with subspaces between $H^1_0(\Omega)$

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and $H^1(\Omega)$. We will comment on the situation in full generality in the final section. Some of the ideas that motivated this analysis are also reminiscent of similar ones in [3]. We have found no more references in this regard.

Let $\Omega \subset \mathbb{R}^3$ be a domain as smooth as we may need it to be. We can think of $\Omega$ as a ball, to make things more transparent. For an arbitrary function $w \in H^1(\Omega)$ with range $J \equiv \text{image} w \subset \mathbb{R}$, an interval, consider the following subspaces of $H^1(\Omega)$

$$\mathbb{L}_w = \{\psi(w) : \psi : J \to \mathbb{R}, \text{ measurable}, \psi(w) \in H^1(\Omega)\},$$
$$\mathbb{H}_w = \mathbb{L}_w + H_0^1(\Omega).$$

Functions of the form $\psi(w)$ for $\psi \in W^{1,\infty}(J)$ belong to $\mathbb{L}_w$ if $\Omega$ is bounded, for example. Note how $\mathbb{L}_w$ is a subspace of $H^1(\Omega)$ given the conditions assumed on feasible functions $\psi$ in $\mathbb{L}_w$. It will be more precisely defined below (see Sect. 3). It is elementary to realize that indeed

$$H_0^1(\Omega) \subset \mathbb{L}_w + H_0^1(\Omega) \subset H^1(\Omega),$$

for any $w \in H^1(\Omega) \setminus H_0^1(\Omega)$. Intuitively, functions $v \in \mathbb{H}_w$ are such that their traces on $\partial\Omega$ only depend on $w$ in the sense that $v = \psi(w)$ on $\partial\Omega$ for arbitrary real functions $\psi$.

Any other additional, given function $v_0 \in H^1(\Omega)$, determines the linear manifold $v_0 + \mathbb{H}_w$. In fact, if $v_0 \in \mathbb{H}_w$, then such linear manifold yields back the subspace $\mathbb{H}_w$. To avoid this special situation in which $v_0 + \mathbb{H}_w$ is, in fact, the same subspace $\mathbb{H}_w$, we must make sure that $v_0 \notin \mathbb{H}_w$. We want to understand the variational problem

$$\text{Minimize in } v \in v_0 + \mathbb{H}_w : \quad \frac{1}{2}\int_\Omega |\nabla v(\boldsymbol{x})|^2 \, d\boldsymbol{x}, \tag{1.1}$$

or more generally

$$\text{Minimize in } v \in v_0 + \mathbb{H}_w : \quad \frac{1}{2}\int_\Omega \phi(\nabla v(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) \, d\boldsymbol{x}, \tag{1.2}$$

for a convex integrand

$$\phi(\mathbf{v}, v, \boldsymbol{x}) : \mathbb{R}^3 \times \mathbb{R} \times \Omega \to \mathbb{R}$$

under the additional standard quadratic growth condition

$$c(|\mathbf{v}|^2 + v^2 - 1) \le \phi(\mathbf{v}, v, \boldsymbol{x}) \le C(|\mathbf{v}|^2 + v^2 + 1).$$

There are three main initial points of interest:

(1) existence of optimal solutions $v \in v_0 + \mathbb{H}_w$;
(2) form of boundary requirements for such a minimizer $v$;
(3) numerical approximation of $v$.

We will focus, in this first contribution, on the first two issues. For the third one, new ideas are necessary as the usual finite element software packages are not typically prepared to deal with this kind of boundary conditions. Our main results are the following.

**Theorem 1.1** *Under the specified assumptions, there are minimizers $v \in v_0 + \mathbb{H}_w$ for problem (1.2). If, in addition, the integrand $\phi$ is strictly convex in pairs $(\mathbf{v}, v)$ for each individual $\boldsymbol{x} \in \Omega$, then the minimizer is unique.*

The form of the associated boundary condition for a minimizer $v$ is quite appealing and unexpected.

**Theorem 1.2** *Under technical conditions that are specified below, a minimizer* $v \in v_0 + \mathbb{H}_w$ *of problem* (1.2) *is a weak solution of*

$$-\text{div}[\nabla_{\mathbf{v}}\phi(\nabla v, v, \boldsymbol{x})] + \phi_v(\nabla v, v, \boldsymbol{x}) = 0 \quad in\ \Omega,$$

*together with*

$$\int_{\{w=\lambda\}\cap\partial\Omega} \nabla_{\mathbf{v}}\phi(\nabla v(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, d\boldsymbol{x} = 0 \tag{1.3}$$

*for every* $\lambda \in J$ (*recall* $J = \text{image} w$), *where* $\boldsymbol{n}$ *is the outer, unit normal to* $\partial\Omega$.

Note how optimality condition (1.3) is an average condition on the normal component over level sets of $w$ restricted to $\partial\Omega$.

To see more clearly the structure of such problems, and gain some initial intuition with this kind of boundary conditions, let us state briefly the conclusions of our results for the following particular case:

(1) domain $\Omega = \mathbf{B}$, the unit ball of $\mathbb{R}^3$;
(2) quadratic integrand

$$\phi(\mathbf{v}, v, \boldsymbol{x}) = \frac{1}{2}|\mathbf{v}|^2 + \frac{1}{2}v^2 - f(\boldsymbol{x})v,$$

for a certain $f \in L^2(\Omega)$;
(3) $w(\boldsymbol{x}) = x_1$, $v_0 \equiv 0$.

It is elementary to conclude that we are looking at the problem

$$\text{Minimize in } v \in H^1(\mathbf{B}) : \quad \int_{\mathbf{B}} \left[\frac{1}{2}|\nabla v(\boldsymbol{x})|^2 + \frac{1}{2}v(\boldsymbol{x})^2 - f(\boldsymbol{x})v(\boldsymbol{x})\right] d\boldsymbol{x}$$

subject to the condition that the restriction of $v$ to $\partial\mathbf{B}$ is a function of $x_1$ alone. There is a unique minimizer $v$ for such a problem, according to our results below. Optimality conditions lead to the PDE-problem

$$-\Delta v + v = f \text{ in } \mathbf{B},$$

together with

$$v(\boldsymbol{x}) = \psi(x_1) \text{ on } \partial\mathbf{B}$$

for a certain unknown function $\psi$ of a single variable, that is determined through the condition

$$\int_{\{x_1=\lambda\}\cap\partial\Omega} \nabla v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, d\boldsymbol{x} = 0$$

for every $\lambda \in [-1, 1]$.

Before starting with our analysis, it is worth to devote some time to explain the reasons that motivate such boundary conditions, beyond its purely mathematical interest. It will also help us in better appreciating its nature and its geometrical meaning.

We know that when the normal derivative is involved in differential problems, one has to pay attention to the translation-invariant issue. We briefly comment on such situation too. The last section indicates the changes in a higher-dimensional scenario in full generality.

## 2 Inverse problems in conductivity

Our discussion in this section is, for the most part, informal. Our only aim is to motivate how the condition $v \in v_0 + \mathbb{H}_w$ arises in some problems, and how it can be interpreted geometrically.

A typical Calderón problem in dimension 3 (for one measurement) reads as follows:

For a pair of boundary data $(u^\circ, v^\circ)$ taken from a suitable class on $\partial\Omega$, find a conductivity coefficient

$$\gamma(\boldsymbol{x}) : \Omega \to \mathbb{R}^+$$

such that the unique solution $u$ of the problem

$$\operatorname{div}(\gamma \nabla u) = 0 \quad \text{in } \Omega, \quad u = u^\circ \quad \text{on } \partial\Omega, \tag{2.1}$$

complies with

$$\gamma \nabla u \cdot \boldsymbol{n} = v^\circ \quad \text{on } \partial\Omega \tag{2.2}$$

as well.

The literature on this problem is quite abundant (check, for instance, the two recent general sources [1], [5]). The so-called Dirichlet-to-Neumann operator is one main tool in this analysis.

A variational perspective on this problem, as developed in [7] for the 2-dimensional case, aims at determining the conductivity coefficient $\gamma$ through the solution of a non-linear, non-convex vector variational problem. In the 3-dimensional case, problem (2.1) can be formally interpreted as

$$\gamma \nabla u = \nabla v \wedge \nabla w \quad \text{in } \Omega, \quad u = u^\circ \quad \text{on } \partial\Omega, \tag{2.3}$$

for suitable functions $v$ and $w$, while the Neumann condition (2.2) becomes

$$(\nabla v \wedge \nabla w) \cdot \boldsymbol{n} = v^\circ \quad \text{on } \partial\Omega.$$

Some times the functions $v$ and $w$ in (2.3) are referred to as Clebsch potentials. If we multiply (2.3) by $\nabla w$, and divide through by $\gamma$, we find

$$\nabla u \wedge \nabla w = \frac{1}{\gamma}(\nabla v \wedge \nabla w) \wedge \nabla w.$$

This identity is informing us that

$$\operatorname{div}\left[\frac{1}{\gamma}(\nabla v \wedge \nabla w) \wedge \nabla w\right] = 0 \quad \text{in } \Omega. \tag{2.4}$$

If the coefficient $\gamma$ is unknown, we can always try to recover it through the quotient

$$\gamma = \frac{|\nabla v \wedge \nabla w|}{|\nabla u|}.$$

If we replace this formula in the two equations (2.1) and (2.4), we arrive at the two coupled equations

$$\operatorname{div}\left[\frac{|\nabla v \wedge \nabla w|}{|\nabla u|}\nabla u\right] = 0 \quad \text{in } \Omega, \tag{2.5}$$

$$\operatorname{div}\left[\frac{|\nabla u|}{|\nabla v \wedge \nabla w|}(\nabla v \wedge \nabla w) \wedge \nabla w\right] = 0 \quad \text{in } \Omega, \tag{2.6}$$

that must be completed with the boundary conditions

$$u = u^\circ, \ (\nabla v \wedge \nabla w) \cdot \boldsymbol{n} = v^\circ \quad \text{on } \partial\Omega.$$

It turns out that the two previous PDEs correspond exactly to the Euler-Lagrange equations for the functional

$$E(u, v) = \int_\Omega |\nabla u(\boldsymbol{x})| \, |\nabla v(\boldsymbol{x}) \wedge \nabla w(\boldsymbol{x})| \, d\boldsymbol{x} \tag{2.7}$$

where the function $w(\boldsymbol{x})$ is assumed to be given. The boundary condition for $u$ is clear as it is a standard Dirichlet condition. The one for $v$ as

$$(\nabla v \wedge \nabla w) \cdot \boldsymbol{n} = v^\circ \quad \text{on } \partial\Omega \tag{2.8}$$

is not so. The strategy to determine or approximate the unknown conductivity coefficient $\gamma$ from the measurement $(u^\circ, v^\circ)$ is then to solve system (2.5)–(2.6) as the optimality system of the functional $E$ in (2.7) under the given boundary condition $u = u^\circ$ for $u$, and (2.8) for $v$. We would like to argue that this boundary condition is of the form described in the Introduction.

Suppose $v_0$ is a particular function such that (2.8) holds

$$(\nabla v_0 \wedge \nabla w) \cdot \boldsymbol{n} = v^\circ \quad \text{on } \partial\Omega.$$

**Proposition 2.1** *The boundary condition* (2.8) *for $v$ is equivalent to*

$$v \in v_0 + \mathbb{H}_w,$$

*with the notation introduced earlier.*

***Proof*** Thanks to the linearity of condition (2.8) with respect to $v$, it suffices to check that

$$v \in \mathbb{H}_w \iff (\nabla v \wedge \nabla w) \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega.$$

We can write (2.8), for $v^\circ = 0$, in the form

$$\nabla v \cdot (\nabla w \wedge \boldsymbol{n}) = 0 \quad \text{on } \partial\Omega. \tag{2.9}$$

For vectors $\mathbf{a} \in \mathbb{R}^3$, we will write

$$\mathbf{a} = \mathbf{a} \cdot \boldsymbol{n}\,\boldsymbol{n} + \mathbf{a}^{\boldsymbol{t}}$$

for the decomposition of $\mathbf{a}$ in the $\boldsymbol{n}$-direction, and its corresponding orthogonal component to $\boldsymbol{n}$ for any unit vector $\boldsymbol{n} \in \mathbb{R}^3$. If at points $\boldsymbol{x}$ on $\partial\Omega$, $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ indicates the outer, unit normal, (2.9) becomes

$$\nabla v^{\boldsymbol{t}} \cdot (\nabla w^{\boldsymbol{t}} \wedge \boldsymbol{n}) = 0 \quad \text{on } \partial\Omega,$$

because the contribution in this equation of both $\nabla v$ and $\nabla w$ along the normal direction $\boldsymbol{n}$ vanishes. Since now both vectors $\nabla v^{\boldsymbol{t}}$, $\nabla w^{\boldsymbol{t}}$ are orthogonal to $\boldsymbol{n}$, the previous equation amounts to

$$\nabla v^{\boldsymbol{t}} \parallel \nabla w^{\boldsymbol{t}} \quad \text{on } \partial\Omega,$$

and this in turn implies that there is a functional relationship between $v$ and $w$ on $\partial\Omega$, i.e.

$$v = \psi(w) \quad \text{on } \partial\Omega.$$

This exactly means that $v \in \mathbb{H}_w$. □

The conclusion is then

$$\begin{aligned} v \in \mathbb{H}_w \iff & v = \psi(w) \quad \text{on } \partial\Omega \\ \iff & \nabla v \cdot (\nabla w \wedge \boldsymbol{n}) = 0 \quad \text{on } \partial\Omega \iff \nabla v^{\boldsymbol{t}} \parallel \nabla w^{\boldsymbol{t}} \quad \text{on } \partial\Omega. \end{aligned}$$

Note how these identifications furnish the right way to interpret the boundary condition on left-hand side: $v \in \mathbb{H}_w$ when the level curves of $v$, restricted to $\partial\Omega$, are the same as those of $w$. This boundary condition restricts, in this way, the level curves of feasible functions on $\partial\Omega$.

The 3-dimensional inverse-conductivity problem for a single measurement $(u^\circ, v^\circ)$ can be treated in three steps as follows.

(1) Select a non-constant function $w \in H^1(\Omega)$, and find $v_0 \in H^1(\Omega)$ such that

$$v^\circ = \nabla v_0 \cdot (\nabla w \wedge \boldsymbol{n}) \quad \text{on } \partial\Omega.$$

(2) Find a solution $(u, v)$ of the coupled, non-linear system of PDEs

$$\operatorname{div}\left[\frac{|\nabla v \wedge \nabla w|}{|\nabla u|}\nabla u\right] = 0 \quad \text{in } \Omega, \tag{2.10}$$

$$\operatorname{div}\left[\frac{|\nabla u|}{|\nabla v \wedge \nabla w|}(|\nabla w|^2\mathbf{1} - \nabla w \otimes \nabla w)\nabla v\right] = 0 \quad \text{in } \Omega, \tag{2.11}$$

under the boundary conditions

$$u - u^\circ \in H^1_0(\Omega), \quad v - v_0 \in \mathbb{H}_w \tag{2.12}$$

on $\partial\Omega$.

(3) Put

$$\gamma = \frac{|\nabla v \wedge \nabla w|}{|\nabla u|} \quad \text{in } \Omega.$$

The interesting fact is that the pair $(u, v)$ solution of the impressive non-linear system (2.10)–(2.11) under boundary conditions (2.12) can be sought (in fact this is the only way one can deal with such a system) as a minimizer for the variational problem

$$\text{Minimize in } (u, v) \in H^1(\Omega; \mathbb{R}^2) : \quad \int_\Omega |\nabla u|\,|\nabla v \wedge \nabla w|\,d\boldsymbol{x} \tag{2.13}$$

under

$$u - u^\circ \in H^1_0(\Omega), \quad v - v_0 \in \mathbb{L}_w + H^1_0(\Omega).$$

The description of this procedure to solve the inverse-conductivity problem is purely formal since it is not clear how to handle either Steps 1 or 2. Note that problem (2.13) is a vector, non-(quasi)convex, non-coercive problem with an unfamiliar boundary conditions for $v$. It is a specific problem under the most adverse of circumstances. Yet some interesting things can be tried out as in the 2-dimensional [7], and even the 3-dimensional cases [2]. We plan to address the analysis of that particular problem from this perspective in the future.

## 3 The subspace $\mathbb{L}_w$

Once we have motivated the interest of such a boundary condition

$$v \in v_0 + \mathbb{L}_w + H_0^1(\Omega)$$

for a fixed, given function $w \in H^1(\Omega)$, we would like to investigate the two fundamental issues related to the corresponding variational problem

$$\text{Minimize in } v \in v_0 + \mathbb{L}_w + H_0^1(\Omega) : \quad \int_\Omega \phi(\nabla v(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x})\, d\boldsymbol{x} \tag{3.1}$$

for a certain integrand

$$\phi(\mathbf{v}, v, \boldsymbol{x}) : \mathbb{R}^3 \to \mathbb{R}$$

which is $\mathcal{C}^1$ with respect to variables $(\mathbf{v}, v)$. Those two basic questions are the existence of minimizers, and the form of optimality conditions that such minimizers should comply with.

Before we prove existence of minimizers, we ought to define more precisely the subspace $\mathbb{L}_w$. To this end, we use Sobolev spaces with weights as introduced and considered in many places (see the classic monographs [6] or [8]). We do it in a very particular situation.

**Definition 3.1** (1) A weight $\omega(\lambda)$ in $J \subset \mathbb{R}$ is a measurable strictly positive function in $J$.
(2) For a weight $\omega(\lambda) : J \to \mathbb{R}^+$, we define the weighted Lebesgue space

$$L^2_\omega(J) = \left\{\psi,\ \text{measurable} : \int_J \phi(\lambda)^2 \omega(\lambda)\, d\lambda < \infty\right\}.$$

(3) For two given weights $\omega_i(\lambda) : J \to \mathbb{R}^+$, $i = 0, 1$, we define the corresponding weighted Sobolev space

$$H^1_{\omega_0,\omega_1}(J) = \{\psi \in L^2_{\omega_0}(J) : \psi' \in L^2_{\omega_1}(J)\}.$$

Associated with the function $w \in H^1(\Omega)$ with range $J$, we define its two corresponding weights

$$\omega_0(\lambda) = |\{w = \lambda\} \cap \Omega| = \int_{\{w=\lambda\}\cap\Omega} dS(\boldsymbol{x}), \tag{3.2}$$

$$\omega_1(\lambda) = \int_{\{w=\lambda\}\cap\Omega} |\nabla w(\boldsymbol{x})|^2\, dS(\boldsymbol{x}), \tag{3.3}$$

where $dS(\boldsymbol{x})$ is the appropriate dimensional Lebesgue measure.

To have a more intuitive feeling about these spaces, let us look at some simple situations.

***Example 3.1*** Consider the class of functions in $H^1(\mathbf{B})$, for the unit ball $\mathbf{B}$ of $\mathbb{R}^3$, that are functions $u = u(x_1)$ of $x_1$ alone. It is not difficult to conclude that the property characterizing these functions is

$$\int_{-1}^1 u(x_1)^2(1 - x_1^2)\, dx_1 + \int_{-1}^1 u'(x_1)^2(1 - x_1^2)\, dx_1 < \infty.$$

In this particular case both $\omega_0$ and $\omega_1$ are equal to

$$\omega(\lambda) = (1 - \lambda^2),$$

and $H^1_{\omega,\omega}(-1, 1)$ is precisely the class of functions $u = u(x_1)$ of one variable $x_1$ in the interval $(-1, 1)$ that, in fact, belong to $H^1(\mathbf{B})$.

***Example 3.2*** In the same domain $\mathbf{B}$, the unit ball in $\mathbb{R}^3$, take

$$w(\boldsymbol{x}) = w(x_1, x_2, x_3) = \frac{x_3}{1 + x_1^2 + x_2^2}.$$

The level surfaces are paraboloids of various apertures. For this case, we have

$$\omega_0(\lambda) = |\{x_3 = \lambda(1 + x_1^2 + x_2^2)\} \cap \mathbf{B}|$$

while

$$\omega_1(\lambda) = \int_{\{x_3=\lambda(1+x_1^2+x_2^2)\}\cap\mathbf{B}} |\nabla w(\boldsymbol{x})|^2 \, dS(\boldsymbol{x}),$$

for $\lambda \in (-1, 1)$. The class of real functions $\psi(\lambda)$ for $\lambda \in (-1, 1)$ such that the composition $\psi \circ w$ belongs to $H^1(\mathbf{B})$ is just $H^1_{\omega_0,\omega_1}(-1, 1)$.

Computing these two weights $\omega_0$ and $\omega_1$ is an interesting exercise in Multivariate Calculus. For instance

$$\omega_0(\lambda) = \begin{cases} \frac{\pi}{6\lambda^2}\left[(1 + 4\lambda^2 r(\lambda)^2)^{3/2} - 1\right], & \lambda \in [-1, 1] \setminus \{0\}, \\ \pi, & \lambda = 0, \end{cases}$$

where

$$r(\lambda) = 1 - \left(\sqrt{2 + \frac{1}{4\lambda^2}} - \frac{1}{2\lambda}\right)^2.$$

A similar computation can lead to a more explicit formula for

$$\omega_1(\lambda) = \int_{\{x_3=\lambda(1+x_1^2+x_2^2)\}\cap\mathbf{B}} \frac{1 + 4x_3^2(x_1^2 + x_2^2)}{(1 + x_1^2 + x_2^2)^2} \, dS(\boldsymbol{x}).$$

**Definition 3.2** For a function $w \in H^1(\Omega)$ with range $J \subset \mathbb{R}$, we put

$$H^1_w(J) = H^1_{\omega_0,\omega_1}(J)$$

for weights given in (3.2) and (3.3).

From Measure Theory, we know that

$$d\boldsymbol{x}|_\Omega = dS(\boldsymbol{x})|_{\{w=\lambda\}\cap\Omega} \otimes d\lambda|_J \,,$$

and thus integrals of the form

$$\int_\Omega G(\boldsymbol{x})\phi(w(\boldsymbol{x}))\, d\boldsymbol{x} \tag{3.4}$$

can be computed through the decomposition

$$\int_J \phi(\lambda) \left[\int_{\{w=\lambda\}\cap\Omega} G(\boldsymbol{x})\, dS(\boldsymbol{x})\right] d\lambda, \tag{3.5}$$

for a measurable $G$ such that the product $G(\boldsymbol{x})\psi(w(\boldsymbol{x}))$ turns out to be integrable in $\Omega$. In particular,

$$\int_\Omega |\psi \circ w(\boldsymbol{x})|^2 \, d\boldsymbol{x} = \int_J |\psi(\lambda)|^2 \left[ \int_{\{w=\lambda\}\cap\Omega} 1 \, dS(\boldsymbol{x}) \right] d\lambda$$
$$= \int_J |\psi(\lambda)|^2 \omega_0(\lambda) \, d\lambda$$

according to (3.2), while

$$\int_\Omega |\nabla(\psi \circ w)(\boldsymbol{x})|^2 \, d\boldsymbol{x} = \int_\Omega \psi'(w(\boldsymbol{x}))^2 |\nabla w(\boldsymbol{x})|^2 \, d\boldsymbol{x}$$
$$= \int_J \psi'(\lambda)^2 \left[ \int_{\{w=\lambda\}\cap\Omega} |\nabla w(\boldsymbol{x})|^2 \, dS(\boldsymbol{x}) \right] d\lambda$$
$$= \int_J \psi'(\lambda)^2 \omega_1(\lambda) \, d\lambda,$$

according to (3.3). These calculations clearly show that

$$H^1_w(J) \subset \mathbb{L}_w.$$

**Definition 3.3** We take

$$\mathbb{L}_w \equiv H^1_w(J)$$

which is a closed subspace of $H^1(\Omega)$.

## 4 Existence

We are now ready to deal with our basic existence result.

**Theorem 4.1** *Let the density*

$$\phi(\mathbf{v}, v, \boldsymbol{x}) : \mathbb{R}^3 \times \mathbb{R} \times \Omega \to \mathbb{R}$$

*be convex in the* $\mathbf{v}$*-variable, and coercive in the sense*

$$c(|\mathbf{v}|^2 + v^2 - 1) \le \phi(\mathbf{v}, v, \boldsymbol{x}), \quad \mathbf{v} \in \mathbb{R}^3, v \in \mathbb{R}, \boldsymbol{x} \in \Omega, c > 0. \tag{4.1}$$

*Then problem (3.1) admits, at least, one optimal solution. If, in addition,* $\phi$ *is assumed to be strictly convex in pairs* $(\mathbf{v}, v)$*, the solution is unique.*

Though the proof may look straightforward, it requires a preliminary interesting discussion because the subspace $\mathbb{H}_w$ of $H^1(\Omega)$, as the sum of two additional closed subspaces $\mathbb{L}_w$ and $H^1_0(\Omega)$, may not be a direct sum, i.e. the intersection might not be the trivial subspace. This, in particular, would imply that $\mathbb{H}_w$ might not be weakly closed, or simply closed for that matter being a subspace. Once this trouble is overcome, the proof is elementary along the direct method of the Calculus of Variations.

***Proof*** Suppose first that the function $w \in H^1(\Omega)$ is such that the two subsets of $\mathbb{R}$

$$J \equiv \{w(\boldsymbol{x}) : \boldsymbol{x} \in \Omega\}, \quad J_\partial \equiv \{w(\boldsymbol{x}) : \boldsymbol{x} \in \partial\Omega\}, \tag{4.2}$$

are the same one. Note that this condition is impossible in dimension N=1 because $\partial\Omega$ would be a discrete subset. We claim that, in this situation, the intersection $\mathbb{L}_w \cap H^1_0(\Omega)$ is trivial.

Indeed, if $\psi(w) \in H_0^1(\Omega)$, then we must have $\psi|_{J_\partial} = 0$, which, under our hypothesis, implies $\psi|_J = 0$. This clearly means that $\psi(w) \equiv 0$.

On the other hand, both subspaces, independently, are closed in $H^1(\Omega)$, and then one can define the two continuous (non-orthogonal) projections

$$\pi_1 : \mathbb{L}_w + H_0^1(\Omega) \mapsto \mathbb{L}_w, \quad \pi_2 : \mathbb{L}_w + H_0^1(\Omega) \mapsto H_0^1(\Omega),$$

in such a way that

$$\|\pi_i u\| \le C\|u\|, \quad i = 1, 2, u \in \mathbb{H}_w, C > 0.$$

We claim that $\mathbb{H}_w$ is then closed (or weakly closed) in $H^1(\Omega)$. To briefly check this, suppose that

$$u_j = \psi_j(w) + v_j \to u \text{ in } H^1(\Omega).$$

By the just indicated properties of the projections and the fact that both $\mathbb{L}_w$ and $H_0^1(\Omega)$ are closed, for suitable subsequences that we do not care to relabel, we would have

$$\psi_j(w) \rightharpoonup \psi(w), \quad v_j \rightharpoonup v,$$

for some $\psi \in H_w^1(J)$, and $v \in H_0^1(\Omega)$. By the uniqueness of limits, we must necessarily have $u = \psi(w) + v$ and $u \in \mathbb{H}_w$. Again, since $\mathbb{H}_w$ is a subspace, it is also weakly closed.

With this information at our disposal, the proof of our result is now straightforward. Indeed, if $\{v_0 + v_j\}$ is minimizing, then by the coercivity (4.1), $\{v_j\}$ is uniformly bounded in $\mathbb{H}_w$ in such a way that a non-relabeled subsequence converges weakly to some $v \in \mathbb{H}_w$ by our discussion above. The sum $v_0 + v$ is feasible for our problem, and the convexity implies that it is a minimizer.

It remains to remove the constraint related to the two subsets $J$ and $J_\partial$ in (4.2). But this is easy if we realize that the subspace $\mathbb{H}_w$ only depends upon $w$ through its values at the boundary $\partial\Omega$. This is precisely the effect of adding $H_0^1(\Omega)$ to $\mathbb{L}_w$. Said differently,

$$w_1 - w_2 \in H_0^1(\Omega) \quad \text{implies} \quad \mathbb{H}_{w_1} = \mathbb{H}_{w_2}. \tag{4.3}$$

In particular, given $w$ we can change it to $\hat{w}$ respecting the same boundary conditions but enjoying property (4.2) without changing $\mathbb{H}_w$. This is always possible, thanks to the classic maximum principle, as we can take $\hat{w}$ to be, for instance, the harmonic function sharing with $w$ the boundary values along $\partial\Omega$.

The uniqueness under strict convexity is standard. □

According to (4.3), we will always assume, without further notice, that the function $w$ is taken to enjoy (4.2).

## 5 Optimality

Optimality conditions for the minimizer $u$ of problem (3.1) is the second most important issue to be understood. Since the relevant part of the functional concerning optimality and how it is reflected on a condition on the boundary is the dependence of the integrand $\phi$ on variable $\mathbf{v} = \nabla v$, we will reduce the main argument to such a simplified situation.

**Theorem 5.1** *Suppose $\phi(\mathbf{v}) : \mathbb{R}^3 \to \mathbb{R}$ is a $\mathcal{C}^1$-integrand for which,*

$$c(|\mathbf{v}|^2 - 1) \le \phi(\mathbf{v}) \le C(|\mathbf{v}|^2 + 1),$$

$$|\nabla\phi(\mathbf{v})| \le C(|\mathbf{v}|+1).$$

*If $v$ is a minimizer of the problem*

$$\textit{Minimize in } v \in v_0 + \mathbb{H}_w : \quad \int_\Omega \phi(\nabla v(\boldsymbol{x}))\,d\boldsymbol{x}$$

*then $v \in v_0 + \mathbb{H}_w$ is a weak solution of*

$$\operatorname{div}[\nabla\phi(\nabla v)] = 0 \textit{ in } \Omega,$$

*together with*

$$\int_{\{w=\lambda\}\cap\partial\Omega} \nabla\phi(\nabla v(\boldsymbol{x}))\cdot\boldsymbol{n}(\boldsymbol{x})\,d\boldsymbol{x} = 0$$

*for every $\lambda \in J$, where $\boldsymbol{n}$ is the outer, unit normal to $\partial\Omega$.*

***Proof*** As usual, one can perform variations of the form $v + \epsilon V$ for arbitrary $V \in \mathbb{H}_w$, and demand that

$$0 = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_\Omega \phi(\nabla v(\boldsymbol{x}) + \epsilon\nabla V(\boldsymbol{x}))\,d\boldsymbol{x}.$$

Under our hypotheses in the form of bounds on both $\phi$ and its gradient $\nabla\phi$, it is legitimate to take the derivative under the integral sign and conclude

$$0 = \int_\Omega \nabla\phi(\nabla v(\boldsymbol{x}))\cdot\nabla V(\boldsymbol{x})\,d\boldsymbol{x}$$

for all such $V$. We can select first $V \in H^1_0(\Omega)$ (by taking $\psi \equiv 0$), and conclude, through a usual integration by parts, that

$$\operatorname{div}[\nabla\phi(\nabla v)] = 0 \quad \text{in } \Omega.$$

Once we have this information, for a general function of the form $V = \psi(w)$, we would conclude, again by a usual integration by parts, that

$$\int_{\partial\Omega} \nabla\phi(\nabla v(\boldsymbol{x}))\cdot\boldsymbol{n}(\boldsymbol{x})\psi(w(\boldsymbol{x}))\,d\boldsymbol{x} = 0 \tag{5.1}$$

for all feasible $\psi$. If we put

$$G(\boldsymbol{x}) \equiv \nabla\phi(\nabla v(\boldsymbol{x}))\cdot\boldsymbol{n}(\boldsymbol{x}),$$

a similar discussion as the one related to (3.4) and (3.5) but restricted to $\partial\Omega$ instead of $\Omega$, leads to expressing (5.1) in the form

$$\int_J \int_{\{w=\lambda\}\cap\partial\Omega} G(\boldsymbol{x})\psi(w(\boldsymbol{x}))\;\left.dS(\boldsymbol{x})\right|_{\{w=\lambda\}}\,d\lambda.$$

Recall that $w$ is assumed to have been selected so that $J$ and $J_\partial$ are the same subset of $\mathbb{R}$. Those integrals therefore become

$$0 = \int_J \psi(\lambda)\mathbb{G}(\lambda)\,d\lambda, \quad \mathbb{G}(\lambda) = \int_{\{w=\lambda\}\cap\partial\Omega} G(\boldsymbol{x})\;\left.dS(\boldsymbol{x})\right|_{\{w=\lambda\}}.$$

The arbitrariness of $\psi \in H^1_w(J)$ forces, if all these integrals vanish, to

$$0 = \int_{\{w=\lambda\}\cap\partial\Omega} G(\boldsymbol{x})\;\left.dS(\boldsymbol{x})\right|_{\{w=\lambda\}},$$

for all $\lambda \in J$. This condition is nothing but

$$\int_{\{w=\lambda\}\cap\partial\Omega} \nabla\phi(\nabla v(\boldsymbol{x})) \cdot \boldsymbol{n}(\boldsymbol{x})\, d\boldsymbol{x} = 0$$

for all $\lambda \in J$. □

## 6 The situation of translation-invariance

Because of the relevance of problem (1.1), we would like to address its solution. The particular ingredient to be taken into account is the fact that $\mathbb{L}_w$ is translation invariant (adding a constant to an element of $\mathbb{L}_w$, keeps the function in the same subspace) if $\Omega$ is bounded. This implies that an additional normalization constraint must be enforced in case functionals are also translation invariant, like the one in problem (1.1), much in the same way as with Neumann boundary conditions. In such situations, we put

$$L^2_0(\Omega) = \left\{ u \in L^2(\Omega) : \int_\Omega u(\boldsymbol{x})\, d\boldsymbol{x} = 0 \right\},$$

and regard $\mathbb{L}_w$ as incorporating this integral constraint

$$\mathbb{L}_w \mapsto \mathbb{L}_w \cap L^2_0(\Omega).$$

Other normalization conditions can be used in the same way.

Once this peculiarity has been taken care of, the application of our results on existence and optimality are exactly like the ones in the previous section. The following is a direct corollary of the previous results. It refers to problem (1.1)

$$\text{Minimize in } v \in v_0 + \mathbb{H}_w : \quad \frac{1}{2}\int_\Omega |\nabla v(\boldsymbol{x})|^2\, d\boldsymbol{x} \tag{6.1}$$

where

$$\mathbb{H}_w = \mathbb{L}_w \cap L^2_0(\Omega) + H^1_0(\Omega).$$

**Corollary 6.1** (1) *There is a unique minimizer $v \in v_0 + \mathbb{H}_w$ for problem* (6.1).
(2) *The minimizer $v \in v_0 + \mathbb{H}_w$ is harmonic in $\Omega$, $\Delta v = 0$, and*

$$\int_{\{w=\lambda\}\cap\partial\Omega} \nabla v(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x})\, d\boldsymbol{x} = 0$$

*for all $\lambda \in J$.*

## 7 A more general framework

Once the main idea for the previous analysis has been settled, it is not difficult to treat other more general situations.

Let $\Omega \subset \mathbb{R}^N$ be a domain, and let

$$\mathbf{w}(\boldsymbol{x}) : \Omega \to \mathbb{R}^n, \quad n \le N,$$

be a mapping such that:

(1) $\mathbf{w} \in H^1(\Omega; \mathbb{R}^n)$;
(2) $D = \mathbf{w}(\Omega)$ is a domain in $\mathbb{R}^n$; and
(3) $D = \mathbf{w}(\partial\Omega)$, as well.

**Definition 7.1** We define the space

$$H^1_{\mathbf{w}}(D) = \{\Psi \in L^2_{\omega_0}(D) : \nabla\Psi \in L^2_{\omega_1}(D; \mathbb{R}^n)\}$$

where this time

$$\omega_0(\boldsymbol{y}) = |\{\mathbf{w} = \boldsymbol{y}\} \cap \Omega| = \int_{\{\mathbf{w}=\boldsymbol{y}\}\cap\Omega} 1\, dS(\boldsymbol{x}), \quad \boldsymbol{y} \in D, \tag{7.1}$$

$$\omega_1(\boldsymbol{y}) = \int_{\{\mathbf{w}=\boldsymbol{y}\}\cap\Omega} \nabla\mathbf{w}(\boldsymbol{x})\nabla\mathbf{w}(\boldsymbol{x})^T\, dS(\boldsymbol{x}) \in \mathbb{R}^{n\times n}, \quad \boldsymbol{y} \in D. \tag{7.2}$$

The space $L^2_{\mathbf{A}}(D; \mathbb{R}^n)$, for a positive-definite matrix field $\mathbf{A}(\boldsymbol{y}) \in \mathbb{R}^{n\times n}$, contains all measurable fields $\mathbf{u} : D \to \mathbb{R}^n$ such that

$$\int_D \mathbf{u}(\boldsymbol{y})^T\mathbf{A}(\boldsymbol{y})\mathbf{u}(\boldsymbol{y})\, d\boldsymbol{y} < \infty.$$

Mimicking the previous discussion, we introduce the following spaces.

**Definition 7.2** We introduce the closed subspace

$$\mathbb{L}_{\mathbf{w}} = \{\Psi \circ \mathbf{w} : \Psi \in H^1_{\mathbf{w}}(D)\}$$

of $H^1(\Omega; \mathbb{R}^n)$. Even more, if $\mathbb{W}$ is itself a closed subspace of $H^1_{\mathbf{w}}(D)$, we can also consider

$$\mathbb{L}_{\mathbf{w},\mathbb{W}} = \{\Psi \circ \mathbf{w} : \Psi \in \mathbb{W}\}$$

as a closed subspace of $H^1(\Omega; \mathbb{R}^n)$.

For a typical example where the subspace $\mathbb{W}$ could be taken as a proper subspace of $H^1_{\mathbf{w}}(D)$, one can think of

$$\mathbb{W} = \{\Psi \in H^1_{\mathbf{w}}(D) : \Psi|_{\hat{D}} \equiv \mathbf{0}\}$$

for a subset $\hat{D} \subset \overline{D}$. In the particular case $n = 1$, where $D = J$ is an interval of $\mathbb{R}$,

$$\mathbb{W} = \{\psi \in H^1_w(J) : \psi(\hat{J}) = 0\}$$

for $\hat{J} \subset \overline{J}$. $\hat{J}$ could be just one single point, or a discrete subset.

Suppose one such $\mathbf{w} \in H^1(\Omega; \mathbb{R}^n)$ has been chosen, and a suitable subspace $\mathbb{W}$ of $H^1_{\mathbf{w}}(D)$ selected, as described above. For a density

$$\phi(\mathbf{u}, u, \boldsymbol{x}) : \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$$

that is convex in the variable $\mathbf{u}$ and coercive in the sense

$$c(|\mathbf{v}|^2 + v^2 - 1) \le \phi(\mathbf{v}, v, \boldsymbol{x}), \quad \mathbf{v} \in \mathbb{R}^N, v \in \mathbb{R}, \boldsymbol{x} \in \Omega, c > 0,$$

we consider the variational problem

$$\text{Minimize in } v \in v_0 + \mathbb{H}_{\mathbf{w}} : \quad \int_\Omega \phi(\nabla v(\boldsymbol{x}), v(\boldsymbol{x}), \boldsymbol{x})\, d\boldsymbol{x}, \tag{7.3}$$

where $\mathbb{H}_{\mathbf{w}} = \mathbb{L}_{\mathbf{w}} + H^1_0(\Omega)$, and an additional $v_0 \notin \mathbb{H}_{\mathbf{w}}$ has been suitably selected.

The following two results have identical proofs than their earlier versions.

**Theorem 7.1** *Under the assumptions given, there is a minimizer $v \in v_0 + \mathbb{H}_{\mathbf{w}}$ of problem* (7.3)*. If, in addition, $\phi(\mathbf{v}, v, \boldsymbol{x})$ is strictly convex in pairs $(\mathbf{v}, v)$, then the minimizer is unique.*

**Theorem 7.2** *Suppose $\phi(\mathbf{v}) : \mathbb{R}^N \to \mathbb{R}$ is a $\mathcal{C}^1$-integrand for which,*

$$c(|\mathbf{v}|^2 - 1) \le \phi(\mathbf{v}) \le C(|\mathbf{v}|^2 + 1),$$
$$|\nabla\phi(\mathbf{v})| \le C(|\mathbf{v}| + 1).$$

*If $v$ is a minimizer of the problem*

$$\text{Minimize in } v \in v_0 + \mathbb{H}_{\mathbf{w}} : \quad \int_\Omega \phi(\nabla v(\boldsymbol{x}))\, d\boldsymbol{x}$$

*then $v \in v_0 + \mathbb{H}_{\mathbf{w}}$ is a weak solution of*

$$\text{div}[\nabla\phi(\nabla v)] = 0 \text{ in } \Omega,$$

*together with*

$$\int_{\{\mathbf{w}=\boldsymbol{y}\}\cap\partial\Omega} \nabla\phi(\nabla v(\boldsymbol{x})) \cdot \boldsymbol{n}(\boldsymbol{x})\, d\boldsymbol{x} = 0$$

*for every $\boldsymbol{y} \in D$, where $\boldsymbol{n}$ is the outer, unit normal to $\partial\Omega$.*

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