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ORIGINAL PAPER

On the spherical maximal function on finite graphs

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Abstract

This paper is devoted to studying the mapping properties for the spherical maximal operator $\mathbf{S}_G$ defined on finite connected graphs G. Some operator norms of $\mathbf{S}_G$ on the $\ell^p(G)$, $\ell^{p,\infty}(G)$ and the spaces of bounded p-variation functions defined on G are investigated. Particularly, as some special examples of finite connected graphs, the complete graph K_n and star graph S_n are discussed.

Keywords Finite connected graph · Spherical maximal function · Bounded variation · Operator norm

Mathematics Subject Classification Primary 42B25 · 05C12; Secondary 46E35 · 05C75

1 Introduction

Let $G = (V_G, E_G)$ be an undirected combinatorial graph with the set of vertices V_G and the set of edges E_G. We say that two vertices $x,\ y \in V_G$ are neighbors if they are connected by an edge in E_G, which is denoted by $x \sim y$. If $x \sim y$, then $y \sim x$ and we set $x \sim y = y \sim x$. We denote by $N_G(v)$ the set of neighbors of v for any $v \in V_G$. The graph G is called finite if $|V_G| < \infty$. The graph is called connected if for any distinct $x,\ y \in V_G$, there is a finite sequence of vertices $\{x_i\}_{i=0}^k$, $k \in \mathbb{N}$, such that $x = x_0 \sim x_1 \sim \cdots \sim x_k = y$.

In what follows, we always assume that the graph $G = (V_G, E_G)$ is a finite connected graph with $n\ (n \geq 2)$ vertices. Let d_G be the metric induced by the edges in E_G. That is, given $u, v \in V_G$, the distance $d_G(u, v)$ is the number of edges in a shortest path connecting u and v. Let $B_G(v, r)$ be the ball centered at v, with radius r on the graph, i.e.

$$B_G(v, r) = \{u \in V_G : d_G(u, v) \leq r\}.$$

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For example, $B_G(v,r)=\{v\}$ if $0\le r<1$ and $B_G(v,r)=\{v\}\cup N_G(v)$ if $1\le r<2$. Let $|B_G(v,r)|$ be the cardinality of $B_G(v,r)$. For a function $f:V_G\to\mathbb{R}$, the Hardy–Littlewood maximal operator on G is given by

$$M_G f(v)=\sup_{r\ge 0}\frac{1}{|B_G(v,r)|}\sum_{w\in B_G(v,r)}|f(w)|. \tag{1.1}$$

Since G has n ($n\ge 2$) vertices, the maximal operator M_G can be redefined by

$$M_G f(v)=\max_{k=0,\ldots,n-1}\frac{1}{|B_G(v,k)|}\sum_{w\in B_G(v,k)}|f(w)|.$$

Over the last several years the Hardy–Littlewood maximal operators on graphs has been studied by many authors (see [2,6,24,26,27]). This operator defined in (1.1) was first introduced and studied by Korányi and Picardello [24] who used the maximal operator to investigate the boundary behaviour of eigenfunctions of the Laplace operator on trees. Later on, Cowling et al. [6] further studied the Hardy–Littlewoood maximal operators on homogeneous trees. Some weighted norm inequalities for the Hardy–Littlewoood maximal operators on infinite graphs were established by Badr and Martell [2]. Recently, Soria and Tradacete [26] studied the ℓ^p-norm for the Hardy–Littlewood maximal operators on finite connected graphs.

Definition 1.1 ($\ell^p(G)$ *spaces*) Let $G=(V_G,E_G)$ be a graph with the set of vertices V_G and the set of edges E_G. For $0<p\le\infty$, let $L^p(G)$ be the set of all functions $f:V_G\to\mathbb{R}$ satisfying $\|f\|_{\ell^p(G)}<\infty$, where $\|f\|_{\ell^p(G)}=(\sum_{v\in V_G}|f(v)|^p)^{1/p}$ for all $0<p<\infty$ and $\|f\|_{\ell^\infty(G)}=\sup_{v\in V_G}|f(v)|$.

Soria and Tradacete [26] studied the ℓ^p-norm of M_G:

$$\|M_G\|_p:=\sup_{\|f\|_{\ell^p(G)}\neq 0}\frac{\|M_G f\|_{\ell^p(G)}}{\|f\|_{\ell^p(G)}},\quad \text{for } 0<p\le\infty.$$

We now introduce partial results of [26] as follows:

Theorem A [26] *Let $G=(V_G,E_G)$ be a graph with n vertices and $0<p<1$. Let K_n be the complete graph with n vertices, i.e. $|N_{K_n}(v)|=n-1$ for any $v\in V_{K_n}$ and let S_n be the star graph of n vertices, i.e. there exists an unique $v\in V_{S_n}$ such that $|N_{S_n}(v)|=n-1$ and $|N_{S_n}(w)|=1$ for every $w\in V_{S_n}\backslash\{v\}$. Then,*

$$\left(1+\frac{n-1}{n^p}\right)^{1/p}\le\|M_G\|_p\le\left(1+\frac{n-1}{2^p}\right)^{1/p}.$$

Moreover,

(i) *If $0<p\le 1$, then $\|M_G\|_p=(1+\frac{n-1}{n^p})^{1/p}$ if and only if $G=K_n$.*
(ii) *If $0<p\le 1$, then $\|M_G\|_p=(1+\frac{n-1}{2^p})^{1/p}$ if and only if G is isomorphic to S_n.*
(iii) *If $1<p<\infty$, then*

$$\left(1+\frac{n-1}{n^p}\right)^{1/p}\le\|M_{K_n}\|_p\le\left(1+\frac{n-1}{n}\right)^{/p}.$$

(iv) *If $1<p<\infty$, then*

$$\left(1+\frac{n-1}{2^p}\right)^{1/p}\le\|M_{S_n}\|_p\le\left(\frac{n+5}{2}\right)^{1/p}.$$

Very recently, González-Riquelme and Madrid [7] obtained the best constants for $\|M_{K_n}\|_2$ and $\|M_{S_n}\|_2$. In [26], Soria and Tradacete [26] also investigated the weak-type $\ell^{p,\infty}$-norm of M_G:

$$\|M_G\|_{\ell^{p,\infty}(G)} := \sup_{\|f\|_{\ell^p(G)}\neq 0} \frac{\|M_G f\|_{\ell^{p,\infty}(G)}}{\|f\|_{\ell^p(G)}}, \quad \text{for } 0 < p < \infty,$$

where

$$\|f\|_{\ell^{p,\infty}(G)} := \sup_{t>0} t|\{j \in V_G : |f(j)| > t\}|^{1/p}.$$

It is clear that $\|f\|_{\ell^{p,\infty}(G)} \leq \|f\|_{\ell^p(G)}$.

Soria and Tradacete [26] proved the following result.

Theorem B [26] *Let* $0 < p < \infty$*, then*

$$\|M_{K_n}\|_{p,\infty} = \begin{cases} n^{1/p-1}, & \text{if } 0 < p \leq 1; \\ 1, & \text{if } p \geq 1. \end{cases}$$

Moreover,

$$\max\left\{\frac{n^{1/p}}{2}, 1\right\} \leq \|M_{S_n}\|_{p,\infty} \leq n^{1/p}.$$

In particular, for every connected graph G with n vertices,

$$2n^{1/p} \geq \|M_G\|_{p,\infty} \geq \begin{cases} n^{1/p-1}, & \text{if } 0 < p \leq 1; \\ 1, & \text{if } p \geq 1. \end{cases}$$

It is well known that the spherical maximal function

$$\mathcal{S}f(x) = \sup_{t>0} \left| \int_{S^{n-1}} f(x - t\theta) d\sigma(\theta) \right|$$

has played a key role in harmonic analysis and partial differential equations. This introduction was motivated by some special spherical averages, which are some solutions of certain partial differential equations, such as wave equation, Darboux's equation and so on. We can consult [3,20–22] for their history, background and applications. Stein [21] first established the L^p bounds for $\mathcal{S}$ with $\frac{n}{n-1} < p \leq \infty$ when $n \geq 3$. Other proofs for the case $n \geq 3$ can be found in [5,20]. It is more remarkable that the more difficult case $n = 2$ was first settled by Bourgain in [3]. Alternative proofs for the case $n = 2$ were given by Mockenhaupt, Seeger and Sogge [19] as well as Schlag [25]. Other interesting works can be consulted [11,18] for the discrete spherical maximal functions as well as [8,10,14] for the Sobolev regularity of the spherical maximal functions.

The main objective of this paper is the spherical maximal function on finite graphs. Let us introduce one definition.

Definition 1.2 (*Spherical maximal operator on graphs*) Let $G = (V_G, E_G)$ be a graph with n vertices. For $v \in V_G$ and $r \geq 0$, let $S_G(v, r)$ denote the sphere of center v and radius r on the graph G, i.e.

$$S_G(v, r) = \{u \in V_G : d_G(v, u) = r\}.$$

Specially, $S_G(v,1) = N_G(v)$ for all $v \in V_G$. The Spherical maximal operator $\mathbf{S}_G$ is given by

$$\mathbf{S}_G f(v) = \sup_{r\geq 0} \frac{1}{|S_G(v,r)|} \sum_{u\in S_G(v,r)} |f(u)|.$$

Since G has n ($n \geq 2$) vertices, the Spherical maximal operator $\mathbf{S}_G$ can be rewritten by

$$\mathbf{S}_G f(v) = \max_{r=0,1,\ldots,n-1} \frac{1}{|S_G(v,r)|} \sum_{u\in S_G(v,r)} |f(u)|.$$

Clearly, $|f(v)| \leq \mathbf{S}_G f(v) \leq \|f\|_\infty$ for all $v \in V_G$. It follows that

$$1 \leq \|\mathbf{S}_G\|_p \leq n^{1/p}, \quad 0 < p \leq \infty. \tag{1.2}$$

It was pointed out in [27] that

$$M_G f(x) \leq \mathbf{S}_G f(x), \quad x \in V_G.$$

Based on (1.2) and the facts concerning the best constants of $\|M_G\|_p$ and $\|M_G\|_{p,\infty}$, it is interesting and natural to study the best constants of $\|\mathbf{S}_G\|_p$ and $\|\mathbf{S}_G\|_{p,\infty}$, which is one of main motivations in this work.

On the other hand, the regularity theory of maximal operators has been the subject of many recent research papers in harmonic analysis. The first work related to this topic was due to Kinnunen [12] in 1997 when he established the boundedness for the centered Hardy–Littlewood maximal operator $\mathcal{M}$ on the first order Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ with $1 < p \leq \infty$. Since then, Kinnunen's result was extended to various versions (see [4,13,14]). Since Kinnunen's result does not include the case $p = 1$, the $W^{1,1}$-regularity for $\mathcal{M}$ is a certainly more delicate issue. A complete solution was obtained only in dimension $n = 1$ (see [1,15,23]) and partial progress on the general dimension $n \geq 2$ was given by Hajłasz and Malý [9] and Luiro [17]. In particular, Aldaz and Pérez Lázaro [1] proved that if f is of bounded variation on $\mathbb{R}$, the uncentered Hardy–Littlewood maximal function $\widetilde{\mathcal{M}} f$ is absolutely continuous and

$$\operatorname{Var}\left(\widetilde{\mathcal{M}} f\right) \leq \operatorname{Var}(f),$$

where $\operatorname{Var}(f)$ denotes the total variation of f on $\mathbb{R}$.

Very recently, in order to generalize the endpoint regularity of maximal operators in [1] to the graph setting, Liu and Xue [16] introduced the following BV_p spaces on graphs.

Definition 1.3 ($\mathrm{BV}_p(G)$ **spaces**) Let $G = (V_G, E_G)$ and $0 < p \leq \infty$. We define the spaces of bounded p-variation functions on graph G by

$$\mathrm{BV}_p(G) := \{f : V_G \to \mathbb{R};\ \|f\|_{\mathrm{BV}_p(G)} := \operatorname{Var}_p(f) < \infty\},$$

where $\operatorname{Var}_p(f)$ represents the p-variation of f defined by

$$\operatorname{Var}_p(f) = \left(\sum_{u\sim v\in E_G} |f(u) - f(v)|^p\right)^{1/p}, \quad \text{for } 0 < p < \infty$$

and

$$\operatorname{Var}_\infty(f) = \sup_{u\sim v\in E_G} |f(u) - f(v)|.$$

When $p = 1$, we denote $\mathrm{BV}_p(G) = \mathrm{BV}(G)$ and $\operatorname{Var}_p(f) = \operatorname{Var}(f)$.

One can easily check that

$$\mathrm{Var}_q(f) \le \mathrm{Var}_p(f) \le \left(\frac{n(n-1)}{2}\right)^{1/p-1/q} \mathrm{Var}_q(f), \quad \text{for } \ 0 < p \le q \le \infty.$$

and

$$\mathrm{Var}_p(f) \le C_{p,n}\|f\|_{\ell^p(G)}, \quad 0 < p \le \infty.$$

However, there does not exist a constant $C_{p,n} > 0$ such that

$$\|f\|_{\ell^p(G)} \le C_{p,n}\mathrm{Var}_p(f),$$

for all $0 < p \le \infty$ and any functions $f \in \mathrm{BV}_p(G)$. We also note that

$$\ell^p(G) = \mathrm{BV}_p(G), \quad 0 < p \le \infty, \tag{1.3}$$

where G is a finite graph.

Liu and Xue [16] investigated the boundedness for M_G on $\mathrm{BV}_p(G)$ and the BV_p-norm of M_G

$$\|M_G\|_{\mathrm{BV}_p} := \sup_{\substack{f:V\to\mathbb{R} \\ \mathrm{Var}_p(f)\neq 0}} \frac{\mathrm{Var}_p(M_G f)}{\mathrm{Var}_p(f)}.$$

To be more precise, Liu and Xue [16] proved the following result.

Theorem C [16] *Let G be a simple, finite and connected graph with $n \ge 2$ vertices. Then*

(i) *If $n = 2$ and $0 < p \le \infty$, then $\|M_G\|_{\mathrm{BV}_p} = 1/2$;*
(ii) *If $n = 3$ and $0 < p \le 1$, then $\|M_G\|_{\mathrm{BV}_p} = 2/3$;*
(iii) *If $n \ge 3$, then $1 - 1/n \le \|M_{K_n}\|_{\mathrm{BV}_p} < 1$.*
(iv) *If $n \ge 3$, then $1 - 1/n \le \|M_{S_n}\|_{\mathrm{BV}_p} < 1$;*
(v) *M_G is bounded from $\mathrm{BV}_p(G)$ to $\mathrm{BV}_p(G)$ for all $0 < p \le \infty$. Specially, for any $f \in \mathrm{BV}_p(G)$, it holds that*

$$\mathrm{Var}_p(M_G f) \le \left(\frac{n}{2}\right)^{1/p} (n-1)^{\max\{1,1/p\}}\mathrm{Var}_p(f).$$

Gonzalez-Riquelme and Madrid [7] improved partial results in Theorem C. More precisely, they established the following

Theorem D [7] *Let $n \ge 3$ and $0 < p \le \infty$.*

(i) *Then $\|M_{K_n}\|_{\mathrm{BV}_p} = 1 - 1/n$ if one of the following conditions holds:*
   (a) *$p \ge 1$;*
   (b) *$0 < p < 1$ and $n = 4$;*
   (c) *$n \ge 3$ and $\frac{\ln 4}{\ln 6} \le p < 1$.*

(ii) *If $1 < p \le \infty$, then $\|M_{S_n}\|_{\mathrm{BV}_p} = \frac{(1+2^{p/(p-1)})^{(p-1)/p}}{3}$. Moreover, the equality $\|M_{S_n}\|_{\mathrm{BV}_p} = 1 - 1/n$ holds if one of the following conditions holds:*
(a') *$p = 1$;*
(b') *$0 < p < 1$ and $n = 4$;*
(c') *$1/2 \le p \le 1$ and $n \ge 5$.*

In light of (1.2) and (1.3), the boundedness of $\mathbf{S}_G$ on $\mathrm{BV}_p(G)$ is trivial. However, it is interesting and natural to investigate $\|\mathbf{S}_G\|_{\mathrm{BV}_p}$, which is another motivation of this work.

This paper will be organized as follows. In Sect. 2 we shall present certain ℓ^p-norm estimates for the spherical maximal function as well as two restricted type estimates. Section 3 is devoted to establishing the optimal constants for $\|\mathbf{S}_{K_n}\|_{p,\infty}$ and $\|\mathbf{S}_{S_n}\|_{p,\infty}$. Finally, the BV_p-norm of $\mathbf{S}_G$ is discussed in Sect. 4.

2 Estimates for $\|\mathbf{S}_G\|_p$ and two restricted type estimates

This section is devoted to studying the ℓ^p-norm of the spherical maximal operator. We start with the following observation.

Theorem 2.1 *Let $G = (V_G, E_G)$ be graph with n ($n \geq 2$) vertices and $0 < p < \infty$. Then*

(i) $1 \leq \|\mathbf{S}_G\|_p \leq n^{1/p}$;
(ii) *If $n = 2$ and G is connected, then*

$$\|\mathbf{S}_G\|_p = 2^{1/p}.$$

Proof The claim (i) follows from (1.2). We now prove part (ii). Let $n = 2$ and $G = (V_G, E_G)$ with $V_G = \{1, 2\}$ and $E_G = \{1 \sim 2\}$. Given a function $f : V_G \to \mathbb{R}$, it holds that $\mathbf{S}_G f(1) = \mathbf{S}_G f(2) = \max\{|f(1)|, |f(2)|\}$. Then we have

$$\frac{\|\mathbf{S}_G f\|^p_{\ell^p(G)}}{\|f\|^p_{\ell^p(G)}} = \frac{2(\max\{|f(1)|, |f(2)|\})^p}{|f(1)|^p + |f(2)|^p}.$$

Without loss of generality we may assume that $|f(1)| \geq |f(2)|$ and $\beta = \frac{|f(2)|}{|f(1)|}$. It is clear that $\beta \in [0, 1]$ and

$$\|\mathbf{S}_G\|_p = \sup_{\beta\in[0,1]} \left(\frac{2}{1+\beta^p}\right)^{1/p} = 2^{1/p}.$$

This completes the proof. □

Remark 2.1 Let $G = (V_G, E_G)$ be a connected graph with set of vertices $V_G = \{u, v\}$ and $0 < p \leq \infty$. Then $\|\mathbf{S}_G f\|_{\ell^p(G)} = 2^{1/p}\|f\|_{\ell^p(G)}$ if and only if $f(u)f(v) = 0$.

In order to establish next results, let us introduce an useful lemma.

Lemma 2.2 [26] *Let $G = (V_G, E_G)$ be graph with n ($n \geq 2$) vertices, and $T : \ell^p(G) \to \ell^p(G)$ be a sublinear operator, with $0 < p \leq 1$. Then*

$$\|T\|_p = \max_{k\in V_G} \|T\delta_k\|_{\ell^p(G)}.$$

We are going to present the estimates for $\|\mathbf{S}_{K_n}\|_p$.

Theorem 2.3 *Let $n \geq 3$.*

(i) *If $0 < p \leq 1$, then*

$$\|\mathbf{S}_{K_n}\|_p = (1 + (n-1)^{1-p})^{1/p}.$$

(ii) *If $1 < p < \infty$, then*

$$(1 + (n-1)^{1-p})^{1/p} \leq \|\mathbf{S}_{K_n}\|_p < 2^{1/p}.$$

Proof Without loss of generality we may assume that $K_n = (V_{K_n}, E_{K_n})$, where $V_{K_n} = \{1, \ldots, n\}$. Given $k \in V_{K_n}$. One can easily check that $\mathbf{S}_{K_n}(\delta_k)(k) = 1$ and $\mathbf{S}_{K_n}(\delta_k)(i) = \frac{1}{n-1}$ for all $i \in V_{K_n} \backslash \{k\}$. Then we have

$$\|\mathbf{S}_{K_n}\delta_k\|_{\ell^p(K_n)} = (1 + (n-1)^{1-p})^{1/p}.$$

By Lemma 2.2, one has

$$\|\mathbf{S}_{K_n}\|_p = (1 + (n-1)^{1-p})^{1/p}, \quad \text{for } \ 0 < p \leq 1.$$

This proves part (i).

Now we prove part (ii). Note that $\|\delta_1\|_{\ell^p(K_n)} = 1$. Then we have

$$\|\mathbf{S}_{K_n}\|_p \geq \frac{\|\mathbf{S}_{K_n}\delta_1\|_{\ell^p(K_n)}}{\|\delta_1\|_{\ell^p(K_n)}} = (1 + (n-1)^{1-p})^{1/p}, \quad \text{for } 0 < p < \infty.$$

Next we shall prove

$$\|\mathbf{S}_{K_n}\|_p < 2^{1/p}, \quad \text{for } 1 < p < \infty. \tag{2.1}$$

To prove (2.1), it suffices to show that

$$\|\mathbf{S}_{K_n} f\|_{\ell^p(K_n)} < 2^{1/p}, \quad \text{for } 1 < p < \infty. \tag{2.2}$$

for all nonnegative functions $f : V_{K_n} \to \mathbb{R}$ with $\|f\|_{\ell^p(K_n)} = 1$.

Given a function $f = \sum_{i=1}^n a_i\delta_i$ with $a_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n a_i^p = 1$. We write

$$\mathbf{S}_{K_n}(f)(i) = \max\left\{a_i, \frac{1}{n-1}\left(\sum_{j=1}^n a_j - a_i\right)\right\}, \quad i = 1, \ldots, n.$$

Since $p > 1$, by Jensen's inequality

$$\begin{aligned} \|\mathbf{S}_{K_n} f\|_{\ell^p(K_n)}^p &= \sum_{i=1}^n \left(\max\left\{a_i, \frac{1}{n-1}\left(\sum_{j=1}^n a_j - a_i\right)\right\}\right)^p \\ &\leq \sum_{i=1}^n \max\left\{a_i^p, \frac{1}{n-1}(1 - a_i^p)\right\}. \end{aligned} \tag{2.3}$$

We set

$$\begin{aligned} N_1 &:= \left\{i : i \in \{1, \ldots, n\}, \ \ a_i^p \leq \frac{1}{n}\right\}, \\ N_2 &:= \left\{i : i \in \{1, \ldots, n\}, \ \ a_i^p > \frac{1}{n}\right\}. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{i=1}^n \max\left\{a_i^p, \frac{1}{n-1}(1 - a_i^p)\right\} &= \sum_{i \in N_1} \frac{1}{n-1}(1 - a_i^p) + \sum_{i \in N_2} a_i^p \\ &= \frac{|N_1|}{n-1} + 1 - \left(\frac{1}{n-1} + 1\right)\sum_{i \in N_1} a_i^p =: g(N_1) \end{aligned}$$

Notice that $1 \leq |N_1| \leq n$. One can get $g(N_1) \leq 2$. Moreover, $g(N_1) = 2$ if there exists an unique $j_0 \in \{1, \ldots, n\}$ such that $a_{j_0} = 1$ and $a_j = 0$ for all $j \in \{1, \ldots, n\} \backslash \{j_0\}$.

Actually, if there exists an unique $j_0 \in \{1, \dots, n\}$ such that $a_{j_0} = 1$ and $a_j = 0$ for all $j \in \{1, \dots, n\}\backslash\{j_0\}$, we can get $\|\mathbf{S}_{K_n} f\|_{\ell^p(K_n)} = (1 + (n-1)^{1-p})^{1/p} < 2^{1/p}$ when $p > 1$. These together with (2.3) yields (2.2). □

The following result focuses the estimates for $\|\mathbf{S}_{S_n}\|_p$.

Theorem 2.4 *Let $n \geq 3$ and $0 < p < \infty$. Then*

(i) $\|\mathbf{S}_{S_n}\|_p = n^{1/p}$;
(ii) *Let $S_n = (V_{S_n}, E_{S_n})$ with $V_{S_n} = \{1, \dots, n\}$ and $E_{S_n} = \{1 \sim 2, \dots, 1 \sim n\}$, then*

$$\frac{\|\mathbf{S}_{S_n} f\|_{\ell^p(S_n)}}{\|f\|_{\ell^p(S_n)}} = n^{1/p}$$

if and only if $f = \pm\alpha\delta_1$ for all $\alpha \in \mathbb{R}\backslash\{0\}$.

Proof We may assume without loss of generality that $S_n = (V_{S_n}, E_{S_n})$ with $V_{S_n} = \{1, \dots, n\}$ and $E_{S_n} = \{1 \sim 2, \dots, 1 \sim n\}$. Clearly, $\mathbf{S}_{S_n}\delta_1(i) = 1$ for all $i \in V_{S_n}$. Hence, $\|\mathbf{S}_{S_n}\delta_1\|_{\ell^p(S_n)} = n^{1/p}$. Then we have

$$\|\mathbf{S}_{S_n}\|_p \geq n^{1/p}, \quad \text{for } 0 < p < \infty.$$

This together with part (i) of Theorem 2.1 implies

$$\|\mathbf{S}_{S_n}\|_p = n^{1/p}, \quad \text{for } 0 < p < \infty.$$

This proves part (i) of Theorem 2.4.

We now prove part (ii). It suffices to show that if there exists a nonnegative function f such that $\|f\|_{\ell^p(S_n)} = 1$ and $\|\mathbf{S}_{S_n} f\|_{\ell^p(S_n)} = n^{1/p}$, then $f = \delta_1$. Let $f = \sum_{i=1}^n a_i\delta_i$ with $a_i \geq 0\,(i = 1, 2, \dots, n)$ and $\sum_{i=1}^n a_i^p = 1$. Then we have

$$\mathbf{S}_{S_n} f(i) = \begin{cases} \max\left\{a_1, \dfrac{1}{n-1}\displaystyle\sum_{j=2}^n a_j\right\}, & i = 1; \\ \max\left\{a_1, a_i, \dfrac{1}{n-2}\left(\displaystyle\sum_{j=2}^n a_j - a_i\right)\right\}, & i = 2, \dots, n. \end{cases}$$

Observe that $\mathbf{S}_{S_n} f(i) \leq 1$ for all $i = 1, \dots, n$. By our assumption $\|\mathbf{S}_{S_n} f\|_{\ell^p(S_n)} = n^{1/p}$, we have $\mathbf{S}_{S_n} f(i) = 1$ for all $i = 1, \dots, n$. Since

$$\mathbf{S}_{S_n} f(1) = \max\left\{a_1, \frac{1}{n-1}\sum_{j=2}^n a_j\right\} = 1,$$

it follows that $a_1 = 1$ or $\frac{1}{n-1}\sum_{j=2}^n a_j = 1$. If $a_1 = 1$, then $a_i = 0$ for all $i = 2, \dots, n$, there is nothing to do. If $\frac{1}{n-1}\sum_{j=2}^n a_j = 1$, then $a_i = 1$ for all $i = 2, \dots, n$, there is impossible since $\sum_{i=1}^n a_i^p = 1$. This completes the proof. □

Remark 2.2 The ℓ^p-norm of $\mathbf{S}_G$ relies on strictly the structure of the graph G when $0 < p < \infty$. Even for the connected graph G with three vertices, we can't obtain an uniform estimate for $\|\mathbf{S}_G\|_p$ with $0 < p < \infty$. For example, if G is a connected graph with three vertices, then $G = K_3$ or $G = S_3$, it follows from Theorem 2.3 that $\|\mathbf{S}_{K_3}\|_p = (1 + 2^{1-p})^{1/p}$ for $0 < p \leq 1$ and $(1 + 2^{1-p})^{1/p} \leq \|\mathbf{S}_{K_3}\|_p < 2^{1/p}$ for $1 < p < \infty$. However, $\|\mathbf{S}_{S_3}\|_p = 3^{1/p}$ for all $0 < p < \infty$.

Next we shall consider the following restricted type estimate:

$$\|\mathbf{S}_G\|_{p,\mathrm{rest}} = \max_{A\subset V_G} \frac{\|\mathbf{S}_G(\chi_A)\|_{\ell^p(G)}}{\|\chi_A\|_{\ell^p(G)}}.$$

It is clear that

$$\|\mathbf{S}_G\|_{p,\mathrm{rest}} \le \|\mathbf{S}_G\|_p. \tag{2.4}$$

Motivated by the idea in the proof of Theorem 2.5 in [26], we can get the following result.

Theorem 2.5 *Let $n \ge 2$ and $1 < p < \infty$.*

(i) *If $n \le p'$, then*

$$\|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} = (1 + (n-1)^{1-p})^{1/p}.$$

(ii) *If $n \le p$, then*

$$\|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} = (1 + (n-1)^{-1})^{1/p}.$$

(iii) *If $n > \max\{p, p'\}$, $p = p_1/p_2 \in \mathbb{Q}$ and p_1 divides n. Then*

$$\|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} = \left(1 + \frac{n^p(p-1)^{p-1}}{(n-1)^p p^p}\right)^{1/p}.$$

(iv) *Let $[x]$ be the integer part of x. If $n > \max\{p, p'\}$ and p is not a rational number or $p = p_1/p_2 \in \mathbb{Q}$ and p_1 is not divide n. Then*

$$\|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} = \left(1 + \frac{1}{(n-1)^p} \max\{(n-[n]_p)([n]_p)^{p-1}, (n-1-[n]_p)([n]_p+1)^{p-1}\}\right)^{1/p},$$

where $[n]_p = [n/p']$.

(v) *If $n > \max\{p, p'\}$, then*

$$\max\{(1 + (n-1)^{-1})^{1/p}, (1 + (n-1)^{1-p})^{1/p}\} \le \|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} < 2^{1/p}.$$

Proof Given a set $A \subset V_{K_n}$ with $|A| = k \le n$. one has that $\mathbf{S}_{K_n}(\chi_A)(j) = 1$ if $j \in A$ and $\mathbf{S}_{K_n}(\chi_A)(j) = \frac{k}{n-1}$ if $j \notin A$. It follows that

$$\|\mathbf{S}_{K_n}(\chi_A)\|_{\ell^p(K_n)} = \left(\sum_{j=1}^{n} \mathbf{S}_{K_n}(\chi_A)(j)^p\right)^{1/p} = \left(k + \frac{(n-k)k^p}{(n-1)^p}\right)^{1/p}.$$

Notice that $\|\chi_A\|_{\ell^p(K_n)} = k^{1/p}$. Therefore,

$$\|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} = \left(1 + \frac{1}{(n-1)^p} \max_{1\le k\le n-1} (n-k)k^{p-1}\right)^{1/p}. \tag{2.5}$$

Let $\varphi(x) = (n-x)x^{p-1}$ for $x \in (0,\infty)$. Observing that φ is increasing on $(0, n/p')$ and is decreasing on $(n/p', \infty)$. We discuss the following cases:

(i) If $n \le p'$, then the function φ is decreasing on $[1, n-1]$. We have $\varphi(x) \le \varphi(1)$ for all $x \in [1, n-1]$. Hence, we get

$$\|\mathbf{S}_{K_n}\|_{p,\mathrm{rest}} = (1 + (n-1)^{1-p})^{1/p}.$$

(ii) If $n \le p$, then $n/p' \ge n-1$ and φ is increasing on $[1, n-1]$. Thus, $\varphi(x) \le \varphi(n-1)$ for all $x \in [1, n-1]$. It holds that

$$\|\mathbf{S}_{K_n}\|_{p,\text{rest}} = (1 + (n-1)^{-1})^{1/p}.$$

(iii) If $n > \max\{p, p'\}$, $p = p_1/p_2 \in \mathbb{Q}$ and p_1 divides n. In this case we have n/p' is an integer and $n/p' \in [1, n-1]$. Thus we have $\varphi(x) \le \varphi(n/p')$ for all $x \in [1, n-1]$. Hence,

$$\|\mathbf{S}_{K_n}\|_{p,\text{rest}} = \left(1 + \frac{n^p (p-1)^{p-1}}{(n-1)^p p^p}\right)^{1/p}.$$

(iv) If $n > \max\{p, p'\}$ and p is not of the previous form. In this case we have $n/p' \in [1, n-1]$, but it is not an integer. Then

$$\|\mathbf{S}_{K_n}\|_{p,\text{rest}}^p = 1 + \frac{1}{(n-1)^p} \max\{(n - [n]_p)([n]_p)^{p-1}, (n-1-[n]_p)([n]_p+1)^{p-1}\}.$$

The conclusion of part (v) follows easily from (2.4), (2.5) and part (ii) of Theorem 2.3.

□

We end this section by presenting the estimate for $\|\mathbf{S}_{S_n}\|_{p,\text{rest}}$.

Theorem 2.6 *Let $n \ge 2$ and $0 < p < \infty$. Then we have*

$$\|\mathbf{S}_{S_n}\|_{p,\text{rest}} = n^{1/p}.$$

Proof We may assume without loss of generality that $S_n = (V_{S_n}, E_{S_n})$ with $V_{S_n} = \{1, \ldots, n\}$ and $E_{S_n} = \{1 \sim 2, \ldots, 1 \sim n\}$. Let $A = \{1\}$. Then we have $\mathbf{S}_{S_n}(\chi_A)(1) = 1$ for all $i \in V$. Hence, $\frac{\|\mathbf{S}_{S_n}(\chi_A)\|_{\ell^p(S_n)}}{\|\chi_A\|_{\ell^p(S_n)}} = n^{1/p}$ and $\|\mathbf{S}_{S_n}\|_{p,\text{rest}} \ge n^{1/p}$. This together with (2.4) and part (i) of Theorem 2.4 yields $\|\mathbf{S}_{S_n}\|_{p,\text{rest}} = n^{1/p}$. □

Remark 2.3 Let $n \ge 2$ and $S_n = (V_{S_n}, E_{S_n})$, where $V_{S_n} = \{1, \ldots, n\}$ and $E_{S_n} = \{1 \sim 2, \ldots, 1 \sim n\}$ and $A \subset V_{S_n}$. It should be pointed out that $\frac{\|\mathbf{S}_{S_n}(\chi_A)\|_{\ell^p(S_n)}}{\|\chi_A\|_{\ell^p(S_n)}} = 2 + \frac{1}{n-1}$ if and only if $A = \{1\}$.

3 Estimates for $\|\mathbf{S}_G\|_{p,\infty}$

This section is devote to investigating the term $\|\mathbf{S}_G\|_{p,\infty}$. Let us begin with the following result.

Theorem 3.1 *Let $G = (V_G, E_G)$ be graph with n $(n \ge 2)$ vertices and $0 < p < \infty$. Then*

(i) $n^{-1/p} \le \|\mathbf{S}_G\|_{p,\infty} \le n^{1/p}$.
(ii) *If $n = 2$ and G is connected, then* $\|\mathbf{S}_G\|_{p,\infty} = 1$.

Proof Given a function $f : V \to \mathbb{R}$ with $\|f\|_p = 1$, by the fact that $|f(v)| \le \mathbf{S}_G f(v)$ for all $v \in V$, we have

$$\|\mathbf{S}_G\|_{p,\infty} \ge \sup_{t>0} t |\{j \in V : |f(j)| > t\}|^{1/p}.$$

Let $\alpha = \max_{i \in V} |f(v)|$. Then $\alpha \geq n^{-1/p}$ because of $\|f\|_{\ell^p(G)} = 1$. Noting that $|\{j \in V : |f(j)| > t\}| \geq 1$ when $t \in (0, \alpha)$. Then we have

$$\sup_{t>0} t|\{j \in V : |f(j)| > t\}|^{1/p} \geq \alpha \geq n^{-1/p}.$$

It follows that

$$\|\mathbf{S}_G\|_{p,\infty} \geq n^{-1/p}.$$

On the other hand, by part (i) of Theorem 2.1 and the fact that $\|\mathbf{S}_G\|_{p,\infty} \leq \|\mathbf{S}_G\|_p$, it holds that $\|\mathbf{S}_G\|_{p,\infty} \leq n^{1/p}$. This proves part (i) of Theorem 3.1.

We now prove (ii). Let $V_G = \{1, 2\}$ and $E_G = \{1 \sim 2\}$. Given a function $f : V_G \to \mathbb{R}$ with $|f(1)|^p + |f(2)|^p = 1$, it holds that $\mathbf{S}_G f(1) = \mathbf{S}_G f(2) = \max\{|f(1)|, |f(2)|\}$. We consider the following cases:

1. If $|f(1)| = |f(2)|$. One can easily check that $|f(1)| = |f(2)| = 2^{-1/p}$ and $\|\mathbf{S}_G f\|_{p,\infty} = 2^{-1/p}2^{1/p} = 1$.
2. If $|f(1)| \neq |f(2)|$. Without loss of generality we may assume that $|f(1)| > |f(2)|$. Then we have

$$\|\mathbf{S}_G f\|^p_{\ell^{p,\infty}(G)} = \max\{2|f(2)|^p, |f(1)|^p\} = \max\{2(1 - |f(1)|^p), |f(1)|^p\} = 1.$$

Hence, we have

$$\|\mathbf{S}_G\|_{p,\infty} = 1.$$

This proves part (ii) of Theorem 3.1

□

Remark 3.1 Let $G = (V_G, E_G)$ be a connected graph with two vertices. Then we have that $\|\mathbf{S}_G f\|_{\ell^{p,\infty}(G)} = \|f\|_p$ for any arbitrary functions $f : V_G \to \mathbb{R}$.

Now we present some optimal constants for $\|\mathbf{S}_{K_n}\|_{p,\infty}$.

Theorem 3.2 *If* $0 < p < \infty$, *then*

$$\|\mathbf{S}_{K_n}\|_{p,\infty} = \begin{cases} \frac{n^{1/p}}{n-1}, & \text{if } 0 < p < \frac{\ln n}{\ln(n-1)}; \\ 1, & \text{if } \frac{\ln n}{\ln(n-1)} \leq p < \infty. \end{cases}$$

Proof Let $K_n = (V_{K_n}, E_{K_n})$, where $V_{K_n} = \{1, \dots, n\}$. Given $k \in V_{K_n}$. One can easily check that $\mathbf{S}_{K_n}\delta_1(1) = 1$ and $\mathbf{S}_{K_n}\delta_1(i) = \frac{1}{n-1}$ for $i = 2, \dots, n$. It follows that

$$\{j \in V : \mathbf{S}_{K_n}\delta_1(j) > t\} = \begin{cases} \{1, \dots, n\}, & \text{if } t \in (0, \frac{1}{n-1}); \\ \{1\}, & \text{if } t \in [\frac{1}{n-1}, 1); \\ \emptyset, & \text{if } t \in [1, \infty). \end{cases}$$

Thus we have

$$\|\mathbf{S}_{K_n}\delta_1\|_{\ell^{p,\infty}(K_n)} = \sup_{t>0} |\{j \in V : \mathbf{S}_{K_n}\delta_1(j) > t\}|^{1/p} = \max\left\{\frac{n^{1/p}}{n-1}, 1\right\}.$$

It follows that

$$\|\mathbf{S}_{K_n}\|_{p,\infty} \geq \begin{cases} \frac{n^{1/p}}{n-1}, & \text{if } 0 < p < \frac{\ln n}{\ln(n-1)}; \\ 1, & \text{if } p \geq \frac{\ln n}{\ln(n-1)}. \end{cases}$$

We now prove

$$\|\mathbf{S}_{K_n}\|_{p,\infty} \le \begin{cases} \frac{n^{1/p}}{n-1}, & \text{if } 0 < p < \frac{\ln n}{\ln(n-1)}; \\ 1, & \text{if } p \ge \frac{\ln n}{\ln(n-1)} \end{cases}$$

Given a function $f : V_{K_n} \to \mathbb{R}$ with $f \ge 0$ and $\|f\|_{\ell^p(K_n)} = 1$. Then we have

$$\mathbf{S}_{K_n} f(i) = \max\left\{ f(i), \frac{1}{n-1}(\|f\|_{\ell^1(K_n)} - f(i)) \right\}, \quad i = 1, \dots, n.$$

Without loss of generality we may assume that

$$0 \le f(1) \le f(2) \le f(3) \le \cdots \le f(n-1) \le f(n).$$

We consider the following cases:

1. $f(1) > 0$. Then $\{j \in V_{K_n} : \mathbf{S}_{K_n} f(j) > t\} = V$ when $t \in (0, f(1))$. If $f(1) < f(2)$. Then $\{j \in V_{K_n} : \mathbf{S}_{K_n} f(j) > t\} = \{2, \dots, n\}$ when $t \in [f(1), f(2))$. If $f(1) = f(2) < f(3)$, then $\{j \in V_{K_n} : \mathbf{S}_{K_n} f(j) > t\} = \{3, \dots, n\}$ when $t \in [f(1), f(3))$. Reasoning as above, we have that if there exists $j_0 \in \{1, \dots, n-1\}$ such that $f(j_0) < f(j_0 + 1)$, then we have $\{j \in V_{K_n} : \mathbf{S}_{K_n} f(j) > t\} = \{j_0 + 1, \dots, n\}$ when $t \in [f(j_0), f(j_0 + 1))$. If there exists $j_0 \in \{1, \dots, n-1\}$ and $m \ge 1$ such that $f(j_0) = f(j_0 + 1) = \cdots = f(j_0 + m - 1) < f(j_0 + m)$, then $\{j \in V_{K_n} : \mathbf{S}_{K_n} f(j) > t\} = \{j_0 + m, j_0 + m + 1, \dots, n\}$ when $t \in [f(j_0), f(j_0 + m))$. Thus we have

   $$\|\mathbf{S}_{K_n} f\|_{\ell^{p,\infty}(G)} = \sup_{t>0} |\{j \in V : \mathbf{S}_{K_n} f(j) > t\}|^{1/p} = \max_{1 \le i \le n-1} f(i)(n + 1 - i)^{1/p}.$$

   Fix $i \in \{1, \dots, n-1\}$, one can easily check that

   $$f(i)(n + 1 - i)^{1/p} \le \left(\sum_{j=i}^{n} f(i)^p \right)^{1/p} \le \left(\sum_{j=i}^{n} f(j)^p \right)^{1/p} \le 1.$$

2. There exists $i_0 \in \{2, \dots, n\}$ such that $f(i_0 - 1) = 0$ and $f(i_0) > 0$. This case can be dealt by the similar arguments as in getting the case (1) essentially. We omit the details.

□

We end this section by presenting the estimate for $\|\mathbf{S}_{S_n}\|_{p,\infty}$.

Theorem 3.3 *Let $n \ge 3$. Then*

$$\|\mathbf{S}_{S_n}\|_{p,\infty} = n^{1/p}.$$

Proof We may assume without loss of generality that $S_n = (V_{S_n}, E_{S_n})$ with $V_{S_n} = \{1, \dots, n\}$ and $E_{S_n} = \{1 \sim 2, \dots, 1 \sim n\}$. Clearly, $\mathbf{S}_{S_n} \delta_1(i) = 1$ for all $i \in V_{S_n}$. Then we have

$$\{j \in V : \mathbf{S}_{S_n} \delta_1(j) > t\} = \begin{cases} V, & t \in (0, 1); \\ \emptyset, & t \in [1, \infty) \end{cases}$$

It follows that

$$\|\mathbf{S}_{K_n} \delta_1\|_{\ell^{p,\infty}(S_n)} = \sup_{t>0} |\{j \in V_{S_n} : \mathbf{S}_{S_n} \delta_1(j) > t\}|^{1/p} = n^{1/p}.$$

Hence,

$$\|\mathbf{S}_{S_n}\|_{p,\infty} \ge n^{1/p}.$$

On the other hand, invoking Theorem 2.4,

$$\|\mathbf{S}_{S_n}\|_{p,\infty} \le \|\mathbf{S}_{S_n}\|_p \le n^{1/p}.$$

This completes the proof of Theorem 3.3. □

4 Estimates for $\|\mathbf{S}_G\|_{\mathrm{BV}_p}$

In this section we study the estimates for $\|\mathbf{S}_G\|_{\mathrm{BV}_p}$. At first, we present the following observation.

Theorem 4.1 *Let $n \ge 2$ and G be a connected graph with n vertices and $0 < p < \infty$. Then*

(i) *If $n = 2$, then $\|\mathbf{S}_G\|_{\mathrm{BV}_p} = 0$.*
(ii) *If $n \ge 3$, then*

$$\|\mathbf{S}_G\|_{\mathrm{BV}_p} \le \left(\frac{n}{2}\right)^{1/p} (n-1)^{\max\{1,1/p\}}.$$

Proof Let $n = 2$ and $G = (V_G, E_G)$ with $V_G = \{1, 2\}$ and $E = \{1 \sim 2\}$. Given a function $f : V_G \to \mathbb{R}$, we have that $\mathbf{S}_G f(1) = \mathbf{S}_G f(2) = \max\{|f(1)|, |f(2)|\}$. Then $\mathrm{Var}_p(\mathbf{S}_G f) = 0$ and $\|\mathbf{S}_G\|_{\mathrm{BV}_p} = 0$. This proves part (i). By the arguments similar to those used to derive Theorem 1.2 in [16], we can get the conclusion of part (ii). The details are omitted. □

The following result focuses on the estimate of $\|\mathbf{S}_{K_n}\|_{\mathrm{BV}_p}$.

Theorem 4.2 *Let $n \ge 2$ and $0 < p \le \infty$. Then*

(i) *If $p \ge 1$, then*

$$\|\mathbf{S}_{K_n}\|_{\mathrm{BV}_p} = \frac{n-2}{n-1}.$$

(ii) *If $n = 2$ or $n = 3$, then*

$$\|\mathbf{S}_{K_n}\|_{\mathrm{BV}_p} = \frac{n-2}{n-1}.$$

(iii) *If $0 < p < 1$ and $n \ge 4$, then*

$$\frac{n-2}{n-1} \le \|\mathbf{S}_{K_n}\|_{\mathrm{BV}_p} \le \min\left\{1, \frac{(n-2)^{1/p}}{n-1}\right\}.$$

Proof Let $K_n = (V_{K_n}, E_{K_n})$, where $V_{K_n} = \{1, \dots, n\}$. Given $k \in V_{K_n}$. One can easily check that $\mathbf{S}_{K_n}\delta_1(1) = 1$ and $\mathbf{S}_{K_n}\delta_1(i) = \frac{1}{n-1}$ for all $i \in V_{K_n}\backslash\{1\}$. Then we have

$$\|\mathbf{S}_{K_n}\|_{\mathrm{BV}_p} \ge \frac{\mathrm{Var}_p(\mathbf{S}_{K_n}\delta_1)}{\mathrm{Var}_p(\delta_1)} = \frac{\left((n-1)\left(1-\frac{1}{n-1}\right)^p\right)^{1/p}}{(n-1)^{1/p}} = \frac{n-2}{n-1}. \tag{4.1}$$

Given a function $f : V_{K_n} \to \mathbb{R}$, we have

$$\mathbf{S}_{K_n} f(i) = \max\left\{|f(i)|, \frac{1}{n-1}\left(\sum_{j=1}^{n} |f(j)| - |f(i)|\right)\right\}, \quad i = 1, \dots, n.$$

Without loss of generality we may assume that f is non-negative. Let

$$m=\frac{\sum_{i=1}^{n} f(i)}{n}.$$

We also assume that

$$f(1)\le\cdots\le f(r-1)<m\le f(r)\le\cdots\le f(n).$$

For $i=1,\ldots,n$, we set

$$m_i=\frac{\sum\limits_{l=1}^{n} f(l)-f(i)}{n-1}.$$

Observe that

$$m_i>f(i),\quad \forall i<r \ \text{ and } \ m_i\le f(i),\quad \forall i\ge r. \tag{4.2}$$

It follows that

$$\mathbf{S}_{K_n}f(i)=m_i,\quad \forall i<r \ \text{ and } \ \mathbf{S}_{K_n}f(i)=f(i),\quad \forall i\ge r.$$

Therefore,

$$\begin{aligned}(\mathrm{Var}_p(\mathbf{S}_{K_n}f))^p&=\sum_{1\le i<j\le n}|\mathbf{S}_{K_n}f(i)-\mathbf{S}_{K_n}f(j)|^p\\&=\sum_{1\le i<j<r}(m_i-m_j)^p+\sum_{r\le i<j\le n}(f(j)-f(i))^p\\&\quad+\sum_{1\le i<r\le j\le n}(f(j)-m_i)^p.\end{aligned} \tag{4.3}$$

We first prove part (i). The proof is motivated by the idea in [7]. By (4.2) and (4.3), we have

$$\begin{aligned}(\mathrm{Var}_p(\mathbf{S}_{K_n}f))^p&\le\sum_{1\le i<j<r}(m_i-f(j))^p+\sum_{r\le i<j\le n}(f(j)-m_i)^p\\&\quad+\sum_{1\le i<r\le j\le n}(f(j)-m_i)^p\\&\le\sum_{1\le i<j\le n}|f(j)-m_i|^p.\end{aligned} \tag{4.4}$$

Fix $1\le i<j\le n$. By Hölder's inequality, one has

$$|f(j)-m_i|\le\frac{1}{n-1}\sum_{\substack{1\le l\le m\\ l\neq i,j}}|f(l)-f(j)|\le\frac{(n-2)^{1-1/p}}{n-1}\left(\sum_{\substack{1\le l\le m\\ l\neq i,j}}|f(l)-f(j)|^p\right)^{1/p},$$

since $p \geq 1$. This together with (4.4) leads to

$$(\mathrm{Var}_p(\mathbf{S}_{K_n} f))^p \leq \frac{(n-2)^{p-1}}{(n-1)^p} \sum_{1\leq i<j\leq n} \sum_{\substack{1\leq l\leq m \\ l\neq i,j}} |f(l)-f(j)|^p$$
$$\leq \frac{(n-2)^{p-1}}{(n-1)^p} \sum_{1\leq i<j\leq n} (n-2)|f(i)-f(j)|^p$$
$$\leq \left(\frac{n-2}{n-1}\right)^p (\mathrm{Var}_p(f))^p,$$

which combining with (4.1) implies the desired conclusion of part (i).

Next we shall prove part (ii). When $n=2$, the conclusion follows from part (i) of Theorem 4.1. When $n=3$, let $K_3=(V_{K_3}, E_{K_3})$, where $V_{K_3}=\{1,2,3\}$. Given $f=\sum_{i=1}^{3} a_i\delta_i$ with $|a_1-a_2|^p+|a_1-a_3|^p+|a_2-a_3|^p>0$, we want to show that

$$\mathrm{Var}_p(\mathbf{S}_{K_3} f) \leq \frac{1}{2}\mathrm{Var}_p(f). \tag{4.5}$$

Without loss of generality we may assume that all $a_i \geq 0$. It is clear that

$$\mathbf{S}_{K_3}(f)(i) = \max\left\{a_i, \frac{1}{2}\left(\sum_{j=1}^{3} a_j - a_i\right)\right\}, \quad i=1,2,3.$$

We only prove (4.5) for the case $a_1 \geq a_2 \geq a_3$ and $a_1 > a_3$, since other cases are analogous. If $a_2 < \frac{1}{2}(a_1+a_3)$. In this case we have $\mathbf{S}_{K_3} f(1)=a_1$, $\mathbf{S}_{K_3} f(2)=\frac{1}{2}(a_1+a_3)$ and $\mathbf{S}_{K_3} f(3)=\frac{1}{2}(a_1+a_2)$. Then we have

$$(\mathrm{Var}_p(\mathbf{S}_{K_3} f))^p = \left(\frac{1}{2}(a_1-a_3)\right)^p + \left(\frac{1}{2}(a_2-a_3)\right)^p + \left(\frac{1}{2}(a_1-a_2)\right)^p$$
$$= \frac{1}{2^p}(\mathrm{Var}_p(f))^p.$$

If $a_2 \geq \frac{1}{2}(a_1+a_3)$. For convenience, we set

$$\alpha := a_1 - a_2, \quad \beta := a_2 - a_3.$$

Then we have $\beta \geq \alpha$ and $\beta > 0$. In this case we have $\mathbf{S}_{K_3} f(1)=a_1$, $\mathbf{S}_{K_3} f(2)=a_2$ and $\mathbf{S}_{K_3} f(3)=\frac{1}{2}(a_1+a_2)$. We write

$$\frac{(\mathrm{Var}_p(\mathbf{S}_{K_3} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{(a_1-a_2)^p + \frac{1}{2^p}(a_1-a_2)^p + \frac{1}{2^p}(a_1-a_2)^p}{(a_1-a_2)^p+(a_2-a_3)^p+(a_1-a_3)^p}$$
$$= \frac{(1+2^{1-p})\alpha^p}{\alpha^p+\beta^p+(\alpha+\beta)^p} = (1+2^{1-p})\frac{(\frac{\alpha}{\beta})^p}{1+(\frac{\alpha}{\beta})^p+(1+\frac{\alpha}{\beta})^p}.$$

Note that $\frac{\alpha}{\beta} \in [0,1]$. Let $\varphi(x)=\frac{x^p}{1+x^p+(1+x)^p}$, $x\in[0,1]$. It is clear that $\varphi'(x)\geq 0$. Then $\varphi(x)\leq\varphi(1)=\frac{1}{2+2^p}$. Hence,

$$\frac{(\mathrm{Var}_p(\mathbf{S}_{K_3} f))^p}{(\mathrm{Var}_p(f))^p} \leq \frac{1}{2^p}.$$

It remains to prove part (iii). We get from (4.2) and (4.3) that

$$\begin{aligned}(\mathrm{Var}_p(\mathbf{S}_{K_n}f))^p &\le \sum_{1\le i<j<r}\left(\frac{f(j)-f(i)}{n-1}\right)^p + \sum_{r\le i<j\le n}(f(j)-f(i))^p \\ &\quad + \sum_{1\le i<r\le j\le n}(f(j)-f(i))^p \\ &\le \sum_{1\le i<j\le n}|f(i)-f(j)|^p = (\mathrm{Var}_p(f))^p,\end{aligned}$$

which leads to

$$\mathrm{Var}_p(\mathbf{S}_{K_n}f) \le \mathrm{Var}_p(f). \tag{4.6}$$

On the other hand, by Jensen's inequality, one has

$$|f(j)-m_i|^p \le \frac{1}{(n-1)^n}\left(\sum_{\substack{1\le l\le m\\ l\ne i,\,j}}|f(l)-f(j)|\right)^p \le \frac{1}{(n-1)^p}\sum_{\substack{1\le l\le m\\ l\ne i,\,j}}|f(l)-f(j)|^p,$$

since $0<p<1$. This together with (4.4) implies that

$$\begin{aligned}(\mathrm{Var}_p(\mathbf{S}_{K_n}f))^p &\le \frac{1}{(n-1)^p}\sum_{1\le i<j\le n}\sum_{\substack{1\le l\le m\\ l\ne i,\,j}}|f(l)-f(j)|^p \\ &\le \frac{1}{(n-1)^p}\sum_{1\le i<j\le n}(n-2)|f(i)-f(j)|^p \\ &\le \frac{n-2}{(n-1)^p}(\mathrm{Var}_p(f))^p.\end{aligned}$$

This gives that

$$\mathrm{Var}_p(\mathbf{S}_{K_n}f) \le \frac{(n-2)^{1/p}}{n-1}\mathrm{Var}_p(f). \tag{4.7}$$

Inequality (4.7) together with (4.1) and (4.6) implies the desired conclusion of part (iii). This finishes the proof of Theorem 4.2. □

Before presenting the corresponding result for $\mathbf{S}_{S_n}$, let us introduce the following lemma.

Lemma 4.3 [16] *Let $l\ge 2$ and $\{a_i\}_{i=1}^l$, $\{b_i\}_{i=1}^l$ be two finite sequences of nonnegative real numbers. For $2\le k\le l$, we define*

$$M_k=\max\{a_i;\ i=1,2,\dots,k\}\ \text{ and }\ N_k=\max\{b_i;\ i=1,2,\dots,k\}.$$

Then, for any $2\le k\le l$, we have

$$|M_k-N_k|\le\sum_{i=1}^k|a_i-b_i|.$$

The estimates for $\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p}$ can be formulated as follows.

Theorem 4.4 *Let $n\ge 3$ and $1<p<\infty$. Then*

(i) *If* $0 < p \le 1$, *then*

$$\frac{n-2}{n-1}(1+(n-2)^{1-2p})^{1/p} \le \|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} \le \left(1+\frac{2(n-2)^{1-p}}{(n-1)^p}\right)^{1/p}.$$

Clearly, when $p = 1$, *then*

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} = 1.$$

(ii) *If* $1 < p < \infty$, *then*

$$\frac{n-2}{n-1}(1+(n-2)^{1-2p})^{1/p} \le \|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} \le \frac{n+1}{(n-1)^{1/p}}.$$

(iii) *If* $p = \infty$, *then*

$$\frac{n-2}{n-1} \le \|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} \le \frac{n+1}{n-1}.$$

Particularly, when $n = 3$, *then*

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_\infty} = 1.$$

(iv) *If* $0 < p \le 1$ *and* $n = 3$, *then*

$$\|\mathbf{S}_{S_3}\|_{\mathrm{BV}_p} = 2^{1/p-1}.$$

Proof We may assume without loss of generality that $S_n = (V_{S_n}, E_{S_n})$, where $V_{S_n} = \{1,\dots,n\}$ and $E_{S_n} = \{1\sim 2,\dots,1\sim n\}$. It is clear that $\mathbf{S}_{S_n}\delta_2(1) = \frac{1}{n-1}$, $\mathbf{S}_{S_n}\delta_2(2) = 1$ and $\mathbf{S}_{S_n}\delta_2(i) = \frac{1}{n-2}$ for $i \in V_{S_n}\backslash\{1,2\}$. Then we have

$$\begin{aligned}\mathrm{Var}_p(\mathbf{S}_{S_n}\delta_2) &= \left(\left(1-\frac{1}{n-1}\right)^p + (n-2)\left(\frac{1}{n-2}-\frac{1}{n-1}\right)^p\right)^{1/p}\\ &= \frac{n-2}{n-1}(1+(n-2)^{1-2p})^{1/p}.\end{aligned}$$

Hence,

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} \ge \frac{n-2}{n-1}(1+(n-2)^{1-2p})^{1/p}.$$

On the other hand, fix a function $f : V_{S_n} \to \mathbb{R}$. We write

$$\mathbf{S}_{S_n} f(i) = \begin{cases} \max\left\{|f(1)|, \dfrac{1}{n-1}\displaystyle\sum_{j=2}^n |f(j)|\right\}, & i = 1;\\ \max\left\{|f(i)|, |f(1)|, \dfrac{1}{n-2}\left(\displaystyle\sum_{j=2}^n |f(j)| - |f(i)|\right)\right\}, & i = 2,\dots,n.\end{cases}$$

Invoking Lemma 4.3,

$$\begin{aligned}|\mathbf{S}_{S_n} f(i) - \mathbf{S}_{S_n} f(1)| &\le ||f(i)| - |f(1)|| + \left|\frac{1}{n-2}\left(\sum_{j=2}^n |f(j)| - |f(i)|\right) - \frac{1}{n-1}\sum_{j=2}^n |f(j)|\right|\\ &\le |f(i) - f(1)| + \frac{1}{(n-2)(n-1)}\sum_{j=2}^n |f(j) - f(i)|. \qquad (4.8)\end{aligned}$$

for any $i \in \{2, \dots, n\}$.

When $p \in (0, 1]$, we get from (4.8) that

$$|\mathbf{S}_{S_n} f(i) - \mathbf{S}_{S_n} f(1)|^p \le |f(i) - f(1)|^p + \left(\frac{1}{(n-2)(n-1)}\right)^p \sum_{j=2}^{n} |f(j) - f(i)|^p$$

for all $i \in \{2, \dots, n\}$. Then we have

$$\begin{aligned}
(\mathrm{Var}_p(\mathbf{S}_{S_n} f))^p &\le \sum_{i=2}^{n} |f(i) - f(1)|^p + \left(\frac{1}{(n-2)(n-1)}\right)^p \sum_{i=2}^{n}\sum_{j=2}^{n} |f(j) - f(i)|^p \\
&\le \mathrm{Var}_p(f)^p + \frac{1}{((n-2)(n-1))^p} \sum_{i=2}^{n}\sum_{\substack{j=2 \\ j\neq i}}^{n} (|f(j) - f(1)|^p + |f(i) - f(1)|^p) \\
&\le \mathrm{Var}_p(f)^p + \frac{2(n-2)}{((n-2)(n-1))^p} \mathrm{Var}_p(f)^p \\
&= \left(1 + \frac{2(n-2)^{1-p}}{(n-1)^p}\right) \mathrm{Var}_p(f)^p.
\end{aligned}$$

It follows that

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} \le \left(1 + \frac{2(n-2)^{1-p}}{(n-1)^p}\right)^{1/p}.$$

When $1 < p < \infty$, we get from (4.8) that

$$\begin{aligned}
\mathrm{Var}_p(\mathbf{S}_{S_n} f) &= \left(\sum_{i=2}^{n} |\mathbf{S}_{S_n} f(i) - \mathbf{S}_{S_n} f(1)|^p\right)^{1/p} \\
&\le \sum_{i=2}^{n} |f(i) - f(1)| + \frac{1}{(n-2)(n-1)} \sum_{i=2}^{n}\sum_{j=2}^{n} |f(j) - f(i)| \\
&\le \left(1 + \frac{2}{n-1}\right) \sum_{i=2}^{n} |f(i) - f(1)| \\
&\le \left(1 + \frac{2}{n-1}\right) (n-1)^{1-1/p} \mathrm{Var}_p(f).
\end{aligned}$$

It follows that

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_p} \le \frac{n+1}{(n-1)^{1/p}}.$$

This proves parts (i) and (ii) of Theorem 4.4.

When $p = \infty$. It is clear that $\mathrm{Var}_p(\mathbf{S}_{S_n} \delta_2) = \frac{n-2}{n-1}$ and $\mathrm{Var}_p(\delta_2) = 1$. Then we have

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_\infty} \ge \frac{n-2}{n-1}.$$

On the other hand, from (4.8) we see that for each $i \in \{2, \ldots, n\}$,

$$\begin{aligned}|\mathbf{S}_{S_n} f(i) - \mathbf{S}_{S_n} f(1)| &\le |f(i) - f(1)| + \frac{1}{(n-2)(n-1)} \sum_{j=2}^{n} |f(j) - f(i)| \\ &\le \mathrm{Var}_\infty(f) + \frac{1}{(n-2)(n-1)} \sum_{\substack{j=2 \\ j \neq i}}^{n} (|f(j) - f(1)| + |f(i) - f(1)|) \\ &\le \frac{n+1}{n-1} \mathrm{Var}_\infty(f).\end{aligned}$$

This gives

$$\|\mathbf{S}_{S_n}\|_{\mathrm{BV}_\infty} \le \frac{n+1}{n-1}.$$

Now we consider the case $n = 3$. Let $S_3 = (V_{S_3}, E_{S_3})$ with $V_{S_3} = \{1, 2, 3\}$ and $E_{S_3} = \{1 \sim 2, 2 \sim 3\}$. Let $g : V_{S_3} \to \mathbb{R}$ satisfy $|g(1)| > |g(2)| > |g(3)|$ and $|g(2)| = \frac{1}{2}(|g(1)| + |g(3)|)$. Then we have $\mathbf{S}_{S_3} g(1) = \mathbf{S}_{S_3} g(3) = |g(1)|$ and $\mathbf{S}_{S_3} g(2) = |g(2)|$. It follows that

$$\|\mathbf{S}_{S_3}\|_{\mathrm{BV}_\infty} \ge \frac{\mathrm{Var}_\infty(\mathbf{S}_{S_3} f)}{\mathrm{Var}_\infty(f)} = \frac{|g(1)| - |g(2)|}{|g(1)| - |g(2)|} = 1.$$

On the other hand, we get from part (i) that $\|\mathbf{S}_{S_3}\|_{\mathrm{BV}_p} \ge 2^{1/p-1}$ for any $p \in (0, 1]$. Hence, to get the desired conclusions of parts (iii) and (iv), it suffices to show that

$$\mathrm{Var}_p(\mathbf{S}_{S_3} f) \le 2^{1/p-1} \mathrm{Var}_p(f), \quad \text{if } \ 0 < p \le 1 \tag{4.9}$$

and

$$\mathrm{Var}_\infty(\mathbf{S}_{S_3} f) \le \mathrm{Var}_\infty(f). \tag{4.10}$$

for all functions $f : V_{S_3} \to \mathbb{R}$ with $\mathrm{Var}_p(f) > 0$.

Fix $f = \sum_{i=1}^{3} a_i \delta_i$ with $a_i \ge 0$. It is clear that

$$\mathbf{S}_{S_3} f(1) = \mathbf{S}_{S_3} f(3) = \max\{a_1, a_2, a_3\}, \quad \mathbf{S}_{S_3} f(2) = \max\left\{a_2, \frac{1}{2}(a_1 + a_3)\right\}.$$

For convenience, we set

$$\alpha := |a_1 - a_2|, \quad \beta := |a_2 - a_3|.$$

1. ($a_1 \ge a_2 \ge a_3$ and $a_1 > a_3$). If $a_2 \ge \frac{1}{2}(a_1 + a_3)$. Then we have $a_2 - a_3 \ge a_1 - a_2$. Then $\mathbf{S}_{S_3} f(1) = \mathbf{S}_{S_3} f(3) = a_1$ and $\mathbf{S}_{S_3} f(2) = a_2$. In this case we get

$$\frac{(\mathrm{Var}_p(\mathbf{S}_{S_3} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{2(a_1 - a_2)^p}{(a_1 - a_2)^p + (a_2 - a_3)^p} \le 1$$

for $p \in (0, 1]$. Moreover,

$$\frac{\mathrm{Var}_\infty(\mathbf{S}_{S_3} f)}{\mathrm{Var}_\infty(f)} = \frac{a_1 - a_2}{a_2 - a_3} \le 1.$$

If $a_2 < \frac{1}{2}(a_1 + a_3)$. Then $\mathbf{S}_{S_3} f(1) = \mathbf{S}_{S_3} f(3) = a_1$ and $\mathbf{S}_{S_3} f(2) = \frac{1}{2}(a_1 + a_3)$. Therefore,

$$\frac{(\mathrm{Var}_p(\mathbf{S}_{S_3} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{2(\frac{1}{2}(a_1 - a_3))^p}{(a_1 - a_2)^p + (a_2 - a_3)^p} = 2^{1-p} \frac{(a_1 - a_3)^p}{(a_1 - a_2)^p + (a_2 - a_3)^p} \le 2^{1-p},$$

for $p \in (0, 1]$. Moreover,

$$\frac{\mathrm{Var}_\infty(\mathbf{S}_{S_3} f)}{\mathrm{Var}_\infty(f)} = \frac{\frac{1}{2}(a_1 - a_3)}{a_1 - a_2} < 1.$$

This proves (4.9) and (4.10) in this case.

2. $(a_1 \geq a_3 \geq a_2$ and $a_1 > a_2)$. Then $\mathbf{S}_{S_3} f(1) = \mathbf{S}_{S_3} f(3) = a_1$, $\mathbf{S}_{S_3} f(2) = \frac{1}{2}(a_1 + a_3)$ and

$$\frac{(\mathrm{Var}_p(\mathbf{S}_{S_3} f))^p}{(\mathrm{Var}_p(f))^p} = \frac{2(\frac{1}{2}(a_1 - a_3))^p}{(a_1 - a_2)^p + (a_2 - a_3)^p} = 2^{1-p} \frac{(a_1 - a_3)^p}{(a_1 - a_2)^p + (a_2 - a_3)^p} < 2^{1-p},$$

for $p \in (0, 1]$. Moreover,

$$\frac{\mathrm{Var}_\infty(\mathbf{S}_{S_3} f)}{\mathrm{Var}_\infty(f)} = \frac{\frac{1}{2}(a_1 - a_3)}{a_1 - a_2} < \frac{1}{2}.$$

This proves (4.9) and (4.10) in this case.

3. $(a_2 \geq a_1 \geq a_3$ and $a_2 > a_3)$. Then we have $\mathbf{S}_{S_3} f(1) = \mathbf{S}_{S_3} f(2) = \mathbf{S}_{S_3} f(3) = a_2$. In this case we have $\mathrm{Var}_p(\mathbf{S}_{S_3} f) = 0$ and $\mathrm{Var}_\infty(\mathbf{S}_{S_3} f) = 0$. This proves (4.9) and (4.10) in this case.
4. $(a_2 \geq a_3 \geq a_1$ and $a_2 > a_1)$. The case is similar to the case (3).
5. $(a_3 \geq a_2 \geq a_1$ and $a_3 > a_1)$. The case is similar to the case (1).
6. $(a_3 \geq a_1 \geq a_2$ and $a_3 > a_2)$. The case is similar to the case (2).

This completes the proof of Theorem 4.4. □

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