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ORIGINAL PAPER

Dual and canonical dual K-Bessel sequences in quaternionic Hilbert spaces

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Abstract
In this paper, we derive a precise description to the concept of dual K-Bessel sequences of a given K-frame in quaternionic Hilbert spaces. After that, we introduce the notion of canonical dual K-Bessel sequence. We study its existence and uniqueness and we investigate some properties related to this concept.

Keywords Frames · K-frames · Dual K-Bessel sequence · Canonical dual K-Bessel sequence · Quaternionic Hilbert spaces

Mathematics Subject Classification Primary 42C15; Secondary 41A58

1 Introduction

The theory of frames is a useful tool to expand functions with respect to a system of functions which is, in general, non-orthogonal and overcomplete. Frames were first introduced in 1952 by Duffin and Schaeffer [13] in connection with nonharmonic Fourier series. However, among many others, the pioneering works of Daubechies et al. [11] in 1986 brought appropriate attention to frames. The study of frames has exploded in recent years [4,5,8,10,17,19,22,25, 26], and it was shown that this concept is important in many applications in digital signal processing and other areas of physical and engineering problems.
Frames serve as a replacement for orthonormal and Riesz bases in Hilbert spaces which are extremely studied in literature [6–8,15,16,22], that guarantee canonical reconstruction of every element of the Hilbert space by the reconstruction formula, however, giving up linear independence of the elements of the generating frame sequence. This redundancy of frames is the key to their success in applications since redundancy gives greater design flexibility which allows frames to be constructed to fit a particular problem in a manner not possible by a set of linearly independent vectors.
However, there exist some problems arising in sampling theory that can not be solved by using frames. They need some systems of functions generating proper subspaces even though they do not belong to them. These families, called local atoms, are introduced by Feichtinger

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and Werther in [18] and are extended by Găvruţa [19] in 2012 who used the notion of atomic system which not only generalizes those of frames and of atomic systems for a subspace but also turns out to be equivalent to that of K-frames.

It is interesting to note that K-frames are more general than ordinary frames in the sense that the lower frame bound only holds for the elements in the range of K, where K is a bounded linear operator in a separable Hilbert space. This generalization of frames allows to reconstruct elements from the range of a linear and bounded operator in a Hilbert space. In general, range is not a closed subspace.

In the literature there are many further studies or variations of [19] as [9,14,21,23,25]. Mainly in [9], Charfi and Ellouz extends the results developed in [19] to quaternionic Hilbert spaces. They allow us to write every element from the range of a linear and bounded operator in a quaternionic Hilbert space as a superposition of elements which do not necessarily belong to its range.

It should be noted here that quaternionic Hilbert spaces are generalizations of Hilbert spaces by allowing the inner product to take values in the field of quaternions rather than in the field of complex or real numbers. Unlike the fields $\mathbb{R}$ or $\mathbb{C}$ which are associative and commutative, the quaternions form non-commutative associative algebra and this feature highly restricted mathematicians to work out a well-formed theory of functional analysis on quaternionic Hilbert spaces. Further, due to the non commutativity there are two different types of Hilbert spaces on quaternions, the left quaternionic Hilbert space and the right quaternionic Hilbert space depending on positions of quaternions.

In the present paper, we are mainly concerned with the dual and canonical dual K-Bessel sequences of a K-frame in right quaternionic Hilbert spaces. More precisely, we describe first the notion of dual K-Bessel sequences and we develop some characterizations relative to this concept. Next, we investigate an explicit dual K-Bessel sequence the so-called canonical dual K-Bessel sequence of a K-frame as a generalization of the classical dual of a frame in quaternionic Hilbert spaces. Indeed, the frame operator for a K-frame may not be invertible and consequently there is no classical canonical dual for a K-frame. So, we study not only the existence and uniqueness of the canonical dual K-Bessel sequence but also we develop some properties. Further, we provide a sufficient condition for a Bessel sequence to recover an element from $N(K)^\perp$ by vectors from the range of K, where the kernel and the range of K are denoted by $N(K)$ and $R(K)$, respectively. The motivation of this result is given by some specific applications in encoding and decoding problems.

2 Mathematical preliminaries

In order to make the paper self-contained, we recall some facts about quaternions which may not be well known. For more details, we refer the reader to [1,20].

2.1 Quaternions

Let $\mathfrak{Q}$ denotes the skew field of quaternions. A general quaternion can be written as

$$q = q_0 + q_1 i + q_2 j + q_3 k, \quad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where i, j, k are the three quaternionic imaginary units, satisfying

$$i^2 = j^2 = k^2 = -1, \;\; ij = -ji = k, \;\; jk = -kj = i, \quad \text{and} \quad ki = -ik = j.$$

The quaternionic conjugate of q is

$$\overline{q} = q_0 - iq_1 - jq_2 - kq_3,$$

while $|q| = (q\overline{q})^{\frac{1}{2}}$ denotes the usual norm of the quaternion q. If q is the non-zero element, it has inverse $q^{-1} = \dfrac{\overline{q}}{|q|^2}$.

2.2 Right quaternionic Hilbert spaces

In this subsection, we discuss right quaternionic Hilbert spaces. For more information, we refer the reader to [1,20].
Let $V_R(\mathfrak{Q})$ be a vector space under right multiplication by quaternions. For $u, v, w \in V_R(\mathfrak{Q})$ and $p, q \in \mathfrak{Q}$, the inner product

$$\langle\, ,\, \rangle : V_R(\mathfrak{Q}) \times V_R(\mathfrak{Q}) \to \mathfrak{Q}$$

satisfies the following properties:

(i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$,
(ii) $\|u\|^2 = \langle u, u\rangle > 0$ unless $u = 0$,
(iii) $\langle u, vp + wq\rangle = \langle u, v\rangle p + \langle u, w\rangle q$,
(iv) $\langle uq, v\rangle = \overline{q}\langle u, v\rangle$,

where $\overline{q}$ stands for the quaternionic conjugate. It is always assumed that the space $V_R(\mathfrak{Q})$ is complete under the norm given above and separable. Then, together with $\langle ., .\rangle$, this defines a right quaternionic Hilbert space. Quaternionic Hilbert spaces share many of the standard properties of complex Hilbert spaces such as Hilbert basis. Let us recall the following results:

Proposition 2.1 [20] *Let $V_R(\mathfrak{Q})$ be a right quaternionic Hilbert space and N be a subset of $V_R(\mathfrak{Q})$ such that, for $z, z' \in N$, $\langle z, z'\rangle = 0$ if $z \neq z'$ and $\langle z, z'\rangle = 1$. Then, the following assertions are equivalent:*

(i) *For every $u, v \in V_R(\mathfrak{Q})$, the series $\sum_{z\in N}\langle u, z\rangle\langle z, v\rangle$ converges absolutely and it holds:*

$$\langle u, v\rangle = \sum_{z\in N}\langle u, z\rangle\langle z, v\rangle.$$

(ii) $\|u\|^2 = \sum_{z\in N} |\langle z, u\rangle|^2$ *for every* $u \in V_R(\mathfrak{Q})$.
(iii) $N^{\perp} := \{v \in V_R(\mathfrak{Q}) : \ \langle v, z\rangle = 0, \ \forall z \in N\} = \{0\}$.
(iv) *$Span(N)$ is dense in $V_R(\mathfrak{Q})$.* ◇

Remark 2.2 The subset N in Proposition 2.1 is called a Hilbert basis. ◇

Proposition 2.3 [20] *Every right quaternionic Hilbert space admits a Hilbert basis, and two Hilbert bases have the same cardinality. Furthermore, if N is a Hilbert basis of $V_R(\mathfrak{Q})$, then every $u \in V_R(\mathfrak{Q})$ can be uniquely decomposed as follows*

$$u = \sum_{z\in N} z\langle z, u\rangle,$$

where the series $\sum_{z\in N} z\langle z, u\rangle$ converges absolutely in $V_R(\mathfrak{Q})$. ◇

Remark 2.4 It is worth mentioning that the absolute convergence of the series given in Proposition 2.1 relies on the fact that absolute convergence is equivalent to unconditional convergence. For more details, see [20,24]. $\diamond$

2.3 Right quaternionic linear operators

Now, we shall define right $\mathfrak{Q}$-linear operators and recall some basic properties.

Definition 2.5 [20] Let $V_R(\mathfrak{Q})$ and $U_R(\mathfrak{Q})$ be two quaternionic Hilbert spaces. A mapping $T : \mathcal{D}(T) \subseteq V_R(\mathfrak{Q}) \longrightarrow U_R(\mathfrak{Q})$, where $\mathcal{D}(T)$ stands for the domain of T, is said to be a right linear operator if

$$T(up + v) = (Tu)p + Tv, \quad \text{if} \quad u, v \in \mathcal{D}(T) \quad \text{and} \quad p \in \mathfrak{Q}.$$

$\diamond$

We have the following elementary result that permits the introduction of the notion of bounded operator.

Theorem 2.6 [20] *Let $V_R(\mathfrak{Q})$ and $U_R(\mathfrak{Q})$ be two quaternionic Hilbert spaces. A linear operator T is called bounded if there exists $K \geq 0$ such that*

$$\|Tu\| \leq K\|u\|, \quad \forall u \in \mathcal{D}(T).$$

$\diamond$

As in the complex case, if $T : \mathcal{D}(T) \subseteq V_R(\mathfrak{Q}) \longrightarrow U_R(\mathfrak{Q})$ is any right linear operator, we define $\|T\|$ by setting

$$\|T\| := \sup_{u \in \mathcal{D}(T)\setminus\{0\}} \frac{\|Tu\|}{\|u\|} = \inf\{K > 0;\ \|Tu\| \leq K\|u\|, \quad \forall u \in \mathcal{D}(T)\}. \tag{2.1}$$

The set of all bounded right linear operators from $V_R(\mathfrak{Q})$ to $U_R(\mathfrak{Q})$ is denoted by $\mathcal{L}(V_R(\mathfrak{Q}), U_R(\mathfrak{Q}))$, and if $V_R(\mathfrak{Q}) = U_R(\mathfrak{Q})$, then $\mathcal{L}(V_R(\mathfrak{Q}), U_R(\mathfrak{Q}))$ is replaced by $\mathcal{L}(V_R(\mathfrak{Q}))$.

It was shown in [20] that the set of all bounded right linear operators is a complete normed space with the norm defined by (2.1).
We close this part with the following definition of the notion of adjoint operator which is similar to that for complex Hilbert spaces.

Definition 2.7 [20] Let $V_R(\mathfrak{Q})$ and $U_R(\mathfrak{Q})$ be two right quaternionic Hilbert spaces and let $T : \mathcal{D}(T) \subseteq V_R(\mathfrak{Q}) \to U_R(\mathfrak{Q})$ be an operator with dense domain. The adjoint $T^* : \mathcal{D}(T^*) \subseteq U_R(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ of T is the unique operator with the following properties:

$$\mathcal{D}(T^*) := \{u \in U_R(\mathfrak{Q}) \ \ \text{such that } \exists w_u \in V_R(\mathfrak{Q}) \ \text{ with } \langle w_u, v\rangle = \langle u, Tv\rangle \ \forall v \in \mathcal{D}(T)\}$$

and

$$\langle T^*u, v\rangle = \langle u, Tv\rangle, \quad \text{for all } v \in \mathcal{D}(T), u \in \mathcal{D}(T^*). \tag{2.2}$$

$\diamond$

It is worth noting that if $T \in \mathcal{L}(V_R(\mathfrak{Q}), U_R(\mathfrak{Q}))$, then requirement (2.2) alone automatically determines T^* as an element of $\mathcal{L}(U_R(\mathfrak{Q}), V_R(\mathfrak{Q}))$.

3 Main results

In this part, we introduce the concept of dual and canonical dual K-Bessel sequences for a given K-frames in a separable right quaternionic Hilbert space $V_R(\mathfrak{Q})$ and we derive some characterizations relative to these notions, where K is a bounded linear operator on $V_R(\mathfrak{Q})$. Throughout this paper, $I \subseteq \mathbb{N}$ denotes a finite or countable index set.
Let's begin with the definition of frame and Bessel sequence generalized by Sharma and Goel in [24] to separable right quaternionic Hilbert spaces $V_R(\mathfrak{Q})$.

Definition 3.1 [24] A family $\{f_n\}_{n\in I}$ is said to be a frame for $V_R(\mathfrak{Q})$, if there exist two positive constants $0 < A \leq B$ such that

$$A\|f\|^2 \leq \sum_{n\in I} |\langle f_n, f\rangle|^2 \leq B\|f\|^2, \quad \text{for all } \ x \in V_R(\mathfrak{Q}). \tag{3.1}$$

The constants A and B are called lower and upper frame bounds. If only the right inequality of Eq. (3.1) holds, $\{f_n\}_{n\in I}$ is called a Bessel sequence. ◇

For a Bessel sequence $\{f_n\}_{n\in I}$, we define its synthesis operator $T : l^2(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ by

$$Tq = \sum_{n\in I} f_n q_n, \quad q = \{q_n\} \in l^2(\mathfrak{Q}).$$

The adjoint operator of T, $T^* : V_R(\mathfrak{Q}) \to l^2(\mathfrak{Q})$ defined by $T^*f = \{\langle f_n, f\rangle\}_{n\in I}$ for $f \in V_R(\mathfrak{Q})$, is called the analysis operator. By composing T with its adjoint T^* we obtain the frame operator

$$S : V_R(\mathfrak{Q}) \to V_R(\mathfrak{Q}), \quad Sf = TT^*f = \sum_{n\in I} f_n\langle f_n, f\rangle.$$

Now, we recall the concept of K-frames introduced in [9].

Definition 3.2 Suppose that $K \in \mathcal{L}(V_R(\mathfrak{Q}))$. A family $\{f_n\}_{n\in I}$ of $V_R(\mathfrak{Q})$ is said to be a K-frame for $V_R(\mathfrak{Q})$, if there exist $A, B > 0$ such that

$$A\|K^*f\|^2 \leq \sum_{n\in I} |\langle f_n, f\rangle|^2 \leq B\|f\|^2, \quad \forall f \in V_R(\mathfrak{Q}). \tag{3.2}$$

The constants A and B are called lower and upper K-frame bounds. ◇

Proposition 3.3 [9] *Let $\{f_n\}_{n\in I} \subset V_R(\mathfrak{Q})$. Then, the following statements are equivalent:*

(i) *$\{f_n\}_{n\in I}$ is a K-frame for $V_R(\mathfrak{Q})$.*
(ii) *$\{f_n\}_{n\in I}$ is a Bessel sequence and there exists a Bessel sequence $\{g_n\}_{n\in I}$ such that*

$$Kf = \sum_{n\in I} f_n\langle g_n, f\rangle, \quad f \in V_R(\mathfrak{Q}). \tag{3.3}$$

(iii) *$\{f_n\}_{n\in I}$ is a Bessel sequence and $R(K) \subset R(T)$, where T is the synthesis operator of $\{f_n\}_{n\in I}$.* ◇

The following result is a generalization of [3, Lemma 2.2] to right quaternionic Hilbert spaces. It shows that, under a sufficient condition, a Bessel sequence can be a K-frame.

Lemma 3.4 *Let $\{f_n\}_{n\in I}$ and $\{g_n\}_{n\in I}$ be two Bessel sequences satisfying Eq. (3.3). Then, $\{f_n\}_{n\in I}$ and $\{g_n\}_{n\in I}$ are a K-frame and a K^*-frame, respectively.* ◇

Proof Let $f \in V_R(\mathfrak{Q})$. It follows from Eq. (3.3) that

$$\begin{aligned}\|Kf\|^4 &= |\langle Kf, Kf\rangle|^2 \\ &= \left|\left\langle \sum_{n\in I} f_n\langle g_n, f\rangle, Kf\right\rangle\right|^2 \\ &\leq \sum_{n\in I} |\overline{\langle g_n, f\rangle}|^2 \sum_{n\in I} |\langle f_n, Kf\rangle|^2 \\ &\leq B\|Kf\|^2 \sum_{n\in I} |\langle g_n, f\rangle|^2,\end{aligned}$$

where B is the upper bound of $\{f_n\}_{n\in I}$. This implies that $\{g_n\}_{n\in I}$ is a K^*-frame for $V_R(\mathfrak{Q})$. To prove that $\{f_n\}_{n\in I}$ is a K-frame for $V_R(\mathfrak{Q})$, it suffices to see that

$$K^*f = \sum_{n\in I} g_n\langle f_n, f\rangle$$

and repeat the above argument for K^* instead of K. □

Now, we introduce a formal definition of the dual K-Bessel sequence of a K-frame.

Definition 3.5 Assume that $\{f_n\}_{n\in I}$ is a K-frame for $V_R(\mathfrak{Q})$. A Bessel sequence $\{g_n\}_{n\in I}$ for $V_R(\mathfrak{Q})$ is called a dual K-Bessel sequence of $\{f_n\}_{n\in I}$ if

$$Kf = \sum_{n\in I} f_n\langle g_n, f\rangle, \quad f \in V_R(\mathfrak{Q}).$$

◇

Using the operator decompositions, we characterize K-frame in the next result. Further, we present a sufficient condition for a sequence to be a dual K-Bessel sequence of a K-frame.

Theorem 3.6 *Suppose that $\{f_n\}_{n\in I}$ is a Bessel sequence for $V_R(\mathfrak{Q})$. Then $\{f_n\}_{n\in I}$ is a K-frame for $V_R(\mathfrak{Q})$ if and only if there exists a bounded operator $M \in \mathcal{L}(V_R(\mathfrak{Q}), l^2(\mathfrak{Q}))$ such that $K = TM$, where T denotes the synthesis operator of $\{f_n\}_{n\in I}$. Further, if $g_n = M^*e_n$ then $\{g_n\}_{n\in I}$ is a K-dual Bessel sequence of $\{f_n\}_{n\in I}$, where $\{e_n\}_{n\in I}$ denotes the standard Hilbert basis of $l^2(\mathfrak{Q})$.* ◇

To prove our result, we need the following lemma which is a slight modification of [12, Theorem 1]. The proof of this Lemma is similar to the one in complex case.

Lemma 3.7 *Let $L_1 \in \mathcal{L}(V_{1,R}(\mathfrak{Q}), V_R(\mathfrak{Q}))$, $L_2 \in \mathcal{L}(V_{2,R}(\mathfrak{Q}), V_R(\mathfrak{Q}))$ be two bounded operators, where $V_R(\mathfrak{Q})$, $V_{1,R}(\mathfrak{Q})$ and $V_{2,R}(\mathfrak{Q})$ stand for right quaternionic Hilbert spaces. The following statements are equivalent:*

(1) $R(L_1) \subset R(L_2)$*;*
(2) $L_1L_1^* \leq \lambda^2 L_2L_2^*$ *for some* $\lambda \geq 0$*;*
(3) *there exists a bounded linear operator* $M \in \mathcal{L}(V_{1,R}(\mathfrak{Q}), V_{2,R}(\mathfrak{Q}))$ *so that* $L_1 = L_2M$.

Moreover, if (1), (2) and (3) are satisfied, then there exists a unique operator M such that

(i) $\|M\|^2 = \inf\{\mu : L_1L_1^* \leq \mu L_2L_2^*\}$*;*
(ii) $N(L_1) = N(M)$*; and*
(iii) $R(M) \subset \overline{R(L_2^*)}$. ◇

Proof of Theorem 3.6. $\Longrightarrow$. Suppose that $\{f_n\}_{n\in I}$ is a K-frame for $V_R(\Omega)$. Then, Proposition 3.3 entails that $R(K) \subset R(T)$. As $T : l^2(\Omega) \to V_R(\Omega)$ and $K : V_R(\Omega) \to V_R(\Omega)$, Lemma 3.7 asserts the existence of a bounded linear operator $M : V_R(\Omega) \to l^2(\Omega)$ such that $K = TM$.
$\Longleftarrow$. Assume that there exists a bounded linear operator $M \in \mathcal{L}(V_R(\Omega), l^2(\Omega))$ such that $K = TM$. Then, it follows from Lemma 3.7 that $R(K) \subset R(T)$. So, $\{f_n\}_{n\in I}$ is a K-frame by Proposition 3.3.
Further, if $K = TM$ and $g_n = M^*e_n$ then for $g \in V_R(\Omega)$ we have

$$\begin{aligned}\sum_{n\in I} |\langle g_n, g\rangle|^2 &= \sum_{n\in I} |\langle M^*e_n, g\rangle|^2\\ &= \sum_{n\in I} |\langle e_n, Mg\rangle|^2\\ &= \|Mg\|^2\\ &\le \|M\|^2\|g\|^2.\end{aligned}$$

Hence, $\{g_n\}_{n\in I}$ is a Bessel sequence for $V_R(\Omega)$. Now, it remains to show that $\{g_n\}_{n\in I}$ is a K-dual of $\{f_n\}_{n\in I}$. To this interest, let $g \in V_R(\Omega)$. Thus, we have

$$\begin{aligned}Kf &= TMg\\ &= T\left(\sum_{n\in I} e_n\langle e_n, Mg\rangle\right)\\ &= \sum_{n\in I} Te_n\langle M^*e_n, g\rangle. \end{aligned} \tag{3.4}$$

Since T is the synthesis operator of $\{f_n\}_{n\in I}$, we obtain

$$T(\{q_n\}) = \sum_{n\in I} f_n q_n, \quad \{q_n\}_{n\in I} \in l^2(\Omega).$$

Therefore, we get

$$T(e_n) = f_n, \quad \forall n \in I. \tag{3.5}$$

Combining Eqs. (3.4) and (3.5), we obtain

$$Kf = \sum_{n\in I} f_n\langle g_n, g\rangle.$$

Consequently, we claim that $\{g_n\}_{n\in I}$ is a K-dual Bessel sequence of $\{f_n\}_{n\in I}$. □

In Theorem 3.6, we present the sufficient condition ensuring the construction of a dual K-Bessel sequence from a K-frame. Now, we show its necessity.

Theorem 3.8 *Suppose that $\{f_n\}_{n\in I}$ is a K-frame and $\{g_n\}_{n\in I} \subset V_R(\Omega)$. Then $\{g_n\}_{n\in I}$ is a K-dual Bessel sequence of $\{f_n\}_{n\in I}$ if and only if there exists $M \in \mathcal{L}(V_R(\Omega), l^2(\Omega))$ such that $K = TM$ and $g_n = M^*e_n$ for any $n \in I$, where T is the synthesis operator of $\{f_n\}_{n\in I}$ and $\{e_n\}_{n\in I}$ is the standard Hilbert basis of $l^2(\Omega)$.* ◇

Proof The sufficient condition has been proved in Theorem 3.6. Now, we show that the necessary condition holds. Suppose that $\{g_n\}_{n\in I}$ is a K-dual Bessel sequence of $\{f_n\}_{n\in I}$.

Then, Proposition 3.3 implies that

$$Kg = \sum_{n\in I} f_n \langle g_n, g\rangle, \quad \forall g \in V_R(\mathfrak{Q}).$$

Let M be the analysis operator of $\{g_n\}_{n\in I}$, hence

$$Mg = \sum_{n\in I} e_n \langle g_n, g\rangle, \quad \forall g \in V_R(\mathfrak{Q})$$

and $M \in \mathcal{L}(V_R(\mathfrak{Q}), l^2(\mathfrak{Q}))$. Further, since $M^*g = \sum_{n\in I} g_n\langle e_n, g\rangle$, we have $M^*e_n = g_n$ for any $n \in I$. On the other hand, as T is the synthesis operator of $\{f_n\}_{n\in I}$, we get $Te_n = f_n$ for all $n \in I$. So

$$\begin{aligned} Kg &= \sum_{n\in I} f_n \langle g_n, g\rangle \\ &= \sum_{n\in I} Te_n \langle M^*e_n, g\rangle \\ &= T\left(\sum_{n\in I} e_n \langle e_n, Mg\rangle\right) \\ &= TMg, \quad g \in V_R(\mathfrak{Q}). \end{aligned}$$

Hence, $K = TM$. □

In the next theorem, we prove that for any K-frame, there is a unique dual K-Bessel sequence whose analysis operator has the minimal norm of the set of the norms of analysis operators of all dual K-Bessel sequences of the K-frame.

Theorem 3.9 *Suppose that $F = \{f_n\}_{n\in I}$ is a K-frame with A as its optimal lower K-frame bound. If $G = \{g_n\}_{n\in I}$ is a K-dual Bessel sequence of $\{f_n\}_{n\in I}$, then $\|T_G^*\|^2 \geq A$, where T_G denotes the synthesis operator of G. Moreover, there exists a unique K-Bessel sequence $H = \{h_n\}_{n\in I}$ of $\{f_n\}_{n\in I}$ such that $\|T_H^*\|^2 = A$, where T_H denotes the synthesis operator of H.* ◇

Remark 3.10 (i) It should be mention here that the canonical K-dual Bessel sequence is the K-dual Bessel sequence whose analysis operator has minimal operator norm in all the K-dual Bessel sequences. More precisely, the norm of its analysis operator is equal to the optimal lower K-frame bound.

(ii) Any K-frame can admits an infinite numbers of dual K-Bessel sequences. However, it has only a unique canonical K-dual Bessel sequence. ◇

Proof of Theorem 3.9. Let $C > 0$ be the lower K-frame bound of $F = \{f_n\}_{n\in I}$. Then, for $f \in V_R(\mathfrak{Q})$ we have

$$\|T_F^* f\|^2 = \sum_{n\in I} |\langle f_n, f\rangle|^2 \geq C\|K^* f\|^2$$

and so

$$\|K^* f\|^2 \leq C^{-1}\|T_F^* f\|^2.$$

As A is the optimal lower K-frame bound of F, i.e.,

$$A = \max\ \{\lambda > 0 : \lambda\|K^* f\|^2 \leq \|T_F^* f\|^2, \forall f \in V_R(\mathfrak{Q})\},$$

we get

$$A = \inf \{\mu > 0 : \|K^* f\|^2 \leq \mu \|T_F^* f\|^2, \forall f \in V_R(\mathfrak{Q})\}.$$

Since $\{g_n\}_{n\in I}$ is a K-dual Bessel sequence of $\{f_n\}_{n\in I}$, we obtain

$$Kf = \sum_{n\in I} f_n \langle g_n, f\rangle = T_F T_G^* f, \quad \forall f \in V_R(\mathfrak{Q}).$$

So, $K = T_F T_G^*$. Thus,

$$KK^* = T_F T_G^* T_G T_F^* \leq \|T_G^*\|^2 T_F T_F^*.$$

Then, for $f \in V_R(\mathfrak{Q})$ we get

$$\begin{aligned}\|K^* f\|^2 &= \langle K^* f, K^* f\rangle \\ &= \langle KK^* f, f\rangle \\ &\leq \|T_G^*\|^2 \langle T_F T_F^* f, f\rangle \\ &= \|T_G^*\|^2 \|T_F^*\|^2.\end{aligned}$$

Hence, $\|T_G^*\|^2 \geq A$. As $\{f_n\}_{n\in I}$ is a K-frame, then $R(K) \subset R(T_F)$. By Lemma 3.7, there exists a unique operator $M \in \mathcal{L}(V_R(\mathfrak{Q}), l^2(\mathfrak{Q}))$ such that $K = T_F M$ and

$$\|M\|^2 = \inf\{\mu : \|K^* f\|^2 \leq \mu \|T_F^* f\|^2, \forall f \in V_R(\mathfrak{Q})\} = A.$$

Setting $h_n = M^* e_n$. Clearly, $H = \{h_n\}_{n\in I}$ is a Bessel sequence. Now, it remains to show that H is a K-dual Bessel sequence of F. So, let $f \in V_R(\mathfrak{Q})$. We have

$$\begin{aligned}Kf &= T_F M f \\ &= T_F \sum_{n\in I} e_n \langle e_n, Mf\rangle \\ &= \sum_{n\in I} T_F e_n \langle e_n, Mf\rangle \\ &= \sum_{n\in I} f_n \langle h_n, f\rangle,\end{aligned}$$

therefore H is a K-dual Bessel sequence of F. On the other hand, we have

$$\begin{aligned}T_H^* f &= \sum_{n\in I} e_n \langle h_n, f\rangle \\ &= \sum_{n\in I} e_n \langle M^* e_n, f\rangle \\ &= \sum_{n\in I} e_n \langle e_n, Mf\rangle \\ &= Mf, \quad f \in V_R(\mathfrak{Q}).\end{aligned}$$

Then, $T_H^* = M$. Hence, $\|T_H^*\|^2 = \|M\|^2 = A$. □

We close this part with the following theorem which allows, under sufficient condition, to a Bessel sequence to recover a vector in $N(K)^\perp$ by elements from the range of K.

Theorem 3.11 *Suppose that K has closed range and $\{f_n\}_{n\in I}$ is a Bessel sequence of $V_R(\mathfrak{Q})$. Then there exists a Bessel sequence $\{g_n\}_{n\in I}$ for $R(K)$ such that*

$$f = \sum_{n\in I} f_n \langle g_n, Kf\rangle, \quad \forall f \in N(K)^{\perp}$$

if and only if $\{f_n\}_{n\in I}$ is a $K^\dagger$-frame for $R(K)$, where $K^\dagger$ denotes the pseudo-inverse of K. ◇

The following Lemma is a key tool for the proof of our result. We omit its proof since it follows the lines of the complex case given in [10, Lemma 2.5.1, Lemma 2.5.2].

Lemma 3.12 *Let $V_R(\mathfrak{Q})$ and $V_{1,R}(\mathfrak{Q})$ be two quaternionic Hilbert spaces and suppose that $U : V_R(\mathfrak{Q}) \to V_{1,R}(\mathfrak{Q})$ is a bounded operator with closed range $R(U)$. Then, there exists a bounded operator $U^\dagger : V_{1,R}(\mathfrak{Q}) \to V_R(\mathfrak{Q})$ for which*

$$N(U^\dagger) = R(U)^\perp, \quad R(U^\dagger) = N(U)^\perp \ \text{ and } \ UU^\dagger x = x, \quad \forall x \in R(U).$$

Further, we have: (i) *The orthogonal projection of $V_{1,R}(\mathfrak{Q})$ onto $R(U)$ is given by $UU^\dagger$.* (ii) *The orthogonal projection of $V_R(\mathfrak{Q})$ onto $R(U^\dagger)$ is given by $U^\dagger U$.* ◇

Proof of Theorem 3.11. $\Longrightarrow$. Suppose that there exists a Bessel sequence $\{g_n\}_{n\in I}$ for $R(K)$ such that

$$f = \sum_{n\in I} f_n \langle g_n, Kf\rangle, \quad \forall f \in N(K)^{\perp}.$$

Then, for $f \in V_R(\mathfrak{Q})$ we have

$$Pf = \sum_{n\in I} f_n \langle g_n, KPf\rangle,$$

where P denotes the orthogonal projection from $V_R(\mathfrak{Q})$ onto $N(K)^\perp$. Consequently, we get

$$\begin{aligned} Pf &= \sum_{n\in I} f_n \langle g_n, KPf\rangle + \sum_{n\in I} f_n \langle g_n, K(I-P)f\rangle \\ &= \sum_{n\in I} f_n \langle g_n, K(P+(I-P))f\rangle \\ &= \sum_{n\in I} f_n \langle g_n, Kf\rangle \\ &= T_F T_G^* K f. \end{aligned}$$

Hence, $P = T_F T_G^* K$. Therefore, Lemma 3.12 entails that $T_F T_G^* = K^\dagger$. Thus, for any $f \in R(K)$ we have

$$K^\dagger f = T_F T_G^* f = \sum_{n\in I} f_n \langle g_n, f\rangle.$$

Then, Proposition 3.3 implies that $\{f_n\}_{n\in I}$ is a $K^\dagger$-frame for $R(K)$.
$\Longleftarrow$. Suppose that $\{f_n\}_{n\in I}$ is a $K^\dagger$-frame for $R(K)$. Then, $R(T_F) \supset R(K^\dagger) = N(K)^\perp$. Hence, Lemma 3.7 yields the existence of a bounded linear operator $M : R(K) \to l^2(\mathfrak{Q})$ such that $K^\dagger = T_F M$. So, $P = K^\dagger K = T_F MK$. Thus, for $f \in N(K)^\perp$ we have

$$\begin{aligned} f &= Pf \\ &= T_F MKf \end{aligned}$$

$$= T_F \left(\sum_{n\in I} e_n \langle e_n, MKf \rangle \right)$$
$$= \sum_{n\in I} T_F e_n \langle M^* e_n, Kf \rangle$$
$$= \sum_{n\in I} f_n \langle g_n, Kf \rangle.$$

Since $g_n = M^* e_n$ for all $n \in I$ and M is a bounded operator from $R(K)$ to $l^2(\mathfrak{Q})$, hence $\{g_n\}_{n\in I}$ is a Bessel sequence for $R(K)$. □

Remark 3.13 The outcomes developed in this note can be considered as a generalization of the results given in [21] to right quaternionic Hilbert spaces $V_R(\mathfrak{Q})$. ◇

References

1. Adler, S.L.: Quaternionic Quantum Mechanics and Quantum Fields. Oxford University Press, New York (1995)
2. Antoine, J.-P., Balazs, P., Stoeva, D.T.: Classification of general sequences by frame related operators. Sampling Theory Signal Image Proc. **10**, 151–170 (2011)
3. Arabyani Neyshaburi, F., Arefijamaal, A.A.: Some constructions of K-frames and their duals. Rocky Mt. J. Math. **47**, 1749–1764 (2017)
4. Cahill, J., Casazza, P.G., Li, S.: Non-orthogonal fusion frames and the sparsity of fusion frame operators. J. Fourier Anal. Appl. **18**, 287–308 (2012)
5. Casazza, P.G., Kutyniok, V.: Frames of Subspaces. Wavelets, Frames and Operator Theory, Contemp. Math., vol. 345. Amer. Math. Soc., Providence, pp. 87–113 (2004)
6. Charfi, S., Jeribi, A., Walha, I.: Riesz basis property of families of nonharmonic exponentials and application to a problem of a radiation of a vibrating structure in a light fluid. Numer. Funct. Anal. Optim. **32**(4), 370–382 (2011)
7. Charfi, S., Ellouz, H.: Riesz basis of eigenvectors for analytic families of operators and application to a non-symmetrical Gribov operator. Mediterr. J. Math., 15, no. 6, Paper No. 223, 16 pp (2018). https://doi.org/10.1007/s00009-018-1265-y
8. Charfi, S., Ellouz, H.: Frame of exponentials related to analytic families operators and application to a non-self adjoint problem of radiation of a vibrating structure in a light fluid. Complex Anal. Oper. Theory, 13, no. 3, 839–858 (2019). https://doi.org/10.1007/s11785-018-0807-4
9. Charfi, S., Ellouz, H.: Frames for operators in quaternionic Hilbert spaces **(submitted)**
10. Christensen, O.: An Introduction to Frames and Riesz Bases. Applied and Numerical Harmonic Analysis, 2nd edn. Birkhäuser/Springer, Basel (2016)
11. Daubechies, I., Grossmann, A., Meyer, Y.: Painless nonorthogonal expansions. J. Math. Phys. **24**, 1271–1283 (1986)
12. Douglas, R.G.: On majorization, factorization and range inclusion of operators on Hilbert space. Proc. Am. Math. Soc. **17**, 413–415 (1966)
13. Duffin, R.J., Schaeffer, A.C.: A class of nonharmonic Fourier series. Trans. Am. Math. Soc. **72**, 341–366 (1952)
14. Ellouz, H.: Some properties of K-frames in quaternionic Hilbert spaces Complex Anal. Oper. Theory, 14, no. 1, Paper No. 8, 19 pp (2020)
15. Ellouz, H., Feki , I., Jeribi, A.: On a Riesz basis of exponentials related to the eigenvalues of an analytic operator and application to a non-selfadjoint problem deduced from a perturbation method for sound radiation. J. Math. Phys. **54**, 112101 (2013)
16. Ellouz, H., Feki, I., Jeribi, A.: On a Riesz basis of exponentials related to a family of analytic operators and application. J. Pseudo-Differ. Oper. Appl. **10**, 999–1014 (2019)
17. Ellouz, H., Feki, I., Jeribi, A.: Non-orthogonal fusion frames of an analytic operator and application to a one-dimensional wave control system. Mediterr. J. Math. 16 (2), Paper No. 52, 23 pp (2019)
18. Feichtinger, H.G., Werther, T.: Atomic systems for subspaces. In: Zayed, L. (ed.) Proceedings SampTA 2001, pp. 163–165, Orlando (2001)
19. Găvruţa, L.: Frames for operators. Appl. Comput. Harmon. Anal. **32**, 139–144 (2012)

20. Ghiloni, R., Moretti, V., Perotti, A.: Continuous slice functional calculus in quaternionic Hilbert spaces. Rev. Math. Phys. **25**, 1350006 (2013)
21. Guo, X.: Canonical dual K-Bessel sequences and dual K-Bessel generators for unitary systems of Hilbert spaces. J. Math. Anal. Appl. **444**, 598–609 (2016)
22. Jeribi, A.: Denseness, Bases and Frames in Banach Spaces and Applications. De Gruyter, Berlin (2018)
23. Jia, M., Zhu, Y.-C.: Some results about the operator perturbation of a K-frame. Results Math. **73**(4), Art. 138 (2018)
24. Sharma, S.K., Goel, S.: Frames in quaternionic Hilbert spaces. J. Math. Phys. Anal. Geom. **15**, 395–411 (2019)
25. Xiao, X., Zhu, Y., Găvruţa, L.: Some properties of K-frames in Hilbert spaces. Results Math. **63**, 1243–1255 (2013)
26. Young, R.M.: An Introduction to Nonharmonic Fourier Series. Academic Press, London (1980)