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ORIGINAL PAPER

Pietsch's variants of s-numbers for multilinear operators

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Abstract
We study variants of s-numbers in the context of multilinear operators. The notion of an $s^{(k)}$-scale of k-linear operators is defined. In particular, we shall deal with multilinear variants of the $s^{(k)}$-scales of the approximation, Gelfand, Hilbert, Kolmogorov and Weyl numbers. We investigate whether the fundamental properties of important s-numbers of linear operators are inherited to the multilinear case. We prove relationships among some $s^{(k)}$-numbers of k-linear operators with their corresponding classical Pietsch's s-numbers of a generalized Banach dual operator, from the Banach dual of the range space to the space of k-linear forms, on the product of the domain spaces of a given k-linear operator.

Keywords Multilinear operators · s-Numbers · Adjoint operator

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1 Introduction

A natural question that appears in Functional Analysis and theory of operators is whether there are variants of some classical properties of linear operators in the setting of multilinear operators. We note that in many cases dealing with multilinear, instead of linear mappings, has proved to be a subtle subject and require different methods, and the proofs are more complicated. Moreover, many linear results are no longer true in the setting of multilinear operators. For instance, the linear Marcinkiewicz multiplier theorem, whose natural bilinear version fails, as shown by Grafakos and Kalton in [6]. Furthermore, in some questions, completely new definitions and new techniques need to be developed.

An axiomatic approach to s-numbers and Banach operator ideals of linear operators was developed by Pietsch [17]. There are many of these s-numbers sequences. The approximation numbers are the largest (under coordinate-wise ordering as sequences) s-numbers on Banach spaces. These numbers have proved to give a very useful information about the degree of compactness of operators acting between Banach spaces. The Gelfand and Kolmogorov numbers play an important role in the study of the local theory of Banach spaces. Pietsch [17, p. 220] shows that an operator T acting between Banach spaces is compact if and only if $s_n(T) \to 0$ as $n \to \infty$ for (s_n) either the sequence of Gelfand numbers or the sequence of Kolmogorov numbers. The Weyl numbers, defined by Pietsch, are truly important in the study of Riesz operators in Banach spaces. We refer to [4,7,8,10,21,23] for an in-depth study of s-numbers and their applications.

We point out that multilinear operators appear naturally in harmonic analysis and functional analysis, including the theory of ideals of operators in Banach spaces. In recent times singular multilinear operators have been intensively studied, including the famous bilinear Hilbert transform (see [6,9]).

In 1983 Pietsch [20] proposed and sketched a theory of ideals and s-numbers of multilinear functionals. In his work, he wrote "Therefore this paper should be mainly considered as a research program for the future". Pietsch's motivation was the extension of the theory of s-numbers and operator ideals to the nonlinear case. Pietsch's paper is the origin of studies of several authors on some classes of ideals of multilinear operators. We refer to the study of such ideals to [2,11,12] and also to the survey paper [16] and references therein.

While the properties of s-numbers of linear operators are well-known, Pietsch's research problem on a theory of s-numbers of multilinear operators had not been studied extensively. This is the main motivation to continue the study of variants of s-numbers in the context of multilinear operators initiated in the paper [5] and continued in [1] in the setting of polynomials acting between Banach spaces.

We introduce a notion of an $s^{(k)}$-scale of k-linear operators. Exploring ideas of Pietsch developed in the linear setting we investigate whether the fundamental properties of important s-numbers ideals of linear operators are inherited to the multilinear case. It should be noted that whereas the work is based on some ideas from the theory of s-numbers ideals of bounded linear operators, some proofs may be extended from the linear case to multilinear operators, and others require some new ideas and methods. The difficulty comes from the fact that, even in the bilinear case, the range or the kernel of a multilinear operator is not necessarily a linear subspace. In particular, as a consequence, the well-known relations between the dimensions of the kernel and the range in the linear case are not true in general, in the multilinear case.

The organization of the paper is as follows: in Sect. 2, we introduce a modified variant of the notion of s-numbers in the setting of k-linear operators. This notion is more general that the one given in [5]. We also introduce the notion of symmetric, injective, surjective $s^{(k)}$-scale, and also the mixed multiplicative property. In Sect. 3, we consider the $s^{(k)}$-scales of approximation numbers and as application, we study the approximation numbers of bilinear diagonal operators. In other sections, we study in detail multilinear variants of important classical Pietsch's s-numbers which are shown in the of Contents of the paper.

2 *s*-Numbers of multilinear operators

We shall use the notation and terminology commonly used in Banach space theory. If X is a Banach space we denote by X^* its dual Banach space, and by U_X, $\mathring{U}_X$, S_X the closed, the open unit ball and the unit sphere of X, respectively. As usual, $J_X\colon X \to \ell_\infty(U_{X^*})$ denotes the metric injection defined by $J_X(x) := (x^*(x))_{x^*\in U_{X^*}}$ with values in the space $\ell_\infty(U_{X^*})$ of bounded sequences. The canonical embedding of X to the bidual X^{**} of X is denoted by κ_X. $\mathbb{K}$ denotes the field of all scalars (complex or real) and $\mathbb{N}$ the set of all positive integers and for each $n \in \mathbb{N}$, $[n] := \{1, \dots, n\}$. For $1 \le p \le \infty$ and a nonempty set Γ, we let $\ell_p(\Gamma)$ be the space of all p-summable sequences $(x_\gamma)_{\gamma\in\Gamma}$. If $\Gamma = [n]$ (resp., $\Gamma = \mathbb{N}$) we use the standard notion ℓ_p^n (resp., ℓ_p) instead of $\ell_p([n])$ (resp., $\ell_p(\mathbb{N})$). The standard unit vector basis in c_0 is denoted by (e_i). If no confusion arises we also use $(e_i)_{i=1}^n$ to denote the unit basis vectors in $\mathbb{R}^n$. The product $X_1 \times \cdots \times X_k$ of Banach spaces is equipped with the standard norm $\|(x_j)\| := \max_{1\le j\le k} \|x_j\|_{X_j}$, for all $(x_j) \in X_1 \times \cdots \times X_k$. The Banach space of all continuous k-linear mappings T from $X_1 \times \cdots \times X_k$ to a Banach space Y is endowed with the norm

$$\|T\| := \sup\left\{\|T(x_1, \dots, x_k)\|_Y;\ (x_1, \dots, x_k) \in U_{X_1} \times \cdots \times U_{X_k}\right\},$$

and it is denoted by $\mathcal{L}(X_1, \dots, X_k; Y)$. As usual, in the case $k = 1$, the space of all bounded linear operators from X_1 to Y is denoted by $\mathcal{L}(X_1; Y)$. If $k \ge 2$ and $Y = \mathbb{K}$, then we shorten by $\mathcal{L}(X_1, \dots, X_k)$, which is the space of all k-linear forms. If in addition $X_1 = \cdots = X_k = X$ (resp., $Y = \mathbb{K}$), we write $\mathcal{L}(^k X; Y)$ (resp., $\mathcal{L}(^k X)$).

From now on we assume that all spaces which will appear are Banach spaces. Let $k \ge 2$ be an integer and let $T \in \mathcal{L}(X_1, \dots, X_k; Y)$. For each $i \in [k]$, $1 < i \le k$, and every $x^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) \in \prod_{j=1, j\ne i}^k X_j$, the associated operator $T_{x^i} \in \mathcal{L}(X_i; Y)$ is defined by

$$T_{x^i} x := T(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_k), \quad x \in X_i$$

with an obvious modification for $i = 1$ and $i = k$.

For every k-linear operator $T \in \mathcal{L}(X_1, \dots, X_k; Y)$, its rank is given by

$$\operatorname{rank} T := \dim\left([T(X_1 \times \cdots \times X_k)]\right),$$

where $[A]$ denotes the linear subspace generated by a nonempty subset $A \subset Y$.

Throughout the paper, for each $n \in \mathbb{N}$, $I_n\colon \ell_2^n \to \ell_2^n$ denotes the identity map. For each $k, n \in \mathbb{N}$ we let $\otimes_k I_n$ to denote the k-linear operator, from $\prod_{i=1}^k \ell_2^n$ to $\ell_2([n]^k)$, defined for all $x_1 = (x_1(j))_{j=1}^n, \ldots, x_k = (x_k(j))_{j=1}^n \in \ell_2^n$ by

$$\otimes_k I_n(x_1, \ldots, x_k) := x_1 \otimes \cdots \otimes x_k,$$

where $x_1 \otimes \cdots \otimes x_k := ((x_1 \otimes \cdots \otimes x_k)(j))_{j \in [n]^k}$ and

$$(x_1 \otimes \cdots \otimes x_k)(j) := x_1(j_1) \cdots x_k(j_k), \quad j = (j_1, \ldots, j_k) \in [n]^k.$$

Now, we are ready to introduce a modified variant of the notion of s-numbers in the setting of k-linear operators, which appeared in [5].

A rule $s^{(k)} = (s_n^{(k)}) \colon \mathcal{L}(X_1, \ldots, X_k; Y) \to [0, \infty)^{\mathbb{N}}$, assigning to every operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ a non-negative scalar sequence $(s_n^{(k)}(T))$, is said to be an $s^{(k)}$-scale if the following conditions are satisfied:

(S1) *Monotonicity*: For every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$,

$$\|T\| = s_1^{(k)}(T) \geq s_2^{(k)}(T) \geq \cdots \geq 0.$$

(S2) *Additivity*: For every $S, T \in \mathcal{L}(X_1, \ldots, X_k; Y)$,

$$s_{m+n-1}^{(k)}(S + T) \leq s_m^{(k)}(S) + s_n^{(k)}(T).$$

(S3) *Ideal-property*: For every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, $S \in \mathcal{L}(Y; Z)$, $R_1 \in \mathcal{L}(W_1; X_1)$, $\ldots$, $R_k \in \mathcal{L}(W_k; X_k)$,

$$s_n^{(k)}(S \circ T \circ (R_1, \ldots, R_k)) \leq \|S\| \, s_n^{(k)}(T) \, \|R_1\| \cdots \|R_k\|.$$

(S4) *Rank-property*: For every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ with $\operatorname{rank}(T) < n$, $s_n^{(k)}(T) = 0$.

(S5) *Norming property*: For each $n \in \mathbb{N}$ one has $s_n^{(k)}(\otimes_k I_n) = 1$.

We note that in the setting of linear operators the property (S5) is equivalent to the original property defined by Pietsch [19],

$$s_n(I_n) = 1, \quad n \in \mathbb{N},$$

where $I_n \colon \ell_2^n \to \ell_2^n$ denotes the identity map.

Following the classical case in the linear setting, we call $s_n^{(k)}(T)$ the n-th $s^{(k)}$-*number* of the k-linear operator T. To show the domain $X_1 \times \cdots \times X_k$ and the range space Y, we write $s_n^{(k)}(T \colon X_1 \times \cdots \times X_k \to Y)$.

We will use an obvious estimate, which follows from condition (S2):

$$|s_n^{(k)}(S) - s_n^{(k)}(T)| \leq \|S - T\|, \quad S, T \in \mathcal{L}(X_1, \ldots, X_k; Y).$$

Given an $s^{(k)}$-scale $(s_n^{(k)})$, we also introduce the following definitions:

(J1) $(s_n^{(k)})$ is called injective if, given any metric injection $J \in \mathcal{L}(Y; Z)$, that is, $\|Jy\| = \|y\|$ for all $y \in Y$, $s_n^{(k)}(T) = s_n^{(k)}(JT)$ for all $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and all Banach spaces $X_1, \ldots, X_k$.

(J2) $(s_n^{(k)})$ is called injective in the strict sense if $s_n^{(k)}(T) = s_n^{(k)}(J_Y T)$ for all $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$.

(S) $(s_n^{(k)})$ is called surjective if, given any metric surjections $Q_j \in \mathcal{L}(Y_j; X_j)$, (i.e., $Q_j(\overset{\circ}{U}_{Y_j}) = \overset{\circ}{U}_{X_j}$ for each $j \in [k]$), $s_n^{(k)}(T) = s_n^{(k)}(T(Q_1, \ldots, Q_k))$ for all $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and any Banach space Y.

Now we are ready to give the following definition:

(M) If (s_n) is an s-scale and $(s_n^{(k)})$ is an $s^{(k)}$-scale, then the pair $((s_n), (s_n^{(k)}))$ has the *mixed multiplicative* property, if for any $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and $S \in \mathcal{L}(Y; Z)$,

$$s_{m+n-1}^{(k)}(ST) \leq s_m(S)\, s_n^{(k)}(T), \quad m, n \in \mathbf{N}.$$

A sequence $(s^{(k)})$ of $s^{(k)}$-scales is said to be multiplicative if, for each $k \in \mathbb{N}$ the pair $((s_n^{(1)}), (s_n^{(k)}))$ has the mixed multiplicative property.

Given $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, we define the *generalized adjoint* (adjoint for short) operator $T^\times : Y^* \to \mathcal{L}(X_1, \ldots, X_k)$ by

$$(T^\times y^*)x := y^*(Tx), \quad y^* \in Y^*,\ x \in X_1 \times \cdots \times X_k.$$

This operator was introduced in [24], where a variant of Schauder Theorem is proved, which states that T is compact if and only if $T^\times$ is compact.

Note that, for any $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and $S \in \mathcal{L}(Y; Z)$, we have

$$(ST)^\times = T^\times S^*,$$

where S^* is the classical linear adjoint of S.

In what follows for a k-tuple $(X_1, \ldots, X_k)$ of Banach spaces, we define a k-linear mapping $\widehat{\kappa}_{X_1 \times \cdots \times X_k} : X_1 \times \cdots \times X_k \to \mathcal{L}(X_1, \ldots, X_k)^*$ given by

$$\widehat{\kappa}_{X_1 \times \cdots \times X_k} x(S) := Sx, \quad x \in X_1 \times \cdots \times X_k, \quad S \in \mathcal{L}(X_1, \ldots, X_k).$$

Following Pietsch [19], we recall that in the setting of linear operators an s-scale (s_n) is said to be *symmetric* (resp., *fully symmetric*) if, for every operator S, $(s_n(S)) \geq (s_n(S^*))$ (resp., $(s_n(S)) = (s_n(S^*))$).

Let (s_n) be an s-scale and $(s_n^{(k)})$ be an $s^{(k)}$-scale. The following notion is motivated by the above definition of Pietsch.

(S6) A pair $(s, s^{(k)})$ is said to be symmetric (resp., fully symmetric) if, for every operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and each $n \in \mathbb{N}$,

$$s_n(T^\times) \leq s_n^{(k)}(T) \quad (\text{resp.}, s_n(T^\times) = s_n^{(k)}(T)).$$

If $(s^{(k)})_{k=1}^\infty$ is a sequence of $s^{(k)}$-scales then, it is said to be symmetric (resp., fully symmetric) if for each $k \in \mathbb{N}$ the pair $(s^{(1)}, s^{(k)})$ is symmetric (resp., fully symmetric).

Proposition 2.1 *Suppose that an s-scale and $s^{(k)}$-scale form a symmetric pair $(s, s^{(k)})$. Then, for any k-linear operator with $s_n^{(k)}(T) = 0$, we have rank $(T) < n$.*

Proof By our hypothesis $s_n(T^\times) \leq s_n^{(k)}(T)$ and so $s_n(T^\times) = 0$. Since $T^\times$ is a linear operator, it follows that rank $(T^\times) < n$. Applying [5, Lemma 3.1], we conclude that $\text{rank}(T^\times) = \text{rank}(T)$ and the result follows. □

Since the adjoint operator $T^\times$ of a k-linear operator reflects some properties of T, as quoted before, and it is a linear operator, if (s_n) is an s-scale defined for linear operators, then following [17], we define for every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ the numbers

$$s_n^{(\times k)}(T) := s_n(T^\times), \quad n \in \mathbb{N}.$$

In what follows $s^{(\times k)} := (s_n^{(\times k)})$ is said to be an adjoint of s-scale, whenever the conditions (S1), (S2), (S4) and (S5) are satisfied, and in addition
$(S3')$ $\quad s_n^{(\times k)}(ST) \leq \|S\| s_n^{(\times k)}(T)$.

3 Approximation numbers

The n-th *approximation number* of a k-linear operator $T \in \mathcal{L}(X_1, \dots, X_k; Y)$ is defined by

$$a_n^{(k)}(T) := \inf\{\|T - A\|;\ A \in \mathcal{L}(X_1, \dots, X_k; Y),\ \mathrm{rank}(A) < n\}.$$

In the study of these numbers and others $s^{(k)}$-scales, we will use without any references an obvious fact that, for an arbitrary operator $S\colon X \to Y$ between Banach spaces such that rank $(S) < \dim(X)$, there exists $x \in S_X$ with $Sx = 0$.

We prove that $(a_n^{(k)})$ forms an $s^{(k)}$-scale which is called the $a^{(k)}$-scale of approximation numbers. To show this, we prove a preliminary result which we will also need later.

Proposition 3.1 *Let $T \in \mathcal{L}(X_1, \dots, X_k; Y)$ with rank $(T) \geq n$. Suppose that $\|T\| = 1$ and there exists $x^i \in \prod_{j=1,\, j\neq i}^k U_{X_j}$ such that the associate operator $T_{x^i} \in \mathcal{L}(X_i; Y)$ is a metric injection. Then $a_n^k(T) = 1$.*

Proof Clearly, $a_n^{(k)}(T) \leq \|T\| = 1$. We claim that $a_n^{(k)}(T) = 1$. Suppose, on the contrary that

$$a_n^{(k)}(T) < 1.$$

Then, there exists a k-linear operator $A\colon X_1 \times \cdots \times X_k \to Y$ with rank $(A) < n$ such that $\|T - A\| < 1$. In particular, we have

$$\sup_{x \in U_{X_i}} \|T_{x^i}x - A_{x^i}x\|_Y < 1.$$

Since $A_{x^i} : X_i \to Y$ is a bounded operator with

$$\mathrm{rank}\,(A_{x^i}) \leq \mathrm{rank}\,(A) < n,$$

we can find $x \in S_{X_i}$ such that $A_{x^i}x = 0$. Combining with the hypothesis that $T_{x^i} : X_i \to Y$ is a metric injection yields

$$1 > \|T_{x^i}x - A_{x^i}x\|_Y = \|T_{x^i}x\|_Y = \|x\|_{X_i} = 1.$$

This is a contradiction which proves the claim. □

As an application we obtain the following corollary.

Corollary 3.2 *For each $k, n \in \mathbb{N}$ and every $1 \leq p \leq \infty$, one has*

$$a_n^{(k)}(\otimes_k I_n \colon \ell_p^n \times \cdots \times \ell_p^n \to \ell_p([n]^k)) = 1\,.$$

Proof For each $k \in \mathbb{N}$ and for all $(x_1, \dots, x_k) \in \ell_p^n \times \cdots \times \ell_p^n$ one has

$$\| \otimes_k I_n(x_1, \dots, x_k)\|_{\ell_p([n]^k)} = \|x_1\|_{\ell_p^n} \cdots \|x_k\|_{\ell_p^n}.$$

This implies that, if we let $e^i := (e_1, \dots, e_1) \in \prod_{j=1, j\neq i}^k U_{\ell_p^n}$, then the associate operator

$$(\otimes_k I_n)_{e^i} : \ell_p^n \to \ell_p([n]^k)$$

is a metric injection. Since rank $(\otimes_k I_n) \geq n$, the required statement follows by Proposition 3.1. □

In consequence we get the following statement. We omit the simple proof.

Lemma 3.3 *For each k, the sequence* $(a_n^{(k)})$ *of approximation numbers is a fully symmetric* $s^{(k)}$*-scale.*

We have an elementary multilinear variant of a very useful multiplicativity property of an arbitrary s-number sequence called mixing multiplicativity due to Carl [3], which states: *If* $(s_n^{(k)})$ *is an arbitrary* $s^{(k)}$*-scale, then for every* $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ *and* $S \in \mathcal{L}(Y; Z)$*, we have*

$$s_{m+n-1}^{(k)}(ST) \leq a_m(S)\, s_n^{(k)}(T), \quad m, n \in \mathbb{N},$$

that is, the pair $((a_n), (s_n^{(k)}))$ has the mixed multiplicative property.

We note the following straightforward statement, which is a multilinear variant of a well known Pietsch's result in the linear setting.

Lemma 3.4 *For each* $k \in \mathbb{N}$*, the sequence* $(a_n^{(k)})_{n=1}^{\infty}$ *is the largest sequence of* $s^{(k)}$*-scales and satisfy, for all* $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ *and* $S \in \mathcal{L}(Y; Z)$*, the multiplicativity property*:

$$a_{m+n-1}^{(k)}(ST) \leq a_m(S)\, a_n^{(k)}(T), \quad m, n \in \mathbb{N}.$$

We need to calculate the n-th approximation numbers of special k-linear operators on the product of finite dimensional ℓ_p^n-spaces with $1 \leq p < \infty$. Let $n, r \in \mathbb{N}$ with $r \leq n^k$ and let $\sigma_k\colon [n]^k \to [n^k]$ be a bijection, then $I_{\sigma_k}^{(r)}\colon \prod_{i=1}^k \ell_p^n \to \ell_p^r$ denotes the k-linear mapping defined by the formula:

$$I_{\sigma_k}^{(r)}(x_1, \ldots, x_k) := (z_1, \ldots, z_r), \quad x_\nu = (x_\nu(1), \ldots, x_\nu(n)) \in \ell_p^n,\ \ \nu \in [k],$$

where for each $j \in [r]$ we let $z_j := x_1(i_1) \cdots x_k(i_k)$ for the unique $(i_1, \ldots, i_k) \in [n]^k$ such that $\sigma_k(i_1, \ldots, i_k) = j$. Note that if $k = 1$ and $\sigma_1\colon [n] \to [n]$ is given by $\sigma_1(j) := j$ for each $j \in [n]$, then $I_{\sigma_1}^{(n)}\colon \ell_p^n \to \ell_p^n$ is the identity map I_n on ℓ_p^n.

We claim that $\|I_{\sigma_k}^{(r)}\| \leq 1$. Clearly, the case $k = 1$ is trivial, so let $k \geq 2$. A standard calculation shows that, for $r = n^k$, one has

$$\begin{aligned}\|I_{\sigma_k}^{(n^k)}(x_1, \ldots, x_k)\|_{\ell_p^{n^k}}^p &= \sum_{j=1}^{n^k} |z_j|^p = \sum_{(i_1,\ldots,i_k)\in[n]^k} |x_1(i_1)|^p \cdots |x_k(i_k)|^p \\ &= \|x_1\|_{\ell_p^n}^p \cdots \|x_k\|_{\ell_p^n}^p,\end{aligned}$$

so $\|I_{\sigma_k}^{(n^k)}\| = 1$. If $r < n^k$, then for all $x_1, \ldots, x_k \in \ell_p^n$, we have

$$\|I_{\sigma_k}^{(r)}(x_1, \ldots, x_k)\|_{\ell_p^r} \leq \|I_{\sigma_k}^{(n^k)}(x_1, \ldots, x_k)\|_{\ell_p^{n^k}} = \|x_1\|_{\ell_p^n} \cdots \|x_k\|_{\ell_p^n}$$

and hence $\|I_{\sigma_k}^{(r)}\| \leq \|I_{\sigma_k}^{(n^k)}\| = 1$.

Since we are interested in the multilinear case, in what follows for each positive integer $k \geq 2$ we will consider the standard bijection $\sigma_k\colon [n]^k \to [n^k]$ given by the formula:

$$\sigma_k(i_1, \ldots, i_k) = n^{k-1}(i_1 - 1) + n^{k-2}(i_2 - 1) + \cdots + n(i_{k-1} - 1) + i_k,$$

for each $(i_1, \ldots, i_k) \in [n]^k$.
We will need the following lemma.

Lemma 3.5 *Let $k, n, r \in \mathbb{N}$ with $k \geq 2$, $n \leq r \leq n^k$ and let $\sigma_k \colon [n]^k \to [n^k]$ be the standard bijection. Then, for every $1 \leq p < \infty$ one has*

$$a_n^{(k)}\big(I_{\sigma_k}^{(r)} \colon \ell_p^n \times \cdots \times \ell_p^n \to \ell_p^r\big) = 1.$$

Proof Fix $k \geq 2$. We claim that, for each $y \in \ell_p^n$ one has

$$\|I_{\sigma_k}^{(r)}(e_1, \ldots, e_1, y)\|_{\ell_p^r} = \|y\|_{\ell_p^n}.$$

Indeed, we have that

$$I_{\sigma_k}^{(r)}(u_1, \ldots, u_{k-1}, y) := (z_1, \ldots, z_r),$$

where $u_j = e_1 = (x_1(j), \ldots, x_n(j)) = (1, 0, \ldots, 0) \in \ell_p^n$ for $j \in [k-1]$, $y = (y_1, \ldots, y_n)$ and for each $\nu \in [r]$ with $\nu = \sigma_k(i_1, \ldots, i_k)$, $z_\nu := x_1(i_1) \ldots x_k(i_k)$. From definition of σ_k, it follows that $\sigma_k(1, \ldots, 1, i_k) = i_k$.

Since $n \leq r \leq n^k$, the first n elements of $(z_1, \ldots, z_r)$ are $y_1, \ldots, y_n$. Indeed, by the formula for the standard bijection σ_k it follows by $1 = \sigma_k(1, 1, \ldots, 1)$ that $z_1 = 1 \cdot 1 \cdot 1 \cdots 1 \cdot y_1 = y_1$. We conclude in a similar fashion that by $n = \sigma_k(1, 1, \ldots, n)$, $z_n = 1 \cdot 1 \cdot 1 \cdots 1 \cdot y_n = y_n$. For all other z_j in $(z_1, \ldots, z_r)$ with $n < j \leq r$, it follows by $z_j := x_1(i_1) \ldots x_k(i_k)$ that at least one factor in the product is zero. Hence

$$I_{\sigma_k}^{(r)}(u_1, \ldots, u_{k-1}, y) = (y_1, \ldots, y_n, 0, \ldots, 0)$$

and this proves the claim. Thus, the statement follows by Proposition 3.1. □

We calculate now the r-th approximation number of the bilinear diagonal operators on the products of ℓ_p-spaces. Let $r \in \mathbb{N}$ be fixed and let $n \in \mathbb{N}$ be the least number such that $r \leq n^2$. Given the standard bijection $\sigma_2 \colon [n] \times [n] \to [n^2]$, let $\widetilde{\sigma}_2 \colon \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection which is an extension of σ_2 such that $\widetilde{\sigma}_2|_{[n]\times[n]} = \sigma_2$. We point out that the extension $\widetilde{\sigma}_2$ can be obtained using an inductive procedure, considering an arbitrary fixed bijection $\beta \colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$. In what follows, we will use the bijection $\widetilde{\sigma}_2$ without any references.

Given a bounded sequence $\lambda := (\lambda_k)_{k=1}^\infty$, for each $k \in \mathbb{N}$, we find a unique $(i_k, j_k) \in \mathbb{N}\times\mathbb{N}$ such that $k = \widetilde{\sigma}_2(i_k, j_k)$.

As usual we let $\omega(\mathbb{N})$ to denote the space of all sequences modeled on $\mathbb{N}$. In what follows for a fixed bijection $\widetilde{\sigma}_2$ defined above, we define a mapping

$$\omega(\mathbb{N}) \times \omega(\mathbb{N}) \ni ((x_i), (y_j)) \mapsto ((x \tilde{*} y)_k)_{k=1}^\infty \in \omega(\mathbb{N}),$$

where, for each $k \in \mathbb{N}, (x \tilde{*} y)_k := x_{i_k} y_{j_k}$ with $\widetilde{\sigma}_2(i_k, j_k) = k$. Given $1 \leq p < \infty$ and a bounded sequence $\lambda = (\lambda_k)_{k=1}^\infty$, we define a mapping $D_\lambda \colon \ell_p \times \ell_p \to \ell_p$ by

$$D_\lambda(x, y) := (\lambda_k (x \tilde{*} y)_k)_{k=1}^\infty, \quad x, y \in \ell_p.$$

Let $n \in \mathbb{N}$. Similarly as above, for each positive integer $r \leq n^2$, we define a bilinear operator $D_\lambda^{(r)} \colon \ell_p^n \times \ell_p^n \to \ell_p^r$ by

$$D_\lambda^{(r)}(x, y) := (\lambda_k (x \tilde{*} y)_k)_{k=1}^r, \quad x = (x_i), y = (y_j) \in \ell_p^n,$$

where, for each $k \in [r]$ we let $(x \tilde{*} y)_k := x_{i_k} y_{j_k}$ with $\widetilde{\sigma}_2(i_k, j_k) = k$.

Given $\alpha := (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r$, with $\alpha_j > 0$ for each $j \in [r]$, we also define a linear operator $R_\alpha^{(r)} \colon \ell_p^r \to \ell_p^r$ by

$$R_\alpha^{(r)}(x_1, \ldots, x_r) := (\alpha_1^{-1} x_1, \ldots, \alpha_r^{-1} x_r), \quad (x_1, \ldots, x_r) \in \mathbb{K}^r.$$

Observe that for every $x, y \in \ell_p^n$, one has

$$R_\alpha^{(r)} D_\alpha^{(r)}(x, y) = I_{\sigma_2}^{(r)}(x, y).$$

In what follows, we will use the following standard finite dimensional operators $J_n \colon \ell_p^n \to \ell_p$ and $Q_n \colon \ell_p \to \ell_p^n$ defined for each $n \in \mathbb{N}$ by

$$J_n\xi := \sum_{j=1}^{n} \xi_j\, e_j, \quad \xi = (\xi_j) \in \ell_p^n,$$
$$Q_n x := (x_1, \ldots, x_n), \quad x = (x_j) \in \ell_p.$$

Theorem 3.6 *Let $\lambda := (\lambda_j)$ be a sequence such that $\lambda_1 \geq \lambda_2 \geq \cdots > 0$, and let $r, n \in \mathbb{N}$ with $1 \leq r \leq n^2$. Then, for the bilinear operator $D_\lambda \colon \ell_p \times \ell_p \to \ell_p$ with $1 \leq p \leq \infty$ one has*

$$\lambda_r = a_r^{(2)}(D_\lambda).$$

Proof The case $r = 1$ is obvious. Thus, we may assume that $1 < r \leq n^2$. We claim that $\lambda_r \leq a_r^{(2)}(D_\lambda)$. Since $(Q_r D_\lambda(J_n, J_n)) = D_\lambda^{(r)}$, Lemma 3.5 yields

$$\begin{aligned} 1 = a_r^{(2)}(I_{\sigma_2}^{(r)}) &= a_r^{(2)}(R_\lambda^{(r)} \circ D_\lambda^{(r)}) \leq \|R_\lambda^{(r)}\|\, a_r^{(2)}(D_\lambda^{(r)}) \\ &\leq \lambda_r^{-1}\, a_r^{(2)}(D_\lambda^{(r)}) = \lambda_r^{-1}\, a_r^{(2)}(Q_r D_\lambda(J_n, J_n)) \\ &\leq \lambda_r^{-1} \|Q_r\|\, a_r^{(2)}(D_\lambda)\, \|J_n\|\, \|J_n\| \leq \lambda_r^{-1}\, a_r^{(2)}(D_\lambda), \end{aligned}$$

which implies the required inequality.

We now show that $a_r^{(2)}(D_\lambda) \leq \lambda_r$. Let us consider the composition

$$J_{r-1} D_\lambda^{(r-1)}(Q_n,\ Q_n) \colon \ell_p \times \ell_p \xrightarrow{(Q_n,\, Q_n)} \ell_p^n \times \ell_p^n \xrightarrow{D_\lambda^{(r-1)}} \ell_p^{r-1} \xrightarrow{J_{r-1}} \ell_p.$$

Then, for all $x, y \in \ell_p$ one has

$$\begin{aligned} &\|D_\lambda(x, y) - J_{r-1} D_\lambda^{(r-1)}(Q_n,\ Q_n)(x, y))\|_p^p \\ &= \Big\| \sum_{k=1}^{\infty} \lambda_k (x \tilde{*} y)_k\, e_k - \sum_{k=1}^{r-1} \lambda_k (x \tilde{*} y)_k\, e_k \Big\|_p^p \\ &= \sum_{k=r}^{\infty} |\lambda_k (x \tilde{*} y)_k|^p \leq \lambda_r^p \sum_{k=1}^{\infty} |(x \tilde{*} y)_k|^p \leq \lambda_r^p \|x\|_p^p\, \|y\|_p^p. \end{aligned}$$

This implies that $\|D_\lambda - J_{r-1} D_\lambda^{(r-1)}(Q_n, Q_n)\| \leq \lambda_r$. Since

$$\operatorname{rank}(J_{r-1} D_\lambda^{(r-1)}(Q_n, Q_n)) = \operatorname{rank}(D_\lambda^{(r-1)}(Q_n, Q_n))$$

and $[(D_\lambda^{(r-1)}(Q_n, Q_n))(\ell^p \times \ell^p)] \subset \ell_p^{r-1}$,

$$\operatorname{rank}(J_{r-1} D_\lambda^{(r-1)}(Q_n, Q_n)) < r.$$

In consequence $a_r^{(2)}(D_\lambda) \leq \lambda_r$ and so this completes the proof. □

4 Gelfand numbers

Given an operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, following [5], the n-th *Gelfand number* $(c_n^{(k)}(T))$ of $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ is given by

$$c_n^{(k)}(T) := a_n^{(k)}(J_Y T).$$

Recall that $J_Y : Y \to \ell_\infty(U_{Y^*})$ is the metric injection given by $J_Y y := (y^*(y))_{y^* \in U_{Y^*}}$ for all $y \in Y$.

Lemma 4.1 *The sequence $(c_n^{(k)})$ of Gelfand numbers is an $s^{(k)}$-scale.*

Proof Properties (S1) and (S2) follow from the definition. For $(S3)$, since $\|J_Y\| = 1$, it follows by (S3) of $a_n^{(k)}$. The property (S4) also follows from (S4) of $a_n^{(k)}$. We prove the norming property (S5). Clearly, for each $n \in \mathbb{N}$, we have $\| \otimes_k I_n\| = 1$, which implies $c_n^{(k)}(\otimes_k I_n) \leq 1$. We need to show that $c_n^{(k)}(\otimes_k I_n) = 1$.

Given $\varepsilon > 0$, we can find $A \in \mathcal{L}(\ell_2^n, \ldots, \ell_2^n; \ell_\infty(U_{\ell_2([n]^k)^*})$, with $\operatorname{rank}(A) < n$, such that,

$$\|J_{\ell_2([n]^k)} \circ (\otimes_k I_n) - A\| < a_n^{(k)}(J_{\ell_2([n]^k)} \circ (\otimes_k I_n)) + \varepsilon.$$

We define $B : \ell_2^n \to \ell_\infty(U_{\ell_2([n]^k)^*})$ by $By := A(e_1, \ldots, e_1, y)$ for all $y \in \ell_2^n$. Since

$$\operatorname{rank}(B) \leq \operatorname{rank}(A) < n,$$

there exists $\xi \in S_{\ell_2^n}$, such that, $B\xi = 0$. Thus, letting $\Gamma := U_{\ell_2([n]^k)^*}$, we get

$$\begin{aligned}
\varepsilon + a_n^{(k)}(J_{\ell_2([n]^k)} \circ (\otimes_k I_n)) &> \|J_{\ell_2([n]^k)} \circ (\otimes_k I_n - A)\| \\
&\geq \|J_{\ell_2([n]^k)} \circ (\otimes_k I_n - A)(e_1, \ldots, e_1, \xi)\|_{\ell_\infty(\Gamma)} \\
&= \|J_{\ell_2([n]^k)} \circ (\otimes_k I_n)(e_1, \ldots, e_1, \xi) - B\xi\|_{\ell_\infty(\Gamma)} \\
&= \|J_{\ell_2([n]^k)} \circ (\otimes_k I_n)(e_1, \ldots, e_1, \xi)\|_{\ell_\infty(\Gamma)} \\
&= \| \otimes_k I_n(e_1, \ldots, e_1, \xi)\|_{\ell_2([n]^k)} = \|\xi\|_{\ell_2^n} = 1.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $c^{(k)}(\otimes_k I_n) = a_n^{(k)}(J_{\ell_2^n} \circ (\otimes_k I_n)) \geq 1$. This completes the proof. □

Proposition 4.2 *Let $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ such that $\operatorname{rank}(T) \geq n$, and suppose that $T^\times$ is a metric injection. Then, we have $c_n^{(k)}(T) = 1$.*

Proof We first observe that, for every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$,

$$(J_Y T)^\times = T^\times J_Y^* : \ell_\infty(U_{Y^*})^* \xrightarrow{J_Y^*} Y^* \xrightarrow{T^\times} \mathcal{L}(X_1, \ldots, X_k).$$

Suppose $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ satisfies the hypotheses. Then, one has

$$\begin{aligned}
c_n^{(k)}(T) &= \inf\{\|J_Y T - A\|;\ \operatorname{rank}(A) < n\} \\
&= \inf\{\|(J_Y T)^\times - A^\times\|;\ \operatorname{rank}(A) < n\},
\end{aligned}$$

where the infimum is taken over all $A \in \mathcal{L}(X_1, \ldots, X_k; \ell_\infty(U_{Y^*}))$. Since $\operatorname{rank}(A^\times) = \operatorname{rank}(A) < n$, we conclude that

$$a_n((J_Y T)^\times) \leq c_n^{(k)}(T).$$

Clearly, $J_Y^*\colon \ell_\infty(U_{Y^*})^* \to Y^*$ is a metric surjection since $J_Y\colon Y \to \ell_\infty(U_{Y^*})$ is a metric injection. As $(J_Y T)^\times = T^\times J_Y^*$, it follows that $(J_Y T)^\times$ is a metric injection. Now, observe that $\operatorname{rank}((J_Y T)^\times) = \operatorname{rank}(J_Y T) = \operatorname{rank}(T) \geq n$. Combining with Proposition 3.1, we get

$$1 = a_n((J_Y T)^\times) \leq c_n^{(k)}(T).$$

Since $c_n^{(k)}(T) \leq \|T\| = 1$, the statement follows. □

Lemma 4.3 *For every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ one has*

$$c_n(T^\times) \leq a_n^{(k)}(T), \quad n \in \mathbb{N}.$$

Proof We observe that for any operator $A \in \mathcal{L}(X_1, \ldots, X_k; Y)$, we have $J_Z A^\times\colon Y^* \to \ell_\infty(U_{Z^*})$ with $\operatorname{rank}(J_Z A^\times) = \operatorname{rank}(A^\times)$, where $Z := \mathcal{L}(X_1, \ldots, X_k)$. This implies that, for every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, we get

$$\begin{aligned} c_n(T^\times) &= a_n(J_Z T^\times) = \inf\{\|J_Z T^\times - L\|;\ L\colon Y^* \to \ell_\infty(U_{Z^*}),\ \operatorname{rank}(L) < n\} \\ &\leq \inf\{\|J_Z T^\times - J_Z A^\times\|;\ A \in \mathcal{L}(X_1, \ldots, X_k; Y),\ \operatorname{rank}(A) < n\} \\ &\leq \inf\{\|T^\times - A^\times\|;\ A \in \mathcal{L}(X_1, \ldots, X_k; Y),\ \operatorname{rank}(A) < n\} \\ &= \inf\{\|T - A\|;\ A \in \mathcal{L}(X_1, \ldots, X_k; Y),\ \operatorname{rank}(A) < n\} = a_n^{(k)}(T) \end{aligned}$$

as required. □

Similarly as in the linear setting (see [21, Proposition 11.5.3]) one can show that the following statement is true for multilinear operators: if a Banach space F has the metric extension property, then for any $T \in \mathcal{L}(X_1, \ldots, X_k; F)$ we have

$$c_n^{(k)}(T) = a_n^{(k)}(T), \quad n \in \mathbb{N}.$$

As an application, we have the following multilinear variant of Pietsch's result [19, Theorem 11.5.5], that the Gelfand numbers (c_n) forms the largest injective s-scale in the setting of linear operators.

Proposition 4.4 $(c_n^{(k)})$ *is the largest injective $s^{(k)}$-scale.*

Proof Let $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and $J \in \mathcal{L}(Y; Z)$ be an injection. Since $\ell_\infty(U_{Y^*})$ satisfies the metric extension property, there is a linear operator $L\colon Z \to \ell_\infty(U_{Y^*})$, with $\|L\| = \|J_Y\| = 1$ such that $J_Y = LJ$. Then using the fact mentioned above yields

$$\begin{aligned} c_n^{(k)}(T) &= a_n^{(k)}(J_Y T) = a_n^{(k)}(LJT) = c_n^{(k)}(LJT) \\ &\leq \|L\|\, c_n^{(k)}(JT) \leq c_n^{(k)}(JT). \end{aligned}$$

Since $c_n^{(k)}(JT) \leq c_n^{(k)}(T)$, $(c_n^{(k)})$ is an injective $s^{(k)}$-scale.

Next, we observe that the above gives that, if $(s_n^{(k)})$ is an arbitrary injective $s^{(k)}$-scale, then for any $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$,

$$s_n^{(k)}(T) = s^{(k)}(J_Y T) \leq a_n^{(k)}(J_Y T) = c_n^{(k)}(T), \quad n \in \mathbb{N}$$

and the result follows. □

As an application we obtain the following corollary.

Corollary 4.5 *For an arbitrary injective $s^{(k)}$-scale, the pair $((c_n), (s_n^{(k)}))$ has the mixed multiplicative property.*

Proof As we have noticed, the pair $((a_n), (s_n^{(k)}))$ has the mixed multiplicative property for an arbitrary $s^{(k)}$-scale. This fact combined with Proposition 4.4 yields that if $(s_n^{(k)})$ is an injective $s^{(k)}$-scale, then for any $T \in \mathcal{L}(X_1, \dots, X_k; Y)$ and $S \in \mathcal{L}(Y; Z)$, we have

$$s_{m+n-1}^{(k)}(ST) = s_{m+n-1}^{(k)}(J_Z ST) \le a_m(J_Z S)\, s_n^{(k)}(T) = c_m(S)\, s_n^{(k)}(T).$$

□

5 Kolmogorov numbers

Let N be a closed subspace of a Banach space Y, then Q_N^Y (Q_N for short) denotes the canonical quotient map from Y onto Y/N. The n-th Kolmogorov number of $T \in \mathcal{L}(X_1, \dots, X_k; Y)$ is defined by

$$d_n^{(k)}(T) = \inf\left\{\|Q_N T\|;\ \ N \subset Y,\ \dim(N) < n\right\}.$$

We shall use the following formula (see [5, Theorem 4.1]):

$$d_n^{(k)}(T) = a_n^{(k)}(T(Q_1, \dots, Q_k)), \tag{$*$}$$

where Q_j denotes the canonical metric surjection from $\ell_1(U_{X_j})$ onto X_j, $j \in [k]$.

Lemma 5.1 *Let $T \in \mathcal{L}(X_1, \dots, X_k; Y)$ be a k-linear operator. Then, $d_n^{(k)}(T) = 0$ if and only if rank $(T) < n$.*

Proof Clearly, $\operatorname{rank}(T) < n$ implies that, for $N := [T(X_1 \times \cdots \times X_k)]$, we have $\dim(N) < n$ and whence $\|Q_N T\| = 0$. In consequence $d_n^{(k)}(T) = 0$.

Now assume that $d_n^{(k)}(T) = 0$. We apply the above formula $(*)$ to get

$$a_n^{(k)}(T(Q_1, \dots, Q_k)) = 0,$$

where $Q_j \colon \ell_1(U_{X_j}) \to X_j$ are canonical metric surjections, $j \in [k]$. By Theorem 3.3 and Proposition 2.1, we have $\operatorname{rank}(T(Q_1, \dots, Q_k)) < n$. Combining,

$$\begin{aligned}\dim\,[T(X_1 \times \cdots \times X_k)] &= \dim\,[T(Q_1(\ell_1(U_{X_1})) \times \cdots \times Q_k(\ell_1(U_{X_k})))]\\ &= \dim\,[T(Q_1, \dots, Q_k)(\ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_k}))]\\ &= \operatorname{rank}\,(T(Q_1, \dots, Q_k)) < n,\end{aligned}$$

yields $\operatorname{rank}(T) < n$, and this completes the proof. □

Lemma 5.2 *Let H be a Hilbert space and $T \in \mathcal{L}(X_1, \dots, X_k; H)$ be a k-linear operator. Then, for each $n \in \mathbb{N}$, we have*

$$a_n^{(k)}(T) \le \inf\{\|T - PT\|;\ \ P \colon H \to H,\ \operatorname{rank}(P) < n\} \le d_n^{(k)}(T),$$

where the infimum is taken over all orthogonal projections in H.

Proof By the above formula $(*)$, it follows that for a given $\varepsilon > 0$, there exists $A \in \mathcal{L}(\ell_1(U_{X_1}), \dots, \ell_1(U_{X_k}); H)$, such that $\operatorname{rank}(A) < n$ and

$$\|T(Q_1, \dots, Q_k) - A\| \le (1 + \varepsilon)\, d_n^{(k)}(T).$$

Define $V := [A(\ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_k}))] \subset H$ and let P be the orthogonal projection onto V. Then, we have $\operatorname{rank}(P) < n$ and

$$\begin{aligned}\|T - PT\| &= \|(I-P)T(Q_1, \dots, Q_k)\| = \|(I-P)(T(Q_1, \dots, Q_k) - A)\| \\ &\leq \|T(Q_1, \dots, Q_k) - A\|.\end{aligned}$$

This implies that

$$a_n^{(k)}(T) \leq \|T - PT\| \leq \|T(Q_1, \dots, Q_k) - A\| \leq (1+\varepsilon) d_n^{(k)}(T).$$

Since $\varepsilon > 0$ is arbitrary the proof is completed. □

Proposition 5.3 *For each k, the sequence $(d_n^{(k)})$ of the Kolmogorov numbers forms an $s^{(k)}$-scale.*

Proof Again, for simplicity of presentation we only consider the case $k = 2$. The properties $(S1) - (S2)$ are obvious. Let $T \in \mathcal{L}(X_1, X_2; Y)$, $S \in \mathcal{L}(Y; Z)$ and $R_1 \in \mathcal{L}(W_1; X_1)$, $R_2 \in \mathcal{L}(W_2; X_2)$. Then, for any subspace $N \subset Y$ with $\dim(N) < n$ one has

$$\|Q_{S(N)}^Z ST(R_1, R_2)\| \leq \|S\| \, \|Q_N^Y T\| \, \|R_1\| \, \|R_2\|.$$

Since $\dim(S(N)) \leq \dim(N) < n$ and subspace N is arbitrary, this yields

$$d_n^{(2)}(ST(R_1, R_2)) \leq \|S\| \, d_n^{(2)}(T) \, \|R_1\| \, \|R_2\|,$$

so the property (S3) is satisfied. The property (S4) follows from Lemma 5.1.

To finish we observe that from Lemma 5.2, we have

$$a_n^{(2)}(\otimes_2 I_n) \leq d_n^{(2)}(\otimes_2 I_n) \leq \| \otimes_2 I_n\| = 1, \quad n \in \mathbb{N}.$$

and so, we conclude by Corollary 3.2 that

$$d_n^{(2)}(\otimes_2 I_n) = 1, \quad n \in \mathbb{N}.$$

Thus the property (S5) is also satisfied and this completes the proof. □

Since $(a_n^{(k)})$ is the largest $s^{(k)}$-scale, we obtain from Lemma 5.2 and Proposition 5.3 a multilinear variant of the linear result by Pietsch (see [19, Proposition 11.6.2]).

Corollary 5.4 *Let H be a Hilbert space and $T \in \mathcal{L}(X_1, \dots, X_k; H)$ be a k-linear operator. Then, for each $n \in \mathbb{N}$, we have*

$$a_n^{(k)}(T) = d_n^{(k)}(T) = \inf\{\|T - PT\|; \ P \colon H \to H, \ \operatorname{rank}(P) < n\},$$

where the infimum is taken over all orthogonal projections in H.

We conclude this section with a corollary which follows from the formula $(*)$.

Corollary 5.5 *$(d_n^{(k)})$ is the largest surjective $s^{(k)}$-scale.*

6 Symmetrized approximation numbers

Symmetrized approximation numbers (t_n) were introduced by Pietsch in [19, Proposition 11.7.9]. For any operator $T \in \mathcal{L}(X; Y)$, these are defined by

$$(t_n(T)) := (a_n(J_Y T Q_X)),$$

where Q_X denotes the canonical metric surjection from $\ell_1(U_X)$ onto X.

Note that (t_n) are the largest injective and surjective s-numbers with the property

$$t_n(T^*) = t_n(T).$$

It is worth noting that we have here a refined version of Schauder's theorem (see [3, p. 84]), which states that an operator T between arbitrary Banach spaces X and Y is compact if and only if $\lim_{n\to\infty} t_n(T) = 0$. Then, by the above formula, it follows that the degree of compactness of T and T^* are the same when they are measured by the symmetrized approximation numbers t_n.

Following the linear case, we introduce a variant of symmetrized approximation numbers in the multilinear setting. For each $k \in \mathbb{N}$, we define a rule $(t^{(k)})\colon \mathcal{L}^k \to [0,\infty)^{\mathbb{N}}$ assigning to every k-linear operator $T\colon X_1 \times \cdots \times X_k \to Y$ a non-negative scalar sequence $(t_n^{(k)}(T))$ given by

$$t_n^{(k)}(T) := a_n^{(k)}(J_Y T(Q_{X_1}, \ldots, Q_{X_k})), \quad n \in \mathbb{N}.$$

Clearly, this definition is equivalent to

$$t_n^{(k)}(T) = d_n^{(k)}(J_Y T),$$

as well as to

$$t_n^{(k)}(T) = c_n^{(k)}(T(Q_{X_1}, \ldots, Q_{X_k})).$$

Our aim is to prove the following main result of this section.

Theorem 6.1 *For each $k \in \mathbb{N}$, and any operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, one has*

$$t_n(T^\times) = t_n^{(k)}(T), \quad n \in \mathbb{N}.$$

To prove Theorem 6.1 we need two preliminary results. Before we state these results, we define a special operator between spaces of k-linear forms. Given a positive integer $k \geq 2$ and operators $A_1 \in \mathcal{L}(Y_1; X_1), \ldots, A_k \in \mathcal{L}(Y_k; X_k)$, we define the mapping $\Phi_{A_1,\ldots,A_k}$ from $\mathcal{L}(X_1, \ldots, X_k)$ to $\mathcal{L}(Y_1, \ldots, Y_k)$ by

$$\Phi_{A_1,\ldots,A_k} S(y_1, \ldots, y_k) := S(A_1 y_1, \ldots, A_k y_k),$$

for all $S \in \mathcal{L}(X_1, \ldots, X_k)$ and $(y_1, \ldots, y_k) \in Y_1 \times \cdots \times Y_k$.

Under the above notation, we have the following lemma.

Lemma 6.2 *We have that $\Phi_{A_1,\ldots,A_k}\colon \mathcal{L}(X_1, \ldots, X_k) \to \mathcal{L}(Y_1, \ldots, Y_k)$ is a bounded linear operator, and for any operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, it holds*

$$(T(A_1, \ldots, A_k))^\times = \Phi_{A_1,\ldots,A_k} \circ T^\times.$$

Proof The first statement is obvious. For any $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$,

$$(T(A_1, \ldots, A_k))^{\times} \colon Y^* \to \mathcal{L}(Y_1, \ldots, Y_k)$$

is a bounded operator. Thus, for all $y^* \in Y^*$ and all $(y_1, \ldots, y_k) \in Y_1 \times \cdots \times Y_k$,

$$\begin{aligned}(T(A_1, \ldots, A_k))^{\times} y^*(y_1, \ldots, y_k) &= y^* \circ T(A_1 y_1, \ldots, A_k y_k) \\ &= T^{\times} y^*(A_1 y_1, \ldots, A_k y_k) \\ &= \Phi_{A_1,\ldots,A_k}(T^{\times} y^*)(y_1, \ldots, y_k),\end{aligned}$$

and so the required formula follows. □

To state the next result, recall that given Banach spaces $X_1, \ldots, X_k$, we let $\widehat{\kappa}_{X_1\times\cdots\times X_k}$ to denote the k-linear operator from $X_1 \times \cdots \times X_k$ to $\mathcal{L}(X_1, \ldots, X_k)^*$ given, for all $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$, by

$$\langle S, \widehat{\kappa}_{X_1\times\cdots\times X_k}(x_1, \ldots, x_k)\rangle := S(x_1, \ldots, x_k), \quad S \in \mathcal{L}(X_1, \ldots, X_k).$$

The following factorization result is a preliminary to Theorem 6.1, but is of interest in itself.

Proposition 6.3 *For any Banach spaces $X_1, \ldots, X_k$, let $X := X_1 \times \cdots \times X_k$ and, for each $j \in [k]$, let $Q_{X_j} \colon \ell_1(U_{X_j}) \to X_j$ be the canonical surjection. Then, the operator $\Phi := \Phi_{Q_{X_1},\ldots,Q_{X_k}}$ admits the following factorization with $E := \mathcal{L}(X_1, \ldots, X_k)$:*

$$\Phi \colon E \xrightarrow{J_E} \ell_\infty(U_{E^*}) \xrightarrow{P} \ell_\infty(U_X) \xrightarrow{R} \mathcal{L}(\ell_1(U_{X_1}), \ldots, \ell_1(U_{X_k})),$$

where the norms of operators P and R are less than or equal 1.

Proof We first observe that for any $S \in \mathcal{L}(X_1, \ldots, X_k)$ and $(\lambda_{x_j}) \in \ell_1(U_{X_j})$, $j \in [k]$, we have

$$\begin{aligned}\Phi S((\lambda_{x_1}), \ldots, (\lambda_{x_k})) &= S(Q_{X_1}(\lambda_{x_1}), \ldots, Q_{X_k}(\lambda_{x_k})) \\ &= \sum_{x_1 \in U_{X_1}} \cdots \sum_{x_k \in U_{X_k}} \lambda_{x_1} \cdots \lambda_{x_k} S(x_1, \ldots, x_k).\end{aligned} \tag{$*$}$$

We define on $\ell_\infty(U_{\mathcal{L}(X_1,\ldots,X_k)^*})$ a mapping P by

$$Pf := f|_{\widehat{\kappa}_X(U_X)}, \quad f \in \ell_\infty(U_{\mathcal{L}(X_1,\ldots,X_k)^*}).$$

Clearly, $P \colon \ell_\infty(U_{\mathcal{L}(X_1,\ldots,X_k)^*}) \to \ell_\infty(U_X)$ is a bounded operator with $\|P\| \le 1$.

We also define an operator $R \colon \ell_\infty(U_{X_1\times\cdots\times X_k}) \to \mathcal{L}(\ell_1(U_{X_1}), \ldots, \ell_1(U_{X_k}))$ with norm $\|R\| \le 1$ given by

$$R\xi((\lambda_{x_1}), \ldots, (\lambda_{x_k})) := \sum_{x_1 \in U_{X_1}} \cdots \sum_{x_k \in U_{X_k}} \xi_{(x_1,\ldots,x_k)} \lambda_{x_1} \cdots \lambda_{x_k},$$

for all $\xi \in \ell_\infty(U_{X_1\times\cdots\times X_k})$ and $((\lambda_{x_1}), \ldots, (\lambda_{x_k})) \in \ell_1(U_{X_1}) \times \cdots \times \ell_1(U_{X_k})$.

Finally, we see that, for all $S \in \mathcal{L}(X_1, \ldots, X_k)$ and all $x \in U_X$, we have

$$PJ_{\mathcal{L}(X_1,\ldots,X_k)}(S) = P\big((\langle S, \varphi\rangle)_{\varphi\in\mathcal{L}(X_1,\ldots,X_k)^*}\big) = (\langle S, \widehat{\kappa}_X x\rangle)_{x\in U_X} = (Sx)_{x\in U_X}.$$

Combining formulas $(*)$ and R yields the required factorization

$$\Phi_{Q_{X_1},\ldots,Q_{X_k}} = RPJ_{\mathcal{L}(X_1,\ldots,X_k)},$$

and this completes the proof. □

Now, we are ready to prove the main result of this section.

Proof of Theorem 6.1 We first recall that if Y is a Banach space and $J_Y\colon Y \to \ell_\infty(U_{Y^*})$ is the canonical injection, then for the dual operator $J_Y^*\colon \ell_\infty(U_{Y^*})^* \to Y^*$, we have

$$Q_{Y^*} = J_Y^* \widehat{\kappa}_{\ell_1(U_{Y^*})}.$$

Given $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, we have $T^\times\colon Y^* \to \mathcal{L}(X_1, \ldots, X_k)$. Combining with the above formula and [5, Theorem 5.1] yield

$$\begin{aligned} t_n(T^\times) &= c_n(T^\times Q_{Y^*}) = c_n(T^\times J_Y^* \widehat{\kappa}_{\ell_1(U_{Y^*})}) \le c_n(T^\times J_Y^*) \\ &= c_n((J_Y T)^\times) \le d_n^{(k)}(J_Y T) = t_n^{(k)}(T). \end{aligned}$$

To finish, we apply Proposition 6.3 and [5, Theorem 5.1] to get (by $\|P\|, \|R\| \le 1$)

$$\begin{aligned} t_n^{(k)}(T) &= c_n^{(k)}(T(Q_{X_1}, \ldots, Q_{X_k})) = d_n((T(Q_{X_1}, \ldots, Q_{X_k}))^\times) \\ &= d_n\big(\Phi_{Q_{X_1}, \ldots, Q_{X_k}} T^\times\big) = d_n(RPJ_{\mathcal{L}(X_1, \ldots, X_k)} T^\times) \\ &\le d_n(J_{\mathcal{L}(X_1, \ldots, X_k)} T^\times) = t_n(T^\times), \end{aligned}$$

and this completes the proof. □

The following theorem is a consequence of Borsuk antipodal theorem (see, e.g., [22, Theorem 1.4]).

Theorem 6.4 *Let Y and Z be closed subspaces of a Banach space X, where Z is finite dimensional, and* $\dim Y > \dim Z$. *Then, there exists $y \in Y$ such that* $\|y\|_Y = 1 = \operatorname{dist}(y, Z)$.

As an application of Theorem 6.4 we get the following result.

Proposition 6.5 *Let $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ be a surjective operator with* $\operatorname{rank}(T) \ge n$. *If $T^\times\colon Y^* \to \mathcal{L}(X_1, \ldots, X_k)$ is a metric injection, then* $t_n^{(k)}(T) = 1$.

Proof For simplicity of notation, we let $F := \mathcal{L}(X_1, \ldots, X_k)$. By Proposition 6.1, we have

$$t_n^{(k)}(T) = t_n(T^\times) = d_n(J_F T^\times).$$

Clearly, $d_n(J_F T^\times) \le 1$. We claim that $d_n(J_F T^\times) = 1$. To prove this, we apply Proposition 2.2.2 from [4], which says that the nth Kolgomorov number $d_n(S)$ of an operator $S \in \mathcal{L}(E; F)$ can be expressed as

$$d_n(S) = \inf\big\{\varepsilon > 0;\ S(U_E) \subset N_\varepsilon + \varepsilon U_F,\ \text{ where } N_\varepsilon \subset F \text{ with } \dim N_\varepsilon < n\big\}.$$

Suppose $d_n(J_F T^\times) < 1$. Then, by the above formula, we can find $\gamma \in (0, 1)$ and a subspace $N_\gamma \subset \ell_\infty(U_{F^*})$ with $\dim N_\gamma < n$, such that

$$J_F T^\times(U_{Y^*}) \subset N_\gamma + \gamma\, U_{\ell_\infty(U_{F^*})}.$$

Since $\operatorname{rank}(T) \ge n$ and $T^\times$ is a metric injection, $\operatorname{rank}(J_F T^\times(Y^*)) \ge n$. Combining with Theorem 6.4, we deduce that for $\varepsilon = \gamma^{-1} - 1$, there exists $y^* \in Y^*$ such that

$$\|y^*\|_{Y^*} = 1 = \|J_F T^\times(y^*)\|_{\ell_\infty(U_{F^*})}$$

and, for all $v \in N_\gamma$, we have

$$\|J_F T^\times(y^*) - v\|_{\ell_\infty(U_{F^*})} > 1/(1+\varepsilon) = \gamma.$$

This a contradiction with the above inclusion. □

We conclude this section with a remark that it is an immediate consequence of the properties of approximation numbers $a_n^{(k)}$ that the sequence of symmetrized approximation numbers $(t_n^{(k)})$ satisfy the corresponding properties (S1), (S2), (S3) and (S4).

7 Hilbert numbers

The n-th *Hilbert number* of a k-linear operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ is defined by

$$h_n^{(k)}(T) := \sup\{a_n^{(k)}(BT(A_1, \ldots, A_k))\},$$

where the supremum is taken over all linear operators $B \in U_{\mathcal{L}(Y;\ell_2)}$ and $A_1 \in U_{\mathcal{L}(\ell_2;X_1)}, \ldots, A_k \in U_{\mathcal{L}(\ell_2;X_k)}$.

We have the following theorem.

Theorem 7.1 *Let $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ be a k-linear operator. Then, $h_n^{(k)}(T) = 0$ implies rank $(T) < n$.*

Proof Fix $n \in \mathbb{N}$ and assume $h_n^{(k)}(T) = 0$. Then, for all operators $B \in \mathcal{L}(Y; \ell_2)$ and $A_i \in \mathcal{L}(\ell_2; X_i)$ with $i \in [k]$ one has

$$a_n^{(k)}(BT(A_1, \ldots, A_k)) = 0.$$

Since $(a_n^{(k)})_{n=1}^{\infty}$ is a fully symmetric $s^{(k)}$-scale, it follows from Proposition 2.1 that for all operators $A_1, \ldots, A_k$ and B as above,

$$\operatorname{rank}(BT(A_1, \ldots, A_k)) < n.$$

We claim that rank $(T) < n$. Suppose this is false. Then, there are $(x_1^j, \ldots, x_k^j) \in X_1 \times \cdots \times X_k$, $j \in [n]$, such that $(y_j) := (T(x_1^j, \ldots, x_k^j))_{j=1}^n$ forms a basis in $[T(X_1 \times \cdots \times X_k)]$. Let (y_j^*) be a set of biorthogonal functionals to the basis (y_j), that is,

$$y_i^*(y_j) = \delta_{ij}, \quad i, j \in [n].$$

For each $i \in [k]$, we define the operator $A_i \in \mathcal{L}(\ell_2; X_i)$ by

$$A_i\xi = \sum_{j=1}^{n} \xi_j x_i^j, \quad \xi = (\xi_j) \in \ell_2.$$

We also define $B \in \mathcal{L}(Y; \ell_2)$ by

$$By = \sum_{i=1}^{n} y_i^*(y)e_i, \quad y \in Y.$$

Then, for each $j \in [n]$, we have

$$\begin{aligned} BT(A_1, \ldots, A_k)(e_j, \ldots, e_j) &= B(T(A_1e_j, \ldots, A_ke_j) \\ &= B(T(x_1^j, \ldots, x_k^j)) = By_j = e_j. \end{aligned}$$

Hence, rank $(BT(A_1, \ldots, A_k)) \geq n$, and so we arrive a contradiction which completes the proof. ◻

It turns out that the Hilbert numbers for k-linear operators are $s^{(k)}$-numbers in the sense given in Sect. 2.

Theorem 7.2 *For each $k \in \mathbb{N}$, the sequence $(h_n^{(k)})$ of Hilbert numbers is an $s^{(k)}$-scale.*

Proof Let $T \in \mathcal{L}(X_1, \dots, X_k; Y)$. We claim that the property (S1) holds. Clearly, $(h_n^{(k)}(T))$ is a non-increasing sequence. We show that $h_1^{(k)}(T) = \|T\|$.

Let $(x_1, \dots, x_k) \in U_{X_1} \times \cdots \times U_{X_k}$. By the Hahn–Banach theorem, we can find a norm one functional $y^* \in Y^*$ such that

$$\langle T(x_1, \dots, x_k), y^* \rangle = \|T(x_1, \dots, x_k)\|_Y.$$

We consider the operators $B \in U_{\mathcal{L}(Y;\ell_2)}$ and $A_1 \in U_{\mathcal{L}(\ell_2;X_1)}, \dots, A_k \in U_{\mathcal{L}(\ell_2;X_k)}$, given, for all $y \in Y$ and for all $(\xi_j) \in \ell_2$, by

$$By := \langle y, y^* \rangle e_1, \quad A_i(\xi_j) := \xi_i x_i, \quad i \in [k].$$

Since $BT(A_1, \dots, A_k)(e_1, \dots, e_k) = \|T(x_1, \dots, x_k)\|_Y \, e_1$, we get

$$\begin{aligned} \|T(x_1, \dots, x_k))\|_Y &= \|BT(A_1, \dots, A_k)(e_1, \dots, e_k))\|_{\ell_2} \le \|BT(A_1, \dots, A_k)\|_Y \\ &= a_1^{(k)}(BT(A_1, \dots, A_k) \le h_1^{(k)}(T). \end{aligned}$$

This proves that $\|T\| \le h_1^{(k)}(T)$. Since the opposite inequality is obvious, the claim is proved.

Using properties of $(a_n^{(k)})$, we deduce that the properties (S2) and (S3) hold. To prove (S4), we observe that $\operatorname{rank}(T) < n$ implies that, for all $B \in U_{\mathcal{L}(Y;\ell_2)}$ and $A_1 \in U_{\mathcal{L}(\ell_2;X_1)}, \dots, A_k \in U_{\mathcal{L}(\ell_2;X_k)}$, we have $\operatorname{rank}(BT(A_1, \dots, A_k)) < n$. We have seen that $(a_n^{(k)})$ is an $s^{(k)}$-scale for each k. Hence, $a_n^{(k)}(T(A_1, \dots, A_k)) = 0$ and so $h_n^{(k)}(T) = 0$ as required.

To finish we need to prove the property (S5). Fix $n \in \mathbb{N}$ and define operators $A_1 = \cdots = A_k := P_n \in U_{\mathcal{L}(\ell_2;\ell_2^n)}$ and $B \in U_{\mathcal{L}(\ell_2([n]^k);\ell_2)}$ by

$$P_n(\xi_i) := (\xi_1, \dots, \xi_n), \quad (\xi_i) \in \ell_2,$$

$$B(x_j) := \sum_{i=1}^{n^k} z_i e_i, \quad (x_j) \in \ell_2([n]^k),$$

where, for each $i \in [n^k]$ with $i = \sigma_k(j)$ for the unique $j := (j_1, \dots, j_k) \in [n]^k$, we take $z_i := x_j$. Now observe that, for $r := n^k$, we have

$$B \circ (\otimes_k I_n)(A_1, \dots, A_k) = J_r \circ I_{\sigma_k}^{(r)},$$

where $J_r \colon \ell_2^r \to \ell_2$ is a metric injection given by

$$J_r \xi := \sum_{i=1}^{r} \xi_i e_i, \quad \xi = (\xi_1, \dots, \xi_r) \in \ell_2^r.$$

Since $\operatorname{rank}(J_r \circ I_{\sigma_k}^{(r)}) \ge n$ and for all $x \in \ell_2^n$,

$$\|J_r \circ I_{\sigma_k}^{(r)}(e_1, \dots, e_1, x)\|_{\ell_2} = \|I_{\sigma_k}^{(r)}(e_1, \dots, e_1, x)\|_{\ell_2^n} = \|x\|_{\ell_2^r},$$

it follows from Proposition 3.1 that $a_n^{(k)}(J_r \circ I_{\sigma_k}^{(r)}) = 1$. In consequence, for $r = n^k$ we conclude that,

$$\begin{aligned} 1 = a_n^{(k)}(J_r \circ I_{\sigma_k}^{(r)}) &= a_n^{(k)}(B \circ (\otimes_k I_n(A_1, \dots, A_k))) \\ &\le h_n^{(k)}(\otimes_k I_n) \le \| \otimes_k I_n\| = 1. \end{aligned}$$

This completes the proof. □

8 Weyl and Chang numbers

An important role is played in the theory of eigenvalues of operators in Banach spaces by the famous Weyl numbers defined by Pietsch [18]. We introduce the Weyl numbers in the setting of multilinear operators. The n-th *Weyl number* of a k-linear operator $T: X_1 \times \cdots \times X_k \to Y$ is defined by

$$x_n^{(k)}(T) = \sup\{ a_n^{(k)}(T(R_1, \ldots, R_k))\,;\ R_j \in U_{\mathcal{L}(\ell_2; X_j)},\, j \in [k]\}.$$

We have the following observation.

Proposition 8.1 *For each $k \in \mathbb{N}$, the sequence $(x_n^{(k)})$ of Weyl numbers is an $s^{(k)}$-scale.*

Proof Since $(a_n^{(k)})$ is an $s^{(k)}$-scale, the properties (S1)–(S3) easily follows. If $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ is such that $\operatorname{rank}(T) < n$, then $\operatorname{rank}(T(A_1, \ldots, A_k)) < n$ and $a_n^{(k)}(T(A_1, \ldots, A_k)) = 0$, for all $A_j \in U_{\mathcal{L}(\ell_2; X_j)}$, $j \in [k]$. Consequently $x_n^{(k)}(T) = 0$. This shows that the property (S4) is satisfied.

Clearly, for each $n \in N$, we have $h_n^{(k)}(T) \leq x_n^{(k)}(T)$. Since $(h_n^{(k)})$ is an $s^{(k)}$-scale,

$$x_n^{(k)}(\otimes_k I_n) = 1, \quad n \in \mathbb{N},$$

and so the property (S5) is also satisfied. □

Following the proof of Theorem 7.1, we get the following.

Theorem 8.2 *Let $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ be a k-linear operator. Then, $x_n^{(k)}(T) = 0$ implies rank $(T) < n$.*

The n-th *Chang number* of an operator $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ is given by

$$y_n^{(k)}(T) := \sup\left\{a_n^{(k)}(ST)\,;\ S \in U_{\mathcal{L}(Y;\ell_2)}\right\}.$$

Proposition 8.3 *For each $k \in \mathbb{N}$, the sequence $(y_n^{(k)})$ of Chang numbers is an $s^{(k)}$-scale, which has the property: $y_n^{(k)}(T) = 0$ implies rank $(T) < n$.*

Proof The properties (S1)–(S3) are easily verified. To show the property (S4), we fix $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ with $\operatorname{rank}(T) < n$. Then, $\operatorname{rank}(ST) < n$ for all $S \in U_{\mathcal{L}(Y;\ell_2)}$. Thus $a_n^{(k)}(ST) = 0$ for all $S \in U_{\mathcal{L}(Y;\ell_2)}$ yields $y_n(T) = 0$.

To finish observe that, for every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ one has

$$h_n^{(k)}(T) \leq y_n^{(k)}(T), \quad n \in \mathbb{N}.$$

This clearly combined with the fact that $(h_n^{(k)})$ is a $s^{(k)}$-scale implies that the property (S5) is satisfied for $(y_n^{(k)})$. □

It is well known that Hilbert numbers fail to be multiplicative (see [17, Remark 2.9.19]). This is a consequence of [17, Proposition 2.9.19], which states that

$$h_n(I: \ell_1 \to \ell_1) \asymp n^{-1/2}, \quad n \in \mathbb{N}.$$

However, the following inequality is true (see [17, Lemma 2.6.6]) for any operators $T \in \mathcal{L}(X; Y)$ and $S \in \mathcal{L}(Y; Z)$

$$h_{m+n-1}(ST) \leq y_m(S) h_n(T), \quad m, n \in \mathbb{N}.$$

We have a multilinear variant of this inequality. The proof is similar to the linear case, but we include a proof for the sake of completeness.

Lemma 8.4 *For each $k \in \mathbb{N}$ the pair $((y_n), (h_n^{(k)}))$ satisfies property (M), that is, for every $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$ and $S \in \mathcal{L}(Y; Z)$, we have*

$$h_{m+n-1}^{(k)}(ST) \leq y_m(S)\, h_n^{(k)}(T), \quad m, n \in \mathbb{N}.$$

Proof Fix positive integers $k \geq 2$, m and n. Let $A_j \in \mathcal{L}(\ell_2; X_j)$, $j \in [k]$ and $Q \in \mathcal{L}(Z; \ell_2)$ with norm less or equal 1. Given $\varepsilon > 0$, let $B \in \mathcal{L}(Y; \ell_2)$ be such that rank $(B) < m$ and

$$\|QS - B\| \leq (1 + \varepsilon)\, a_m(QS).$$

Now, let $A \in \mathcal{L}(^k\ell_2; \ell_2)$ with rank $(A) < n$ be such that

$$\|(QS - B)T(A_1, \ldots, A_k) - A\| \leq (1 + \varepsilon) a_n^{(k)}((QS - B)T(A_1, \ldots, A_k)).$$

Clearly, rank $(BT(A_1, \ldots, A_k) + A) \leq$ rank (B) + rank $(A) < m + n - 1$. Thus letting $R := (QS - B)/\|QS - B\| \in U_{\mathcal{L}(Y;\ell_2)}$, we obtain

$$\begin{aligned} a_{m+n-1}^{(k)}&(QST(A_1, \ldots, A_k)) \leq \|QST(A_1, \ldots, A_k) - (BT(A_1, \ldots, A_k) + A)\| \\ &= \|(QS - B)T(A_1, \ldots, A_k) - A\| \leq (1 + \varepsilon)\, a_n^{(k)}((QS - B)T(A_1, \ldots, A_k)) \\ &\leq (1 + \varepsilon)\, \|QS - B\|\, a_n^{(k)}(RT(A_1, \ldots, A_k)) \leq (1 + \varepsilon)\, \|QS - B\|\, h_n^{(k)}(T) \\ &\leq (1 + \varepsilon)^2\, a_m(QS) h_n^{(k)}(T). \end{aligned}$$

Since $\varepsilon > 0$, Q and $A_1, \ldots, A_k$ are arbitrary, the desired estimate follows. □

For next theorem we need the following known result.

Lemma 8.5 *If an operator $T \in \mathcal{L}(Y^*; X^*)$, then $T = S^*$ for some $S \in \mathcal{L}(X; Y)$ whenever T is weak*-weakly continuous. In particular the statement is true if Y is a reflexive space.*

Theorem 8.6 *If $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$, then for each $n \in \mathbb{N}$ we have*

(i) $x_n(T^\times) \leq y_n^{(k)}(T)$;
(ii) $y_n^{(k)}(T) \leq x_n(T^\times)$.

Proof (i). We have that $T^\times : Y^* \to \mathcal{L}(X_1, \ldots, X_k)$. Let $V : \ell_2 \to Y^*$ with $\|V\| \leq 1$. By Lemma 8.5, there is $R : Y \to \ell_2$ such that $R^* = V$. Thus, by [5, Proposition 3.2], we obtain

$$\begin{aligned} a_n(T^\times V) = a_n(T^\times R^*) &= a_n((RT)^\times) \leq a_n^{(k)}(RT) \\ &\leq \sup\{a_n^{(k)}(ST);\ \|S : Y \to \ell_2\| \leq 1\} = y_n^{(k)}(T). \end{aligned}$$

(ii). Let $S : Y \to \ell_2$ with $\|S\| \leq 1$. Since ℓ_2 is reflexive, it follows from [5, Proposition 3. 3] that

$$a_n^{(k)}(ST) = a_n((ST)^\times) = a_n(T^\times S^*).$$

Since S is arbitrary, $y_n^{(k)}(T) \leq x_n(T^\times)$ as required. □

9 Bernstein numbers

In the theory of Banach operator ideals a closed ideal of super strictly singular (or finitely strictly singular) operator is an interesting class of operators, which contains compact operators and is contained in the class of strictly singular operators. Recall that an operator is strictly singular if its restriction to any infinite-dimensional subspace is not an isomorphism.

Super strictly operators were introduced implicitly by Mityagin and Pełczyński in [15], and explicitly by Milman in [13,14]. Recall that an operator $T \in \mathcal{L}(X; Y)$ is said to be *super strictly singular*, if there does not exist a number $\gamma > 0$ and a sequence (E_n) of subspaces, with dim $(E_n) = n$, such that

$$\|Tx\|_Y \geq \gamma \|x\|_X, \quad x \in \cup_n E_n.$$

Thus, T is super strictly singular if and only if the *Bernstein* numbers $b_n(T) \to 0$, as $n \to \infty$, where

$$b_n(T) := \sup \inf_{x \in S_{E_n}} \|Tx\|_Y,$$

where the supremum is taken over all n-dimensional subspaces of X.

We define a variant of Bernstein's numbers for bilinear operators. We start with some notations from the theory of linear operators. We recall that if $T : E \to F$ is an operator between Banach spaces, then the *injection modulus* of T is given by

$$j_1(T) := \inf\{\|Tx\|_F;\ \|x\|_E = 1\}.$$

An operator T is called an injection if $j(T) > 0$. Clearly, an injection can be characterized as a one-to-one operator from E into F with closed range.

Recall that the *surjection modulus* of T is given by

$$q_1(T) := \sup\{\tau > 0;\ \tau B_F \subset T(B_E)\}.$$

An operator T is called a *surjection* if $q_1(T) > 0$, which is equivalent to $T(E) = F$.

The above modules are important characteristics in the theory of linear operators, and they are used in the study of isomorphic embeddings, quotients of Banach spaces and, in particular, in the study of isomorphic classification of Banach spaces by the fact that both $j_1(T) > 0$ and $q_1(T) > 0$ if and only if T is an isomorphism.

In what follows, if X is a non-trivial Banach space, then we let $\mathcal{F}in(X)$ to denote the set of all non-trivial finite-dimensional subspaces of X. If $E \in \mathcal{F}in(X)$ with dim $(E) = n$, then we write $E \in \mathcal{F}in_n(X)$.

Let $T \in \mathcal{L}(X_1, \ldots, X_k; Y)$. Following the linear case, for every closed subspaces $N_1 \subset X_1, \ldots, N_k \subset X_k$, we let

$$j_k^{N_1 \times \cdots \times N_k}(T) := \inf \left\{ \|T(x_1, \ldots, x_k)\|_Y;\ (x_1, \ldots, x_k) \in S_{N_1} \times \cdots \times S_{N_k} \right\}.$$

We call $j_k^{X_1 \times \cdots \times X_k}(T)$ the modulus of injection of T and denote it by $j_k(T)$.
In a similar fashion, we define the *surjection modulus* of T by

$$q_k(T) := \sup\{\tau \geq 0;\ T(B_{X_1} \times \cdots \times B_{X_k}) \supset \tau B_Y\}$$

(we put $q_k(O) := 0$, where O is the null operator). T is said to be a surjection if $q_k(T) > 0$, that is, T maps $X_1 \times \cdots \times X_k$ onto Y. If $\|T\| = q_k(T) = 1$, then T is said to be a *metric surjection*. This means that T maps $U_{X_1} \times \cdots \times U_{X_k}$ onto U_Y.

In what follows we restrict our discussion to the bilinear operators. At first we note that the modules satisfy the following properties:

Lemma 9.1 *The following statements are true*:

(ii) *If* $S, T \in \mathcal{L}(X, Y; Z)$, *then* $j_2(S+T) \leq j_2(S) + j_2(T)$ *and* $q_2(S+T) \leq q_2(S) + \|T\|$.
(ii) *If* $T \in \mathcal{L}(X, Y; Z)$ *and* $S \in \mathcal{L}(Z; W)$, *then* $j_2(ST) \leq \|S\| j_2(T)$ *and* $q_2(ST) \leq q_1(S)\|T\|$. *Moreover if* T *is surjective, then* $j_2(ST) \leq j_1(S)\|T\|$, *while if* S *is surjective, then* $q_2(ST) \leq \|S\| q_2(T)$.

(iii) *If* $T \in \mathcal{L}(X, Y; Z)$, $R_1 \in \mathcal{L}(X_0; X)$ *and* $R_2 \in \mathcal{L}(Y_0; Y)$, *then* $j_2(T(R_1, R_2)) \leq \|T\| j_1(R_1) j_1(R_2)$. *Moreover, if* R_1 *and* R_2 *are surjective, then* $j_2(T(R_1, R_2)) \leq j_2(T)\|R_1\| \, \|R_2\|$.

These properties can be verified easily following the definitions.

Suppose that X, Y and Z are Banach spaces with $\dim(X) \geq N$ and $\dim(Y) \geq N$. Then, for each $n \in [N]$, the n-th Bernstein number of every bilinear operator $T : X \times Y \to Z$ is given by

$$b_n^{(2)}(T) := \sup\left\{ j_2^{M \times N}(T);\ \ M \times N \in \mathcal{F}in_n(X) \times \mathcal{F}in_n(Y) \right\}.$$

Thus, if both X and Y are infinite-dimensional Banach spaces, then $(b_n^{(2)}(T))$ is well defined for each $n \in \mathbb{N}$.

Proposition 9.2 *The sequence* $(b_n^{(2)})$ *of Bernstein's numbers satisfies the following properties*: (S1), (S2′) *and* (S3), *where for* (S2′) *we mean* $b_n^{(2)}(S + T) \leq b_n^{(2)}(S) + \|T\|$ *for all* $S, T \in \mathcal{L}(X, Y; Z)$. *In addition, for all* $T \in \mathcal{L}(X, Y; Z)$ *with* $\operatorname{rank}(T) < n$ *one has* $b_n^{(2)}(T) = 0$.

Proof Let $T \in \mathcal{L}(X, Y; Z)$. Clearly, the sequence $(b_n^{(2)}(T))$ is non-increasing with $b_n^{(2)}(T) \leq \|T\|$ for each $n \in \mathbb{N}$. We claim that $b_1^{(2)}(T) = \|T\|$. In fact, for each $n \in \mathbb{N}$, we can find $(x_n, y_n) \in X \times Y$ such that $\|x_n\|_X = \|y_n\|_Y = 1$ and $\|T(x_n, y_n)\|_Z \to \|T\|$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $V_n = \{\alpha x_n;\ \alpha \in \mathbb{K}\}$ and $W_n = \{\beta y_n;\ \beta \in \mathbb{K}\}$. Clearly, $\dim(V_n) = \dim(W_n) = 1$ and so

$$b_1^{(2)}(T) \geq j_2^{V_n \times W_n}(T) = \|T(x_n, y_n)\|_Z,$$

and this proves the claim. Since for every $M \times N \in \mathcal{F}in(X) \times \mathcal{F}in(Y)$,

$$j_2^{M \times N}(S + T) \leq j_2^{M \times N}(S) + \|T\|,$$

then the property (S2') follows.

In what follows, for simplicity, we write $J_{M \times N}$ instead of the inclusion $J_{M \times N}^{X \times Y}$ whenever the spaces X and Y are clear. To prove the property (S3), we take any operators $R_1 \in \mathcal{L}(X_0; X)$, $R_2 \in \mathcal{L}(Y_0; Y)$, $S \in \mathcal{L}(Z; W)$ and fix $0 < \varepsilon < b_n^{(2)}(ST(R_1, R_2))$. Then, there is a subspace $M_0 \times N_0 \in \mathcal{F}in_n(X_0) \times \mathcal{F}in_n(Y_0)$, for which

$$b_n^{(2)}(ST(R_1, R_2)) - \varepsilon \leq j_2\big(ST(R_1, R_2)J_{M_0 \times N_0}\big).$$

Let $A_1 := R_1|_{M_0}$ and $A_2 := R_2|_{N_0}$, and let $M := R_1(M_0)$ and $N := R_2(N_0)$. Then

$$ST(R_1, R_2)J_{M_0 \times N_0} = STJ_{M \times N}(A_1, A_2),$$

and $\|A_1\| \leq \|R_1\|$, $\|A_2\| \leq \|R_2\|$. By Lemma 9.1 (iii), it follows that

$$\begin{aligned} 0 < b_n^{(2)}(ST(R_1, R_2)) - \varepsilon &\leq j_2(STJ_{M \times N}(A_1, A_2)) \\ &\leq \|STJ_{M \times N}\| \, j_1(A_1) j_1(A_2), \end{aligned}$$

which implies that $j_1(A_1) > 0$ and $j_1(A_2) > 0$. In consequence A_1 and A_2 are injective operators, and so $\dim(M) \geq \dim(M_0)$ and $\dim(N) \geq \dim(N_0)$. Thus, $\dim(M \times N) \geq n$. Since A_1 and A_2 are surjective, Lemma 9.1 (ii) and (iii), gives

$$\begin{aligned} b_n^{(2)}(ST(R_1, R_2)) - \varepsilon &\leq j_2\big(STJ_{M \times N}(A_1, A_2)\big) \\ &\leq \|S\| j_2\big(TJ_{M \times N}\big)\|A_1\| \, \|A_2\| \\ &\leq \|S\| b_n^{(2)}(T)\|R_1\| \, \|R_2\|. \end{aligned}$$

which completes the proof that the property (S3) holds.

To show the last property, fix $T \in \mathcal{L}(X, Y; Z)$ with rank $(T) < n$. Let $M \times N \in \mathcal{F}in_n(X) \times \mathcal{F}in_n(Y)$. Then, rank $(TJ_{M\times N}) \leq$ rank $(T) < n$.

Now observe that, for a given $v \in N \setminus \{0\}$, there exists $u \in M \setminus \{0\}$ such that $T(u, v) = 0$. Otherwise, we would have rank $T \geq$ rank $T_v \geq n$, where the mapping $T_v \colon M \to Z$ is defined by $T_v(x) = T(x, v)$ for all $x \in M$. But this is a contradiction. In consequence, we conclude that

$$j_2(TJ_{M\times N}) = \inf\{\|T(x, y)\|;\ \|x\|_M = 1\ \|y\|_N = 1\} = 0.$$

Since M and N are arbitrary, the required statement follows. □

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