



New consistent exponentiality tests based on V-empirical Laplace transforms with comparison of efficiencies

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Abstract

We present new consistent goodness-of-fit tests for exponential distribution, based on the Desu characterization. The test statistics represent the weighted L^2 and L^∞ distances between appropriate V-empirical Laplace transforms of random variables that appear in the characterization. In addition, we perform an extensive comparison of Bahadur efficiencies of different recent and classical exponentiality tests. We also present the empirical powers of new tests.

Keywords Goodness-of-fit · Exponential distribution · Laplace transform · Bahadur efficiency · V-statistics

Mathematics Subject Classification 62G10 · 62G20

1 Introduction

To justify the use of more complicated models for lifetime data, one of the first steps is to reject the most simple one, the exponential. For this purpose numerous tests have been developed and are available in the literature.

The classical, and most commonly used procedure, is to apply one of universal time-honored goodness-of-fit tests based on empirical distribution function, such as Kolmogorov–Smirnov, Cramér–von Mises, Anderson–Darling. To make them applicable to the case of a composite null hypothesis, the Lilliefors modification with estimated rate parameter is frequently used.

Another approach is to use tests tailor-made for testing exponentiality. Such tests usually employ some special properties of the exponential distribution. Various integral transform related properties have been exploited: characteristic functions (see e.g. [17,19,20]); Laplace transforms (see e.g. [18,24,30]); and other integral transforms (see e.g. [25,29]). Other possible properties include maximal correlations (see [15,16,48]), entropy (see [1]), etc.

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An important type of such properties are the characterizations of the exponential distribution. Many of them, being relatively simple, are very suitable for construction of goodness-of-fit tests. This is especially true for the equidistribution-type characterizations. Since the equality in distribution can be expressed in many ways (equality of distribution functions, densities, integral transforms, etc.), many different types of test statistics are available. Tests that use U-empirical and V-empirical distribution functions, of integral-type (integrated difference) and supremum-type, can be found in Nikitin and Volkova [42], Volkova [53], Jovanović et al. [23], Milošević and Obradović [33], Milošević [31], Nikitin and Volkova [43]. Weighted integral-type and L^2 -type tests that use U- or V-empirical Laplace transforms are presented in Milošević and Obradović [32] and in Cuparić et al. [11].

The common approach to explore the quality of tests is to find their power against different alternatives. Several papers are devoted to comparative power studies of exponentiality tests (see e.g. [3,20,51]).

Another popular choice for the quality assessment is the asymptotic efficiency. In this regard, however, no extensive study has been done. In this paper our aim is to compare the exponentiality tests using the approximate Bahadur efficiency (see [5]).

We opt for the approximate Bahadur efficiency since it is applicable to asymptotically non-normally distributed test statistics, and moreover it can distinguish tests better than some other types of efficiencies like Pitman or Hodges–Lehmann (see [36]).

Consider the setting of testing the null hypothesis $H_0 : \theta \in \Theta_0$ against the alternative $H_1 : \theta \in \Theta_1$. Let us suppose that for a test statistic T_n , under H_0 , the limit $\lim_{n \rightarrow \infty} P\{T_n \leq t\} = F(t)$, where F is non-degenerate distribution function, exists. Further, suppose that $\lim_{t \rightarrow \infty} t^{-2} \ln(1 - F(t)) = -\frac{a_T}{2}$, and that the limit in probability $P_\theta \lim_{n \rightarrow \infty} T_n = b_T(\theta) > 0$, exists for $\theta \in \Theta_1$. The relative approximate Bahadur efficiency with respect to another test statistic V_n is

$$e_{T,V}^*(\theta) = \frac{c_T^*(\theta)}{c_V^*(\theta)},$$

where

$$c_T^*(\theta) = a_T b_T^2(\theta) \quad (1)$$

is the approximate Bahadur slope of T_n . Its limit when $\theta \rightarrow 0$ is called the local approximate Bahadur efficiency.

The tests we consider may be classified into three groups according to their limiting distributions: asymptotically normal ones; those whose asymptotic distribution coincides with the supremum of some Gaussian process; and those whose limiting distribution is an infinite linear combination of independent and identically distributed (i.i.d.) chi-squared random variables.

For the first group of tests, the coefficient a_T is the inverse of the limiting variance. For the second, it is the inverse of the supremum of the covariance function of the limiting process (see [28]). For the third group, a_T is the inverse of the largest coefficient in the corresponding linear combination (see [55]), which is also equal to the largest eigenvalue of some integral operator.

The goal of this paper is twofold. First, we propose two new classes of characterization based exponentiality tests. One of them is of weighted L^2 -type, and the other, for the first time, is based on L^∞ distance between two V-empirical Laplace transforms of the random variables that appear in the characterization.

Secondly, we perform an extensive efficiency comparison. Unlike for the remaining two, for the third group of tests, the efficiencies have not been calculated so far. This is due to the fact that the largest eigenvalue in question usually cannot be obtained analytically. We

overcome this problem using a recently proposed approximation procedure from Božin et al. [9].

The rest of the paper is organized as follows. In Sect. 2 we propose new tests and explore their asymptotic properties. In Sect. 3 we give a partial review of test statistics for testing exponentiality, together with their Bahadur slopes. Section 4 is devoted to the comparison of efficiencies. In Sect. 5 we present the powers of new tests. Appendices A and B contain all the proofs, while the tables with Bahadur efficiencies are given in Appendix C.

2 New test statistics

In this section we present two new exponentiality tests based on the following characterization from Desu [12].

Characterization 1 *Let X_1, \dots, X_m be independent copies of a random variable X with pdf $f(x)$. Then for each m , X and $m \cdot \min(X_1, \dots, X_m)$ have the same distribution if and only if for some $\lambda > 0$ $f(x) = \lambda e^{-\lambda x}$, for $x \geq 0$.*

It is accustomed for goodness-of-fit testing purposes to use this characterization for $m = 2$ (see e.g. [37,52]). Although some distributions different from exponential for which $2 \min(X_1, X_2)$ and X_1 are equally distributed exist and can be constructed (see [21]), they are of no practical interest and this special case is often referred to as Desu’s characterization (see [38, Theorem 3]).

Let X_1, X_2, \dots, X_n be a random sample from a non-negative continuous distribution. To test the null hypothesis that the sample comes from the exponential distribution $\mathcal{E}(\lambda)$, with an unknown $\lambda > 0$, we examine the difference $\mathcal{L}_n^{(1)}(t) - \mathcal{L}_n^{(2)}(t)$, of V-empirical Laplace transforms of X and $2 \min(X_1, X_2)$.

Clearly, if null hypothesis is true, the difference $\mathcal{L}_n^{(1)}(t) - \mathcal{L}_n^{(2)}(t)$ will be small for each t . Taking this into account we propose the following two classes of test statistics, with their large values considered significant:

$$M_{n,a}^{\mathcal{D}} = \int_0^\infty \left(\frac{1}{n} \sum_{i_1=1}^n e^{-tY_{i_1}} - \frac{1}{n^2} \sum_{i_1, i_2=1}^n e^{-t2 \min(Y_{i_1}, Y_{i_2})} \right)^2 e^{-at} dt;$$

$$L_{n,a}^{\mathcal{D}} = \sup_{t>0} \left| \left(\frac{1}{n} \sum_{i_1=1}^n e^{-tY_{i_1}} - \frac{1}{n^2} \sum_{i_1, i_2=1}^n e^{-t2 \min(Y_{i_1}, Y_{i_2})} \right) e^{-at} \right|,$$

where $Y_i = \frac{X_i}{X_n}$, $i = 1, 2, \dots, n$, is the scaled sample.

The sample is scaled to make the test statistic ancillary for the parameter λ and the purpose of the tuning parameter a is to magnify different types of deviations from the null distribution.

Remark 1 The supremum in the expression for test statistic $L_{n,a}^{\mathcal{D}}$ is calculated using greed search over fixed interval $[0, A]$ with suitably chosen large value of A .

2.1 Asymptotic properties under H_0

Notice that $M_{n,a}^D$ is a V-statistic with estimated parameter $\hat{\lambda}$, i.e. it can be represented in the form

$$M_{n,a}^D = M_{n,a}^D(\hat{\lambda}_n) = \frac{1}{n^4} \sum_{i_1, i_2, i_3, i_4} H(X_{i_1}, X_{i_2}, X_{i_3}, X_{i_4}; a, \hat{\lambda}_n),$$

where H is

$$H(x_1, x_2, x_3, x_4; a, \hat{\lambda}_n) = \frac{1}{4!} \sum_{\pi \in \Pi(4)} \left(\frac{1}{a + \hat{\lambda}_n x_{\pi(1)} + \hat{\lambda}_n x_{\pi(3)}} - \frac{1}{a + \hat{\lambda}_n x_{\pi(1)} + 2\hat{\lambda}_n \min\{x_{\pi(3)}, x_{\pi(4)}\}} - \frac{1}{\hat{\lambda}_n x_{\pi(3)} + 2\hat{\lambda}_n \min\{x_{\pi(1)}, x_{\pi(2)}\}} + \frac{1}{a + 2\hat{\lambda}_n \min\{x_{\pi(1)}, x_{\pi(2)}\} + 2\hat{\lambda}_n \min\{x_{\pi(3)}, x_{\pi(4)}\}} \right),$$

where $\Pi(4)$ is the set of all permutations of $\{1, 2, 3, 4\}$, and $\hat{\lambda}_n$ is the reciprocal sample mean.

Similarly, for a fixed t , the expression in the absolute parenthesis of the statistics $L_{n,a}^D$ is a V-statistics that can be represented as

$$\frac{1}{n^2} \sum_{i_1, i_2} \Phi(X_{i_1}, X_{i_2}; t, a, \hat{\lambda}_n), \tag{2}$$

where

$$\Phi(x_1, x_2; t, a, \hat{\lambda}_n) = \frac{1}{2} \left(e^{-t\hat{\lambda}_n x_1} + e^{-t\hat{\lambda}_n x_2} - e^{-t\hat{\lambda}_n 2 \min(x_1, x_2)} \right) e^{-at}$$

is a symmetric function of its arguments.

From the law of large number for V-statistics with estimated parameters (see [22]), both statistics converge in probability to zero if and only if the equidistribution in the Desu’s characterization is satisfied, which happens under H_0 . Under H_1 , when the equidistribution does not hold, both test statistics converge to a positive number, from where follows their consistency.

The asymptotic behaviour of $M_{n,a}^D$ is given in the following theorem.

Theorem 2 *Let X_1, \dots, X_n be i.i.d. with exponential distribution. Then*

$$nM_{n,a}^D \xrightarrow{D} 6 \sum_{k=1}^{\infty} \delta_k W_k^2,$$

where $\delta_k, k = 1, 2, \dots$, is the sequence of eigenvalues of the integral operator A defined by $Aq(x) = \int_0^{\infty} \tilde{h}_2(x, y; a)q(y)dF(y)$, with $\tilde{h}_2(x, y) = E(H(\cdot)|X_1 = x, X_2 = y)$ being the second projection of kernel $H(X_1, X_2, X_3, X_4; a, \lambda)$, and $W_k, k = 1, 2, \dots$, are independent standard normal variables.

The asymptotic behaviour of $L_{n,a}^D$ is given in the following theorem.

Theorem 3 Let X_1, \dots, X_n be i.i.d. with exponential distribution. Then

$$\sqrt{n}L_{n,a}^{\mathcal{D}} \xrightarrow{D} \sup_{t>0} |\eta(t)|,$$

where $\eta(t)$ is a centered Gaussian process with the covariance function

$$K(s, t) = \frac{e^{-a(s+t)}st(4 + 8s + 4s^2 + 8t + 15st + 6s^2t + 4t^2 + 6st^2)}{(1 + s)(1 + t)(1 + s + t)(2 + 2s + t)(2 + s + 2t)(3 + 2s + 2t)}, \quad s, t \geq 0. \tag{3}$$

2.2 Approximate Bahadur slope

Let $\mathcal{G} = \{G(x; \theta), \theta > 0\}$ with corresponding densities $\{g(x; \theta)\}$ be a family of alternative distribution functions with finite expectations, such that $G(x, \theta) = 1 - e^{-\lambda x}$, for some $\lambda > 0$, if and only if $\theta = 0$, and the regularity conditions for V-statistics with weakly degenerate kernels from Nikitin and Peaucelle [39, Assumptions WD] are satisfied.

The approximate local Bahadur slopes of $M_{n,a}^{\mathcal{D}}$ and $L_{n,a}^{\mathcal{D}}$, for close alternatives, are derived in the following theorem.

Theorem 4 For the statistics $M_{n,a}^{\mathcal{D}}$ and $L_{n,a}^{\mathcal{D}}$ and a given alternative density $g(x, \theta)$ from \mathcal{G} , the local Bahadur approximate slopes are given by

$$(1) \quad c_M^*(\theta) = \delta_1^{-1} \int_0^\infty \int_0^\infty \tilde{h}_2(x, y)g'_\theta(x; 0)g'_\theta(y; 0)dx dy \cdot \theta^2 + o(\theta^2), \theta \rightarrow 0,$$

where δ_1 is the largest eigenvalue of the integral operator A with kernel \tilde{h}_2 ;

$$(2) \quad c_L^*(\theta) = \frac{1}{\sup_t K(t, t)} \sup \left(2 \int_0^\infty \tilde{\varphi}_1(x; t)g'_\theta(x; 0)dx \right)^2 \cdot \theta^2 + o(\theta^2), \theta \rightarrow 0,$$

where $\tilde{\varphi}_1(x; t) = E(\Phi(\cdot)|X_1 = x)$ with Φ being defined in (2).

Proof See Appendix A. □

To calculate the slope of $M_{n,a}^{\mathcal{D}}$, one needs to find the largest eigenvalue δ_1 . Since it cannot be obtained analytically, we use the approximation introduced in Božin et al. [9]. The procedure utilizes the fact that δ_1 is the limit of the sequence of the largest eigenvalues of linear operators defined by $(m + 1) \times (m + 1)$ matrices $M^{(m)} = ||m_{i,j}^{(m)}||$, $0 \leq i \leq m, 0 \leq j \leq m$, where

$$m_{i,j}^{(m)} = \tilde{h}_2\left(\frac{i}{m}B, \frac{j}{m}B\right) \sqrt{e^{-\frac{i}{m}B} - e^{-\frac{i+1}{m}B}} \cdot \sqrt{e^{-\frac{j}{m}B} - e^{-\frac{j+1}{m}B}} \cdot \frac{1}{1 - e^{-B}}, \tag{4}$$

when m tends to infinity and B is constant chosen in such way that $F(B)$ approaches 1.

3 Other exponentiality tests—a partial review

In this section we present test statistics of some classical and some recent goodness-of-fit tests for the exponential distribution, along with their Bahadur local approximate slopes. For some of the test statistics, the Bahadur local approximate slope (or exact slope which locally

coincides with the approximate one) is available in the literature and for the others we derive them in Appendix B.

As indicated in Introduction, we classify the tests according to their asymptotic distribution. The first group contains asymptotically normally distributed statistics.

- The test proposed by Epps and Pulley [13] based on the expected value of the exponential density, with test statistic

$$EP_n = \sqrt{48} \left(\frac{1}{n} \sum_{j=1}^n e^{-\frac{X_j}{\bar{X}_n}} - \frac{1}{2} \right).$$

Its approximate Bahadur slope is

$$c_{EP}^*(\theta) = 3 \left(\int_0^\infty (4e^{-x} + x) g'_\theta(x; 0) dx \right)^2 \cdot \theta^2 + o(\theta^2)$$

- The score test for the Weibull shape parameter proposed by Cox and Oakes [10]

$$CO_n = 1 + \frac{1}{n} \sum_{i=1}^n \left(1 - \frac{X_i}{\bar{X}_n} \right) \ln \frac{X_i}{\bar{X}_n}.$$

Its approximate slope is

$$c_{CO}^*(\theta) = \frac{6}{\pi^2} \left(\int_0^\infty ((1-x) \ln x + (1-\gamma)x) g'_\theta(x; 0) dx \right)^2 \cdot \theta^2 + o(\theta^2)$$

- A test based on Gini coefficient from Gail and Gastwirth [14]

$$G_n^* = \left| \frac{1}{2n(n-1)\bar{X}_n} \sum_{i,j=1}^n |X_i - X_j| - \frac{1}{2} \right|.$$

The approximate slope is (see [40])

$$c_G^*(\theta) = 12 \left(\int_0^\infty \left(2e^{-x} + \frac{x}{2} \right) g'_\theta(x; 0) dx \right)^2 \theta^2 + o(\theta^2).$$

- The score test for the gamma shape parameter proposed by Moran [35] and Tchirina [49]

$$MO_n = \left| \gamma + \frac{1}{n} \sum_{i=1}^n \ln \frac{X_i}{\bar{X}_n} \right|.$$

Its approximate slope is (see [49])

$$c_{MO}^*(\theta) = \left(\frac{\pi^2}{6} - 1 \right)^{-1} \left(\int_0^\infty (\ln x - x) g'_\theta(x; 0) dx \right)^2 \theta^2 + o(\theta)$$

- Characterization based integral-type tests

Let the relation

$$\omega_1(X_1, \dots, X_m) \stackrel{d}{=} \omega_2(X_1, \dots, X_p), \tag{5}$$

where $X_1, \dots, X_{\max\{m, p\}}$ are i.i.d. random variables, characterize the exponential distribution. Then the following types of test statistics have been proposed:

$$I_n = \int_0^\infty \left(H_n^{(\omega_1)}(t) - H_n^{(\omega_2)}(t) \right) dF_n(t),$$

where $H_n^{(\omega_1)}(t)$ and $H_n^{(\omega_2)}(t)$ are V -empirical distribution functions of ω_1 and ω_2 , respectively, and F_n is the empirical distribution function, and

$$J_{n,a} = \int_0^\infty \left(L_n^{(\omega_1)}(t) - L_n^{(\omega_2)}(t) \right) e^{-at} dt, \tag{6}$$

where $L_n^{(\omega_1)}(t)$ and $L_n^{(\omega_2)}(t)$ are V -empirical Laplace transforms of ω_1 and ω_2 , respectively, applied to the scaled sample, and $a > 0$ is the tuning parameter.

From these groups of tests we take the following representatives

- $I_{n,k}^{(1)}$, proposed in Jovanović et al. [23], based on the Arnold and Villasenor characterization, where $\omega_1(X_1, \dots, X_k) = \max(X_1, \dots, X_k)$ and $\omega_2(X_1, \dots, X_k) = X_1 + \frac{X_2}{2} + \dots + \frac{X_k}{k}$ (see [4,34]);
- $I_n^{(2)}$, proposed in Milošević and Obradović [33], based on the Milošević-Obradović characterization, where $\omega_1(X_1, X_2) = \max(X_1, X_2)$ and $\omega_2(X_1, X_2, X_3) = X_1 + \min(X_2, X_3)$ (see [34]);
- $I_n^{(3)}$, proposed in Milošević [31], based on the Obradović characterization, where $\omega_1(X_1, X_2, X_3) = \max(X_1, X_2, X_3)$ and $\omega_2(X_1, X_2, X_3, X_4) = X_1 + \text{med}(X_2, X_3, X_4)$ (see [45]);
- $I_n^{(4)}$, proposed in Volkova [53], based on the Yanev-Chakraborty characterization, where $\omega_1(X_1, X_2, X_3) = \max(X_1, X_2, X_3)$ and $\omega_2(X_1, X_2, X_3) = \frac{X_1}{3} + \max(X_2, X_3)$ (see [54]);
- $I_n^{\mathcal{D}}$ based on the Desu characterization 1;
- $I_n^{\mathcal{P}}$ based on the Puri–Rubin characterization, where $\omega_1(X_1) = X_1$ and $\omega_2(X_1, X_2) = |X_1 - X_2|$ (see [46]);
- $J_{n,a}^{\mathcal{D}}$, proposed in Milošević and Obradović [32], based on the Desu characterization 1;
- $J_{n,a}^{\mathcal{P}}$, proposed in Milošević and Obradović [32], based on the Puri–Rubin characterization.

Since these statistics are very similar, we give general expressions for their Bahadur approximate slopes.

Statistics I_n are non-degenerate V -statistics with some kernel Ψ and their approximate slope is (see [39])

$$c_I^*(\theta) = \frac{1}{\text{Var}\psi(X_1)} \left(\int \psi(x) g'_\theta(x; 0) dx \right)^2 \cdot \theta^2 + o(\theta^2), \tag{7}$$

where $\psi(x) = E\Psi(\cdot | X_1 = x)$.

Statistics $J_{n,a}$ are, due to the sample scaling, non-degenerate V -statistics with estimated parameters. Nevertheless, the formula (7) is applicable here also, with Ψ being the kernel of the test statistic as if the scaling were done using the real value of λ (see [32] for details).

The second group contains statistics whose limiting distribution is the supremum of some centered Gaussian process.

- Lilliefors modification of the Kolmogorov–Smirnov test

$$KS_n = \sup |F_n(t) - (1 - e^{-\frac{t}{x_n}})|.$$

The approximate slope is (see [41])

$$c_{KS}^*(\theta) = \frac{1}{\sup_{x \geq 0} (e^{-2x}(e^x - x^2 - 1))} \sup_{x \geq 0} \left(x e^x \int_0^\infty G'_\theta(u; 0) du - G(x; 0) dx \right)^2 \cdot \theta^2 + o(\theta^2).$$

- Characterization based supremum-type tests

Using the characterizations of the type (5), another proposed type of test statistics is

$$D_n = \sup_{t > 0} |H_n^{(\omega_1)}(t) - H_n^{(\omega_2)}(t)|.$$

From this group of tests we take the following representatives:

$D_{n,k}^{(1)}$, proposed in Jovanović et al. [23]; $D_n^{(2)}$, proposed in Milošević and Obradović [33]; $D_n^{(3)}$, proposed in Milošević [31]; $D_n^{(4)}$, proposed in Volkova [53], based on the same characterizations as for the respective integral-type statistics, $D_n^{\mathcal{D}}$ based on Desu characterization 1 and $D_n^{\mathcal{P}}$ based on Puri–Rubin characterization [46]. Statistics from this group are asymptotically distributed as a supremum of some non-degenerate V-empirical processes, and the expression in the absolute parenthesis, for a fixed t is a V-statistic with some kernel $\Psi(X_1, \dots, X_{\max\{m,p\}}; t)$. Their approximate slopes is (see [37])

$$c_D^*(\theta) = \frac{1}{\sup_{t > 0} \text{Var} \psi(X_1; t)} \sup_{t > 0} \left(\int \psi(x; t) g'_\theta(x; 0) dx \right)^2 \cdot \theta^2 + o(\theta^2),$$

where $\psi(x; t) = E\Psi(\cdot; t | X_1 = x)$.

The third group contains statistics whose limiting distribution is an infinite linear combination of i.i.d. chi-squared random variables. Each of the presented statistics, except the last one, is of the form

$$T_n = \int_0^\infty U_n^2(t; \hat{\mu}) w(t) dt,$$

where $U_n(t; \hat{\mu})$ is an empirical process of order 1 with estimated parameter. It also can be viewed as a weakly degenerate V-statistics with estimated parameters, with some kernel $\Phi(X_1, X_2; \hat{\mu})$, where $\mu = E_\theta X_1$. Then, the Bahadur approximate slope of such statistic is

$$c^*(\theta) = \frac{1}{2\delta_\Phi} \int_0^\infty \int_0^\infty \left(2\Phi(x, y; 1)g'_\theta(x; 0)g'_\theta(y; 0) + 4\Phi'_{\mu(\theta)}(x, y; 1)\mu'_\theta(0)g'(x; 0)g(y; 0) + \Phi''_{\mu^2(\theta)}(x, y; 1)(\mu'_\theta(0))^2g(x; 0)g(y; 0) \right) dx dy \cdot \theta^2 + o(\theta^2), \tag{8}$$

where δ_Φ is the largest eigenvalue of the integral operator

$$Aq(s) = \int_0^\infty K(s, t)w(t)q(t)dt,$$

where $K(s, t) = \lim_{n \rightarrow \infty} n \text{Cov}(U_n(t), U_n(s))$ is the limiting covariance function.

Hence it suffices to present only the kernels and limiting covariance functions of test statistics. We consider the following tests:

- Lilliefors modification of the Cramér–von Mises test

$$\omega_n^2 = \int_0^\infty (F_n(t) - (1 - e^{-\frac{t}{\bar{X}_n}}))^2 \frac{1}{\bar{X}_n} e^{-\frac{t}{\bar{X}_n}} dt.$$

Its kernel is

$$\Phi_{\omega^2}(x, y; \mu(\theta)) = e^{-\max(\frac{x}{\mu(\theta)}, \frac{y}{\mu(\theta)})} - e^{-\frac{x}{\mu(\theta)}} - e^{-\frac{y}{\mu(\theta)}} + \frac{1}{2}(e^{-2\frac{x}{\mu(\theta)}} + e^{-2\frac{y}{\mu(\theta)}}) + \frac{1}{3};$$

and the covariance function is

$$K_{\omega^2}(s, t) = e^{-\frac{3}{2}s - \frac{3}{2}t}(e^{\min(s,t)} - 1 - st).$$

- Lilliefors modification of the Anderson–Darling test

$$AD_n = \int_0^\infty \frac{(F_n(t) - (1 - e^{-\frac{t}{\bar{X}_n}}))^2}{\bar{X}_n(1 - e^{-\frac{t}{\bar{X}_n}}} dt.$$

Its kernel is

$$\Phi_{AD}(x, y; \mu(\theta)) = \frac{x}{\mu(\theta)} + \frac{y}{\mu(\theta)} - 1 - \ln(e^{\max(\frac{x}{\mu(\theta)}, \frac{y}{\mu(\theta)})} - 1),$$

and the covariance function is

$$K_{AD}(s, t) = \frac{e^{-s-t}(e^{\min(s,t)} - 1 - st)}{\sqrt{(1 - e^{-s})(1 - e^{-t})}}.$$

- A test proposed by Baringhaus and Henze [7]

$$BH_n = \int_0^\infty \left((1+t)\psi'_n(t) + \psi_n(t) \right)^2 e^{-at} dt,$$

where $\psi_n(t)$ is the empirical Laplace transform. Its kernel is

$$\Phi_{BH}(x, y; \mu(\theta)) = \frac{(1-x)(1-y)}{x+y+a\mu(\theta)} - \frac{x\mu(\theta) + y\mu(\theta) - 2xy}{(a\mu(\theta) + x + y)^2} + \frac{2xy\mu(\theta)}{(a\mu(\theta) + x + y)^3};$$

and the covariance function is

$$K_{BH}(s, t) = \frac{1 + s + t + 2st}{(1 + s + t)^3} - \frac{1}{(1 + s)^2(1 + t)^2}.$$

- The test proposed by Henze [17]

$$HE_n = \int_0^\infty \left(\psi_n(t) - \frac{1}{1+t} \right)^2 e^{-at} dt;$$

Its kernel is

$$\begin{aligned} \Phi_{HE}(x, y; \mu(\theta)) = & 1 + \frac{\mu(\theta)}{a\mu(\theta) + x + y} + ae^a Ei(-a) + e^{a+\frac{x}{\mu(\theta)}} Ei\left(-\left(a + \frac{x}{\mu(\theta)}\right)\right) \\ & + e^{a+\frac{y}{\mu(\theta)}} Ei\left(-\left(a + \frac{y}{\mu(\theta)}\right)\right); \end{aligned}$$

and the covariance function is

$$K_{HE}(s, t) = \frac{s^2 t^2}{(s + t + 1)(s + 1)^2(t + 1)^2}.$$

- The test proposed by Henze and Meintanis [18]

$$W_n = \int_0^\infty \left(\psi_n(t) - \frac{1}{1+t} \right)^2 (1+t)^2 e^{-at} dt.$$

Its kernel is

$$\begin{aligned} \Phi_W(x, y; \mu(\theta)) &= \frac{1}{a} - \frac{\mu(\theta)(\mu(\theta)(1+a) + x)}{(a\mu(\theta) + x)^2} - \frac{\mu(\theta)(\mu(\theta)(1+a) + y)}{(a\mu(\theta) + y)^2} \\ &+ \frac{2\mu^3(\theta)}{(a\mu(\theta) + x + y)^3} + \frac{2\mu^2(\theta)}{(a\mu(\theta) + x + y)^2} + \frac{\mu(\theta)}{a\mu(\theta) + x + y}; \end{aligned}$$

and the covariance function is

$$K_W(s, t) = \frac{s^2 t^2}{(s+t+1)(s+1)(t+1)}.$$

- Two tests proposed by Henze and Meintanis [19]

$$HM_n = \int_0^\infty (s_n(t) - tc_n(t))^2 \omega_i(t) dt, \quad i = 1, 2,$$

where $\omega_1(t) = e^{-at}$ and $\omega_2(t) = e^{-at^2}$. Their kernels are

$$\begin{aligned} \Phi_{HM1}(x, y; \mu(\theta)) &= \frac{a\mu^2(\theta)}{2(a^2\mu^2(\theta) + (x-y)^2)} - \frac{a\mu^2(\theta)}{2(a^2\mu^2(\theta) + (x+y)^2)} \\ &+ \frac{a^2\mu^2(\theta) - 3(x-y)^2}{(a^2\mu^2(\theta) + (x-y)^2)^3} + \frac{a^2\mu^2(\theta) - 3(x+y)^2}{(a^2\mu^2(\theta) + (x+y)^2)^3} \\ &- \frac{2a\mu^3(\theta)(x+y)}{(a^2\mu^2(\theta) + (x+y)^2)^2}, \end{aligned}$$

and

$$\begin{aligned} \Phi_{HM2}(x, y; \mu(\theta)) &= \frac{\sqrt{\pi}}{4\sqrt{a}} \left(\left(\frac{1}{2a} - \frac{x+y}{a\mu(\theta)} - \frac{(x-y)^2}{4a^2\mu^2(\theta)} - 1 \right) e^{-\frac{(x+y)^2}{4a\mu^2(\theta)}} \right. \\ &\left. + \left(1 + \frac{1}{2a} - \frac{(x-y)^2}{4a^2\mu^2(\theta)} \right) e^{-\frac{(x-y)^2}{4a\mu^2(\theta)}} \right); \end{aligned}$$

and the covariance function is

$$K_{HM}(s, t) = \frac{st(s^2 + t^2 + 1)}{(1 + (s-t)^2)(1 + (s+t)^2)} - \frac{st}{(1 + s^2)(1 + t^2)}.$$

- Characterization based L^2 -type test proposed by Cuparić et al. [11].

$$M_{n,a}^P = \int_0^\infty \left(L_n^{(1)}(t) - L_n^{(2)}(t) \right)^2 e^{-at} dt. \tag{9}$$

Its slope is

$$\begin{aligned}
 c_M^*(\theta) = & (2\delta_1)^{-1} \int_0^\infty \int_0^\infty \left[\frac{1}{6} e^{a-x-y} \text{Ei}(-a) \left(a(e^x - 2)(e^y - 2) - e^x - e^y + 4 \right) \right. \\
 & + \frac{1}{6} e^{-a-x-y} \left(\text{Ei}(a)(4a + e^x + e^y - 4) - (\text{Ei}(a+x)(4(a+x) - 1) + e^y) \right. \\
 & \left. \left. + \text{Ei}(a+y)(4(a+y) - 1) + e^x \right) - 4(a+x+y-1)\text{Ei}(a+x+y) \right] - \frac{1}{2} \\
 & \left. + \frac{1}{3}(e^{-x} + e^{-y}) + \frac{1}{6(a+x+y)} \right] g'(x; 0)g'(y; 0) dx dy \cdot \theta^2 + o(\theta^2).
 \end{aligned}$$

4 Comparison of efficiencies

In this section we calculate approximate local relative Bahadur efficiencies of test statistics introduced in Sects. 2 and 3 with respect to the likelihood ratio test (see [6]). Likelihood ratio tests are known to have optimal Bahadur efficiencies and they are therefore used as benchmark for comparison.

The alternatives we consider are the following:

- a Weibull distribution with density

$$g(x, \theta) = e^{-x^{1+\theta}} (1 + \theta)x^\theta, \theta > 0, x \geq 0; \tag{10}$$

- a gamma distribution with density

$$g(x, \theta) = \frac{x^\theta e^{-x}}{\Gamma(\theta + 1)}, \theta > 0, x \geq 0; \tag{11}$$

- a linear failure rate (LFR) distribution with density

$$g(x, \theta) = e^{-x-\theta \frac{x^2}{2}} (1 + \theta x), \theta > 0, x \geq 0; \tag{12}$$

- a mixture of exponential distributions with negative weights (EMNW(β)) with density

$$g(x, \theta) = (1 + \theta)e^{-x} - \theta\beta e^{-\beta x}, \theta \in \left(0, \frac{1}{\beta - 1} \right], x \geq 0;$$

On Figs. 1, 2, 3 and 4, there are plots of local approximate Bahadur efficiencies as a function of the tuning parameter. For tests with no such parameter straight lines are drawn. To avoid too many lines on the same plot, there are three separate plots given for each alternative, each corresponding to one of the classes of tests from Sect. 3. The exact values of local Bahadur efficiencies are available in Table 3 in Appendix C.

As a rule we can notice that, in the class of supremum-type statistics, new test $L_{n,a}^D$ is by far the most efficient. On the other hand, supremum-type test based on characterizations that use U-empirical distribution functions, are the least efficient among all considered tests.

The impact of the tuning parameter, for the tests that have got it, is also visible in all the figures. It is interesting to note that this impact is different for various tests in terms of monotonicity of the plotted functions.

Another general conclusion is that the ordering of the tests depends on the alternative and that there is no most efficient test to be recommended in any situation.

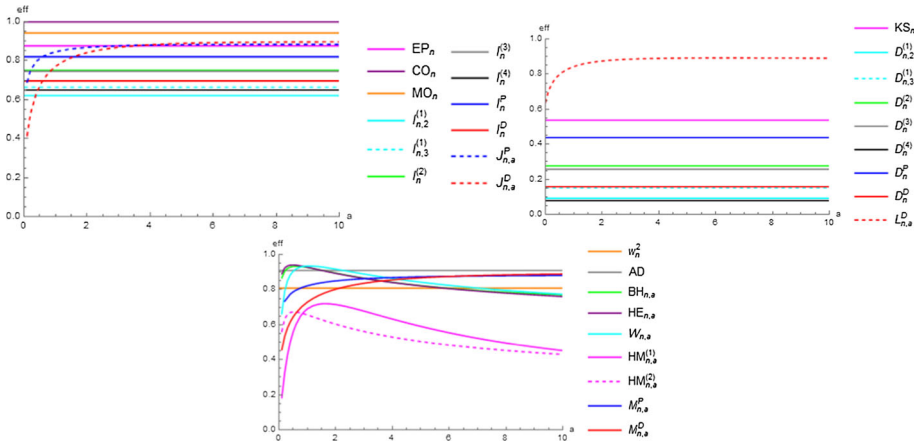


Fig. 1 Local approximate Bahadur efficiencies w.r.t. LRT for a Weibull alternative

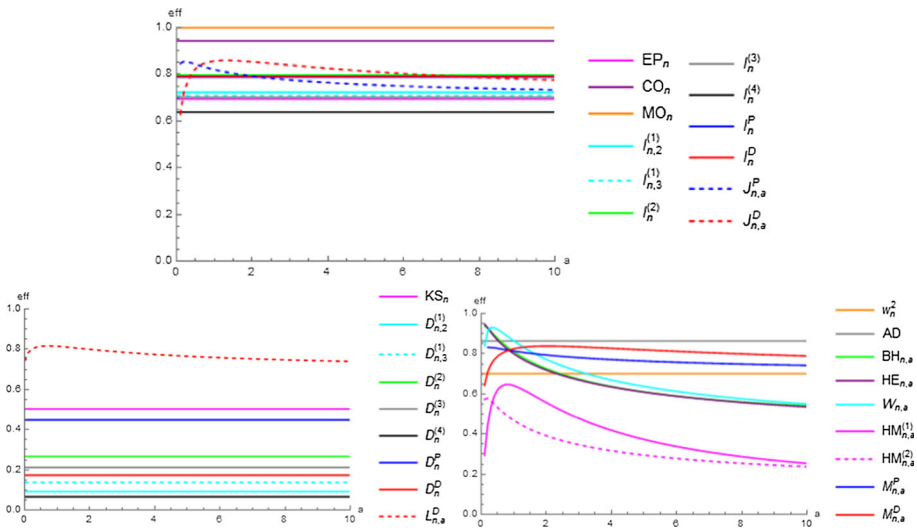


Fig. 2 Local approximate Bahadur efficiencies w.r.t. LRT for a Gamma alternative

The CO and MO tests are known to be locally optimal for Weibull and gamma alternatives, respectively, so they are the most efficient in these cases. However, there are quite a few other tests that perform very well in these cases. In the case of the LFR alternative, the most efficient are EP and $HM_{n,a}^{(1)}$ and $HM_{n,a}^{(2)}$. It is interesting that for other alternatives the latter two tests are among the least efficient.

In the case of the EMNW alternative, the integral and supremum-type tests based on the characterizations via Laplace transforms, as well as most of the L^2 test reach, for some value of the tuning parameter, an efficiency close to one.

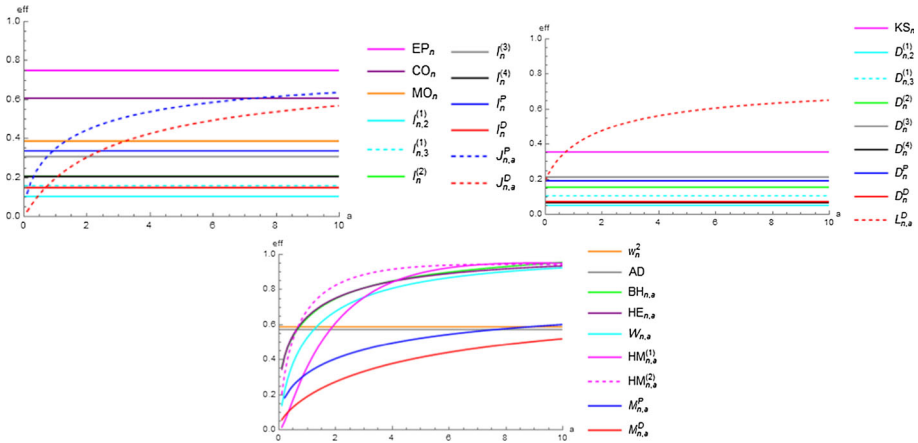


Fig. 3 Local approximate Bahadur efficiencies w.r.t. LRT for a LFR alternative

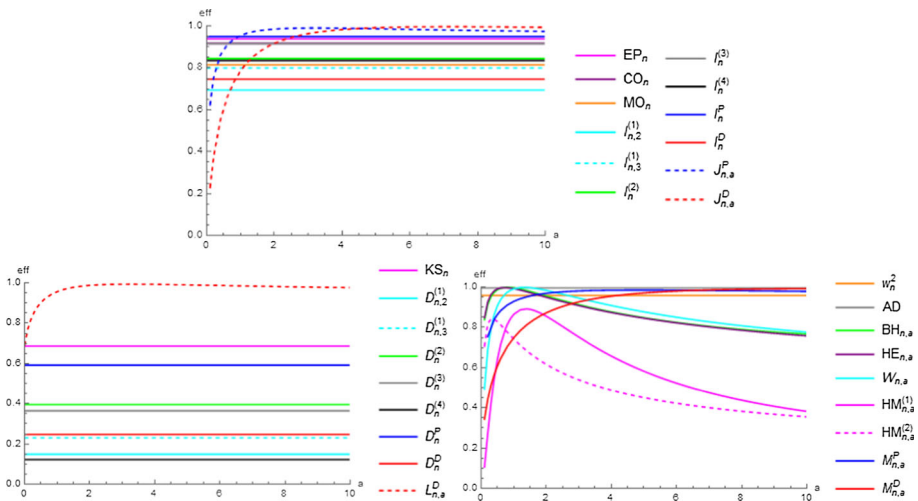


Fig. 4 Local approximate Bahadur efficiencies w.r.t. LRT for a EMNW(3) alternative

5 Powers of new tests

In this section we present the simulated powers of our new tests against different alternatives. The list of alternatives is chosen to be in concordance with the papers with extensive power comparison studies. The alternatives are:

- a Weibull $W(\theta - 1)$ distribution with density (10);
- a gamma $\Gamma(\theta - 1)$ distribution with density (11);
- a half-normal HN distribution with density

$$g(x) = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}, x \geq 0;$$

- a uniform U distribution with density

$$g(x) = 1, 0 \leq x \leq 1;$$

- a Chen's $CH(\theta)$ distribution with density

$$g(x, \theta) = 2\theta x^{\theta-1} e^{x^\theta - 2(1-e^{x^\theta})}, x \geq 0;$$

- a linear failure rate $LF(\theta)$ distribution with density (12);
- a modified extreme value $EV(\theta)$ distributions with density

$$g(x, \theta) = \frac{1}{\theta} e^{\frac{1-e^x}{\theta} + x}, x \geq 0;$$

- a log-normal $LN(\theta)$ distribution with density

$$g(x, \theta) = \frac{1}{x\sqrt{2\pi\theta^2}} e^{-\frac{(\ln x)^2}{2\theta^2}}, x \geq 0;$$

- a Dhillon $DL(\theta)$ distribution with density

$$g(x, \theta) = \frac{\theta + 1}{x + 1} (\ln(x + 1))^\theta e^{-(\ln(x+1))^{\theta+1}}, x \geq 0.$$

The powers, for aforementioned alternatives, and different choices of the tuning parameter are estimated using the Monte Carlo procedure with 10000 replicates at the level of significance 0.05.

The results are presented in Tables 1 and 2. In addition, we provide the bootstrap expected power estimate for data-driven optimal value of the tuning parameter (see [2] for details). Some steps to overcome "random nature" of selected parameters are made in Tenreiro [50], but some questions still remain open and are planned for future research.

We can see from tables that all the sizes of our tests are equal to the level of significance, and that the powers range from reasonable to high. In comparison to the other exponentiality tests (see [11,51]) we can conclude that our tests are serious competitors to the most powerful classical and recent exponentiality tests.

6 Conclusion

In this paper we proposed two new consistent scale-free tests for the exponential distribution. In addition, we performed an extensive comparison of efficiency of recent and classical exponentiality tests.

We showed that our tests are very efficient and powerful and can be considered as serious competitors to other high quality exponentiality tests.

From the comparison study, the general conclusion is that there is no uniformly best test, since the performance is different for different alternatives. However, the tests based on integral transforms, due to their flexibility because of the tuning parameter, generally tend to have higher efficiency, and they are recommended to use.

Table 1 Percentage of rejection of the hypothesis of exponentiality for $n = 20$

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	U	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EV(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$M_{n,0.2}^D$	5	24	45	9	20	72	8	52	13	20	18	58	12	28	63	18	85
$M_{n,0.5}^D$	5	34	55	15	33	67	11	67	20	29	26	61	14	36	76	15	81
$M_{n,1}^D$	5	43	63	21	46	64	15	77	27	39	35	60	18	38	80	12	77
$M_{n,2}^D$	5	44	64	23	56	57	16	82	31	44	42	56	24	38	83	11	74
$M_{n,5}^D$	5	49	64	29	67	53	21	87	37	53	52	46	33	35	80	10	70
$M_{n,10}^D$	5	47	63	30	73	50	21	90	38	54	54	41	41	31	79	10	66
$M_{n,\hat{a}}^D$	5	44	61	27	75	66	20	80	36	50	53	58	40	35	79	13	78
$L_{n,0.2}^D$	5	41	61	21	53	58	15	80	28	42	39	58	18	38	80	11	73
$L_{n,0.5}^D$	5	44	64	24	61	56	17	84	33	47	44	54	23	38	82	10	71
$L_{n,1}^D$	5	47	64	27	66	53	20	86	35	50	50	49	30	35	81	10	69
$L_{n,2}^D$	5	48	64	29	71	53	21	88	37	53	52	42	37	33	80	10	68
$L_{n,5}^D$	5	50	64	32	78	49	23	91	41	57	58	40	48	29	97	12	65
$L_{n,10}^D$	5	50	61	31	77	46	21	91	39	53	22	35	49	29	77	11	62
$L_{n,\hat{a}}^D$	5	46	61	27	73	57	20	87	36	51	53	55	39	35	78	11	72

Table 2 Percentage of rejection of the hypothesis of exponentiality for $n = 50$

Alt.	$Exp(1)$	$W(1.4)$	$\Gamma(2)$	HN	U	$CH(0.5)$	$CH(1)$	$CH(1.5)$	$LF(2)$	$LF(4)$	$EV(1.5)$	$LN(0.8)$	$LN(1.5)$	$DL(1)$	$DL(1.5)$	$W(0.8)$	$\Gamma(0.4)$
$M_{n,0.2}^D$	5	57	89	17	41	97	10	91	26	41	33	98	30	74	98	36	100
$M_{n,0.5}^D$	5	70	93	25	65	97	16	97	38	59	52	98	42	77	99	35	99
$M_{n,1}^D$	5	76	95	35	79	97	21	99	49	70	64	97	55	76	100	34	99
$M_{n,2}^D$	5	81	96	43	90	96	29	100	59	79	76	92	66	73	100	35	99
$M_{n,5}^D$	5	83	96	51	96	95	35	100	67	86	85	82	81	66	100	36	99
$M_{n,10}^D$	5	86	96	58	98	95	41	100	73	89	90	73	86	60	99	38	99
$M_{n,\infty}^D$	5	85	93	62	100	90	45	100	77	91	94	54	92	47	98	38	97
$M_{n,\hat{a}}^D$	5	81	95	56	96	97	41	97	71	87	90	97	87	74	99	36	99
$L_{n,0.2}^D$	5	79	96	41	90	96	27	96	58	78	74	96	62	77	100	34	99
$L_{n,0.5}^D$	5	83	96	45	93	95	30	100	63	82	79	93	69	72	100	32	99
$L_{n,1}^D$	5	85	96	51	96	95	100	100	66	86	85	87	78	69	100	36	99
$L_{n,2}^D$	5	86	96	56	98	94	39	100	72	88	89	79	85	63	99	37	99
$L_{n,5}^D$	5	86	95	59	99	94	43	100	74	90	92	65	89	56	99	39	98
$L_{n,10}^D$	5	84	95	62	99	92	43	100	76	91	92	62	80	53	99	39	98
$L_{n,\hat{a}}^D$	5	83	95	57	96	95	40	100	72	88	89	96	84	72	100	35	99

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Appendix A: Proofs of theorems

Proof (Theorem 2) Our statistic $M_{n,a}^D(\widehat{\lambda}_n)$ can be rewritten as

$$\begin{aligned} M_{n,a}^D(\widehat{\lambda}_n) &= \int_0^\infty \left(\frac{1}{n^2} \sum_{i_1, i_2=1}^n \xi(X_{i_1}, X_{i_2}, t; a\widehat{\lambda}_n) \right)^2 e^{-at} dt \\ &= \int_0^\infty V_n(t, \widehat{\lambda}_n)^2 e^{-at} dt. \end{aligned}$$

Here $V_n(t; \widehat{\lambda}_n)$, for each $t > 0$, is a V -statistic of order 2 with an estimated parameter, and kernel

$$\xi(x_1, x_2, t; a, \widehat{\lambda}_n) = \frac{1}{2} \left(e^{-t\widehat{\lambda}_n x_1} + e^{-t\widehat{\lambda}_n x_2} - e^{-t\widehat{\lambda}_n 2 \min(x_1, x_2)} \right).$$

Since the function $\xi(x_1, x_2, t; a, \gamma)$ is continuously differentiable with respect to γ at the point $\gamma = \lambda$ we may apply the mean value theorem. We have

$$V_n(t; \widehat{\lambda}_n) = V_n(t; \lambda) + (\widehat{\lambda}_n - \lambda) \frac{\partial V_n(t; \gamma)}{\partial \gamma} \Big|_{\gamma=\lambda^*}, \tag{13}$$

for some λ^* between λ and $\widehat{\lambda}_n$. From the Law of large numbers for V -statistics [47, 6.4.2.], the partial derivative $\frac{\partial V_n(t; \gamma)}{\partial \gamma}$ converges to

$$E \left(2t \min\{X_1, X_2\} e^{-2t \min\{X_1, X_2\} \gamma} - t X_1 e^{-t X_1 \gamma} \right) = 0. \tag{14}$$

Since $\sqrt{n}(\widehat{\lambda}_n - \lambda)$ is stochastically bounded, it follows that statistics $\sqrt{n}V_n(t; \widehat{\lambda}_n)$ and $\sqrt{n}V_n(t; \lambda)$ are asymptotically equally distributed. Therefore, $nM_{n,a}^D(\widehat{\lambda}_n)$ and $nM_{n,a}^D(\lambda)$ will have the same limiting distribution. Hence we need to derive limiting distribution of $nM_{n,a}^D(\lambda)$.

First notice that $M_{n,a}^D(\lambda)$ is a V -statistic with symmetric kernel H . Also, since the distribution of $M_{n,a}^D(\lambda)$ does not depend on λ we may assume that $\lambda = 1$.

It is easy to show that its first projection of kernel H on X_1 is equal to zero. After some calculations, we obtain that its second projection on (X_1, X_2) is given by

$$\begin{aligned} \widetilde{h}_2(u, v; a) &= E(H(X_1, X_2, X_3, X_4; a, 1) | X_1 = u, X_2 = v) \\ &= \frac{1}{6} \left(3 + \frac{1}{a+u+v} - \frac{2e^{-u}}{a+2u+v} - \frac{2e^{-v}}{a+u+2v} - (4-a)e^a Ei(-a) \right. \\ &\quad \left. + e^{\frac{a+v}{2}} \left(Ei\left(-\frac{a+v}{2}\right) - Ei\left(-\frac{a+2u+v}{2}\right) \right) + e^{a+u} \left(4Ei(-a-2u) \right. \right. \\ &\quad \left. \left. - Ei(-a-u) \right) + e^{\frac{a+u}{2}} \left(Ei\left(-\frac{a+u}{2}\right) - Ei\left(-\frac{a+u+2v}{2}\right) \right) \right) \\ &\quad + e^{a+v} \left(4Ei(-a-2v) - Ei(-a-v) \right) - 2(e^{-u} + e^{-v}) \\ &\quad + e^{\frac{a}{2}} \left(-(4+a+2u)Ei\left(-\frac{a}{2}-u\right) + (a+4)Ei\left(-\frac{a}{2}\right) \right) \\ &\quad \left. + (a+2(2+u+v))Ei\left(-\frac{a}{2}-u-v\right) - (4+a+2v)Ei\left(-\frac{a}{2}-v\right) \right) \end{aligned}$$

$$+ \frac{e^{-u-v}}{a + 2(u + v)} (2a + 4(1 + u + v)) \Big),$$

where $Ei(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t} dt$ is the exponential integral. The function \tilde{h}_2 is non-constant for any $a > 0$. Hence, kernel h is degenerate with degree 2.

Since the kernel H is bounded and degenerate, from the theorem on asymptotic distribution of U-statistics with degenerate kernels [26, Corollary 4.4.2], and the Hoeffding representation of V-statistics, we get that, $M_{n,a}^{\mathcal{D}}(1)$, being a V-statistic of degree 2, has the following asymptotic distribution

$$nM_{n,a}^{\mathcal{D}}(1) \xrightarrow{d} 6 \sum_{k=1}^{\infty} \delta_k W_k^2, \tag{15}$$

where $\{\delta_k\}$ are the eigenvalues of the integral operator \mathcal{M}_a defined by

$$\mathcal{M}_a q(x) = \int_0^{+\infty} \tilde{h}_2(x, y; a) q(y) dF(y), \tag{16}$$

and $\{W_k\}$ is the sequence of i.i.d. standard Gaussian random variables. □

Proof (Theorem 3) The test statistic can be represented as

$$L_{n,a}^{\mathcal{D}} = \sup_{t \geq 0} |V_n(t; \hat{\lambda}_n) e^{-at}|, \tag{17}$$

where $\{V_n(t; \hat{\lambda}_n)\}$ is a V-empirical process introduced in the proof of Theorem 2.

Substituting $s = e^{-t}$ in (17) we can express our statistic as

$$L_{n,a}^{\mathcal{D}} = \sup_{s \in (0,1)} |V_n(-\ln s; \hat{\lambda}_n) s^a|,$$

thus obtaining, as a core of the statistic, the process $V_n(-\ln s; \hat{\lambda}_n)$, $s \in (0, 1)$, defined on $C[0, 1]$ equipped with supremum norm.

Next we show that the difference between $\sqrt{n}L_{n,a}^{\mathcal{D}}$ and $\sqrt{n} \sup_{s \in (0,1)} |V_n(-\ln s; \lambda) s^a|$ (for a fixed λ) tends uniformly to zero and proceed finding the limiting distribution of the latter.

This is, taking into account (13), equivalent to $\lim_{n \rightarrow \infty} \sup_{s \in (0,1)} R_n(s) = 0$, where

$$\begin{aligned} R_n(s) &= s^a \left| \frac{1}{n^2} \sum_{i,j} \left(-2 \ln s \min\{X_1, X_2\} s^{2 \min\{X_1, X_2\} \gamma} + \ln s X_1 s^{X_1 \gamma} \right) \right| \\ &= -\ln s \cdot s^a \left| \frac{1}{n^2} \sum_{i,j} \left(2 \min\{X_1, X_2\} s^{2 \min\{X_1, X_2\} \gamma} - X_1 s^{X_1 \gamma} \right) \right|. \end{aligned}$$

From (14) we know that the expression in the absolute parentheses tends to zero for each s , and using [44, Lemma 1] we get that its supremum also tends to zero. Since $-\ln s \cdot s^a$ is bounded function, $\sup_s R_n(s)$ tends also to zero.

Since, for a fixed s , $\sqrt{n}V_n(-\ln s; \lambda)$ is a non-degenerate V-statistic, $\sqrt{n}V_n(-\ln s; \lambda)$ is asymptotically normally distributed. The same holds for finite-dimensional distributions of the process $\sqrt{n}V_n(-\ln s; \lambda)$. In addition, it can be shown that $\sqrt{n}V_n(-\ln s; \lambda)$ satisfies conditions from Billingsley [8, Theorem 12.3], and is, therefore, tight in $C[0, 1]$. Therefore it converges weakly to a zero mean Gaussian process $\{\eta^*(s)\}$ with covariance function

$$K^*(u, v) = K(-\ln u, -\ln v),$$

where K is defined in (3).

Since the supremum is continuous on $C[0, 1]$, using the continuous mapping theorem we get that $\sqrt{n} \sup_{s \in (0,1)} |V_n(-\ln s; \lambda) s^a|$ converges to $\sup_{s \in (0,1)} |\eta^*(s) s^a|$, which is the same as $\sup_{t>0} |\eta(t) e^{-at}|$. This completes the proof. \square

Proof (Lemma 4) Using the result of Zolotarev [55], the logarithmic tail behavior of limiting distribution function of $\tilde{M}_{n,a}^D(\hat{\lambda}_n) = \sqrt{n} M_{n,a}^D(\hat{\lambda}_n)$ is

$$\ln(1 - F_{\tilde{M}_a}(t)) = -\frac{t^2}{12\delta_1} + o(t^2), \quad t \rightarrow \infty.$$

Therefore, $a_{\tilde{M}_a} = \frac{1}{6\delta_1}$. The limit in probability P_θ of $\tilde{M}_{n,a}(\hat{\lambda}_n)/\sqrt{n}$ is

$$b_{\tilde{M}_a} = \sqrt{b_M(\theta)}.$$

The expression for $b_M(\theta)$ is derived in the following lemma.

Lemma A1 For a given alternative density $g(x; \theta)$ whose distribution belongs to \mathcal{G} , we have that the limit in probability of the statistic $M_{n,a}^D(\hat{\lambda}_n)$ is

$$b_M(\theta) = 6 \int_0^\infty \int_0^\infty \tilde{h}_2(x, y; a) g'_\theta(x; 0) g'_\theta(y; 0) dx dy \cdot \theta^2 + o(\theta^2), \quad \theta \rightarrow 0.$$

Proof For brevity, denote $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and $\mathbf{G}(\mathbf{x}; \theta) = \prod_{i=1}^4 G(x_i; \theta)$. Since \bar{X}_n converges almost surely to its expected value $\mu(\theta)$, using the Law of large numbers for V -statistics with estimated parameters (see [22]), $M_{n,a}^D(\hat{\lambda}_n)$ converges to

$$\begin{aligned} b_M(\theta) &= E_\theta(H(\mathbf{X}, a; \mu(\theta))) \\ &= \int_{(R^+)^4} \left(\frac{\mu(\theta)}{2 \min\{x_1, x_2\} + 2 \min\{x_3, x_4\} + a\mu(\theta)} - \frac{\mu(\theta)}{x_1 + 2 \min\{x_3, x_4\} + a\mu(\theta)} \right. \\ &\quad \left. - \frac{\mu(\theta)}{x_3 + 2 \min\{x_1, x_2\} + a\mu(\theta)} + \frac{\mu(\theta)}{x_1 + x_3 + a\mu(\theta)} \right) d\mathbf{G}(\mathbf{x}; \theta). \end{aligned}$$

We may assume that $\mu(0) = 1$ since the test statistic is ancillary for λ under the null hypothesis. After some calculations we get that $b'_M(0) = 0$ and that

$$b''(0) = \int_{(R^+)^4} H(\mathbf{x}, a; 1) \frac{\partial^2}{\partial \theta^2} d\mathbf{G}(\mathbf{x}, 0) = 6 \int_{(R^+)^2} \tilde{h}_2(x, y) g'_\theta(x; 0) g'_\theta(y; 0) dx dy.$$

Expanding $b_M(\theta)$ into the Maclaurin series

$$b_M(\theta) = b_M(0) + b'_M(0) \cdot \theta + b''_M(0)\theta^2 + o(\theta^2), \quad \theta \rightarrow 0,$$

we complete the proof. \square

Now we pass to the statistic L_n^D . The tail behaviour of the random variable $\sup_{t>0} |n_t|$ is equal to the inverse of supremum of its covariance function, i.e. the $a_L = \frac{1}{\sup_{t>0} K(t, t)}$ (see [27,36]).

Similarly like before, since \bar{X}_n converges almost surely to its expected value $\mu(\theta)$, using the Law of large numbers for V -statistics with estimated parameters [22], $V_n(t, a; \hat{\lambda}) e^{-at}$ converges to

$$b_L(\theta; t) = E_\theta(\Phi(X_1, X_2; t, a, \mu(\theta))).$$

Expanding $b_L(\theta; t)$ in the Maclaurin series we obtain

$$b_L(\theta; t) = 2 \int_0^\infty \tilde{\varphi}_1(x, t; a) g'_\theta(x; 0) dx \cdot \theta + o(\theta),$$

where $\tilde{\varphi}_1(x, t; a) = E(\Phi(X_1, X_2, t; a, 1) | X_1 = x_1)$.

Since V-empirical Laplace transforms are monotonous functions, a Glivenko-Cantelli-type theorem holds, see Novoa-Muñoz and Jiménez-Gamero [44, Lemma 1]. Hence, the limit in probability under the alternative for statistics $L_{n,a}^D$ is equal to $\sup_{t \geq 0} |b_L(\theta; t)|$. Inserting this into the expression for the Bahadur slope completes the proof. \square

Appendix B: Bahadur approximate slopes

Proof Approximate local Bahadur slope of statistics EP and CO

Those statistics can be represented as

$$T_n = \frac{1}{n} \sum_{i=1}^n \Phi(X; \hat{\mu}),$$

where $\Phi(x; \gamma)$ is continuously differentiable with respect to γ at point $\gamma = \mu$. It was shown that the limiting distribution of $\sqrt{n}T_n$ is zero mean normal with variance σ_Φ^2 (see [10,13]). Hence, the coefficient a_T is equal to $\frac{1}{\sigma_\Phi^2}$.

Further, we have

$$b(\theta) = E_\theta(\Phi(X; \mu(\theta))) = \int_0^\infty \Phi(x; \mu(\theta)) dG(x; \theta)$$

$$b'(\theta) = \int_0^\infty \frac{\partial}{\partial \mu} \Phi(x; \mu(\theta)) \frac{\partial}{\partial \theta} \mu(\theta) dG(x; \theta) + \int_0^\infty \Phi(x; \mu(\theta)) \frac{\partial}{\partial \theta} dG(x; \theta).$$

Then it holds that

$$b(\theta) = b(0) + b'(0)\theta + o(\theta)$$

$$= \left(\mu'(0) \int_0^\infty \Phi'(x; 1) g(x; 0) dx + \int_0^\infty \Phi(x; 1) g'(x; 0) dx \right) \theta + o(\theta).$$

From this we obtain the expression for $c_T(\theta)$. \square

Proof Approximate local Bahadur slope of statistics BH, HE, Wn, HM, ω^2 and AD

Let T be the one of considered statistics. It was shown that the limiting distribution of nT_n is $\sum_{i=1}^{\infty} \delta_i W_i^2$, where $\{W_i\}$ is the sequence of i.i.d. standard normal variables and $\{\delta_i\}$ the sequence of eigenvalues of certain covariance operator. Using the result of Zolotarev in [55], we have that the logarithmic tail behavior of limiting distribution function of $\tilde{T}_n = \sqrt{nT_n}$ is

$$\ln(1 - F_{\tilde{T}}(s)) = -\frac{s^2}{2\delta_1} + o(s^2), s \rightarrow \infty.$$

Next, the limit in probability of \tilde{T}_n/\sqrt{n} is $b_{\tilde{T}}(\theta) = \sqrt{b_T(\theta)}$. Statistic T_n can be represented as

$$T_n = \frac{1}{n^2} \sum_{k,j=1}^n \Phi(X_k, X_j; \hat{\mu}).$$

As before, we may assume that $\mu(0) = 1$. Since the sample mean converges almost surely to its expected value, by using the Law of large numbers for V -statistics with estimated parameters (see [22]), we can conclude that the limit in the probability of statistic T_n is equal to the one of

$$b_T(\theta) = E_{\theta}(\Phi(X_1, X_2; \mu(\theta))) = \int_0^{\infty} \int_0^{\infty} \Phi(x, y; \mu(\theta))g(x; \theta)g(y; \theta)dx dy.$$

We get that $b'_T(0) = 0$ and that

$$\begin{aligned} b''_T(0) &= 2 \int_0^{\infty} \int_0^{\infty} \Phi(x, y; 1)g'_x(x; 0)g'_y(y; 0)dx dy \\ &+ 4\mu'(0) \int_0^{\infty} \int_0^{\infty} \Phi'(x, y; 1)g(x; 0)g'_y(y; 0)dx dy \\ &+ (\mu'(0))^2 \int_0^{\infty} \int_0^{\infty} \Phi''(x, y; 1)g(x; 0)g(y; 0)dx dy, \end{aligned}$$

Expanding $b_T(\theta)$ into Maclaurin series we obtain expression for b_T . □

Appendix C: Tables of efficiencies

Table 3 Relative Bahadur efficiency with respect to LRT

	<i>Weibull</i>	<i>Gamma</i>	<i>LFR</i>	<i>EMNW(3)</i>
EP_n	0.876	0.694	0.750	0.937
CO_n	1	0.943	0.608	0.917
G_n	0.876	0.694	0.750	0.937
MO_n	0.943	1	0.388	0.814
$I_{n,2}^{(1)}$	0.621	0.723	0.104	0.694
$I_{n,3}^{(1)}$	0.664	0.708	0.159	0.799
$I_n^{(2)}$	0.750	0.796	0.208	0.844
$I_n^{(3)}$	0.746	0.701	0.308	0.916
$I_n^{(4)}$	0.649	0.638	0.206	0.835
$I_n^{\mathcal{P}}$	0.821	0.788	0.337	0.949
$I_n^{\mathcal{D}}$	0.697	0.790	0.149	0.746
$J_{n,0.2}^{\mathcal{P}}$	0.750	0.856	0.171	0.751
$J_{n,0.5}^{\mathcal{P}}$	0.812	0.843	0.262	0.888
$J_{n,1}^{\mathcal{P}}$	0.846	0.820	0.349	0.955
$J_{n,2}^{\mathcal{P}}$	0.868	0.792	0.445	0.985
$J_{n,5}^{\mathcal{P}}$	0.882	0.756	0.566	0.987
$J_{n,10}^{\mathcal{P}}$	0.884	0.733	0.637	0.974
$J_{n,0.2}^{\mathcal{D}}$	0.526	0.731	0.053	0.370
$J_{n,0.5}^{\mathcal{D}}$	0.674	0.826	0.117	0.608
$J_{n,1}^{\mathcal{D}}$	0.771	0.857	0.198	0.786
$J_{n,2}^{\mathcal{D}}$	0.842	0.854	0.305	0.917
$J_{n,5}^{\mathcal{D}}$	0.889	0.813	0.465	0.991
$J_{n,10}^{\mathcal{D}}$	0.896	0.775	0.569	0.994
KS	0.538	0.503	0.356	0.686
$D_{n,2}^{(1)}$	0.092	0.093	0.052	0.149
$D_{n,3}^{(1)}$	0.152	0.138	0.106	0.230
$D_n^{(2)}$	0.277	0.267	0.155	0.396
$D_n^{(3)}$	0.258	0.212	0.213	0.364
$D_n^{(4)}$	0.079	0.066	0.067	0.122
$D_n^{\mathcal{P}}$	0.437	0.448	0.192	0.592
$D_n^{\mathcal{D}}$	0.158	0.174	0.073	0.247
ω_n^2	0.808	0.701	0.588	0.958
AD_n	0.909	0.863	0.573	0.996
$BH_{n,0.2}$	0.905	0.928	0.421	0.914

Table 3 continued

	Weibull	Gamma	LFR	EMNW(3)
$BH_{n,0.5}$	0.932	0.877	0.534	0.987
$BH_{n,1}$	0.926	0.810	0.638	0.996
$BH_{n,2}$	0.894	0.726	0.749	0.956
$BH_{n,5}$	0.823	0.611	0.878	0.848
$BH_{n,10}$	0.771	0.542	0.956	0.767
$HE_{n,0.2}$	0.923	0.928	0.420	0.927
$HE_{n,0.5}$	0.940	0.868	0.542	0.991
$HE_{n,1}$	0.928	0.799	0.647	0.992
$HE_{n,2}$	0.893	0.719	0.752	0.949
$HE_{n,5}$	0.822	0.609	0.873	0.846
$HE_{n,10}$	0.761	0.536	0.935	0.758
$W_{n,0.2}$	0.790	0.914	0.224	0.688
$W_{n,0.5}$	0.905	0.922	0.382	0.909
$W_{n,1}$	0.935	0.864	0.528	0.991
$W_{n,2}$	0.917	0.772	0.677	0.983
$W_{n,5}$	0.842	0.638	0.842	0.877
$W_{n,10}$	0.774	0.550	0.924	0.776
$HM_{n,0.2}^{(1)}$	0.324	0.448	0.049	0.271
$HM_{n,0.5}^{(1)}$	0.560	0.621	0.174	0.643
$HM_{n,1}^{(1)}$	0.691	0.642	0.361	0.865
$HM_{n,2}^{(1)}$	0.715	0.557	0.612	0.855
$HM_{n,5}^{(1)}$	0.591	0.373	0.895	0.582
$HM_{n,10}^{(1)}$	0.452	0.254	0.951	0.382
$HM_{n,0.2}^{(2)}$	0.633	0.579	0.320	0.818
$HM_{n,0.5}^{(2)}$	0.673	0.533	0.520	0.828
$HM_{n,1}^{(2)}$	0.656	0.468	0.683	0.742
$HM_{n,2}^{(2)}$	0.603	0.391	0.825	0.616
$HM_{n,5}^{(2)}$	0.504	0.295	0.931	0.451
$HM_{n,10}^{(2)}$	0.430	0.238	0.942	0.355
$M_{n,0.2}^{(P)}$	0.734	0.832	0.183	0.754
$M_{n,0.5}^{(P)}$	0.787	0.827	0.253	0.865
$M_{n,1}^{(P)}$	0.822	0.814	0.324	0.929
$M_{n,2}^{(P)}$	0.850	0.794	0.407	0.969
$M_{n,5}^{(P)}$	0.873	0.764	0.523	0.985

Table 3 continued

	Weibull	Gamma	LFR	EMNW(3)
$M_{n,10}^{(P)}$	0.881	0.742	0.601	0.979
$M_{n,0.2}^{(D)}$	0.533	0.712	0.080	0.443
$M_{n,0.5}^{(D)}$	0.645	0.788	0.130	0.610
$M_{n,1}^{(D)}$	0.729	0.825	0.191	0.750
$M_{n,2}^{(D)}$	0.803	0.838	0.275	0.873
$M_{n,5}^{(D)}$	0.867	0.820	0.413	0.971
$M_{n,10}^{(D)}$	0.889	0.789	0.520	0.992
$L_{n,0.2}^{(D)}$	0.738	0.798	0.259	0.821
$L_{n,0.5}^{(D)}$	0.799	0.815	0.323	0.902
$L_{n,1}^{(D)}$	0.844	0.815	0.394	0.957
$L_{n,2}^{(D)}$	0.875	0.800	0.479	0.988
$L_{n,5}^{(D)}$	0.892	0.766	0.588	0.990
$L_{n,10}^{(D)}$	0.891	0.740	0.652	0.976

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